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PREFACE

In this era of cut throat competition among **JEE** aspirants, only few are lucky enough to make it through. During the preparation, a student needs the most relevant study material to cover all the topics with proper planning to crack **JEE**.

We know there is abundance of study materials floating in the market but what We felt missing, is a self-driven and well-planned Study material. We have tried to bring clarity and Quality in the students by introducing this **Aspire for JEE**

The book covers all the important concepts and theories with a rich amount of practice questions as well. This book is designed for the perfect guidance you need.

The study material is consistent with the syllabus offered in NCERT Books as well and has been curated by industry experts under the careful observation of **Physics Wallah Team**.

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1 Basic Mathematics and Logarithm

NUMBER SYSTEM

(*i*) Natural numbers: The counting numbers 1, 2, 3, 4, ... are called Natural Numbers. The set of natural numbers is denoted by N.

Thus $N = \{1, 2, 3, 4, ...\}$.

(*ii*) Whole numbers: Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W.

Thus $W = \{0, 1, 2, ...\}$

(*iii*) Integers: The numbers $...-3, -2, -1, 0, 1, 2, 3$ .... are called integers and the set is denoted by I or Z. Thus I (or Z) $= \{..-3, -2, -1, 0, 1, 2, 3...\}$

Note :

(*a*) Positive integers $I^+ = \{1, 2, 3 ....\} = N$

(*b*) Negative integers $I^- = \{....., -3, -2, -1\}$.

(*c*) Non-negative integers (whole numbers) $= \{0, 1, 2, ...\}$.

(*d*) Non-positive integers $= \{....., -3, -2, -1, 0\}$.

(*iv*) Even integers: Integers which are divisible by 2 are called even integers.

e.g. $0, \pm 2, \pm 4, ...$

(*v*) Odd integers: Integers which are not divisible by 2 are called odd integers.

e.g. $\pm 1, \pm 3, \pm 5, \pm 7.....$

(*vi*) Prime numbers: Natural numbers which are divisible by 1 and itself only are called prime numbers.

e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(*vii*) Composite number: Let 'a' be a natural number, 'a' is said to be composite if, it has atleast three distinct factors.

e.g. 4, 6, 8, 9, 10, 12, 14, 15 .....

Note :

(*i*) 1 is neither a prime number nor a composite number.

(*ii*) Numbers which are not prime are composite numbers (except 1).

(*iii*) '4' is the smallest composite number.

(*iv*) '2' is the only even prime number.

(*viii*) Co-prime numbers: Two natural numbers (not necessarily prime) are called coprime, if their H.C.F (Highest common factor) is one.

e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc.

These numbers are also called as **relatively prime** numbers.

Note:

(*a*) Two prime number(s) are always co-prime but converse need not be true.

(*b*) Consecutive natural numbers are always co-prime numbers.

(*ix*) Twin prime numbers : If the difference between two prime numbers is two, then the numbers are called twin prime numbers.

e.g. $\{3, 5\}, \{5, 7\}, \{11, 13\}, \{17, 19\}, \{29, 31\}$

Note: Number between twin prime numbers is divisible by 6 (except (3, 5)).

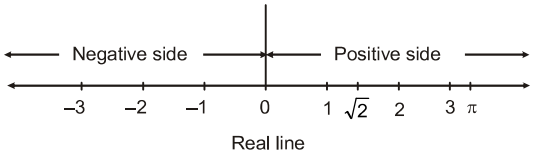
(*x*) Rational numbers: All the numbers that can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q. Thus $Q = \{p/q : p, q \in I \text{ and } q \neq 0\}$. It may be noted that every integer is a rational number since it can be written as p/q. It may be noted that all recurring decimals are rational numbers.

Note : Maximum number of different decimal digits in $\frac{p}{q}$ is equal to q, i.e. $\frac{11}{9}$ will have maximum of 9 different decimal digits.

(*xi*) Irrational numbers : The numbers which can not be expressed in p/q form where $p, q \in I$ and $q \neq 0$ i.e. the numbers which are not rational are called irrational numbers and their set is denoted by Q^c. (i.e. complementary set of Q) e..g. $\sqrt{2}$, $1 + \sqrt{3}$ etc. Irrational numbers can not be expressed as recurring decimals.

Note : $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$ are irrational numbers.

(*xii*) Real numbers: Numbers which can be expressed on number line are called real numbers. The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^c$.



All real numbers follow the order property i.e. if there are two distinct real numbers a and b then either $a < b$ or $a > b$.

Note:

(*a*) Integers are rational numbers, but converse need not be true.

(*b*) Negative of an irrational number is an irrational number.

(*c*) Sum of a rational number and an irrational number is always an irrational number

e.g. $2+\sqrt{3}$

(*d*) The product of a non zero rational number and an irrational number will always be an irrational number.

(*e*) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if $a = 0$.

(*f*) Sum, difference, product and quotient of two irrational numbers need not be a irrational number or we can say, result may be a rational number also.

(*xiii*) Complex number: A number of the form $a + ib$ is called a complex number, where $a, b \in R$ and $i = \sqrt{-1}$. Complex number is usually denoted by Z and the set of complex number is represented by C. Thus $C = \{a + ib : a, b \in R$ and $i = \sqrt{-1}\}$

Note: It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

TRAIN YOUR BRAIN

Example 1: Express the following rational numbers in the form of $\frac{p}{q}$, (where $p, q \in I$)

(*i*) $0.1\overline{2}$ (*ii*) $1.5\overline{23}$

Sol. (*i*) Let $x = 0.1222.....$

$10x = 1.\overline{2}$ ...(i)

$100x = 12.\overline{2}$ ...(ii)

$\Rightarrow \quad 90x = 11 \Rightarrow x = \frac{11}{90}$ (so x is a rational number)

(*ii*) Let $x = 1.5\overline{23}$

$10x = 15.\overline{23}$

$1000x = 1523 \cdot \overline{23}$

$990x = 1508$

$\Rightarrow \quad x = \frac{1508}{990} = \frac{754}{495}$ (so x is a rational number)

DIVISIBILITY TEST

(*i*) A number will be divisible by 2 iff the digit at the unit place is divisible by 2.

(*ii*) A number will be divisible by 3 iff the sum of its digits of the number is divisible by 3.

(*iii*) A number will be divisible by 4 iff last two digits of the number together are divisible by 4.

(*iv*) A number will be divisible by 5 iff digit at the unit place is either 0 or 5.

(*v*) A number will be divisible by 6 iff the digit at the unit place of the number is divisible by 2 & sum of all digits of the number is divisible by 3.

(*vi*) A number will be divisible by 8 iff the last 3 digits, all together, is divisible by 8.

(*vii*) A number will be divisible by 9 iff sum of all it's digits is divisible by 9.

(*viii*) A number will be divisible by 10 iff it's last digit is 0.

(*ix*) A number will be divisible by 11 iff the difference between the sum of the digits at even places and sum of the digits at odd places is a multiple of 11.

e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321, 795432

Example 2: Prove that the difference $10^{25} - 7$ is divisible by 3.

Sol. Write the given difference in the form $10^{25} - 7 = (10^{25} - 1) - 6$. The number $10^{25} - 1 = \overline{\underbrace{99..9}_{25 \text{ digits}}}$ is divisible by 3 (and 9). Since the numbers $(10^{25} - 1)$ and 6 are divisible by 3, the number $10^{25} - 7$, being their difference, is also divisible by 3 without a remainder.

REMAINDER THEOREM

Let $p(x)$ be any polynomial of degree greater than or equal to one and 'a' be any real number. If $p(x)$ is divided by $(x - a)$, then the remainder is equal to $p(a)$.

FACTOR THEOREM

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$. Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

TRAIN YOUR BRAIN

Example 3: Show that $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$.

Sol. Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial. By factor theorem, $(x - a)$ is a factor of a polynomial $p(x)$ iff $p(a) = 0$. Therefore, in order to prove that $x - 3$ is a factor of $p(x)$, it is sufficient to show that $p(3) = 0$. Now,

$p(x) = x^3 - 3x^2 + 4x - 12$

$\Rightarrow \quad p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12$

$= 27 - 27 + 12 - 12 = 0$

Hence, $(x - 3)$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 12$.

TRAIN YOUR BRAIN

Example 4: The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x - 4)$ leave the same remainder. The value of k is

Sol. $P(4) = 64k + 48 - 3 = 64k + 45$

$Q(4) = 128 - 20 + k = k + 108$

given $P(4) = Q(4)$

$\therefore \quad 64k + 45 = k + 108;$

$\Rightarrow \quad 63k = 63 \quad \Rightarrow \quad k = 1$

SOME IMPORTANT IDENTITIES

1. $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$
2. $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$
3. $a^2 - b^2 = (a+b)(a-b)$
4. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
5. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
6. $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (a+b)(a^2 + b^2 - ab)$
7. $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = (a-b)(a^2 + b^2 + ab)$
8. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$
9. $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$
10. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

    $= \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

    If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
11. $a^4 - b^4 = (a+b)(a-b)(a^2 + b^2)$
12. $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

TRAIN YOUR BRAIN

Example 5: Show that the expression, $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx).(z^2 - xy)$ is a perfect square and find its square root.

Sol. $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy) = a^3 + b^3 + c^3 - 3abc$

where $a = x^2 - yz, b = y^2 - zx, c = z^2 - xy$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2)$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[(x^2 - yz - y^2 + zx)^2 + (y^2 - zx - z^2 + xy)^2 + (z^2 - xy - x^2 + yz)^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[\{x^2 - y^2 + z(x-y)\}^2 + \{y^2 - z^2 + x(y-z)\}^2 + \{z^2 - x^2 + y(z-x)\}^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)(x+y+z)^2 [(x-y)^2 + (y-z)^2 + (z-x)^2]$

$= (x+y+z)^2 (x^2 + y^2 + z^2 - xy - yz - zx)^2$

$= (x^3 + y^3 + z^3 - 3xyz)^2$

(which is a perfect square) its square roots are

$\pm (x^3 + y^3 + z^3 - 3xyz)$

INDICES

If 'a' is any non zero real or imaginary number and 'm' is the positive integer, then $a^m = a \cdot a \cdot a. \ldots a$ (m times). Here a is called the base and m is called the index, power or exponent.

Law of indices:

1. $a^0 = 1, \quad (a \neq 0)$
2. $a^{-m} = \frac{1}{a^m}, \quad (a \neq 0)$
3. $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers
4. $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$
5. $(a^m)^n = a^{mn}$
6. $a^{p/q} = \sqrt[q]{a^p}$

TRAIN YOUR BRAIN

Example 6: Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4 \left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

Sol. $a^{9(1/6)(1/3)4} \cdot a^{9(1/3)(1/6)4} = a^2 \cdot a^2 = a^4.$

RATIO

(i) If A and B be two quantities of the same kind, then their ratio is $A : B$; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)

(ii) A ratio may represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots\ldots$ where $m, n, \ldots\ldots$ are non-zero numbers.

(iii) To compare two or more ratio, reduce them to common denominator.

(iv) Ratio between two ratios may be represented as the ratio of two integers

e.g. $\frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ad}{bc}$ or $ad : bc$.

(v) Ratios are compounded by multiplying them together

i.e. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} \ldots\ldots = \frac{ace}{bdf} \ldots\ldots$

(vi) If $a : b$ is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$..... etc.

(vii) If $a : b$ is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

PROPORTION

When two ratios are equal, then the four quantities compositing them are said to be proportional. If $\frac{a}{b}=\frac{c}{d}$, then it is written as $a:b=c:d$ or $a:b::c:d$

(i) 'a' and 'd' are known as extremes and 'b and c' are known as means.

(ii) An important property of proportion Product of extremes = product of means.

(iii) If $a:b=c:d$, then $b:a=d:c$ (Invertando)

(iv) If $a:b=c:d$, then $a:c=b:d$ (Alternando)

(v) If $a:b=c:d$, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)

(vi) If $a:b=c:d$, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo)

(vii) If $a:b=c:d$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

(viii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each $\frac{a+c+e+.....}{b+d+f+.....}$

$$=\frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$$

(ix) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each $=\frac{xa+yc+ze+.....}{xb+yd+zf+.....}$

(x) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each $=\left(\frac{xa^n+yc^n+ze^n}{xb^n+yd^n+zf^n}\right)^{1/n}$

TRAIN YOUR BRAIN

Example 7: If $\frac{x+y}{2}=\frac{y+z}{3}=\frac{z+x}{4}$, then find $x:y:z$.

Sol. Each $=\frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$

$$=\frac{2(x+y+z)}{9}=\frac{x+y+z}{9/2}$$ and therefore each

$$=\frac{(x+y+z)-(y+z)}{\frac{9}{2}-3}=\frac{(x+y+z)-(x+z)}{\frac{9}{2}-4}$$

$$=\frac{(x+y+z)-(x+y)}{\frac{9}{2}-2}$$

$$=\frac{x}{3/2}=\frac{y}{1/2}=\frac{z}{5/2}\Rightarrow x:y:z=3:1:5$$

INTERVALS

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in R$ such that $a < b$, we can define four types of intervals as follows :

Open Interval (a, b)

$\{x : a < x < b\}$ i.e. extreme points are not includes

Closed Interval $[a, b]$

$\{x : a \le x \le b\}$ i.e. extreme points are includes

It can possible when a and b are finite

Semi-Open Interval $(a, b]$

$\{x : a < x \le b\}$ i.e. a is not include and b is include

Semi-Closed Interval $[a, b)$

$\{x : a \le x < b\}$ i.e. a is include and b is not include

Note:

1. The infinite intervals are defined as follows :
   - (i) $(a, \infty) = \{x : x > a\}$
   - (ii) $[a, \infty) = \{x : x \ge a\}$
   - (iii) $(-\infty, b) = \{x : x < b\}$
   - (iv) $(\infty, b] = \{x : x \le b\}$
   - (v) $(-\infty, \infty) = \{x : x \in R\}$
2. $x \in \{1, 2\}$ denotes some particular values of x, i.e. $x = 1, 2$
3. If there is no value of x, then we say $x \in \phi$ (null set)

GENERAL METHOD TO SOLVE INEQUALITIES (Method of intervals (Wavy curve method)

Let $g(x)=\left(\frac{(x-b_1)^{k_1}(x-b_2)^{k_2}---(x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2}---(x-a_n)^{r_n}}\right)$ ... (i)

Where $k_1, k_2 ..... k_n$ and $r_1, r_2 ..... r_n \in N$ and $b_1, b_2 ..... b_n$ and $a_1, a_2 ... a_n$ are real numbers.

Then to solve the inequality following steps are taken.

Steps: Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function.

(ii) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.

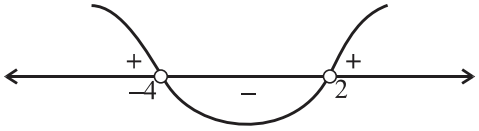
(iii) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

(iv) Thus we consider all the intervals. The solution of the $g(x) > 0$ is the union of the intervals in which we have put the plus sign and the solution of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

TRAIN YOUR BRAIN

Example 8: Solution $\frac{(x-1)^2(x+4)}{(2-x)}<0$ is

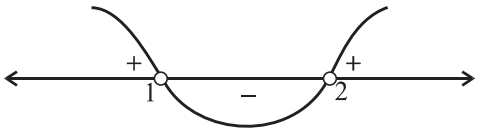
Sol. $\frac{(x-1)^2(x+4)}{(2-x)}<0 \Rightarrow \frac{(x+4)}{(x-2)}>0$



$\Rightarrow (-\infty, -4) \cup (2, \infty)$.

Example 9: The solution of $\frac{\sqrt[3]{(x+4)^4}(x-1)^3}{(x-2)} > 0$ is

Sol. $\frac{(x+4)^{4/3}(x-1)^3}{(x-2)} > 0 \Rightarrow \frac{x-1}{x-2} > 0$



$\Rightarrow (-\infty, 1) \cup (2, \infty)$ Excluding -4.

LOGARITHM FUNCTION

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N.

This number is designated as $\log_a N$.

Hence $\log_a N = x \Leftrightarrow a^x = N, a > 0, a \neq 1$ and $N > 0$

If $a = 10$, then we write log b rather than $\log_{10} b$

$a = e$, we write ln b rather than $\log_e b$

The existence and uniqueness of the number $\log_a N$ follows from the properties of an exponential functions.

Domain :

The existence and uniqueness of the number $\log_a N$ can be determined with the help of set of conditions, $a > 0$ & $a \neq 1$ & $N > 0$.

The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm and any number will be the logarithm of unity.

FUNDAMENTAL IDENTITY

(*i*) $\log_a 1 = 0$ $(a > 0,\ a \neq 1)$

(*ii*) $\log_a a = 1$ $(a > 0,\ a \neq 1)$

(*iii*) $\log_{1/a} a = -1$ $(a > 0,\ a \neq 1)$

Remember

$\log_{10} 2 \approx 0.3010$; $\log_{10} 3 \approx 0.4771$

$\ell n\, 2 \approx 0.693$; $\ell n\, 10 \approx 2.303$

FUNDAMENTAL LOGARITHMIC IDENTITY

$a^{\log_a N} = N, a > 0, a \neq 1$ & $N > 0$

Proof :

$\log_a N = x$ ...(1)

$N = (a)^x$ ...(2)

by equation (1) & (2)

$N = (a)^{\log_e N}$

PRINCIPAL PROPERTIES

Let M & N are arbitrary positive numbers, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and α, β are any real numbers, then:

(*i*) $\log_a (M \cdot N) = \log_a M + \log_a N$;

Proof :

Let $\log_a M = x$ and $\log_a N = y$

$\Rightarrow$ $M = a^x$ and $N = a^y$

Now, $MN = a^x a^y = a^{x+y} \Rightarrow \log_a MN = x + y$

In general

$\log_a (x_1 x_2 \ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_n$

(*ii*) $\log_a (M/N) = \log_a M - \log_a N$

Proof :

Let $\log_a M = x$ and $\log_a N = y$

$\Rightarrow$ $M = a^x$ and $N = a^y$

Now, $M/N = a^x/a^y = a^{x-y} \Rightarrow \log_a M/N = x - y$

(*iii*) $\log_a M^\alpha = \alpha \cdot \log_a M$

BASE CHANGING THEOREM

It states that ratio of logarithm of two numbers is independent of their common base

Symbolically

$$\frac{\log_a M}{\log_a b} = \log_b M \ (a > 0, M > 0, b > 0)$$

Proof :

Let $\log_b M = x$

$\Rightarrow$ $M = b^x$

$\Rightarrow$ $\log_a M = \log_a b^x$

$\Rightarrow$ $\log_a M = x \cdot \log_a b$

$\Rightarrow$ $\frac{\log_a M}{\log_a b} = x = \log_b M$

Important Results

(*i*) **Base power formula:** $\log_{a^k} M = \frac{1}{k} \log_a M$

Proof :

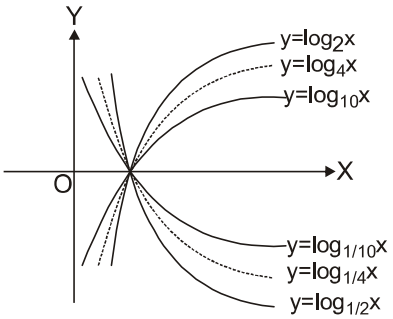
$$\log_{a^k}(m) = \frac{\log_a m}{\log_a a^k} = \frac{\log_a m}{k \log_a a} = \frac{1}{k} \log_a m$$

(*ii*) $a^{\log_b c} = c^{\log_b a}$

Proof : $a^{\log_b c} = a^{\log_a c . \log_b a} = (a^{\log_a c})^{\log_b a} = (c)^{\log_b a}$

GRAPH OF LOGARITHMIC FUNCTIONS

If $a > 0, a \neq 1$, then the function $y = \log_a x, x \in R^+$ (set of positive real numbers) is called the logarithmic Function with base a.



Domain : R^+
Range : R
Nature : one-one

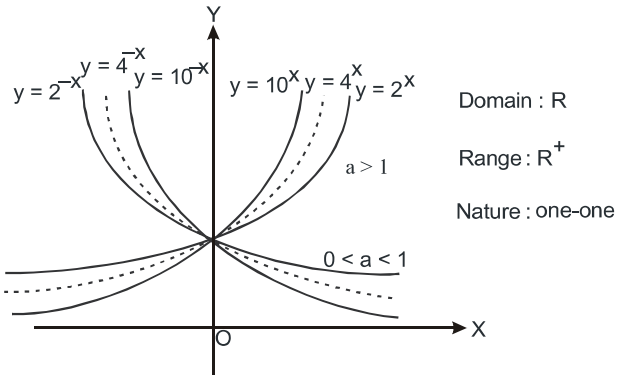
Note:

(*i*) If the number and the base are on the same side of the unity, then the logarithm is positive.

(*ii*) If the number and the base are on the opposite sides of unity, then the logarithm is negative.

EXPONENTIAL FUNCTION

If $a > 0, a \neq 1$ then the function defined by $f(x) = a^x, x \in R$ is called an Exponential Function with base a.



LOGARITHMIC EQUATION

The equality $\log_a x = \log_a y$ is possible if and only if $x = y$

i.e. $\log_a x = \log_a y \Leftrightarrow x = y$

Always check validity of given equation, ($x > 0, y > 0, a > 0, a \neq 1$)

LOGARITHMIC INEQUALITY

Let 'a' is a real number such that

(*i*) If $a > 1$, then $\log_a x > \log_a y \Rightarrow x > y$

(*ii*) If $a > 1$, then $\log_a x < \alpha \Rightarrow 0 < x < a^\alpha$

(*iii*) If $a > 1$, then $\log_a x > \alpha \Rightarrow x > a^\alpha$

(*iv*) If $0 < a < 1$, then $\log_a x > \log_a y \Rightarrow 0 < x < y$

(*v*) If $0 < a < 1$, then $\log_a x < \alpha \Rightarrow x > a^\alpha$

Form - I : $f(x) > 0, g(x) > 0, g(x) \neq 1$

Form	Collection of system
(*a*) $\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow \begin{cases} f(x) \geq 1, & g(x) > 1 \\ 0 < f(x) \leq 1, & 0 < g(x) < 1 \end{cases}$
(*b*) $\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow \begin{cases} f(x) \geq 1 & , \ 0 < g(x) < 1 \\ 0 < f(x) \leq 1 & , \ g(x) > 1 \end{cases}$
(*c*) $\log_{g(x)} f(x) \geq a$	$\Leftrightarrow \begin{cases} f(x) \geq g(x)^a & , \ g(x) > 1 \\ 0 < f(x) \leq g(x)^a & , \ 0 < g(x) < 1 \end{cases}$
(*d*) $\log_{g(x)} f(x) \leq a$	$\Leftrightarrow \begin{cases} 0 < f(x) \leq g(x)^a & , \ g(x) > 1 \\ f(x) \geq g(x)^a & , \ 0 < g(x) < 1 \end{cases}$

From - II : When the inequality of the form

Form	Collection of system
(*a*) $\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x)$	$\Leftrightarrow \begin{cases} f(x) \geq g(x), \ \phi(x) > 1, \\ 0 < f(x) \leq g(x); 0 < \phi(x) < 1 \end{cases}$
(*b*) $\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x)$	$\Leftrightarrow \begin{cases} 0 < f(x) \leq g(x), \phi(x) > 1, \\ f(x) \geq g(x) > 0, \ 0 < \phi(x) < 1 \end{cases}$

COMMON AND NATURAL LOGARITHM

$\log_{10} N$ is referred as a common logarithm and $\log_e N$ is called as natural logarithm of N to the base Napierian and is popularly written as $\ell n\ N$. Note that e is an irrational quantity lying between 2.7 to 2.8 **Note that** $e^{\ell n x} = x$.

CHARACTERISTIC AND MANTISSA

The common logarithm of a number consists of two parts, integral and fractional, of which the integral part may be zero or an integer (+ve or –ve) and the fractional part a decimal, less than one and always positive.

The integral part is called the characteristic and the decimal part is called the mantissa. It should be noted that, if the characteristic of the logarithm of N is p then number of significant digit in $N = p + 1$ if p is the non negative characteristic of log$N$. Number of zeros after decimal before a significant figure start is $p - 1$

ANTILOGARITHM

The positive real number 'n' is called the antilogarithm of a number 'm' if $\log n = m$

Thus, $\log_a n = m \Leftrightarrow n = \text{antilog}_a\, m$

So $\text{antilog}_a\, m = a^m$ ($\because n = a^x$)

e.g. $\text{antilog}_8\left(\frac{2}{3}\right) = 8^{2/3} = 4$

TRAIN YOUR BRAIN

Example 10: Compute the value of $\frac{1}{\log_2 36} + \frac{1}{\log_3 36}$.

Sol. $\frac{1}{\log_2 36} + \frac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = \frac{1}{2}$

Example 11: Find the value of $\log_{100} 0.00001$ by definition.

Sol. Let $y = \log_{100} 0.00001$

$\Rightarrow \quad (100)^y = 0.00001$

$\Rightarrow \quad 10^{2y} = 10^{-5}$

$\Rightarrow \quad 2y = -5$

$\Rightarrow \quad y = -\frac{5}{2}$

Example 12: How many solutions are there for equation

$\log_4(x-1) = \log_2(x-3)$?

Sol. $\log_4(x-1) = \log_2(x-3)$

$\Rightarrow \log_{2^2}(x-1) = \log_2(x-3)$

$\Rightarrow \frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\Rightarrow \log_2(x-1)^{1/2} = \log_2(x-3)$

$\Rightarrow (x-1)^{1/2} = (x-3)$

$\Rightarrow x-1 = x^2-6x+9$

$\Rightarrow (x-2)(x-5) = 0$

$\Rightarrow x = 2, 5$

But $x-1>0$ and $x-3>0$

$x>1$ and $x>3$

So only one solution $x = 5$

Example 13: Solve the logarithmic inequality $\log_{1/5}\frac{4x+6}{x} \geq 0$.

Sol. Since $\log_{1/5} 1 = 0$, the given inequality can be written as.

$$\log_{1/5}\frac{4x+6}{x} \geq \log_{1/5} 1$$

when the domain of the function is taken into account the inequality is equivalent to the system of ineqaulities.

$$\begin{cases} \frac{4x+6}{x} > 0, \\ \frac{4x+6}{x} \leq 1 \end{cases}$$

Solving the ineqaualities by using method of intervals

$$x \in \left[-2, \frac{-3}{2}\right)$$

Example 14: Solve the inequality $\log_{1/3}(5x-1) > 0$.

Sol. By using the basic property of logarithm.

$$\begin{cases} 5x-1<1 \\ 5x-1>0 \end{cases} \Rightarrow \begin{cases} 5x<2 & x<\frac{2}{5} \\ & \Rightarrow \\ 5x>1 & x>\frac{1}{5} \end{cases}$$

$\Rightarrow$ The solution of the inequality is given by $\left(\frac{1}{5}, \frac{2}{5}\right)$

Example 15: If $\log_{x-3}(2x-3)$ is a meaningful quantity then find the interval in which x must lie.

Sol. $x-3>0, x-3\neq 1$ and $2x-3>0 \Rightarrow x>3$, $x\neq 4$ and $x>3/2 \Rightarrow (3,4)\cup(4,\infty)$

Example 16: The value of $\log_{\sqrt{2}}\left(\text{antilog}_{128}\left(\frac{3}{7}\right)\right)$ equals

(*a*) 16 (*b*) 8

(*c*) 6 (*d*) 4

Ans. (*c*)

Sol. $\text{anti}\log_{128}\left(\frac{3}{7}\right) = (128)^{\frac{3}{7}} = (2^7)^{\frac{3}{7}} = 2^3$.

$$\log_{\sqrt{2}}(2^3) = \frac{3}{\frac{1}{2}}\log_2 2 = 6$$

Example 17: For $x \geq 0$, what is the smallest possible value of the expression $\log(x^3-4x^2+x+26) - \log(x+2)$?

Sol. $\log\frac{(x^3-4x^2+x+26)}{(x+2)}$

$= \log\frac{(x^2-6x^2+13)(x+2)}{(x+2)}$

$= \log(x^2-6x+13)$ [$\because x\neq -2$]

$= \log\{(x-3)^2+4\}$

$\therefore$ Minimum value is $\log 4$ when $x=3$

Example 18: Given $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8) = \frac{2}{s^3+1}$. Write $\log_2\frac{a^2b^5}{c^4}$ as a function of 's' ($a, b, c > 0, c\neq 1$).

Sol. Given $\log_2 a = s$ ...(1)

$\log_2 b = 2s^2$ ...(2)

$\log_8 c^2 = \frac{s^3+1}{2}$ ...(3)

$\Rightarrow \frac{2\log c}{3\log 2} = \frac{s^3+1}{2} \Rightarrow 4\log_2 c = 3(s^3+1)$ ...(4)

to find $2\log_2 a + 5\log_2 b - 4\log_2 c$

$\Rightarrow 2s + 10s^2 - 3(s^3+1)$

Example 19: Suppose that a and b are positive real numbers such that $\log_{27} a + \log_9 b = \frac{7}{2}$ and $\log_{27} b + \log_9 a = \frac{2}{3}$. Find the value of the ab.

Sol. $\log_{27} a + \log_9 b = \frac{7}{2} \Rightarrow \frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$; $\log_{27} b + \log_9 a = \frac{2}{3} \Rightarrow \frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$ adding the equation

$$\frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab) = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$$

$$\frac{5}{6}\log_3(ab) = \frac{25}{6} \Rightarrow \log_3(ab) = 5 \Rightarrow ab = 3^5 = 243$$

Example 20: Let $x = (0.15)^{20}$. Find the characteristic and mantissa in the logarithm of x, to the base 10. Assume $\log_{10}2 = 0.301$ and $\log_{10}3 = 0.477$.

Sol. $\log x = \log(0.15)^{20} = 20\log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2]$

$= 20[-1 + \log 3 - \log 2]$

$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824$

$= -16.48 = \overline{17}.52$

Hence characteristic $= -17$

Mantissa $= 0.52$

Example 21: How many digits are contained in the number 2^{75}?

Sol. Computing $\log 2^{75}$, we have $\log 2^{75} = 75.\log 2 \approx 75.(0.3010) = 22.5750$.

Consequently, the characteristic of this common logarithm is equal to 22. Therefore, $2^{75} = a.10^{22}$, where $1 \le a < 10$, a is an integer, and, hence the number 2^{75} has 23 digits.

Example 22: Number of cyphers after decimal before a significant figure starts in $\left(\frac{5}{4}\right)^{-100}$ is equal to [**Use:** $\log_{10}2 = 0.3010$]

Sol. Let $N = \left(\frac{5}{4}\right)^{-100}$

$\Rightarrow \log_{10}N = -100\log_{10}\left(\frac{5}{4}\right) = -100\left(\log_{10}\left(\frac{10}{2}\right) - \log_{10}4\right)$

$= -100(1 - 3\log_{10}2)$

$= -100(1 - 3 \times 0.3010)$

$= -100(1 - 0.9030) = -100 \times 0.0970 = -9.7$

$= -10 + 0.3 = \overline{10}.03$

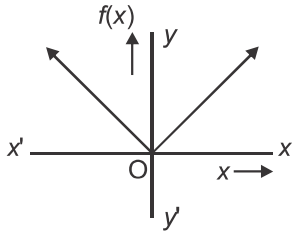
$\therefore$ Number of zeroes $= 9$ **Ans.**

ABSOLUTE VALUE FUNCTION / MODULUS FUNCTION:

This is also known as absolute value function and denoted by

$f(x) = |x|$ i.e. $f(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

Domain of this function is set of all real numbers because $f(x)$ exists for all $x \in R$ but $|x| \ge 0$ so range is all non-negative real numbers.



Domain $= R$; Range $= [0, \infty)$ or $R^+ \cup \{0\}$

Properties of modulus : For any $a, b \in R$

(*i*) $|a| \ge 0$

(*ii*) $|a| = |-a|$

(*a*) $|a|^n = |a^n|$

(*b*) $|a^n| = a^n$, where n is even and $n \in z$

(*iii*) $|a| \ge a, |a| \ge -a$

(*iv*) $|ab| = |a||b|$

(*v*) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Note : $|f(x)| + |g(x)| = |f(x) + g(x)|$

$\Rightarrow$ $f(x) \cdot g(x) \ge 0$

Example 23: Solve the following linear equation

(*i*) $x|x| = 4$

(*ii*) $|x-3| + 2|x+1| = 4$

Sol.

(*i*) $x|x| = 4$

If $x > 0$

$\therefore x^2 = 4 \Rightarrow x = \pm 2 \quad \therefore \quad x = 2 \ (\because x \ge 0)$

If $x < 0 \Rightarrow -x^2 = 4 \Rightarrow x^2 = -4$ which is not possible

(*ii*) $|x-3| + 2|x+1| = 4$

Case-I If $x \le -1$

$\therefore -(x-3) - 2(x+1) = 4$

$\Rightarrow -x + 3 - 2x - 2 = 4 \quad \Rightarrow \quad -3x + 1 = 4$

$\Rightarrow -3x = 3 \quad \Rightarrow \quad x = -1$

Case-II If $-1 < x \le 3$

$\therefore \quad -(x-3) + 2(x+1) = 4$

$\Rightarrow \quad -x + 3 + 2x + 2 = 4$

$\Rightarrow \quad x = -1$ which is not possible

Case-III If $x > 3$

$x - 3 + 2(x+1) = 4$

$3x - 1 = 4 \Rightarrow x = 5/3$ which is not possible

$\therefore \quad x = -1$ **Ans.**

Example 24: The absolute value of sum of real solutions of $\log_2|x^2 + 5x + 4| = \log_2 3 + \log_2|x+1|$ is

(*a*) 8 (*b*) 6 (*c*) 7 (*d*) 5

Ans. (*c*)

Sol. $\log_2 \frac{|(x+1)(x+4)|}{|x+1|} = \log_2 3$

$|x+4| = 3$

$x + 4 = -3, +3$

$x = -7, -1$ (rejected);

$\Rightarrow \quad x = -7$

Example 25: Number of real solutions of $|x-1|=|x-2|+|x-3|$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) more than 2

Ans. (*c*)

Sol. Case-I: $x \le 1$, $1-x=2-x+3-x$

$x=4$ (rejected)

Case-II: $1<x\le 2$, $x-1=2-x+3-x, x=2$

Case-III: $2<x<3$, $x-1=x-2+3-x, x=2$

Case-IV: $x\ge 3$, $x-1=x-2+x-3$

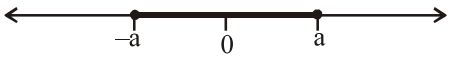
$x=4$

$\Rightarrow$ $x=2,4$

INEQUALITIES INVOLVING ABSOLUTE VALUE

(*i*) $|x|\le a$ (where $a>0$)

It implies those values of x on real number line which are at distance a or less than a from zero.



$\Rightarrow$ $-a\le x\le a$

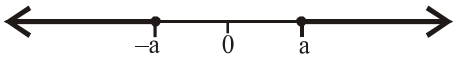
e.g. $|x|\le 2 \Rightarrow -2\le x\le 2$

$|x|<3 \Rightarrow -3<x<3$

In general, $|f(x)|\le a$ (where $a>0$) $\Rightarrow -a\le f(x)\le a$.

(*ii*) $|x|\ge a$ (where $a>0$)

It implies those values of z on real number line which are at distance a or more than a from zero



$\Rightarrow$ $x\le -a$ or $x\ge a$

e.g. $|x|\ge 3 \Rightarrow x\le -3$ or $x\ge 3$

$|x|>2 \Rightarrow x<-2$ or $x>2$

In general, $|f(x)|\ge a$ $\Rightarrow$ $f(x)\le -a$ or $f(x)\ge a$

(*iii*) $a\le|x|\le b$ (where $a, b>0$)

It implies those value of x on real number line whose distance from zero is equal to a or b or lies between a and b

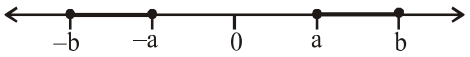


Fig 1.22

$\Rightarrow [-b,-a]\cup[a,b]$

e.g. $2\le|x|\le 4 \Rightarrow x\in[-4,-2]\cup[2,4]$

(*iv*) If $|x+y|=|x|+|y|, xy\ge 0$

If $|x-y|=|x|+|y|, xy\le 0$

If $|x+y|=||x|-|y||, xy\le 0$

If $|x-y|=||x|-|y||, xy\ge 0$

Example 26: Solve $x^2-4|x|+3<0$.

Sol. $x^2-4|x|+3<0$

$\Rightarrow$ $(|x|-1)(|x|-3)<0$

$\Rightarrow$ $1<|x|<3$

$\Rightarrow$ $-3<x<-1$ or $1<x<3$

$\Rightarrow$ $x\in(-3,-1)\cup(1,3)$

Example 27: Solve $1\le|x-2|\le 3$

Sol. $1\le|x-2|\le 3$

$\Rightarrow -3\le x-2\le -1$ or $1\le x-2\le 3$

$\Rightarrow -1\le x\le 1$ or $3\le x\le 5$

$\Rightarrow x\in[-1,1]\cup[3,5]$

Example 28: Solve $||x-1|-2|<5$

Sol. $||x-1|-2|<5$

$\Rightarrow -5<|x-1|-2<5$

$\Rightarrow -3<|x-1|<7$

$\Rightarrow |x-1|<7$

$\Rightarrow -6<x<8$

Example 29: If x satisfies $|x-1|+|x-2|+|x-3|\ge 6$, then

Sol. For $x\le 1$, the given inequation becomes

$1-x+2-x+3-x\ge 6 \Rightarrow -3x\ge 0$

$\Rightarrow$ $x\le 0$ and for $x\ge 3$, the given equation becomes

$x-1+x-2+x-3\ge 6$ $\Rightarrow$ $3x\ge 12$ $\Rightarrow$ $x\ge 4$

For $1<x\le 2$

we get $x-1+2-x+3-x\ge 6$

$\Rightarrow$ $-x+4\ge 6$

i.e. $-x\ge 2$ $\Rightarrow$ $x\le -2$ Not possible

For $2<x<3$,

We get $x-1+x-2+3-x\ge 6$

$\Rightarrow$ $x\ge 6$ not possible

Hence solution set is $(-\infty, 0]\cup[4,\infty)$

i.e. $x\le 0$ or $x\ge 4$

GREATEST INTEGER FUNCTION (STEP-UP FUNCTION)

The function $f(x)=[x]$ is called the greatest integer function and is defined as follows:

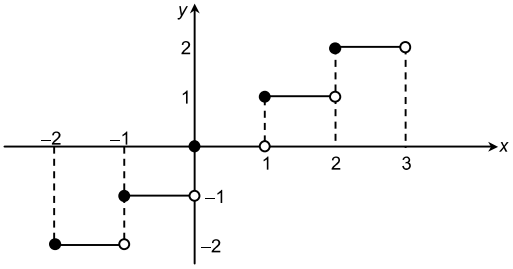
$[x]$ is the greatest integer less than or equal to x.

e.g., $[3]=3, [2.7]=2, [-7.8]=-8, [0.8]=0$

In other words if we list all the integers less then or equal to x, then the integer greatest among them is called greatest integer of x. Greater integer of x is also called integral part of x.

$y=f(x)=[x]$

Domain : **R** ; Range : **I** ;



Remarks

(*i*) $[x] \le x < [x]+1$

(*ii*) $[x+m]=[x]+m; m \in I$

(*iii*) $[x]+[-x]=\begin{cases} 0 & ; \ x \in \text{I} \\ -1 & ; \ x \notin \text{I} \end{cases}$

FRACTIONAL PART FUNCTION

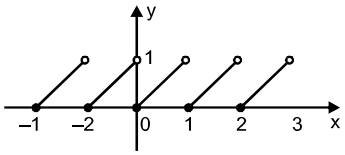
$$y=f(x)=\{x\}=x-[x]$$

Domain : $x \in R$; Range : [0, 1)

E.g. $2.3=2+0.3 \rightarrow$ fractional part

$\downarrow$

Integer part



Remarks

(*i*) Fractional part of any integer is zero.

(*ii*) $\{x+n\}=\{x\}, n \in I$

(*iii*) $\{x\}+\{-x\}=\begin{cases} 0 \ ; & x \in \text{I} \\ 1 \ ; & \text{otherwise} \end{cases}$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

Topicwise Questions

BASIC CONCEPTS AND NUMBER SYSTEM

1. Let $x \in Q$, $y \in Q^c$, Which of the following statement is always WRONG?
(*a*) $xy \in Q^c$
(*b*) $y/x \in Q$, whenever defined
(*c*) $\sqrt{2}x + y \in Q$
(*d*) $x/y \in Q^c$, whenever defined

2. If x and y are two rational numbers such that $(x + y) + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then:
(*a*) x = 1, y = 1
(*b*) x = 2, y = 1
(*c*) x = 5, y = 1
(*d*) x and y can take infinitely many values

3. Which of the following statement is incorrect :
(*a*) rational number + rational number = rational number
(*b*) irrational number + rational number = irrational number
(*c*) integer + rational number = rational number
(*d*) irrational number + irrational number = Irrational number

4. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is :
(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

5. If x – a is a factor of $x^3 - a^2x + x + 2$, then 'a' is equal to
(*a*) 0 (*b*) 2
(*c*) –2 (*d*) 1

6. Every irrational number can be expressed on the number line. This statement is
(*a*) always true
(*b*) never true
(*c*) true subject to some condition
(*d*) None of these

7. The multiplication of a rational number 'x' and an irrational number 'y' is
(*a*) always rational
(*b*) rational except when $y = \pi$
(*c*) always irrational
(*d*) irrational except when x = 0

8. If x, y are integral solutions of $2x^2 - 3xy - 2y^2 = 7$, then value of $|x + y|$ is
(*a*) 2 (*b*) 4
(*c*) 6 (*d*) 2 or 4 or 6

9. If a, b, c are real, then $a(a - b) + b(b - c) + c(c - a) = 0$, only if
(*a*) a + b + c = 0 (*b*) a = b = c
(*c*) a = b or b = c or c = a (*d*) a – b – c = 0

10. If $2x^3 - 5x^2 + x + 2 = (x - 2)(ax^2 - bx - 1)$, then a & b are respectively
(*a*) 2, 1 (*b*) 2, – 1
(*c*) 1, 2 (*d*) –1, 1/2

11. The value of $[e] - [-\pi]$ is, where [.] denotes greatest integer function.
(*a*) 5 (*b*) 6
(*c*) 7 (*d*) 8

12. Number of real solution (x) of the equation $|x - 3|^{3x^2 - 10x + 3} = 1$ is
(*a*) exactly four (*b*) exactly three
(*c*) exactly two (*d*) exactly one

LOGARITHM AND ITS PRINCIPLE PROPERTIES

13. $\frac{1}{\log_{\sqrt{bc}} abc} + \frac{1}{\log_{\sqrt{ca}} abc} + \frac{1}{\log_{\sqrt{ab}} abc}$ has the value equal to
(*a*) 1/2 (*b*) 1
(*c*) 2 (*d*) 4

14. If $\log_x \log_{18}(\sqrt{2} + \sqrt{8}) = \frac{1}{3}$. Then the value of 1000 x is equal to
(*a*) 8 (*b*) 1/8
(*c*) 1/125 (*d*) 125

15. Number of real solutions of the equation $\sqrt{\log_{10}(-x)} = \log_{10}\sqrt{x^2}$ is :
(*a*) none (*b*) exactly 1
(*c*) exactly 2 (*d*) 4

16. Greatest integer less than or equal to the number $\log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 1/6$ is
(*a*) 4 (*b*) 3
(*c*) 2 (*d*) 1

17. The ratio $\frac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$ simplifies to
(*a*) $a^2 - a - 1$ (*b*) $a^2 + a - 1$
(*c*) $a^2 - a + 1$ (*d*) $a^2 + a + 1$

18. If $3^{2\log_3 x} - 2x - 3 = 0$, then the number of values of 'x' satisfying the equation is
(*a*) zero (*b*) 1
(*c*) 2 (*d*) more than 2

19. The number $\log_2 7$ is
(*a*) an integer (*b*) a rational number
(*c*) an irrational number (*d*) a prime number

20. Anti logarithm of 0.75 to the base 16 has the value equal to
(*a*) 4 (*b*) 6
(*c*) 8 (*d*) 12

INEQUALITIES

21. If the solution set of the inequality $\log_{\sqrt{0.9}} \log_5 (\sqrt{x^2+5}+x) > 0$ contains 'n' integral values, then n equals to
(*a*) 7 (*b*) 8
(*c*) 6 (*d*) 10

22. If $\log_{0.5} \log_5 (x^2-4) > \log_{0.5} 1$, then 'x' lies in the interval
(*a*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3)$
(*b*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3\sqrt{5})$
(*c*) $(\sqrt{5}, 3\sqrt{5})$
(*d*) ϕ

23. The set of all the solutions of the inequality $\log_{1-x} (x-2) \geq -1$ is
(*a*) $(-\infty, 0)$ (*b*) $(2, \infty)$
(*c*) $(-\infty, 1)$ (*d*) ϕ

24. Solution set of the inequality $2 - \log_2 (x^2 + 3x) \geq 0$ is :
(*a*) $[-4, 1]$
(*b*) $[-4, -3) \cup (0, 1]$
(*c*) $(-\infty, -3) \cup (1, \infty)$
(*d*) $(-\infty, -4) \cup [1, \infty)$

MODULUS FUNCTION

25. Solutions of $|4x+3| + |3x-4| = 12$ are
(*a*) $x = -\frac{7}{3}, \frac{3}{7}$ (*b*) $x = -\frac{5}{2}, \frac{2}{5}$
(*c*) $x = -\frac{11}{7}, \frac{13}{7}$ (*d*) $x = -\frac{3}{7}, \frac{7}{5}$

26. If $|x^2-2x-8| + |x^2+x-2| = 3|x+2|$, then the set of all real values of x is
(*a*) $[1,4] \cup \{-2\}$ (*b*) $[1,4]$
(*c*) $[-2,1] \cup [4,\infty)$ (*d*) $(-\infty,-2] \cup [1,4]$

27. The complete set of real 'x' satisfying $||x-1|-1| \leq 1$ is:
(*a*) $[0, 2]$ (*b*) $[-1, 3]$
(*c*) $[-1, 1]$ (*d*) $[1, 3]$

28. The number of real roots of the equation $|x|^2 - 3|x| + 2 = 0$ is
(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

Learning Plus

1. If A & B are two rational numbers and AB, A + B and A – B are rational numbers, then A/B is
(*a*) always rational
(*b*) never rational
(*c*) rational when $B \neq 0$
(*d*) rational when $A \neq 0$

2. If a, b, c are real and distinct numbers, then the value of
$$\frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{(a-b)(b-c)(c-a)}$$ is
(*a*) 1 (*b*) a b c
(*c*) 2 (*d*) 3

3. The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by (x – 4) leave the same remainder. The value of k is
(*a*) 2 (*b*) 1
(*c*) 0 (*d*) –1

4. The remainder obtained when the polynomial $1 + x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by x – 1 is
(*a*) 3 (*b*) 5
(*c*) 7 (*d*) 11

5. If $x^{x\sqrt[3]{x}} = \left(x.\sqrt[3]{x}\right)^x$, then $x =$
(*a*) 1 (*b*) –1
(*c*) 0 (*d*) 2

6. The equation $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0$ has the solution
(*a*) $x = \pm 1$ (*b*) $x = 10$
(*c*) $x = \pm\sqrt{2}$ (*d*) $x = \sqrt{3}$

7. Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is
(*a*) 3.6 (*b*) 5
(*c*) 5.6 (*d*) 10

8. The value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$ is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

9. $\frac{1}{1+\log_b a+\log_b c}+\frac{1}{1+\log_c a+\log_c b}+\frac{1}{1+\log_a b+\log_a c}$ has the value equal to

(*a*) abc (*b*) $\frac{1}{abc}$

(*c*) 0 (*d*) 1

10. If $\log_e\left(\frac{a+b}{2}\right)=\frac{1}{2}(\log_e a+\log_e b)$, then relation between *a* and *b* will be

(*a*) $a=b$ (*b*) $a=\frac{b}{2}$

(*c*) $2a=b$ (*d*) $a=\frac{b}{3}$

11. $\log_7 \log_7 \sqrt{7(\sqrt{7\sqrt{7}})}=$

(*a*) $3\log_2 7$ (*b*) $1-3\log_3 7$

(*c*) $1-3\log_7 2$ (*d*) $1-10\log_2 7$

12. The value of $81^{(1/\log_5 3)}+27^{\log_9 36}+3^{4/\log_7 9}$ is equal to

(*a*) 49 (*b*) 625

(*c*) 216 (*d*) 890

13. If $\log_{10} x = y$, then $\log_{1000} x^2$ is equal to

(*a*) y^2 (*b*) 2y

(*c*) 3y/2 (*d*) 2y/3

14. If $x=\log_a(bc)$, $y=\log_b(ca)$, $z=\log_c(ab)$, then which of the following is equal to 1

(*a*) $x+y+z$

(*b*) $(1+x)^{-1}+(1+y)^{-1}+(1+z)^{-1}$

(*c*) xyz

(*d*) $x+y-z$

15. If $a=\log_{24}12$, $b=\log_{36}24$ and $c=\log_{48}36$, then $1+abc$ is equal to

(*a*) 2ab (*b*) 2ac

(*c*) 2bc (*d*) 0

16. The solution of the equation $\log_7 \log_5\left(\sqrt{x^2+5}+x\right)=0$.

(*a*) $x=2$ (*b*) $x=3$

(*c*) $x=4$ (*d*) $x=-2$

17. The value of $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+0.001+.....)}$ is

(*a*) 81 (*b*) $\frac{1}{81}$

(*c*) 20 (*d*) $\frac{1}{20}$

18. The value of $\log_2 .\log_3 ....\log_{100} 100^{99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 100!

19. The number of solution of $\log_2(x+5)=6-x$ is

(*a*) 2 (*b*) 0

(*c*) 3 (*d*) 1

20. If $\log_{10}2=0.30103$, $\log_{10}3=0.47712$, the number of digits in $3^{12}\times 2^8$ is

(*a*) 7 (*b*) 8

(*c*) 9 (*d*) 10

21. Exhaustive set of values of x satisfying $\log_{|x|}(x^2+x+1)\geq 0$ is

(*a*) $(-1,0)$ (*b*) $(-\infty,-1)\cup(1,\infty)$

(*c*) $(-\infty,\infty)-\{-1,0,1\}$ (*d*) $(-\infty,-1)\cup(-1,0)\cup(1,\infty)$

22. The set of real values of *x* satisfying

$\log_{1/2}\left(x^2-6x+12\right)\geq -2$ is

(*a*) $(-\infty,2]$ (*b*) $[2,4]$

(*c*) $[4,+\infty)$ (*d*) $[3,8]$

23. If $\log_{0.04}(x-1)\geq\log_{0.2}(x-1)$ then *x* belongs to the interval

(*a*) $(1,2]$ (*b*) $(-\infty,2]$

(*c*) $[2,+\infty)$ (*d*) $[2,2]$

24. If $\log_{0.3}(x-1)<\log_{0.09}(x-1)$, then *x* lies in the interval

(*a*) $(2,\infty)$ (*b*) $(-2,-1)$

(*c*) $(1,2)$ (*d*) $(-2,2)$

25. The minimum value of $f(x)=|x-1|+|x-2|+|x-3|$ is equal to

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 0

26. The set of real value(s) of p for which the equation $|2x+3|+|2x-3|=px+6$ has more than two solutions is :

(*a*) [0, 4) (*b*) (– 4, 4)

(*c*) R – {4, –4, 0} (*d*) {0}

27. Number of solutions of the equation $[2x]-3\{2x\}=1$ (where [.] and { . } denotes greatest integer and fractional part function respectively)

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 0

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. If x & y are real numbers and $\frac{y}{x} = x$, then 'y' cannot take the value(s)

(*a*) –1 (*b*) 0

(*c*) 1 (*d*) 2

2. Indicate all correct alternatives, where base of the log is 2.

The equation $x^{(3/4)(\log_2 x)^2 + \log_2 x - (5/4)} = \sqrt{2}$ has :

(*a*) at least one real solution

(*b*) exactly three real solutions

(*c*) exactly one irrational solution

(*d*) Imaginary roots

3. Values of x satisfying the equation

$\log_{5^2} x + \log_{5x}\left(\frac{5}{x}\right) = 1$ are

(*a*) 1 (*b*) 5

(*c*) $\frac{1}{25}$ (*d*) 3

4. Let $N = \frac{\log_3 135}{\log_{15} 3} - \frac{\log_3 5}{\log_{405} 3}$. Then N is

(*a*) a natural number (*b*) a prime number

(*c*) a rational number (*d*) an integer

5. The solution set of the system of equations $\log_3 x + \log_3 y = 2 + \log_3 2$ and $\log_{27}(x+y) = \frac{2}{3}$ is

(*a*) (6, 3) (*b*) (3, 6)

(*c*) (6, 12) (*d*) (12, 6)

6. The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has

(*a*) one irrational solution

(*b*) no prime solution

(*c*) two real solutions

(*d*) one integral solution

7. The equation $x^{\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]} = 3\sqrt{3}$ has

(*a*) exactly three real solution

(*b*) at least one real solution

(*c*) exactly one irrational solution

(*d*) complex roots

8. Solution set of the inequality

$(\log_2 x)^4 - \left(\log_{1/2}\frac{x^3}{8}\right)^2 + 9\log_2\left(\frac{32}{x^2}\right) < 4\left(\log_{1/2} x\right)^2$ is

(a, b) $\cup$ (c, d) then the correct statement is

(*a*) a = 2b and d = 2c

(*b*) b = 2a and d = 2c

(*c*) $\log_c d = \log_b a$

(*d*) there are 4 integers in (c, d)

9. If $\frac{1}{2} \le \log_{0.1} x \le 2$, then

(*a*) maximum value of x is $\frac{1}{\sqrt{10}}$

(*b*) x lies between $\frac{1}{100}$ and $\frac{1}{\sqrt{10}}$

(*c*) minimum value of x is $\frac{1}{10}$

(*d*) minimum value of x is $\frac{1}{100}$

10. If $|x-5| + |x+5| = 10$, then

(*a*) the number of integral solutions is 10

(*b*) the number of integral solutions is 11

(*c*) the sum of all the integral solutions is 0

(*d*) all the solutions of the equation are rational numbers

Comprehension – 01 (no. 11 to 13)

Adenotes the product xyz where x, y and z satisfy

$\log_3 x = \log 5 - \log 7$

$\log_5 y = \log 7 - \log 3$

$\log_7 z = \log 3 - \log 5$

Bdenotes the sum of square of solution of the equation

$\log_2(\log_2 x^6 - 3) - \log_2(\log_2 x^4 - 5) = \log_2 3$

Cdenotes characterstic of logarithm

$\log_2(\log_2 3) - \log_2(\log_4 3) + \log_2(\log_4 5) - \log_2(\log_6 5) + \log_2(\log_6 7) - \log_2(\log_8 7)$

11. Find value of A + B + C

(*a*) 18 (*b*) 34

(*c*) 32 (*d*) 24

12. Find $\log_2 A + \log_2 B + \log_2 C$

(*a*) 5 (*b*) 6

(*c*) 7 (*d*) 4

13. Find |A – B + C|

(*a*) –30 (*b*) 32

(*c*) 28 (*d*) 30

Comprehension – 02 (no. 14 to 16)

Let α and β are the solutions of the equation $\left(\sqrt{x}\right)^{\log_5 x-1} = 5$ where $\alpha \in I$ and $\beta \in I$ Then

[**Use:** $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$]

14. The number of significant digits before decimal in $(\alpha)^{10}$ is
(*a*) 13 (*b*) 14
(*c*) 15 (*d*) None

15. Number of zeroes after decimal before a significant digit in $(b)^{10}$ is
(*a*) 5 (*b*) 7
(*c*) 8 (*d*) 6

16. The value of $(\beta)^{\log_{25} 9}$ is
(*a*) $\frac{1}{3}$ (*b*) 5
(*c*) $\frac{1}{5}$ (*d*) 9

17. Match the Column:

Column-I	**Column-II**
(*a*) The value(s) of x, which does not satisfy the equation $\log_2^2(x^2-x)-4\log_2(x-1)\log_2 x = 1$, is (are)	(P) 2
(*b*) The value of x satisfying the equation $2^{\log_2 e^{\ln 5^{\log_5 7^{\log_7 10^{\log_{10}(8x-3)}}}}} = 13$, is	(Q) 3
(*c*) The number $N = \left(\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi}\right)$ is less than	(R) 4
(*d*) Let $l = (\log_3 4 + \log_2 9)^2 - (\log_3 4 - \log_2 9)^2$ and $m = (0.8)\left(1+9^{\log_3 8}\right)^{\log_{65} 5}$ then $(l+m)$ is divisible by	(S) 5
	(T) 6

18. Match the Column:

Column-I	**Column-II**
(*a*) Anti logarithm of $(0.\overline{6})$ to the base 27 has the value equal to	(P) 5
(*b*) Characteristic of the logarithm of 2008 to the base 2 is	
(*c*) The value of b satisfying the equation, $\log_e 2 \cdot \log_b 625 = \log_{10}16 \cdot \log_e 10$ is	(Q) 7
(*d*) Number of naughts after decimal before a significant figure comes in the number $\left(\frac{5}{6}\right)^{100}$, is	(R) 9
	(S) 10

NUMERICAL BASED QUESTIONS

19. Find the number of integral solution of the equation $\log_{\sqrt{x}}(x+|x-2|) = \log_x(5x-6+5|x-2|)$.

20. If $\log_{3x}45 = \log_{4x}40\sqrt{3}$ then find the characteristic of x^3 to the base 7.

21. Let $y = \sqrt{\log_2 3 \cdot \log_2 12 \cdot \log_2 48 \cdot \log_2 192 + 16} - \log_2 12 \cdot \log_2 48 + 10$.

Find $y \in N$.

22. If a, b are co-prime numbers and satisfying $\left(2+\sqrt{3}\right)^{\frac{1}{\log_a(2-\sqrt{3})}+\frac{1}{\log_b\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)}} = \frac{1}{12}$,

then (a + b) can be is equal to

23. Let $N = 10^{3\log 2 - 2\log\left(\log 10^3\right) + \log\left(\left(\log 10^6\right)^2\right)}$ where base of the logarithm is 10. The characteristic of the logarithm of N to the base 3, is equal to

24. If p is the smallest value of x satisfying the equation $2^x + \frac{15}{2^x} = 8$ then the value of 4^p is equal to

25. Positive numbers x, y and z satisfy xyz = 10^{81} and $(\log_{10}x)(\log_{10}yz) + (\log_{10}y)(\log_{10}z) = 468$.
Find the value of
$(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2$

26. If 'x' and 'y' are real numbers such that,
$2\log(2y-3x) = \log x + \log y$, find $\frac{y}{x}$.

27. If (x_1, y_1) and (x_2, y_2) are the solution of the system of equation
$\log_{225}(x) + \log_{64}(y) = 4$
$\log_x(225) - \log_y(64) = 1$,
then find the value of $\log_{30}(x_1y_1x_2y_2)$.

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. The sum of the solutions of the equation $\left|\sqrt{x}-2\right| + \sqrt{x}\left(\sqrt{x}-4\right) + 2 = 0, (x > 0)$ is equal to :

[*JEE Main-2019 (April)*]

(*a*) 4 (*b*) 9
(*c*) 10 (*d*) 12

JEE-ADVANCED PREVIOUS YEAR'S

1. The value of
$$6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}}....}\right)$$
is [*IIT JEE-2012*]

2. If $3^x = 4^{x-1}$, then x = [*JEE Advanced-2013*]

(*a*) $\frac{2\log_3 2}{2\log_3 2 - 1}$

(*b*) $\frac{2}{2-\log_2 3}$

(*c*) $\frac{1}{1-\log_4 3}$

(*d*) $\frac{2\log_2 3}{2\log_2 3 - 1}$

3. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is _____

[*JEE Advanced-2018*]

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

ANSWER KEY

Topicwise Questions

1. (*b*)	**2.** (*b*)	**3.** (*d*)	**4.** (*a*)	**5.** (*c*)	**6.** (*a*)	**7.** (*d*)	**8.** (*b*)	**9.** (*b*)	**10.** (*a*)
11. (*b*)	**12.** (*b*)	**13.** (*b*)	**14.** (*d*)	**15.** (*c*)	**16.** (*c*)	**17.** (*d*)	**18.** (*b*)	**19.** (*c*)	**20.** (*c*)
21. (*b*)	**22.** (*a*)	**23.** (*d*)	**24.** (*b*)	**25.** (*c*)	**26.** (*a*)	**27.** (*b*)	**28.** (*d*)		

Learning Plus

1. (*c*)	**2.** (*d*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*a*)	**7.** (*a*)	**8.** (*b*)	**9.** (*d*)	**10.** (*a*)
11. (*c*)	**12.** (*d*)	**13.** (*d*)	**14.** (*b*)	**15.** (*c*)	**16.** (*c*)	**17.** (*a*)	**18.** (*b*)	**19.** (*d*)	**20.** (*c*)
21. (*d*)	**22.** (*b*)	**23.** (*c*)	**24.** (*a*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)			

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *b*)	**2.** (*a*, *b*, *c*)	**3.** (*a*, *b*, *c*)	**4.** (*a*, *b*, *c*, *d*)	**5.** (*a*, *b*)
6. (*a*, *b*, *c*, *d*)	**7.** (*a*, *b*, *c*, *d*)	**8.** (*b*, *c*)	**9.** (*a*, *b*, *d*)	**10.** (*b*, *c*)
11. (*b*)	**12.** (*a*)	**13.** (*d*)	**14.** (*b*)	**15.** (*d*)
16. (*a*)	**17.** (*a*)-Q, R, S, T; (*b*)-P; (*c*)-Q, R, S, T; (*d*)-P, R, S			
18. (*a*)-R; (*b*)-S; (*c*)-P; (*d*)-Q				
19. [0001]	**20.** [0002]	**21.** [0006]	**22.** [0007]	**23.** [0003]
24. [0009]	**25.** [5625]	**26.** [2.25]	**27.** [12]	

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*c*)

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

JEE-ADVANCED PREVIOUS YEAR'S

1. (*d*) **2.** (*a*, *b*, *c*) **3.** (8)

2 Trigonometric Ratio and Identities

INTRODUCTION

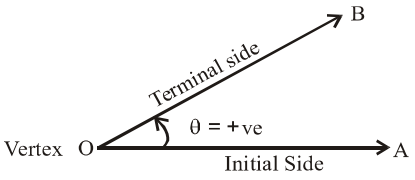
The word trigonometry is derived from three greek words.

Tri	Gon	Metron
↓	↓	↓
three	sides	Measure

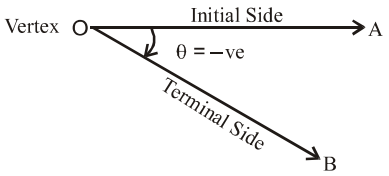
In the ancient time trigonometry defines relations between elements of a triangle. In a triangle there are six basic elements, three sides and three angles. Any three line segments will form a triangle iff they satisfy three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180°. These requirements impose limitations on the manner in which the relations between the elements are defined.

ANGLE

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative.



(i) Positive angle
(anticlockwise measurement)

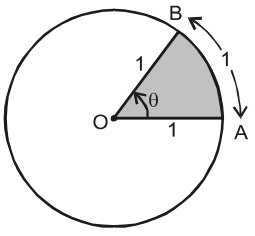


(ii) Negative angle
(clockwise measurement)

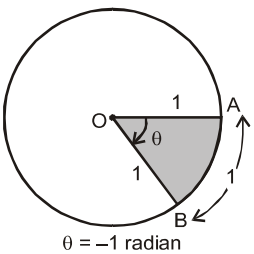
SYSTEMS FOR MEASUREMENT OF ANGLES

An angle can be measured in the following systems.

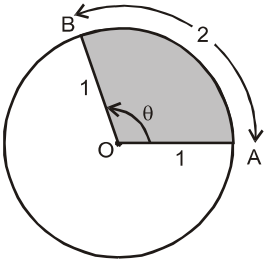
1. **Sexagesimal System (British System):** In this system $\frac{1}{360}$ of a complete circular turn is called a degree (°), $\frac{1}{60}$ of a degree is called a minute (′) and $\frac{1}{60}$ of a minute is called a second (″).
   One right angle = 90°, $1° = 60'$, $1' = 60''$
2. **Centesimal System (French System):** In this system $\frac{1}{400}$ of a complete circular turn is called a grade (^{g}), $\frac{1}{100}$ of a grade is called a minute (′) and $\frac{1}{100}$ of a minute is called a second (″).
   ∴ One right angle = 100^g; $1^g = 100'$; $1' = 100''$
   Note: The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.
3. **Circular System (Radian Measurement):** The angle subtended by an arc of a circle whose length is equal to the radius of the circle at the centre of the circle is called a radian. In this system the unit of measurement is radian (^{c})
   As the circumference of a circle of radius 1 unit is 2π, therefore one complete revolution of the initial side subtends an angle of 2π radian.
   More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length ℓ will subtend an angle whose measure is $\frac{\ell}{r}$ radian. Thus, if in a circle of radius r, arc of length ℓ subtends an angle θ radian at the centre, we have $\theta = \frac{\ell}{r}$ or $\ell = r\theta$.



θ = 1 radian
(i)

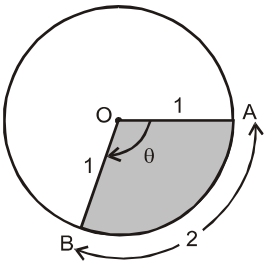


θ = –1 radian
(ii)



θ = 2 radian

(iii)



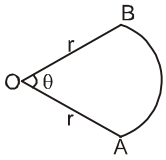
θ = –2 radian

(iv)

Note: If no symbol is mentioned while showing measurement of angle, then it is considered to be measured in radians.

e.g. θ = 15 implies 15 radian

Area of circular sector :



Area = $\frac{1}{2}r^2\theta$ sq. units

Relation between radian, degree and grade:

$\frac{\pi}{2}$ radian = 90° = 100^g

TRAIN YOUR BRAIN

Example 1: Find the radian measure corresponding to –37° 30′.

Sol. 60′ = 1°

$\therefore \quad 30' = \left(\frac{1}{2}\right)^\circ ; \quad \therefore \; -37°\,30' = 37\frac{1}{2}^\circ = -\frac{75^\circ}{2}$

360° = 2π radians

$\therefore \quad \frac{-75^\circ}{2} = -\frac{2\pi}{360}\times\frac{75}{2}$ radians $= \frac{-5\pi}{24}$ radians

Example 2: The minute hand of a clock is 10 cm long. How far does the tip of the hand move in 20 minutes?

Sol. The minute hand moves through 120° in 20 minutes or moves through $\frac{2\pi}{3}$ radians.

Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula $l = r\theta$

$= 10 \times \frac{2\pi}{3} = \frac{20\pi}{3}$ cm.

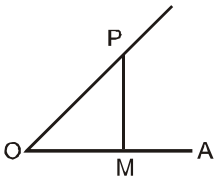
TRIGONOMETRIC RATIOS FOR ACUTE ANGLES

Let a revolving ray OP starts from OA and revolves into the position OP, thus tracing out the angle AOP.

In the revolving ray take any point P and draw PM perpendicular to the initial ray OA.

In the right angle triangle MOP, OP is the hypotenuse, PM is the perpendicular, and OM is the base.

The trigonometrical ratios, or functions, of the angle AOP are defined as follows :



$\frac{MP}{OP}$, i.e., $\frac{\text{Perpendicular}}{\text{Hypotenuse}}$, sin ∠AOP

$\frac{OM}{OP}$, i.e. $\frac{\text{Base}}{\text{Hypotenuse}}$, cos ∠AOP;

$\frac{MP}{OM}$, i.e. $\frac{\text{Perpendicular}}{\text{Base}}$, tan ∠AOP;

$\frac{OM}{MP}$, i.e. $\frac{\text{Base}}{\text{Perpendicular}}$, cot ∠AOP;

$\frac{OP}{OM}$, i.e. $\frac{\text{Hypotenuse}}{\text{Base}}$, sec ∠AOP;

$\frac{OP}{MP}$, i.e. $\frac{\text{Hypotenuse}}{\text{Perpendicular}}$, cosec ∠AOP;

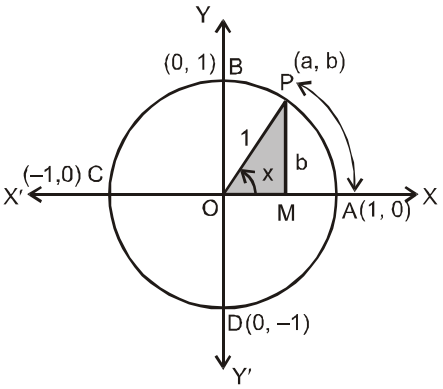
The quantity by which the cosine falls short of unity i.e. 1 – cos ∠AOP, is called the Versed Sine of ∠AOP; also the quantity 1 – sin ∠AOP, by which the sine falls short of unity, is called the Coversed Sine of ∠AOP.

It can be noted that the trigonometrical ratios are all real numbers.

The names of these eight ratios are written, for brevity, sin ∠AOP, cos ∠AOP, tan ∠AOP, cot ∠AOP, cosec ∠AOP, sec ∠AOP, vers ∠AOP, and covers ∠AOP respectively

TRIGONOMETRIC RATIOS FOR ANGLE Q ∈ R

We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions. (also called circular functions)



Consider a unit circle (radius 1 unit) with centre at origin of the coordinate axes. Let P(a, b) be any point on the circle with angle ∠AOP = x radian, i.e., length of arcAP = x

We define cos x = a and sin x = b Since Δ OMP is a right triangle, we have $OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$ Thus, for every point on the unit circle,

we have $a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$\cos 0 = 1$ $\quad$ $\sin 0 = 0$,

$\cos \frac{\pi}{2} = 0$ $\quad$ $\sin \frac{\pi}{2} = 1$

$\cos \pi = -1$ $\quad$ $\sin \pi = 0$

$\cos \frac{3\pi}{2} = 0$ $\quad$ $\sin \frac{3\pi}{2} = -1$

$\cos 2\pi = 1$ $\quad$ $\sin 2\pi = 0$

Now if we take one complete revolution from the position OP, we again come back to same position OP. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$\sin(2n\pi + x) = \sin x$, $n \in Z$, $\cos(2n\pi + x) = \cos x$, $n \in Z$

Further, $\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ ....., i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, ...i.e., cos x vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

$\sin x = 0$ implies $x = n\pi$, where n is any integer

$\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions :

$\operatorname{cosec} x = \frac{1}{\sin x}$, $x \neq n\pi$, where n is any integer.

$\sec x = \frac{1}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer.

$\tan x = \frac{\sin x}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer.

$\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, where n is any integer.

BASIC TRIGONOMETRIC IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$; where $\theta \in R$.
2. $1 + \tan^2\theta = \sec^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ or $\operatorname{cosec}^2\theta - \cot^2\theta = 1$
4. $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta \cdot \cos^2\theta$
5. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta \cdot \cos^2\theta$

Note (i) $(\sec\theta - \tan\theta)$ is reciprocal of $(\sec\theta + \tan\theta)$ and vice-versa.

(ii) $(\operatorname{cosec}\theta - \cot\theta)$ is reciprocal of $(\operatorname{cosec}\theta + \cot\theta)$ and vice-versa.

Using above identities hundreds of other identities can be proved. While proving identities you can use rationalization, factorization and many other similar mathematical operations.

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Example 3: Which of the following reduces to unity for $0 < A < 90^o$?

(*i*) $\cos A \operatorname{cosec} A \sqrt{\sec^2 A - 1}$

(*ii*) $(1 + \tan^2 A)(1 - \sin^2 A)$

(*iii*) $\frac{1}{1 + \sin^2 A} + \frac{1}{1 + \operatorname{cosec}^2 A}$

(*iv*) $\frac{\cot^2 A \cos^2 A}{\cot^2 A - \cos^2 A}$

Sol.

(*i*) $\cos A \frac{1}{\sin A} \tan A = \frac{\cos A}{\sin A} \times \frac{\sin A}{\cos A} = 1$

(*ii*) $(\sec^2 A)(\cos^2 A) = 1$

(*iii*) $\frac{1}{1 + \sin^2 A} + \frac{\sin^2 A}{1 + \sin^2 A} = 1$

(*iv*) $\frac{\frac{\cos^2 A}{\sin^2 A} \times \cos^2 A}{\cos^2 A\left(\frac{1}{\sin^2 A} - 1\right)} = \frac{\frac{\cos^4 A}{\sin^2 A}}{\frac{\cos^4 A}{\sin^2 A}} = 1$

Example 4: Prove that $(\sec\theta + \operatorname{cosec}\theta)(\sin\theta + \cos\theta) = \sec\theta \cdot \operatorname{cosec}\theta + 2$

Sol. $\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} + 2$

$\Rightarrow \frac{1}{\sin\theta\cos\theta} + 2 = \sec\theta\operatorname{cosec}\theta + 2$

Example 5: Prove that $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

Sol. $= \frac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{(1 - \sec A + \tan A)}$

$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{(1 - \sec A + \tan A)} = \frac{1 + \sin A}{\cos A}$

Example 6: If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$ then find the value of $\sin\theta$.

Sol. Given $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$ ...(i)

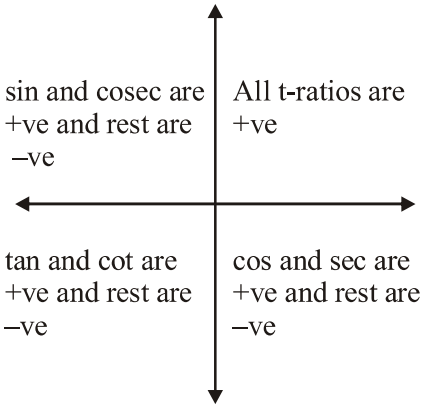
$$\Rightarrow \operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta} = 5 \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2\operatorname{cosec}\theta = 5 + \frac{1}{5} = \frac{26}{5}$$

$$\Rightarrow \operatorname{cosec}\theta = \frac{13}{5} \Rightarrow \sin\theta = \frac{5}{13}$$

SIGN OF THE TRIGONOMETRIC FUNCTIONS



	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}} = 1$
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.

N.D. implies not defined

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cosx and tan x, respectively.

TRIGONOMETRIC RATIOS OF ALLIED ANGLES

If θ is any angle, then $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ etc. are called allied angles.

- **Trigonometric Ratios of $(-\theta)$:**

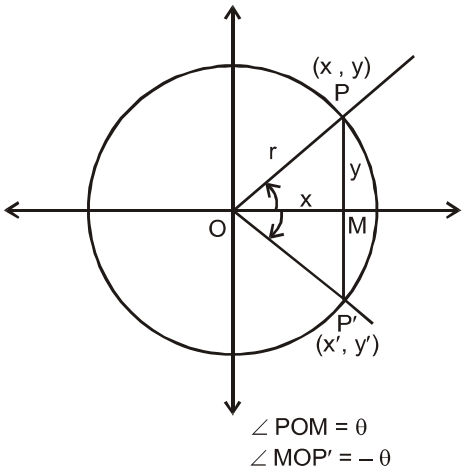
Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre 'O' and radius r in P (x, y).

Let P′ (x′, y′) be the point of intersection of the terminal side of the angle $-\theta$ in the standard position with the circle.

Now $\angle MOP = \angle MOP'$ (numerically) and P & P′ have the same projection M in the x - axis

$\therefore \quad \Delta OMP \equiv \Delta OMP' \Rightarrow x = x'$ and $y = -y'$

$$\therefore \quad \sin(-\theta) = \frac{y'}{r} = \frac{-y}{r} = -\sin\theta.;$$



$$\cos(-\theta) = \frac{x'}{r} = \frac{x}{r} = \cos\theta.$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.;$$

$$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta.$$

$$\sec(-\theta) = \frac{r}{x'} = \frac{r}{x} = \sec\theta.;$$

$$\operatorname{cosec}(-\theta) = \frac{r}{y'} = \frac{r}{-y} = -\operatorname{cosec}\theta.$$

Similarly if θ is in the other quadrants then the above results can also be proved.

- **Trigonometric Ratios of $(\pi - \theta)$**

Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre 'O' and radius r at P (x, y). Let P′ (x′, y′) be the point of intersection of the terminal side of the angle $\pi - \theta$ with the circle. Let M and M′ be the projections of P and P′ respectively in the x-axis. Since $\Delta OM'P' \equiv \Delta OMP$, $x' = -x$, $y' = y$

$$\therefore \sin(\pi - \theta) = \frac{y'}{r} = \frac{y}{r} = \sin\theta.;$$

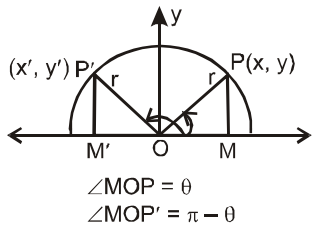
$$\cos(\pi - \theta) = \frac{x'}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\tan(\pi - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta.;$$

$$\cot(\pi - \theta) = \frac{x'}{y'} = -\frac{x}{y} = -\cot\theta.$$

$\sec(\pi-\theta)=\frac{r}{x'}=\frac{r}{-x}=-\sec\theta.$;

$\operatorname{cosec}(\pi-\theta)=\frac{r}{y'}=\frac{r}{y}=\operatorname{cosec}\theta.$



- **Trigonometric Ratios of $\left(\frac{\pi}{2}-\theta\right)$:**

Similarly we can easily prove the following results.

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$, $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta,$

$\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)=\sec\theta,$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta,$

$\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta,$ $\sec\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$

- **Trigonometric Ratios of $\left(\frac{\pi}{2}+\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta,$ $\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta,$

$\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta,$ $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta,$

$\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta,$ $\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta$

- **Trigonometric Ratios of $(\pi+\theta)$**

Similarly we can easily prove the following results.

$\sin(\pi+\theta)=-\sin\theta,$ $\tan(\pi+\theta)=\tan\theta,$

$\operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta,$

$\cos(\pi+\theta)=-\cos\theta,$ $\cot(\pi+\theta)=\cot\theta,$

$\sec(\pi+\theta)=-\sec\theta$

- **Trigonometric Ratios of $\left(\frac{3\pi}{2}-\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta,$ $\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta,$

$\operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$ $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta,$

$\cot\left(\frac{3\pi}{2}-\theta\right)=\tan\theta,$ $\sec\left(\frac{3\pi}{2}-\theta\right)=-\operatorname{cosec}\theta,$

- **Trigonometric Ratios of $\left(\frac{3\pi}{2}+\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta,$ $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta,$

$\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta,$ $\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta,$

$\sec\left(\frac{3\pi}{2}+\theta\right)=\operatorname{cosec}\theta,$ $\operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$

There is an easy way to remember these formulae. First of all think of θ as an acute angle. Angles like $180°\pm\theta$, $360°\pm\theta$, $-\theta$ can be considered as angles associated with the horizontal line, angles like $90°-\theta$, $90°+\theta$, $270°\mp\theta$ can be considered as angles associated with vertical line. When associated with the horizontal line, the magnitude of the function does not change, whereas when associated with the vertical line the function changes to the corresponding complementary value. For example $\sin(180°+\theta)$ will be only $\sin\theta$ (in magnitude) plus or minus and $\cos(180°-\theta)$ will be cosine θ only in magnitude.

To decide upon the sign, consider the quadrant in which the angle falls and decide the sign by the quadrant rule.

For example, $\sin(180°+\theta)$ is $\sin\theta$ (in magnitude) $(180°+\theta)$ lies in third quadrant and hence $\sin(180°+\theta)$ is negative.

$\therefore\quad \sin(180°+\theta)=-\sin\theta$

Again consider $\cos(360°-\theta)$: first of all, it should be $\cos\theta$ (in magnitude); since $(360°-\theta)$ lies in IV quadrant, its cosine is positive.

$\therefore\quad \cos(360°-\theta)=\cos\theta.$

Again consider $\tan(90° + \theta)$: This should be $\cot\theta$ and must have a negative sign since $(90° + \theta)$ is in II quadrant and hence $\tan(90° + \theta)$ is negative.

$\therefore \quad \tan(90° + \theta) = -\cot\theta$

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Example 7: Prove that $\sin420° \cos390° + \cos(-300°) \sin(-330°) = 1$.

Sol. $= \sin(360°+ 60°) \cos(360° + 30°) + \cos(-360° + 60°) \sin(-360° + 30°)$

$= \sin 60° \cos 30° + \cos 60° \sin 30°$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = 1$$

Example 8: Prove that $\sin 240° \sin 510° - \sin330° \cos 390° = 0$

Sol. $\sin(270° - 30°) \sin(540° - 30°) - \sin(360° - 30°) \cos(360° + 30°)$

$= -\cos 30° \sin30° + \sin30° \cos30°$

$= 0$

Example 9: Prove that $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) = 0$

Sol. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A)$

$= \cos A - \cos A - (-\cos A) + (-\cos A) = 0$

Example 10: Find the value of $\tan 35°.\tan 40°.\tan 45°.\tan 50°.\tan 55°$.

Sol. We have,

$\tan 35°.\tan 40°.\tan 45°.\tan 50°.\tan 55°$

$= \{\tan 35° \times \tan 55°\}\{\tan 40° \times \tan 50°\} \times \tan 45°$

$= \{\tan 35° \times \cot 35°\}.\{\tan 40° \times \cot 40°\} \times \tan 45° = 1$

Example 11: Find the value of $\tan(1°).\tan(2°).\tan(3°).....\tan(89°)$.

Sol. We have,

$\tan(1°).\tan(2°).\tan(3°).....\tan(89°)$

$= \tan(1°).\tan(2°).\tan(3°).....\tan(44°)$

$\tan(45°) \tan(46°).....\tan(87°) \tan(88°) \tan(89°)$

$= \{\tan(1°) \times \tan(89°)\}.\{\tan(2°) \times \tan(88°)\}$

$.....\{\tan(44°) \times \tan(46°)\}.\tan(45°) = 1$

Example 12: Find the value of $\cos(1°).\cos(2°).\cos(3°).....\cos(189°)$.

Sol. We have,

$\cos(1°).\cos(2°).\cos(3°).....\cos(189°)$

$= \cos(1°).\cos(2°).\cos(3°).....\cos(89°)$

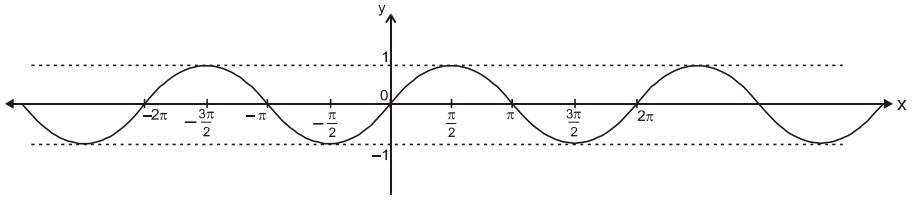
$\cos(90°) \cos(91°).....\cos(189°)$

$= \cos(1°).\cos(2°).\cos(3°).....\cos(89°) \times 0 \times \cos(91°) ..... \cos(189°) = 0$

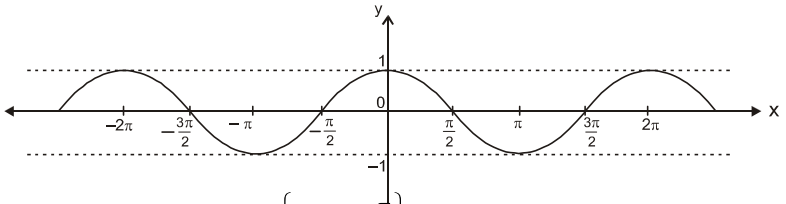
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degree	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	ND

TRIGONOMETRIC FUNCTIONS

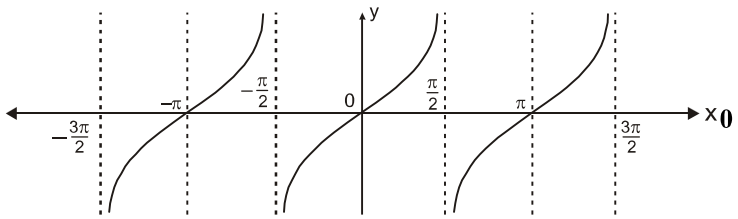
(i) y = sin x Domain : $x \in R$; Range : $y \in [-1, 1]$



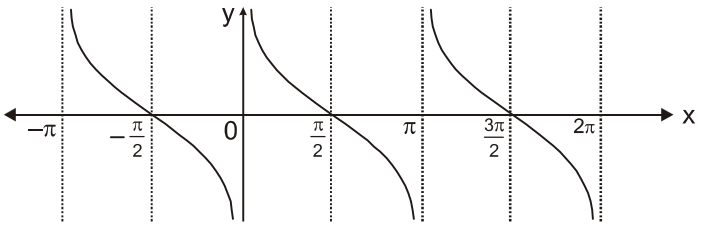
(ii) y = cos x Domain : $x \in R$; Range : $y \in [-1, 1]$



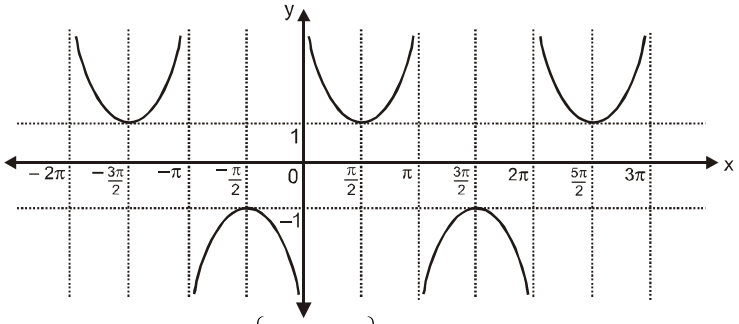
(iii) y = tan x Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$; Range : $y \in R$



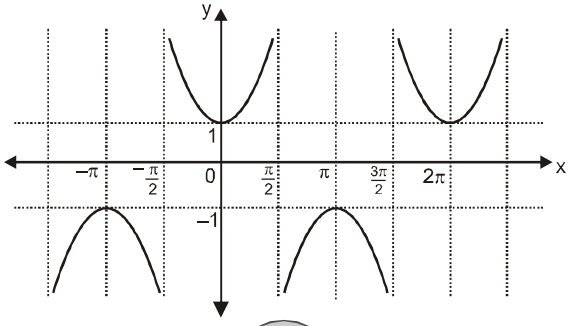
(iv) y = cot x Domain : $x \in R - \{n\pi\}$, $n \in I$; Range : $y \in R$



(v) y = cosec x Domain : $x \in R - \{n\pi\}, n \in I$; Range : $y \in (-\infty, -1] \cup [1, \infty)$



(vi) y = sec x Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$; Range : $y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC RATIOS OF SUM OR DIFFERENCE OF TWO ANGLES

(*a*) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(*b*) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(*c*) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(*d*) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(*e*) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(*f*) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(*g*) $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$

(*h*) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(*i*) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

(*j*) $\tan(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n) = \dfrac{S_1 - S_3 + S_5 - \ldots\ldots}{1 - S_2 + S_4 - \ldots\ldots}$

TRAIN YOUR BRAIN

Example 13: Prove that $\cos(9^\circ) + \sin(9^\circ) = \sqrt{2}\sin(54^\circ)$

Sol. We have, $\cos(9^\circ) + \sin(9^\circ)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos(9^\circ) + \frac{1}{\sqrt{2}}\sin(9^\circ)\right)$$

$$= \sqrt{2}\left(\sin(45^\circ)\cos(9^\circ) + \cos(45^\circ)\sin(9^\circ)\right)$$

$$= \sqrt{2}\left(\sin(45^\circ + 9^\circ)\right) = \sqrt{2}\sin(54^\circ)$$

Example 14: If $A + B = 45^\circ$, then find the value of $(1 + \tan A)(1 + \tan B)$

Sol. We have,

$A + B = 45^\circ$

$\Rightarrow \tan(A+B) = \tan(45^\circ) \qquad \Rightarrow \tan(A+B) = 1$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A.\tan B} = 1$$

$\Rightarrow \tan A + \tan B = 1 - \tan A \,.\, \tan B$

$\Rightarrow \tan A + \tan B + \tan A \,.\, \tan B = 1$

$\Rightarrow 1 + \tan A + \tan B + \tan A \,.\, \tan B = 1 + 1 = 2$

$\Rightarrow (1 + \tan A) + \tan B(1 + \tan A) = 2$

$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$

Example 15: Find the value of $(1 + \tan 20^\circ)(1 + \tan 24^\circ)(1 + \tan 25^\circ)(1 + \tan 21^\circ)$

Sol. We have,

$(1 + \tan 20^\circ)(1 + \tan 24^\circ)(1 + \tan 25^\circ)(1 + \tan 21^\circ)$

$= \{(1 + \tan 20^\circ)(1 + \tan 25^\circ)\} \times \{(1 + \tan 24^\circ)(1 + \tan 21^\circ)\}$

$= 2 \times 2 = 4$

Example 16: If $2\tan\alpha = 3\tan\beta$, then show that,

$$\tan(\alpha - \beta) = \frac{2\sin\beta\cos\beta}{4 + 2\sin^2\beta}$$

Sol. We have, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\tan\beta}$

$$= \frac{\frac{3}{2}\tan\beta - \tan\beta}{1 + \frac{3}{2}\tan\beta\tan\beta} = \frac{\tan\beta}{2 + 3\tan^2\beta}$$

$$= \frac{\frac{\sin\beta}{\cos\beta}}{2 + 3\frac{\sin^2\beta}{\cos^2\beta}} = \frac{\sin\beta\cos\beta}{2\cos^2\beta + 3\sin^2\beta}$$

$$= \frac{\sin\beta\cos\beta}{2 + \sin^2\beta} = \frac{2\sin\beta\cos\beta}{4 + 2\sin^2\beta}$$

TRANSFORMATION FORMULAE

(*i*) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

(*a*) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(*ii*) $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

(*b*) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(*iii*) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

(*c*) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(*iv*) $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

(*d*) $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

TRAIN YOUR BRAIN

Example 17: Prove that $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

Sol. $\dfrac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta}$

$$= \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta}$$

$$= \frac{2\sin 2\theta\cos 5\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta$$

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUB-MULTIPLE ANGLES

(*a*) $\sin 2A = 2\sin A\cos A$

Note : $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ etc.

(*b*) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Note : $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(*c*) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

Note : $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

(*d*) $\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$, $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

(*e*) $\sin 3A = 3\sin A - 4\sin^3 A$

(*f*) $\cos 3A = 4\cos^3 A - 3\cos A$

(*g*) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(*h*) $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

(*i*) $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

(*j*) $\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$

(*k*) $\sin A \sin(60 - A) \cdot \sin(60 + A) = \frac{1}{4}\sin 3A$

(*l*) $\cos A \cdot \cos(60 - A) \cdot \cos(60 + A) = \frac{1}{4}\cos 3A$

(*m*) $\tan A \cdot \tan(60 - A) \cdot \tan(60 + A) = \tan 3A$

(*n*) $\cot\theta - \tan\theta = 2\cot(2\theta)$

TRAIN YOUR BRAIN

Example 18: Prove that

(*i*) $\frac{\sin 2A}{1 + \cos 2A} = \tan A$

(*ii*) $\tan A + \cot A = 2\operatorname{cosec} 2A$

(*iii*) $\frac{1 - \cos A + \cos B - \cos(A + B)}{1 + \cos A - \cos B - \cos(A + B)} = \tan\frac{A}{2}\cot\frac{B}{2}$

(*iv*) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta$

Sol.

(*i*) L.H.S. $\frac{\sin 2A}{1 + \cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \tan A$

(*ii*) L.H.S. $\tan A + \cot A = \frac{1 + \tan^2 A}{\tan A}$

$= 2\left(\frac{1 + \tan^2 A}{2\tan A}\right) = \frac{2}{\sin 2A} = 2\operatorname{cosec} 2A$

(*iii*) L.H.S. $\frac{1 - \cos A + \cos B - \cos(A + B)}{1 + \cos A - \cos B - \cos(A + B)}$

$= \frac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\sin\left(\frac{A}{2} + B\right)}{2\cos^2\frac{A}{2} - 2\cos\frac{A}{2}\cos\left(\frac{A}{2} + B\right)}$

$= \tan\frac{A}{2}\left[\frac{\sin\frac{A}{2} + \sin\left(\frac{A}{2} + B\right)}{\cos\frac{A}{2} - \cos\left(\frac{A}{2} + B\right)}\right]$

$= \tan\frac{A}{2}\left[\frac{2\sin\frac{A + B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A + B}{2}\sin\left(\frac{B}{2}\right)}\right] = \tan\frac{A}{2}\cot\frac{B}{2}$

(*iv*) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta}$

$= \frac{\sin 5\theta\cos 3\theta + \sin 3\theta\cos 5\theta}{\sin 5\theta\cos 3\theta - \sin 3\theta\cos 5\theta}$

$= \frac{\sin 8\theta}{\sin 2\theta} = 4\cos 2\theta \cos 4\theta$

Example 19: Prove that: $\frac{\tan\left(\frac{\pi}{4} + A\right)}{\tan\left(\frac{\pi}{4} - A\right)} = \frac{2\cos A + \sin A + \sin 3A}{2\cos A - \sin A - \sin 3A}$

Sol. L.H.S. $= \frac{\frac{1 + \tan A}{1 - \tan A}}{\frac{1 - \tan A}{1 + \tan A}} = \frac{(1 + \tan A)^2}{(1 - \tan A)^2} = \frac{1 + \tan^2 A + 2\tan A}{1 + \tan^2 A - 2\tan A}$

$= \frac{1 + \frac{2\tan A}{1 + \tan^2 A}}{1 - \frac{2\tan A}{1 + \tan^2 A}} = \frac{1 + \sin 2A}{1 - \sin 2A}$ R.H.S

$= \frac{2\cos A + 2\sin 2A.\cos A}{2\cos A - 2\sin 2A\cos A}$

$= \frac{2\cos A(1 + \sin 2A)}{2\cos A(1 - \sin 2A)} = \frac{1 + \sin 2A}{1 - \sin 2A}$

Both sides reduce to the same result.

Example 20: Prove that $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right) = \frac{1}{8}$

Sol. We have $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right)$

$= \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\pi + \frac{\pi}{7}\right)$

$= -\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$

$$= -\frac{1}{2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)$$

$$= -\frac{\sin\left(\frac{8\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)} = -\frac{\sin\left(\pi+\frac{\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)} = \frac{\sin\left(\frac{\pi}{7}\right)}{8\sin\left(\frac{\pi}{7}\right)} = \frac{1}{8}$$

Example 21: Prove that $\sin(20^o)\sin(40^o)\sin(80^o) = \frac{\sqrt{3}}{8}$

Sol. We have,

$$\sin(20^o)\sin(40^o)\sin(80^o) = \frac{1}{4}(\sin(3.20^o)) = \frac{1}{4}(\sin(60^o)) = \frac{\sqrt{3}}{8}$$

TRIGONOMETRIC RATIOS OF STANDARD ANGLES:

(*a*) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(*b*) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$

or $\sin\frac{5\pi}{12}$; $\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75°$

Values of standard angles :

Angle → / T. Ratio ↓	$\left(\frac{\pi}{12}\right)$	$\left(\frac{\pi}{10}\right)$	$\left(\frac{\pi}{8}\right)$	$\left(\frac{\pi}{5}\right)$	$\left(\frac{3\pi}{8}\right)$	$\left(\frac{5\pi}{12}\right)$
	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$2-\sqrt{3}$	$\frac{1}{\sqrt{(5+2\sqrt{5})}}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{2}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

To find the trigonometrical funcitons of an angle of 18°:

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $2\theta = 90° - 3\theta$.

and therfore

$\sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$

$\therefore$ $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

Hence, either $\cos\theta = 0$, which gives $\theta = 90°$, or

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$

$\therefore$ $4\sin^2\theta + 2\sin\theta = 1$

By solving this quadratic equation, we have

(In our case $\sin\theta$ is necessarily a positive quantity. Hence we take the upper sign, and have)

$$\sin\theta = \frac{\sqrt{5}-1}{4} = \sin 18^o$$

Hence

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$$

The remaining trigonometrical ratio of 18° may be now found.

Since 72° is the complement of 18°, the value of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$$\therefore \cos 36° = 1 - 2\sin^2 18° = \frac{\sqrt{5}+1}{4}$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° may be found.

TRAIN YOUR BRAIN

Example 22: Prove that $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ) = \frac{1}{8}$

Sol. We have, $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ)$

$$= \frac{1}{\sin(72^\circ)}(\sin(12^\circ).\sin(48^\circ).\sin(72^\circ))(\sin(54^\circ))$$

$$= \frac{1}{4\sin(72^\circ)}(4\sin(60^\circ-12^\circ).\sin(12^\circ).\sin(60^\circ+12^\circ))(\cos(36^\circ))$$

$$= \frac{1}{4\sin(72^\circ)}(\sin(36^\circ).\cos(36^\circ))$$

$$= \frac{1}{8\sin(72^\circ)}(2\sin(36^\circ).\cos(36^\circ))$$

$$= \frac{1}{8\sin(72^\circ)}(\sin(72^\circ)) = \frac{1}{8}$$

Example 23: Prove that : $4(\sin(24^\circ)+\cos(6^\circ)) = \left(1+\sqrt{5}\right)$

Sol. We have, $4(\sin(24^\circ)+\cos(6^\circ))$

$$= 4(\sin(24^\circ)+\sin(84^\circ))$$

$$= 4\left(2\sin\left(\frac{24^\circ+84^\circ}{2}\right)\cos\left(\frac{24^\circ-84^\circ}{2}\right)\right)$$

$$= 8(\sin(54^\circ)\cos(30^\circ)) = 8(\cos(36^\circ)\cos(30^\circ))$$

$$= 8\left(\frac{\sqrt{5}+1}{4}\times\frac{1}{2}\right) = \left(\sqrt{5}+1\right)$$

Example 24: Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Sol. L.H.S. $= \sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$

$$= \sin 36^\circ \sin 72^\circ \sin(180^\circ-72^\circ)\sin(108^\circ-36^\circ)$$

$$= \sin 36^\circ \sin 72^\circ \sin 72^\circ \sin 36^\circ = \sin^2 36^\circ \sin^2 72^\circ$$

$$= \frac{5-\sqrt{5}}{8}\times\frac{5+\sqrt{5}}{8} = \frac{25-5}{64} = \frac{20}{64} = \frac{5}{16}.$$

CONDITIONAL IDENTITIES

If $A+B+C=\pi$ then :

(*i*) $\sin2A+\sin2B+\sin2C = 4\sin A\sin B\sin C$

(*ii*) $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(*iii*) $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$

(*iv*) $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(*v*) $\tan A+\tan B+\tan C = \tan A\tan B\tan C$

(*vi*) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1$

(*vii*) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$

(*viii*) $\cot A\cot B+\cot B\cot C+\cot C\cot A = 1$

(*ix*) $A+B+C=\frac{\pi}{2}$ then $\tan A\tan B+\tan B\tan C+\tan C\tan A = 1$

TRAIN YOUR BRAIN

Example 25: If $A+B+C = 180^\circ$, Prove that, $\sin^2A+\sin^2B+\sin^2C = 2+2\cos A\cos B\cos C$.

Sol. Let $S = \sin^2A+\sin^2B+\sin^2C$

so that

$$2S = 2\sin^2A+1-\cos2B+1-\cos2C$$

$$= 2\sin^2A+2-2\cos(B+C)\cos(B-C)$$

$$= 2-2\cos^2A+2-2\cos(B+C)\cos(B-C)$$

$\therefore$ $S = 2+\cos A[\cos(B-C)+\cos(B+C)]$

since $\cos A = -\cos(B+C)$

$\therefore$ $S = 2+2\cos A\cos B\cos C$

Example 26: Prove that $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Sol. $(\cos A+\cos B)+\cos C-1$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C-1$$

$$= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}+\cos C-1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}-1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]$$

$$= 2\sin\frac{C}{2}\,2\sin\frac{A}{2}\sin\frac{B}{2} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Example 27: If A, B, C are the angles of a triangle, show that $\tan A+\tan B+\tan C = \tan A\tan B\tan C$

Sol. We have

$$\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-(\tan B\tan C+\tan C\tan A+\tan A\tan B)}$$

But $\tan(A+B+C) = \tan 180^\circ = 0$

Hence $0 = \tan A+\tan B+\tan C-\tan A\tan B\tan C$

i.e, $\tan A+\tan B+\tan C = \tan A\tan B\tan C$

This may also be proved independently. For

$\tan(A+B) = \tan(180^\circ-C) = -\tan C$.

$\therefore$ $\frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$

$\tan A+\tan B = -\tan C+\tan A\tan B\tan C$

$\tan A+\tan B+\tan C = \tan A\tan B\tan C$

SUMMATION OF TRIGONOMETRIC SERIES

Type I

Sum of the sin and cosine series when the angles are in A.P.

(1) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+....+\sin(\alpha+\overline{n-1}\beta)$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\times\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

Proof : Let $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+....+\sin(\alpha+\overline{n-1}\beta)$

Here angle are in A.P. and common difference of angles $=\beta$

$\therefore$ multiplying both sides by $2\sin\frac{\beta}{2}$, we get

$$2S\sin\frac{\beta}{2}=2\sin\alpha\sin\frac{\beta}{2}+2\sin(\alpha+\beta)\sin\frac{\beta}{2}+.....+2\sin(\alpha+\overline{n-1}\beta)\sin\frac{\beta}{2} \quad ...(i)$$

Now, $2\sin\alpha\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right)$$

... ...

... ...

$$2\sin(\alpha+\overline{n-1}\beta)\sin\frac{\beta}{2}=\cos\left[\alpha+(2n-3)\frac{\beta}{2}\right]-\cos\left[\alpha+(2n-1)\frac{\beta}{2}\right]$$

adding, we get R.H.S. of equation

$$(i)=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left[\alpha+(2n-1)\frac{\beta}{2}\right];$$

hence, $$S=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

(2) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+.....+\cos(\alpha+\overline{n-1}\beta)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\times\cos\left[\alpha+(n-1)\frac{\beta}{2}\right]$

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Example 28: Find the summation of the following series

(*i*) $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

(*ii*) $\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}$

(*iii*) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$

Sol.

(*i*) $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$=\frac{\cos\frac{\left(\frac{2\pi}{7}+\frac{6\pi}{7}\right)}{2}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$$

$$=\frac{\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}=\frac{-\cos\frac{3\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}=-\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}}=-\frac{1}{2}$$

(*ii*) $\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}$

$$=\frac{\cos\left(\frac{\frac{\pi}{7}+\frac{6\pi}{7}}{2}\right)\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}}=\frac{\cos\frac{\pi}{2}\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}}=0$$

(*iii*) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$

$$=\frac{\sin\left(5\times\frac{2\pi}{11\times2}\right)}{\sin\left(\frac{2\pi}{11\times2}\right)}\times\cos\left(\frac{\frac{\pi}{11}+\frac{9\pi}{11}}{2}\right)$$

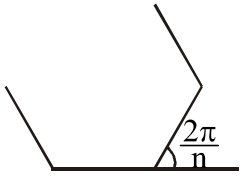
$$=\frac{\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}\times\cos\left(\frac{5\pi}{11}\right)=\frac{2\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{5\pi}{11}\right)}{2\sin\left(\frac{\pi}{11}\right)}$$

$$=\frac{\sin\left(\frac{10\pi}{11}\right)}{2\sin\left(\frac{\pi}{11}\right)}=\frac{1}{2}$$

Type II
For n sided regular polygon
Sum of all exterior angles = 2π

(*i*) The value of one exterior angle = $\frac{2\pi}{n}$

(*ii*) The value of one interior angle = $\left(\pi-\frac{2\pi}{n}\right)=\pi\frac{(n-2)}{n}$



(*iii*) Sum of interior angles = $\pi(n-2)$

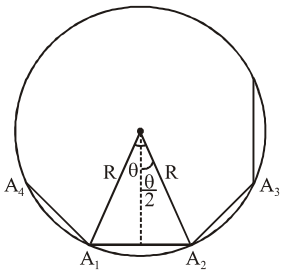
TRAIN YOUR BRAIN

Example 29: In a regular polygon of n-sides with $A_1, A_2, \ldots\ldots A_n$ vertices prove that
$(A_1A_2)^2+(A_1A_3)^2+(A_1A_4)^2+\ldots\ldots+(A_1A_n)^2=2nR^2$
where R is the radius of circumcircle circumscribing it.

Sol. $(A_1A_2)^2+(A_1A_3)^2+\ldots.(A_1A_n)^2=2nR^2$

$\theta=\frac{2\pi}{n}$

By trigonometry



$(A_1A_2)^2=4R^2\cdot\sin^2\frac{\pi}{n}$

$(A_1A_3)^2=4R^2\cdot\sin^2\frac{2\pi}{n}$

$(A_1A_4)^2=4R^2\cdot\sin^2\frac{3\pi}{n}$

...

$(A_1A_n)^2=4R^2\cdot\sin^2\frac{(n-1)\pi}{n}$

$\therefore (A_1A_2)^2+(A_1A_3)^2+\ldots.+(A_1A_n)^2$

$4R^2\left[\sin^2\frac{\pi}{n}+\sin^2\frac{2\pi}{n}+\sin^2\frac{3\pi}{n}+\ldots\sin^2\frac{(n-1)\pi}{n}\right]$

$\frac{1}{2}4R^2$:

$\left[1-\cos\frac{2\pi}{n}+1-\cos\frac{4\pi}{n}+1-\cos\frac{6\pi}{n}+\ldots1-\cos\frac{2(n-1)\pi}{n}\right]$

$2R^2(n-1)-2R^2\left[\frac{\sin\frac{(n-1)\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}\cdot\cos\left(\frac{\frac{2\pi}{n}+\frac{2(n-1)\pi}{n}}{2}\right)\right]$

$\Rightarrow 2R^2(n-1)-2R^2(-1)$

$\Rightarrow 2R^2n-2R^2+2R^2 \quad \Rightarrow 2nR^2$

Type-III of summation of sine/cosine series
Splitting the sum series as difference of 2 terms.

Here each term of trigonometric series is splitted into difference of two so that most of the terms while adding them gets cancelled.

TRAIN YOUR BRAIN

Example 30: $S=\operatorname{cosec}x+\operatorname{cosec}2x+\operatorname{cosec}4x+\ldots\ldots+\operatorname{cosec}2^nx=\cot\frac{x}{2}-\cot 2^nx$.

Sol. Let $\operatorname{cosec}x=\frac{\sin\left(\frac{x}{2}\right)}{\sin\frac{x}{2}\cdot\sin x}=\frac{\sin\left(x-\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cdot\sin x}$

$=\frac{\sin\cdot\cos\frac{x}{2}-\cos x\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin x}=\cot\frac{x}{2}-\cot x$

$S=\left(\cot\frac{x}{2}-\cot x\right)+(\cot x-\cot 2x)+(\cot 2x-\cot 2^2x)+\ldots$

$(\cot 2^{n-1}x-\cot 2^nx)$

$\Rightarrow S=\left(\cot\frac{x}{2}-\cot 2^n\cdot x\right)$

Continued product of cosine Series:

$\cos A\cos 2A\cos 4A\cos 8A\ldots.\cos 2^{n-1}A=\frac{1}{2^n\sin A}\sin(2^nA)$

Proof : Multiplying above and below by $2^n\sin A$

$\therefore$ LHS

$=\frac{2^{n-1}}{2^n\sin A}[2\sin A\cos A\cos 2A\cos 4A\ldots.\cos 2^{n-1}A]$

$=\frac{2^{n-2}}{2^n\sin A}[2\sin 2A\cos 2A\cos 4A\ldots.\cos 2^{n-1}A]$

$=\frac{2^{n-3}}{2^n\sin A}[2\sin 4A\cos 4A\ldots.\cos 2^{n-1}A]$

$=\frac{1}{2^n\sin A}[2\sin 2^{n-1}A\cos 2^{n-1}A]$

$=\frac{1}{2^n\sin A}\sin(2\cdot 2^{n-1}A)=\frac{\sin(2^nA)}{2^n\sin A}$

TRAIN YOUR BRAIN

Example 31: Prove that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

Sol. $\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$, $\cos\frac{7\pi}{15}=-\cos\frac{8\pi}{15}$

$$\text{L.H.S}=\left[\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\left(-\cos\frac{8\pi}{15}\right)\right]$$

$$\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)\cdot\frac{1}{2}=-\frac{1}{2}\cdot\frac{\sin 2^4\left(\frac{\pi}{15}\right)}{2^4\sin\frac{\pi}{15}}\cdot\frac{\sin\left(2^2\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}$$

$$=\frac{1}{128},\ \text{as}\ \sin(\pi+\theta)=-\sin\theta$$

APPLICATION OF TRIGONOMETRY IN MAXIMISING AND MINIMISING I.E. (OPTIMISATION):

1. Maximising and minimising by using the property of boundness of trigonometric functions.
   (*a*) Sine and cosine have bounded values between – 1 and 1.
   (*b*) Tangent and cotangent are unbounded functions.
   (*c*) cosec and sec have values greater than 1 and less than – 1.
   (*d*) $0\le\sin^2x\le1$, $0\le\cos^2x\le1$, $\tan^2x\ge0$, $\sec^2x\ge1$.
   Note : If maximum value of a function is **'b'**, and minimum value **'a'** then range is [a, b].

TYPE-I:

TRAIN YOUR BRAIN

Example 32: Find the max and min values of $f(\theta)=\sin^6\theta+\cos^6\theta$

Sol. We have, $f(\theta)=\sin^6\theta+\cos^6\theta=\left(\sin^2\theta\right)^3+\left(\cos^2\theta\right)^3$

$$=\left(\sin^2\theta+\cos^2\theta\right)^2-3\sin^2\theta\cos^2\theta\left(\sin^2\theta+\cos^2\theta\right)$$

$$=1-3\sin^2\theta\cos^2\theta=1-\frac{3}{4}\left(4\sin^2\theta\cos^2\theta\right)$$

$$=1-\frac{3}{4}(\sin^2 2\theta)=1+\frac{3}{4}\left(-\sin^2 2\theta\right)$$

As we know that, $-1\le(-\sin^2 2\theta)\le0$

$$\Rightarrow-\frac{3}{4}\le\frac{3\left(-\sin^2 2\theta\right)}{4}\le0$$

$$\Rightarrow-\frac{3}{4}\le\frac{3\left(-\sin^2 2\theta\right)}{4}\le0\qquad\Rightarrow\frac{1}{4}\le f(\theta)\le1$$

Hence, the maximum value = 1 and the minimum value = $\frac{1}{4}$

Special Cases:

When angle of sine and cosine are same

$E=a\sin\theta+b\cos\theta$

$$\Rightarrow\ E=\sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta+\frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$

Let $\frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$ & $\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha$

$$\Rightarrow\ E=\sqrt{a^2+b^2}\sin(\theta+\alpha),\ \text{where}\ \tan\alpha=\frac{b}{a}$$

Hence for any real value of θ,

$$-\sqrt{a^2+b^2}\le E\le\sqrt{a^2+b^2}$$

TRAIN YOUR BRAIN

Example 33: Find minimum and maximum value of $7\cos\theta+24\sin\theta$.

Sol. $y=7\cos\theta+24\sin\theta$.

$$-\sqrt{a^2+b^2}\le a\cos\theta+b\sin\theta\le\sqrt{a^2+b^2}$$

$$\therefore\ -\sqrt{7^2+24^2}\le7\cos\theta+24\sin\theta\le\sqrt{7^2+24^2}$$

$$-25\le7\cos\theta+24\sin\theta\le25.$$

TYPE-II:

Argument of sine and cosine are different or a quadratic in sine/cos is given then we make a perfect square in sine/cosine and interpret.

TRAIN YOUR BRAIN

Example 34: $y=\cos2x+3\sin x$. Find range of y.

Sol. $\Rightarrow1-2\sin^2x+3\sin x$

$$\Rightarrow1-2\left[\sin^2x-\frac{3}{2}\sin x+\frac{9}{16}-\frac{9}{16}\right]$$

$$\Rightarrow1-2\left[\sin x-\frac{3}{4}\right]^2+\frac{9}{8};$$

$$y=\frac{17}{8}-2\left[\sin x-\frac{3}{4}\right]^2$$

$y_{max}=\frac{17}{8}$ at $\sin x=\frac{3}{4}$

$y_{min}=-4$ at $\sin x=-1$

$$y\in\left[-4,\frac{17}{8}\right];\ y_{max}=\frac{17}{8};\ y_{min}=-4$$

TYPE-III:

Making use of reciprocal relationship between tan and cot, sin/cosec and cos/sec.

TRAIN YOUR BRAIN

Example 35: If $y = a^2\tan^2x + b^2\cot^2x$ $(a, b \geq 0)$. Find minimum value of y.

Sol. $y = a^2\tan^2x + \dfrac{b^2}{\tan^2 x} - 2ab + 2ab$

$\Rightarrow \left(a\tan x - \dfrac{b}{\tan x}\right)^2 + 2ab \geq 2ab$

$y_{min.} = 2ab$

MISCELLANEOUS PROBLEMS

Example 36: If $x^2 + y^2 = 4$ and $a^2 + b^2 = 8$. Find minimum and maximum value of $(ax + by)$

Sol. Let $x = r_1\cos\theta$, $y = r_1\sin\theta$

and $a = r_2\cos\phi$; $b = r_2\sin\phi$

$\therefore$ $r_1 = 2$, $r_2 = 2\sqrt{2}$

Then $(ax + by) = r_1r_2\cos(\theta - \phi)$

$-r_1r_2 \leq (ax + by) \leq r_1r_2$

$-4\sqrt{2} \leq (ax + by) \leq 4\sqrt{2}$

$y_{max} = 4\sqrt{2}$ and $y_{min} = -4\sqrt{2}$

ELIMINATION

Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice, a considerable amount of artifice and ingenuity is often required in seemingly simple cases.

So, between any three equations involving two unknown quantities, we can theoretically eliminate both of the unknown quantities.

Eliminate θ from the equations

TRAIN YOUR BRAIN

Example 37: Eliminate θ from the equations $a\cos\theta + b\sin\theta = c$ and $b\cos\theta - a\sin\theta = d$.

Sol. $a\cos\theta + b\sin\theta = c$ ...(*i*)

$b\cos\theta - a\sin\theta = d$. ...(*ii*)

Square (*i*) and (*ii*) and adding,

$a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = c^2 + d^2$

or $a^2 + b^2 = c^2 + d^2$.

Example 38: Eliminate θ from the equations

$a\sin\alpha - b\cos\alpha = 2b\sin\theta$, and $a\sin 2\alpha - b\cos 2\theta = a$,

Sol. $a\sin\alpha - b\cos\alpha = 2b\sin\theta$ ...(*i*)

$a\sin 2\alpha - b\cos 2\theta = a$ ...(*ii*)

From (*i*) $\sin\theta = \dfrac{a\sin\alpha - b\cos\alpha}{2b}$

From (*ii*) $a\sin 2\alpha - b(1 - 2\sin^2\theta) = a$

or $a\sin 2\alpha - b\left\{1 - 2\left(\dfrac{a\sin\alpha - b\cos\alpha}{2b}\right)^2\right\} = a$

or $a^2\sin^2\alpha + b^2\cos^2\alpha + 4ab\sin\alpha\cos\alpha - 2ab\sin\alpha\cos\alpha = 2ab + 2b^2$

or $a\sin\alpha + b\cos\alpha = \sqrt{\{2b(a + b)\}}$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

Topicwise Questions

ANGLE AND ITS MEASUREMENT

1. If $\tan\alpha+\cot\alpha=a$ then the value of $\tan^4\alpha+\cot^4\alpha =$
(*a*) a^4+4a^2+2 (*b*) a^4-4a^2+2
(*c*) a^4-4a^2-2 (*d*) None of these

2. The angles of a triangle are in A.P. and the number of degrees in the least to the number of radians in greatest is 60 to p. The angles in degree are
(*a*) $60°, 60°, 60°$ (*b*) $30°, 60°, 90°$
(*c*) $45°, 60°, 75°$ (*d*) $15°, 60°, 105°$

3. The perimeter of a certain sector of a circle is equal to half that of the circle of which it is a sector. The circular measure of one angle of the sector is
(*a*) $(\pi-2)$ radian (*b*) $(\pi+2)$ radian
(*c*) π radian (*d*) $(\pi-3)$ radian

SIGN OF TRIGONOMETRIC RATIO AND ALLIED ANGLE AND REDUCTION FORMULAE

4. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi+\alpha)\right]$ is equal to
(*a*) 0 (*b*) 1
(*c*) 3 (*d*) $\sin 4\alpha+\sin 6\alpha$

5. $\cos(540°-\theta)-\sin(630°-\theta)$ is equal to
(*a*) 0 (*b*) $2\cos\theta$
(*c*) $2\sin\theta$ (*d*) $\sin\theta-\cos\theta$

6. The value of $\sin(\pi+\theta)\sin(\pi-\theta)\operatorname{cosec}^2\theta$ is equal to
(*a*) –1 (*b*) 0
(*c*) $\sin\theta$ (*d*) None of these

7. $\cos 0+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}=$
(*a*) 1/2 (*b*) –1/2
(*c*) 0 (*d*) 1

8. The value of $\cos 10°-\sin 10°$ is
(*a*) Positive (*b*) Negative
(*c*) 0 (*d*) 1

9. Which of the following relations is correct
(*a*) $\sin 1<\sin 1°$ (*b*) $\sin 1>\sin 1^o$
(*c*) $\sin 1=\sin 1°$ (*d*) $\frac{\pi}{180}\sin 1=\sin 1^o$

10. If $\sin\theta+\operatorname{cosec}\theta=2$, the value of $\sin^{10}\theta+\operatorname{cosec}^{10}\theta$ is
(*a*) 10 (*b*) 2^{10}
(*c*) 2^9 (*d*) 2

11. If $\sin\theta=\frac{24}{25}$ and θ lies in the second quadrant, then $\sec\theta+\tan\theta=$
(*a*) -3 (*b*) -5
(*c*) -7 (*d*) -9

12. If A lies in the second quadrant and $3\tan A+4=0$, the value of $2\cot A-5\cos A+\sin A$ is equal to
(*a*) $\frac{-53}{10}$ (*b*) $\frac{-7}{10}$
(*c*) $\frac{7}{10}$ (*d*) $\frac{23}{10}$

13. The value of $e^{\log_{10}\tan 1°+\log_{10}\tan 2°+\log_{10}\tan 3°+.....+\log_{10}\tan 89°}$ is
(*a*) 0 (*b*) e
(*c*) 1/e (*d*) 1

14. If $\tan A+\cot A=4$, then $\tan^4 A+\cot^4 A$ is equal to
(*a*) 110 (*b*) 191
(*c*) 80 (*d*) 194

15. $\cos A+\sin(270°+A)-\sin(270°-A)+\cos(180°+A)=$
(*a*) –1 (*b*) 0
(*c*) 1 (*d*) 2

16. $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=$
(*a*) 1 (*b*) –1
(*c*) 0 (*d*) 2

17. If ABCD is a cyclic quadrilateral, then the value of $\cos A-\cos B+\cos C-\cos D=$
(*a*) 0 (*b*) 1
(*c*) $2(\cos B-\cos D)$ (*d*) $2(\cos A-\cos C)$

TRIGONOMETRIC IDENTITIES (SUM TO PRODUCT AND PRODUCT TO SUM)

18. The value of $\frac{\sin 24°\ \cos 6°-\sin 6°\sin 66°}{\sin 21°\ \cos 39°-\cos 51°\ \sin 69°}$ is
(*a*) -1 (*b*) 1
(*c*) 2 (*d*) None

19. If $3\sin\alpha = 5\sin\beta$, then $\dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$ is equal to

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

20. If tan A and tan B are the roots of the quadratic equation $x^2 - ax + b = 0$, then the value of $\sin^2(A + B)$.

(*a*) $\dfrac{a^2}{a^2+(1-b)^2}$ (*b*) $\dfrac{a^2}{a^2+b^2}$

(*c*) $\dfrac{a^2}{(b+c)^2}$ (*d*) $\dfrac{a^2}{b^2(1-a)^2}$

21. In a triangle ABC if $\tan A < 0$ then:

(*a*) $\tan B.\tan C > 1$ (*b*) $\tan B.\tan C < 1$

(*c*) $\tan B.\tan C = 1$ (*d*) None

22. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot(A - B)$ is equal to

(*a*) $\dfrac{1}{y} - \dfrac{1}{x}$ (*b*) $\dfrac{1}{x} - \dfrac{1}{y}$

(*c*) $\dfrac{1}{x} + \dfrac{1}{y}$ (*d*) None of these

23. If $\tan 25° = x$, then $\dfrac{\tan 155° - \tan 115°}{1 + \tan 155°\tan 115°}$ is equal to

(*a*) $\dfrac{1-x^2}{2x}$ (*b*) $\dfrac{1+x^2}{2x}$

(*c*) $\dfrac{1+x^2}{1-x^2}$ (*d*) $\dfrac{1-x^2}{1+x^2}$

24. If A + B = 225°, then the value of $\left(\dfrac{\cot A}{1+\cot A}\right)\cdot\left(\dfrac{\cot B}{1+\cot B}\right)$ is

(*a*) 2 (*b*) $\dfrac{1}{2}$

(*c*) 3 (*d*) $\dfrac{1}{3}$

25. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then the value of $1 + \cot\alpha\tan\beta$ is

(*a*) 1 (*b*) –1

(*c*) 2 (*d*) None of these

26. The value of $\dfrac{1-\tan^2 15°}{1+\tan^2 15°}$ is

(*a*) 1 (*b*) $\sqrt{3}$

(*c*) $\dfrac{\sqrt{3}}{2}$ (*d*) 2

27. If $\tan^2\theta = 2\tan^2\phi + 1$, then the value of $\cos 2\theta + \sin^2\phi$ is

(*a*) 1 (*b*) 2

(*c*) – 1 (*d*) Independent of ϕ

28. The value of the expression $\left(1+\cos\dfrac{\pi}{10}\right)\left(1+\cos\dfrac{3\pi}{10}\right)\left(1+\cos\dfrac{7\pi}{10}\right)\left(1+\cos\dfrac{9\pi}{10}\right)$ is

(*a*) $\dfrac{1}{8}$ (*b*) $\dfrac{1}{16}$

(*c*) $\dfrac{1}{4}$ (*d*) 0

29. The expression $\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$ is equal to

(*a*) $\cos 2x$ (*b*) $2\cos x$

(*c*) $\cos^2 x$ (*d*) $1 + \cos x$

30. If $\cos(A - B) = 3/5$ and $\tan A\tan B = 2$,

(*a*) $\cos A\cos B = -\dfrac{1}{5}$ (*b*) $\sin A\sin B = -\dfrac{2}{5}$

(*c*) $\cos(A + B) = -\dfrac{1}{5}$ (*d*) $\sin A\cos B = \dfrac{4}{5}$

31. If $\sin A = \dfrac{1}{\sqrt{10}}$ and $\sin B = \dfrac{1}{\sqrt{5}}$, where A and B are positive acute angles, then $A + B =$

(*a*) π (*b*) $\pi/2$

(*c*) $\pi/3$ (*d*) $\pi/4$

32. If $\tan A = 2\tan B + \cot B$, then $2\tan(A - B) =$

(*a*) $\tan B$ (*b*) $2\tan B$

(*c*) $\cot B$ (*d*) $2\cot B$

33. If $\tan A = -\dfrac{1}{2}$ and $\tan B = -\dfrac{1}{3}$, then $A + B =$

(*a*) $\dfrac{\pi}{4}$ (*b*) $\dfrac{3\pi}{4}$

(*c*) $\dfrac{5\pi}{4}$ (*d*) $\dfrac{\pi}{2}$

34. $\dfrac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} =$

(*a*) $\tan(A - B)$ (*b*) $\tan(A + B)$

(*c*) $\cot(A - B)$ (*d*) $\cot(A + B)$

35. $\tan 3A - \tan 2A - \tan A =$

(*a*) $\tan 3A\tan 2A\tan A$

(*b*) $-\tan 3A\tan 2A\tan A$

(*c*) $\tan A\tan 2A - \tan 2A\tan 3A - \tan 3A\tan A$

(*d*) 0

36. If $\tan\theta = \frac{-4}{3}$, then $\sin\theta$ is

(*a*) $\frac{-4}{5}$ but not $\frac{4}{5}$ (*b*) $\frac{-4}{5}$ or $\frac{4}{5}$

(*c*) $\frac{4}{5}$ but not $\frac{5}{4}$ (*d*) $\frac{4}{5}$ and $\frac{5}{4}$ both

37. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ =

(*a*) 3 (*b*) 2

(*c*) 1 (*d*) 0

38. The value of $\cos1° \cdot \cos2° \ldots \cos179°$ is

(*a*) $\frac{1}{\sqrt{2}}$ (*b*) 0

(*c*) 1 (*d*) –1

39. The value of $\sin 1 \cdot \cos 2 \cdot \tan 3 \cdot \cot 4 \cdot \sec 5 \cdot \text{cosec}\, 6$ is

(*a*) Positive

(*b*) Negative

(*c*) Zero

(*d*) May be positive and Negative

40. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$

(*a*) $\frac{1}{2}$ (*b*) $\frac{-1}{2}$

(*c*) $\frac{-1}{4}$ (*d*) 1

41. If $\frac{2\sin\alpha}{1+\sin\alpha+\cos\alpha} = \lambda$ then $\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha}$ is equal to

(*a*) $\frac{1}{\lambda}$ (*b*) λ

(*c*) $1-\lambda$ (*d*) $1+\lambda$

42. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of the equation $8x^2 - 26x + 15 = 0$ then $\cos(\alpha+\beta)$ is equal to

(*a*) $-\frac{627}{725}$ (*b*) $\frac{627}{725}$

(*c*) $-\frac{725}{627}$ (*d*) –1

43. If $\sin A$, $\cos A$ and $\tan A$ are in G.P., then $\cos^3 A + \cos^2 A$ is equal to

(*a*) 1 (*b*) 2

(*c*) 4 (*d*) None of these

44. If $\text{cosec}\, A + \cot A = \frac{11}{2}$, then $\tan A$ is

(*a*) $\frac{21}{22}$

(*b*) $\frac{15}{16}$

(*c*) $\frac{44}{117}$

(*d*) $\frac{117}{43}$

TRIGOMETRIC RATIO OF MULTIPLE ANGLES

45. If $\cos A = 3/4$, then the value of $16\cos^2(A/2) - 32\sin(A/2)\sin(5A/2)$ is

(*a*) – 4 (*b*) – 3

(*c*) 3 (*d*) 4

46. If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$, then $\cos 3\theta$ in terms of 'a' is

(*a*) $\frac{1}{4}\left(a^3 + \frac{1}{a^3}\right)$

(*b*) $\frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

(*c*) $4\left(a^3 + \frac{1}{a^3}\right)$

(*d*) None

47. The numerical value of $\sin 12° \cdot \sin 48° \cdot \sin 54°$ is equal to

(*a*) 1/2 (*b*) 1/4

(*c*) 1/16 (*d*) 1/8

48. If $A = \tan 6° \tan 42°$ and $B = \cot 66° \cot 78°$, then

(*a*) A = 2B (*b*) $A = \frac{1}{3}B$

(*c*) A = B (*d*) 3A = 2B

49. The value of $\cos\frac{\pi}{10}\cos\frac{2\pi}{10}\cos\frac{4\pi}{10}\cos\frac{8\pi}{10}\cos\frac{16\pi}{10}$ is :

(*a*) $\frac{\sqrt{10+2\sqrt{5}}}{64}$

(*b*) $-\frac{\cos(\pi/10)}{8}$

(*c*) $\frac{\cos(\pi/10)}{16}$

(*d*) $-\frac{\sqrt{10+2\sqrt{5}}}{64}$

50. The value of $\cot x + \cot(60^\circ + x) + \cot(120^\circ + x)$ is equal to

(*a*) $\cos 3x$ (*b*) $\tan 3x$

(*c*) $3\tan 3x$ (*d*) $\dfrac{3 - 9\tan^2 x}{3\tan x - \tan^3 x}$

51. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\dfrac{21}{65}$ and $\cos\alpha + \cos\beta = -\dfrac{27}{65}$, then the value of $\cos\dfrac{\alpha-\beta}{2}$ is

(*a*) $-\dfrac{3}{\sqrt{130}}$ (*b*) $\dfrac{3}{\sqrt{130}}$

(*c*) $\dfrac{6}{65}$ (*d*) $-\dfrac{6}{65}$

52. If $\tan x + \tan\left(\dfrac{\pi}{3} + x\right) + \tan\left(\dfrac{2\pi}{3} + x\right) = 3$, then

(*a*) $\tan x = 1$

(*b*) $\tan 2x = 1$

(*c*) $\tan 3x = 1$

(*d*) $\tan x = 2$

53. $\sin 4\theta$ can be written as

(*a*) $4\sin\theta(1 - 2\sin^2\theta)\sqrt{1 - \sin^2\theta}$

(*b*) $2\sin\theta\cos\theta\sin^2\theta$

(*c*) $4\sin\theta - 6\sin^3\theta$

(*d*) $4\sin\theta$

54. $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} =$ [where $\theta \in (0, \pi/4)$]

(*a*) $\cos\theta$ (*b*) $\sin\theta$

(*c*) $2\cos\theta$ (*d*) $2\sin\theta$

55. If $\tan A = \dfrac{1}{2}$, then $\tan 3A =$

(*a*) $\dfrac{9}{2}$ (*b*) $\dfrac{11}{2}$

(*c*) $\dfrac{7}{2}$ (*d*) $-\dfrac{1}{2}$

56. If $x + \dfrac{1}{x} = 2\cos\theta$, then $x^3 + \dfrac{1}{x^3} =$

(*a*) $\cos 3\theta$ (*b*) $2\cos 3\theta$

(*c*) $\dfrac{1}{2}\cos 3\theta$ (*d*) $\dfrac{1}{3}\cos 3\theta$

TRIGONOMETRIC IDENTIES IN A Δ

57. If $\alpha + \beta + \gamma = 2\pi$, then

(*a*) $\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} + \tan\dfrac{\gamma}{2} = \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$

(*b*) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} + \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} + \tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2} = 1$

(*c*) $\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} + \tan\dfrac{\gamma}{2} = -\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$

(*d*) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} + \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} + \tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2} = 0$

58. If $A + B + C = \dfrac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to

(*a*) 1–4cos A cosB cosC (*b*) 4 sinA sinB sinC

(*c*) 1+2 cosA cosB cosC (*d*) 1–4 sinA sinB sinC

59. If $A + B + C = \pi$ and $\cos A = \cos B \cdot \cos C$, then $\tan B \cdot \tan C$ is equal to

(*a*) 1 (*b*) $\dfrac{1}{2}$

(*c*) 2 (*d*) 3

GREATEST AND LEAST VALUE

60. If $f(\theta) = \sin^4\theta + \cos^2\theta$, then range of $f(\theta)$ is

(*a*) $\left[\dfrac{1}{2}, 1\right]$ (*b*) $\left[\dfrac{1}{2}, \dfrac{3}{4}\right]$

(*c*) $\left[\dfrac{3}{4}, 1\right]$ (*d*) None of these

61. Maximum value of f(x) = sin x + cos x is

(*a*) 1 (*b*) 2

(*c*) $\dfrac{1}{\sqrt{2}}$ (*d*) $\sqrt{2}$

Learning Plus

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12}x + 3\cos^{10}x + 3\cos^8 x + \cos^6 x - 1$ is equal to
(*a*) 0 (*b*) 1
(*c*) –1 (*d*) 2

2. The value of $\tan 75° - \cot 75°$ =
(*a*) $2\sqrt{3}$ (*b*) $2+\sqrt{3}$
(*c*) $2-\sqrt{3}$ (*d*) $-2\sqrt{3}$

3. If $\tan\alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1}$, then the expression $\cos 2\alpha + (2+\sqrt{3})\sin 2\alpha$ is
(*a*) $2+\sqrt{3}$ (*b*) –1
(*c*) 1 (*d*) $-(2+\sqrt{3})$

4. The value of $\tan 7\frac{1}{2}°$ is equal to
(*a*) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}$ (*b*) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$
(*c*) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+1}$ (*d*) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}-1}$

5. If $a\sin x + b\cos(c+x) + b\cos(c-x) = \alpha$, $a < \alpha$, then the minimum value of $|\cos c|$ is:
(*a*) $\sqrt{\frac{\alpha^2-a^2}{b^2}}$ (*b*) $\sqrt{\frac{\alpha^2-a^2}{2b^2}}$
(*c*) $\sqrt{\frac{\alpha^2-a^2}{3b^2}}$ (*d*) $\sqrt{\frac{\alpha^2-a^2}{4b^2}}$

6. If $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ and $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$, $0 < A, B < \frac{\pi}{2}$ then $\tan A + \tan B$ is equal to:
(*a*) $\sqrt{\frac{3}{5}}$ (*b*) $\sqrt{\frac{5}{3}}$
(*c*) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{5}}$ (*d*) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}}$

7. The range of $f(\theta) = 3\cos^2\theta - 8\sqrt{3}\cos\theta\cdot\sin\theta + 5\sin^2\theta - 7$ is given by
(*a*) [–7, 7] (*b*) [–10, 4]
(*c*) [–4, 4] (*d*) [–10, 7]

8. The maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ for real values of θ is
(*a*) 3 (*b*) 5
(*c*) 4 (*d*) 2

9. If $\alpha+\gamma = 2\beta$ then the expression $\frac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha}$ simplifies to:
(*a*) $\tan\beta$ (*b*) $-\tan\beta$
(*c*) $\cot\beta$ (*d*) $-\cot\beta$

10. If $\frac{\tan^3 A}{1+\tan^2 A} + \frac{\cot^3 A}{1+\cot^2 A} = p\sec A\,\text{cosec}\,A + q\sin A\cos A$, then:
(*a*) $p = 2, q = 1$ (*b*) $p = 1, q = 2$
(*c*) $p = 1, q = -2$ (*d*) $p = 2, q = -1$

11. If $\log_3\sin x - \log_3\cos x - \log_3(1-\tan x) - \log_3(1+\tan x) = 1$, then $\tan 2x$ is equal to (wherever defined)
(*a*) –2 (*b*) 3/2
(*c*) 2/3 (*d*) 6

12. $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}$ is equal to
(*a*) $\frac{n}{2}$ (*b*) $\frac{n-1}{2}$
(*c*) $\frac{n}{2}-1$ (*d*) $\frac{n+1}{2}$

13. The expression $2\sin 2° + 4\sin 4° + 6\sin 6° + \ldots 180\sin 180°$ equals
(*a*) $\cot 1°$ (*b*) $90\cot 1°$
(*c*) $\sin 1°$ (*d*) $90\cos 1°$

14. The value of $\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec^2\theta)\ldots$ $(1+\sec 2^n\theta)$ is
(*a*) $\tan 2^n\theta$ (*b*) $\tan 2^{n-1}\theta$
(*c*) $\tan 2^{n+1}\theta$ (*d*) $\tan 2^{n-2}\theta$

15. The minimum value of $27^{\cos 3x} \cdot 81^{\sin 3x}$ is

(*a*) 1 (*b*) $\frac{1}{81}$

(*c*) $\frac{1}{243}$ (*d*) $\frac{1}{27}$

16. If $\cos(\alpha+\beta)+\sin(\alpha-\beta)=0$ and $\tan\beta=\frac{1}{2009}$; then $\tan 3\alpha$ is:

(*a*) 2 (*b*) 1

(*c*) 3 (*d*) 4

17. The value of '*a*' for which the equation $\sin x(\sin x + \cos x) = a$ has a real solution are

(*a*) $1-\sqrt{2} \le a \le 1+\sqrt{2}$

(*b*) $2-\sqrt{3} \le a \le 2+\sqrt{3}$

(*c*) $0 \le a \le 2+\sqrt{3}$

(*d*) $\frac{1-\sqrt{2}}{2} \le a \le \frac{1+\sqrt{2}}{2}$

18. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$ lies in the interval

(*a*) $(-\infty, \infty)$ (*b*) $(-2, 2)$

(*c*) $(0, \infty)$ (*d*) $(-1, 1)$

19. If $f(\alpha, \beta) = \cos^2\alpha + \sin^2\alpha \cdot \cos 2\beta$, then which of the following is incorrect?

(*a*) $f\left(\frac{\pi}{5}, \frac{2\pi}{5}\right) \ne f\left(\frac{2\pi}{5}, \frac{\pi}{5}\right)$

(*b*) $f\left(\frac{\pi}{12}, \frac{\pi}{3}\right) = f\left(\frac{\pi}{3}, \frac{\pi}{12}\right)$

(*c*) $3f\left(\frac{\pi}{5}, \frac{\pi}{3}\right) \ne f\left(\frac{\pi}{3}, \frac{\pi}{5}\right)$

(*d*) $f\left(\frac{\pi}{4}, \frac{\pi}{18}\right) \ne 3f\left(\frac{\pi}{18}, \frac{\pi}{4}\right)$

20. Let $f(x) = \cos 10x + \cos 8x + 3\cos 4x + 3\cos 2x$ and $g(x) = 8\cos x \cdot \cos^3 3x$, then for all x we have

(*a*) $f(x) = g(x)$ (*b*) $2f(x) = 3g(x)$

(*c*) $f(x) = 2g(x)$ (*d*) $2f(x) = g(x)$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. The value of $\frac{\sin x + \cos x}{\cos^3 x}$ =

(*a*) $1 + \tan x + \tan^2 x - \tan^3 x$
(*b*) $1 + \tan x + \tan^2 x + \tan^3 x$
(*c*) $1 - \tan x + \tan^2 x + \tan^3 x$
(*d*) $(1 + \tan x)\sec^2 x$

2. If $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$ then each side is equal to
(*a*) 1 (*b*) –1
(*c*) 0 (*d*) None

3. Which of the following is correct ?
(*a*) $\sin 1° > \sin 1$ (*b*) $\sin 1° < \sin 1$
(*c*) $\cos 1° > \cos 1$ (*d*) $\cos 1° < \cos 1$

4. Let $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [.] is the greatest integer function, then

(*a*) $f\left(\frac{\pi}{2}\right) = -1$ (*b*) $f(\pi) = 1$

(*c*) $f(-\pi) = 0$ (*d*) $f\left(\frac{\pi}{4}\right) = 2$

5. The value of $\frac{(\cos 11° + \sin 11°)}{(\cos 11° - \sin 11°)}$ is

(*a*) –tan 304° (*b*) tan 56°
(*c*) cot 214° (*d*) cot 34°

6. For $0 < \theta < \pi/2$, $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ if
(*a*) $\tan\theta = 0$
(*b*) $\tan 2\theta = 0$
(*c*) $\tan 3\theta = 0$
(*d*) $\tan\theta \tan 2\theta = 2$

7. If $\tan x = \frac{2b}{a-c}$, $(a \neq c)$ $y = a\cos^2 x + 2b \sin x \cos x + c \sin^2 x$, $z = a\sin^2 x - 2b \sin x \cos x + c\cos^2 x$, then
(*a*) $y = z$
(*b*) $y + z = a + c$
(*c*) $y - z = a - c$
(*d*) $y - z = (a-c)^2 + 4b^2$

8. The value of $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ is

(*a*) $2\tan^n \frac{A-B}{2}$ (*b*) $2\cot^n \frac{A-B}{2}$: n is even
(*c*) 0 : n is odd (*d*) None

9. $(a + 2)\sin\alpha + (2a - 1)\cos\alpha = (2a + 1)$ if $\tan\alpha =$

(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{3}$

(*c*) $\frac{2a}{a^2+1}$ (*d*) $\frac{2a}{a^2-1}$

10. If $3\sin\beta = \sin(2\alpha+\beta)$, then $\tan(\alpha+\beta) - 2\tan\alpha$ is
(*a*) independent of α
(*b*) independent of β
(*c*) dependent of both α and β
(*d*) independent of α but dependent of β

11. If $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$, then
(*a*) A, B, C may be angles of a triangle
(*b*) $A + B + C$ is an integral multiple of π
(*c*) sum of any two of A, B, C is equal to third
(*d*) none of these

12. In a triangle $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then the values of tan A, tan B and tan C are
(*a*) 1, 2, 3 (*b*) 2, 1, 3
(*c*) 1, 2, 0 (*d*) none

13. If the sides of a right angled triangle are $\{\cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta)\}$ and $\{\sin 2\alpha + \sin 2\beta + 2\sin(\alpha+\beta)\}$, then the length of the hypotenuse is:
(*a*) $2[1+\cos(\alpha-\beta)]$ (*b*) $2[1-\cos(\alpha+\beta)]$

(*c*) $4\cos^2\frac{\alpha-\beta}{2}$ (*d*) $4\sin^2\frac{\alpha+\beta}{2}$

14. An extreme value of $1 + 4\sin\theta + 3\cos\theta$ is
(*a*) –3 (*b*) –4
(*c*) 5 (*d*) 6

15. If $x + y = z$, then $\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x \cos y \cos z$ is equal to
(*a*) $\cos^2 z$ (*b*) $\sin^2 z$
(*c*) $\cos(x + y - z)$ (*d*) 1

16. The equation $\sin^6 x + \cos^6 x = a^2$ has real solution if

(*a*) $a \in (-1, 1)$ (*b*) $a \in \left(-1, -\frac{1}{2}\right)$

(*c*) $a \in \left(-\frac{1}{2}\frac{1}{2}\right)$ (*d*) $a \in \left(\frac{1}{2}, 1\right)$

Comprehenssion – 1 (No. 17 to 19)

Let p be the product of the sines of the angles of a triangle ABC and q is the product of the cosines of the angles.

17. In this triangle $\tan A + \tan B + \tan C$ is equal to

(*a*) $p + q$ (*b*) $p - q$

(*c*) p/q (*d*) none of these

18. $\tan A \tan B + \tan B \tan C + \tan C \tan A$ is equal to

(*a*) $1 + q$ (*b*) $\frac{1+q}{q}$

(*c*) $1 + p$ (*d*) $\frac{1+p}{p}$

19. The value of $\tan^3 A + \tan^3 B + \tan^3 C$ is

(*a*) $\frac{p^3 - 3pq^2}{q^3}$ (*b*) $\frac{q^3}{p^3}$

(*c*) $\frac{p^3}{q^3}$ (*d*) xc$\frac{p^3 - 3pq}{q^3}$

Comprehension – 2 (No. 20 to 22)

If $\cos\alpha + \cos\beta = a$ and $\sin\alpha + \sin\beta = b$ and $2\theta = \alpha + \beta$, then $\sin 2\theta + \cos 2\theta = 1 + \frac{nb(a-b)}{a^2+b^2}$

where n is some integer. Also if a, b, c are 3 non-zeor terms in G.P then $b^2 = ac$. So answer the following questions:

20. The value of n is –

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) –2

21. If for n obtained in above question, $\sin^n A = x$, then $\sin A \sin 2A \sin 3A \sin 4A$ is a polynomial in x, of degree

(*a*) 5 (*b*) 6

(*c*) 7 (*d*) 8

22. If degree of polynomial obtained in previous questins is p and $(p-5) + \sin x, \cos x, \tan x$ are in G.P., then value of $(\cos^9 x + \cos^6 x + 3\cos^5 x - 1)$ is

(*a*) –1

(*b*) 0

(*c*) 1

(*d*) None of these

23. Match the column :

Column - I	**Column - II**
(*a*) $\sin 420° \cos 390° + \cos(-660°) \sin(-330°)$	(p) 0
(*b*) $\tan 315° \cot(-405°) + \cot 495° \tan(-585°)$	(q) 1
(*c*) The value of $\frac{(1+\tan 8°)(1+\tan 37°)}{(1+\tan 22°)(1+\tan 23°)}$ =	(r) 2
(*d*) Value of $\left[\frac{\pi}{4}\right] + \left[\frac{1}{3}\sin^2 x\right]$ is (where [.] represents greatest integer function)	(s) 5

24.

Column – I	**Column – II**
(*a*) If for some real x, the equation $x + \frac{1}{x} = 2\cos\theta$ holds, then $\cos\theta$ is equal to	(p) 2
(*b*) If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^{2008}\theta + \operatorname{cosec}^{2008}\theta$ is equal to	(q) 1
(*c*) Maximum value of $\sin^4\theta + \cos^4\theta$ is	(r) 0
(*d*) Least value of $2\sin^2\theta + 3\cos^2\theta$ is	(s) – 1

NUMERICAL BASED QUESTIONS

25. $\frac{\cos 5A}{\cos A} + \frac{\sin 5A}{\sin A} = a + b\cos 4A$, then find the value of $(a + b)$.

26. If $\cos(\theta - \phi), \cos\theta, \cos(\theta + \phi)$, are in H. P, then the value of $\sqrt{2}\cos\theta.\sec\frac{\phi}{2}$ is $(0 < \theta, \phi < \frac{\pi}{2})$

27. If sin A, cos A and tan A are in geometric progression, then $\cot^6 A - \cot^2 A$= ....

28. If $\operatorname{cosec}\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then $a^2 b^2 (a^2 + b^2) =$

29. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then the value of $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta$ must be

30. The value of $(1 - \cot 23°)(1 - \cot 22°)$ must be equal to

31. The numerical value of $(\tan 20^\circ + \tan 40^\circ + \sqrt{3}\tan 20^\circ \tan 40^\circ)^2$ must be ....

32. The value of

$$\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right) \text{ is}$$

33. Absolute value of $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$ is

34. The maximum value of $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals :

[*JEE Main-2019 (January)*]

(*a*) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

(*b*) $13 - 4\cos^6\theta$

(*c*) $13 - 4\cos^2\theta + 6\cos^4\theta$

(*d*) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

2. Let α and β be the roots of the quadratic equation $x^2\sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$ $(0 < \theta < 45^\circ)$, and $\alpha < \beta$. Then $\sum_{n=0}^{\infty}\left(\alpha^n + \frac{(-1)^n}{\beta^n}\right)$ is equal to :

[*JEE Main-2019 (January)*]

(*a*) $\frac{1}{1-\cos\theta} - \frac{1}{1+\sin\theta}$

(*b*) $\frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta}$

(*c*) $\frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$

(*d*) $\frac{1}{1+\cos\theta} - \frac{1}{1-\sin\theta}$

3. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is:

[*JEE Main-2019 (January)*]

(*a*) $\sqrt{19}$ (*b*) $\frac{\sqrt{79}}{2}$

(*c*) $\sqrt{34}$ (*d*) $\sqrt{31}$

4. If $\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cos\beta$; $\alpha, \beta \in [0, \pi]$, then $\cos(\alpha+\beta) - \cos(\alpha-\beta)$ is equal to :

[*JEE Main-2019 (January)*]

(*a*) 0 (*b*) –1

(*c*) $\sqrt{2}$ (*d*) $-\sqrt{2}$

5. If $\cos(\alpha+\beta) = \frac{3}{5}, \sin(\alpha-\beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to:

[*JEE Main-2019 (April)*]

(*a*) $\frac{21}{16}$ (*b*) $\frac{63}{52}$

(*c*) $\frac{33}{52}$ (*d*) $\frac{63}{16}$

6. The value of $\cos^2 10^\circ - \cos 10^\circ \cos 50^\circ + \cos^2 50^\circ$ is

[*JEE Main-2019 (April)*]

(*a*) $\frac{3}{2}(1+\cos 20^\circ)$ (*b*) $\frac{3}{4}$

(*c*) $\frac{3}{4} + \cos 20^\circ$ (*d*) $\frac{3}{2}$

7. The value of $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$ is :

[*JEE Main-2019 (April)*]

(*a*) $\frac{1}{36}$ (*b*) $\frac{1}{32}$

(*c*) $\frac{1}{18}$ (*d*) $\frac{1}{16}$

8. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the equation.

[*JEE Main-2019 (April)*]

(*a*) $\sin x = |\sin y|$ (*b*) $\sin x = 2\sin y$

(*c*) $2|\sin x| = 3\sin y$ (*d*) $2\sin x = \sin y$

9. The equation $y = \sin x \sin(x+2) - \sin^2(x+1)$ represents a straight line lying in :

[*JEE Main-2019 (April)*]

(*a*) second and third quadrants only

(*b*) third and fourth quadrants only

(*c*) first, third and fourth quadrants

(*d*) first, second and fourth quadrants

10. Let α and β be two real roots of the $(k+1)\tan^2 x - \sqrt{2}\cdot\lambda \tan x = (1-k)$, where $k (\neq -1)$ and λ are equal numbers. If $\tan^2(\alpha+\beta) = 50$, then a value of λ is

[*JEE Main-2020 (January)*]

(*a*) $10\sqrt{2}$ (*b*) $5\sqrt{2}$

(*c*) 10 (*d*) 5

11. If α and β be the coefficient of x^4 and x^2 respectively in the expansion of $\left(x+\sqrt{x^2-1}\right)^6 + \left(x-\sqrt{x^2-1}\right)^6$, then :

[*JEE Main-2020 (January)*]

(*a*) $\alpha+\beta=-30$ (*b*) $\alpha-\beta=60$

(*c*) $\alpha-\beta=-132$ (*d*) $\alpha+\beta=60$

12. If $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7}$ and $\sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, then $\tan(\alpha+2\beta)$ is equal to

[*JEE Main-2020 (January)*]

13. The value of $\cos^3\left(\frac{\pi}{8}\right)\cdot\cos\left(\frac{3\pi}{8}\right) + \sin^3\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)$ is:

[*JEE Main-2020 (January)*]

(*a*) $\frac{1}{2\sqrt{2}}$ (*b*) $\frac{1}{\sqrt{2}}$

(*c*) $\frac{1}{4}$ (*d*) $\frac{1}{2}$

14. The number of distinct solutions of the equation, $\log_{1/2}|\sin x| = 2 - \log_{1/2}|\cos x|$ in the interval $[0, 2\pi]$, is

[*JEE Main-2020 (January)*]

15. If $x = \sum_{n=0}^{\infty}(-1)^n \tan^{2n}\theta$ and $y = \sum_{n=0}^{\infty}\cos^{2n}\theta$, for $0 < \theta < \frac{\pi}{4}$, then:

[*JEE Main-2020 (January)*]

(*a*) $y(1-x)=1$ (*b*) $y(1+x)=1$

(*c*) $x(1+y)=1$ (*d*) $x(1-y)=1$

16. Let $a, b, c \in R$ be such that $a^2+b^2+c^2=1$. If $a\cos\theta = b\cos\left(\theta+\frac{2\pi}{3}\right) = c\cos\left(\theta+\frac{4\pi}{3}\right)$, where $\theta+\frac{\pi}{9}$, then the angle between the vectors $a\hat{i}+b\hat{j}+c\hat{k}$ and $b\hat{i}+c\hat{j}+a\hat{k}$ is :

[*JEE Main-2020 (September)*]

(*a*) 0 (*b*) $\frac{\pi}{9}$

(*c*) $\frac{2\pi}{3}$ (*d*) $\frac{\pi}{2}$

17. The minimum value of $2^{\sin x} + 2^{\cos x}$ is :

[*JEE Main-2020 (September)*]

(*a*) $2^{-1+\sqrt{2}}$ (*b*) $2^{1-\sqrt{2}}$

(*c*) $2^{1-\frac{1}{\sqrt{2}}}$ (*d*) $2^{-1+\frac{1}{\sqrt{2}}}$

18. If $L = \sin^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$ and $M = \cos^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$, then

[*JEE Main-2020 (September)*]

(*a*) $L = -\frac{1}{2\sqrt{2}} + \frac{1}{2}\cos\frac{\pi}{8}$ (*b*) $M = \frac{1}{2\sqrt{2}} + \frac{1}{2}\cos\frac{\pi}{8}$

(*c*) $M = \frac{1}{4\sqrt{2}} + \frac{1}{4}\cos\frac{\pi}{8}$ (*d*) $L = \frac{1}{4\sqrt{2}} - \frac{1}{4}\cos\frac{\pi}{8}$

19. If $e^{(\cos^2 x + \cos^4 x + \cos^6 x + \ldots\infty)\log_e 2}$ satisfies the equation $t^2 - 9t + 8 = 0$, then the value of $\frac{2\sin x}{\sin x + \sqrt{3}\cos x}$ $\left(0 < x < \frac{\pi}{2}\right)$ is

[*JEE Main-2021 (February)*]

(*a*) $\frac{3}{2}$ (*b*) $2\sqrt{3}$

(*c*) $\frac{1}{2}$ (*d*) $\sqrt{3}$

20. The number of integral values of 'k' for which the equation $3\sin x + 4\cos x = k+1$ has a solution, $k \in R$ is

[*JEE Main-2021 (February)*]

21. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha \in R$, then the value of $27\sec^6\alpha + 8\,\text{cosec}^6\alpha$ is equal to:

[*JEE Main-2021 (February)*]

(*a*) 350 (*b*) 500

(*c*) 400 (*d*) 250

22. If $0 < x, y < \pi$ and $\cos x + \cos y - \cos(x+y) = 3/2$, Then, $\sin x + \cos y =$

(*a*) $\frac{1+\sqrt{3}}{2}$ (*b*) $\frac{1}{2}$

(*c*) $\frac{\sqrt{3}}{2}$ (*d*) $\frac{1-\sqrt{3}}{2}$

[*JEE Main-2021 (February)*]

23. If $\tan\left(\frac{\pi}{9}\right)$, x, $\tan\left(\frac{7\pi}{18}\right)$ are in A.P. and $\tan\left(\frac{\pi}{9}\right)$, y, $\tan\left(\frac{5\pi}{18}\right)$ are also in A.P. Then, $|x-2y| =$

[*JEE Main-2021 (July)*]

(*a*) 4 (*b*) 3

(*c*) 0 (*d*) 1

24. The value of $\cot\left(\frac{\pi}{24}\right)$ is:

(*a*) $\sqrt{2}+\sqrt{3}+2-\sqrt{6}$ (*b*) $\sqrt{2}+\sqrt{3}+2+\sqrt{6}$

(*c*) $\sqrt{2}-\sqrt{3}-2+\sqrt{6}$ (*d*) $3\sqrt{2}-\sqrt{3}-\sqrt{6}$

[*JEE Main-2021 (July)*]

25. If $\sin\theta+\cos\theta=\frac{1}{2}$, then $16(\sin(2\theta)+\cos(4\theta)+\sin(6\theta))=$

(*a*) 23 (*b*) –27

(*c*) –23 (*d*) 27

[*JEE Main-2021 (July)*]

26. The value of $2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{5\pi}{8}\right)\sin\left(\frac{6\pi}{8}\right)\sin\left(\frac{7\pi}{8}\right)$ is

(*a*) $\frac{1}{4\sqrt{2}}$ (*b*) $\frac{1}{4}$

(*c*) $\frac{1}{8}$ (*d*) $\frac{1}{8\sqrt{2}}$

[*JEE Main-2021 (August)*]

JEE-ADVANCED PREVIOUS YEAR'S

1. Let $\theta,\varphi\in[0,2\pi]$ be such that **[*JEE-2012*]**

$$2\cos\theta(1-\sin\varphi)=\sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi-1$$

$$\tan(2\pi-\theta)>0 \text{ and } -1<\sin\theta<-\frac{\sqrt{3}}{2}$$

Then φ **cannot** satisfy

(*a*) $0<\varphi<\frac{\pi}{2}$ (*b*) $\frac{\pi}{2}<\varphi<\frac{4\pi}{3}$

(*c*) $\frac{4\pi}{3}<\varphi<\frac{3\pi}{2}$ (*d*) $\frac{3\pi}{2}<\varphi<2\pi$

2. Let α and β be nonzero real numbers such that $2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Then which of the following is/are true? **[*JEE Advanced-2017*]**

(*a*) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

(*b*) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(*c*) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(*d*) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

3. For non-negative integers n, let

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}$ x takes values in $[0,\pi]$, which of the following options is/are correct? **[*JEE Advanced-2019*]**

(*a*) $\sin(7\cos^{-1}f(5))=0$

(*b*) $f(d)=\frac{\sqrt{3}}{2}$

(*c*) $\lim_{x\to\infty}f(n)=\frac{1}{2}$

(*d*) If $\alpha=\tan(\cos^{-1}f(6))$, then $\alpha^2+2\alpha-1=0$

4. In a ∇ PQR let $\angle$ PQR = 30 and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE ?

[*JEE Advanced-2018*]

(*a*) $\angle QPR=45^\circ$

(*b*) The area of the traingel PQR is $25\sqrt{3}$ and $\angle QRP=120^\circ$

(*c*) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$

(*d*) The area of the circumcircle of the triangle PQR is 100π.

5. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

(*a*) Area of $\Delta SOE=\frac{\sqrt{3}}{12}$ **[*JEE Advanced-2019*]**

(*b*) Radius of incircle of $\Delta PQR\frac{\sqrt{3}}{2}(2-\sqrt{3})$

(*c*) Length of $RS=\frac{\sqrt{7}}{2}$

(*d*) Length of $OE=\frac{1}{6}$

6. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE?

(*a*) $\cos P\geq 1-\frac{p^2}{2qr}$

(*b*) $\cos R\geq\left(\frac{q-r}{p+q}\right)\cos P+\left(\frac{p-r}{p+q}\right)\cos Q$

(*c*) $\frac{q+r}{p}<2\frac{\sqrt{\sin Q\sin R}}{\sin P}$

(*d*) If $p<q$ and $p<r$, then $\cos Q>\frac{p}{r}$ and $\cos R>\frac{p}{q}$

ANSWER KEY

Topicwise Questions

1. (*b*) **2.** (*b*) **3.** (*a*) **4.** (*b*) **5.** (*a*) **6.** (*a*) **7.** (*d*) **8.** (*a*) **9.** (*b*) **10.** (*d*)
11. (*c*) **12.** (*d*) **13.** (*d*) **14.** (*d*) **15.** (*b*) **16.** (*d*) **17.** (*a*) **18.** (*a*) **19.** (*d*) **20.** (*a*)
21. (*b*) **22.** (*c*) **23.** (*a*) **24.** (*b*) **25.** (*d*) **26.** (*c*) **27.** (*d*) **28.** (*b*) **29.** (*b*) **30.** (*c*)
31. (*d*) **32.** (*c*) **33.** (*b*) **34.** (*b*) **35.** (*a*) **36.** (*b*) **37.** (*d*) **38.** (*b*) **39.** (*b*) **40.** (*c*)
41. (*b*) **42.** (*a*) **43.** (*a*) **44.** (*c*) **45.** (*c*) **46.** (*b*) **47.** (*d*) **48.** (*c*) **49.** (*d*) **50.** (*d*)
51. (*a*) **52.** (*c*) **53.** (*a*) **54.** (*c*) **55.** (*b*) **56.** (*b*) **57.** (*a*) **58.** (*d*) **59.** (*c*) **60.** (*c*)
61. (*d*)

Learning Plus

1. (*b*) **2.** (*a*) **3.** (*c*) **4.** (*a*) **5.** (*d*) **6.** (*c*) **7.** (*b*) **8.** (*c*) **9.** (*c*) **10.** (*c*)
11. (*c*) **12.** (*c*) **13.** (*b*) **14.** (*a*) **15.** (*c*) **16.** (*b*) **17.** (*d*) **18.** (*a*) **19.** (*a*) **20.** (*a*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (a, d) **2.** (a, b) **3.** (b, c) **4.** (a, c) **5.** (a, b, c, d) **6.** (c, d) **7.** (b, c) **8.** (b, c)
9. (b, d) **10.** (a, b) **11.** (a, b) **12.** (a, b) **13.** (a, c) **14.** (b, d) **15.** (c, d) **16.** (b, d) **17.** (c)
18. (b) **19.** (d) **20.** (c) **21.** (a) **22.** (b) **23.** $(a) \to (q), (b) \to (r), (c) \to (q), (d) \to (p)$
24. $(a) \to (q, s), (b) \to (p), (c) \to (q), (d) \to (p)$ **25.** [0006] **26.** [0002] **27.** [0001] **28.** [0001] **29.** [0004]
30. [0002] **31.** [0003] **32.** [1.5] **33.** [0.0625] **34.** [10]

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*b*) **2.** (*c*) **3.** (*a*) **4.** (*d*) **5.** (*d*) **6.** (*b*) **7.** (*d*) **8.** (*a*) **9.** (*b*) **10.** (*c*)
11. (*c*) **12.** (*a*) **13.** (*a*) **14.** [8] **15.** (*a*) **16.** (*d*) **17.** (*c*) **18.** (*b*) **19.** (*c*) **20.** (11)
21. (*d*) **22.** (*a*) **23.** (*c*) **24.** (*b*) **25.** (*c*) **26.** (*c*)

JEE-ADVANCED PREVIOUS YEAR'S

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

1. (a, c, d) **2.** (b, c) **3.** (a, b, d) **4.** (b, c, d) **5.** (b, c, d) **6.** (a, b)

3 Trigonometric Equation

SOLUTION OF TRIGONOMETRIC EQUATIONS

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, ...$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as:

1. Principal solution
2. General solution.

PRINCIPAL SOLUTIONS

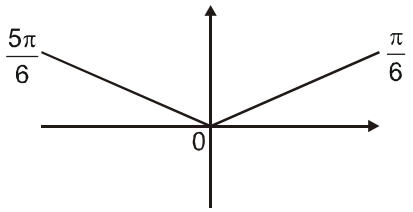
The solutions of trigonometric equation which lie in the interval $[0, 2\pi)$ are called **principal solutions.**

TRAIN YOUR BRAIN

Example 1: Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.

Sol. $\because \quad \sin x = \frac{1}{2}$

$\because$ there exists two values



i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$.

$\therefore$ Principal solutions of the equation

$\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

GENERAL SOLUTION

Since trigonometric functions are periodic function, therefore solution of trigonometric equations can be genralised with the help periodicity of a trigonometrical function. The solution consisting of all possible solution of a trigonometrical equation is called the general solution.

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **General solution.**

Some Important General Solutions of Trigonometric Equations

1. $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in I$
2. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I$
3. $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in I$
4. $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}, n \in I$
5. $\cos\theta = 1 \Rightarrow \theta = 2n\pi, \ n \in I$
6. $\tan\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{4}, n \in I$
7. $\sin\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{2}, n \in I$
8. $\cos\theta = -1 \Rightarrow \theta = (2n-1)\pi, n \in I$
9. $\tan\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{4}, n \in I$

General Solution of Some Standard Trigonometric Equations

(*a*) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where, $n \in I$.
(*b*) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where, $n \in I$.
(*c*) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where, $n \in I$.
(*d*) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
(*e*) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
(*f*) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
[**Note:** α is called the principal angle]
Proof:

(*i*) $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$$\Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}$$

or $\frac{\theta-\alpha}{2} = m\pi$, where $m \in I$.

$\Rightarrow \theta = (2m+1)\pi - \alpha$

or $\theta = 2m\pi + \alpha$, where $m \in I$.

$\Rightarrow \theta = (2m+1)\pi + (-1)^{2m+1}\alpha$

or $\theta = 2m\pi + (-1)^{2m}\alpha$

$\Rightarrow \theta = n\pi + (-1)^n \alpha, n \in I$.

(*ii*) $\cos\theta = \cos\alpha$

$\Rightarrow \quad \cos\alpha - \cos\theta = 0$

$\Rightarrow \quad 2\sin\left(\frac{\alpha+\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \sin\frac{\theta+\alpha}{2} = 0$

or $\quad \sin\frac{\theta-\alpha}{2} = 0, \frac{\theta+\alpha}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \quad \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha \quad n \in I$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$.

(*iii*) $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\theta - \cos\theta.\sin\alpha = 0$

$\Rightarrow \sin(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = n\pi$

$\Rightarrow \theta = n\pi + \alpha$, where $n \in I$

(*iv*) $\sin^2\theta = \sin^2\alpha$

$\sin^2\theta - \sin^2\alpha = \sin(\theta+\alpha)\sin(\theta-\alpha) = 0$

$\sin(\theta+\alpha) = 0$ or $\sin(\theta-\alpha) = 0$

$\theta + \alpha = n\pi$ or $\theta - \alpha = n\pi, n \in I$

$\theta = n\pi \pm \alpha, n \in I$

(*v*) $\cos^2\theta = \cos^2\alpha \Rightarrow 1 - \sin^2\theta = 1 - \sin^2\alpha$

$\Rightarrow \sin^2\theta - \sin^2\alpha$

$\theta = n\pi \pm \alpha, n \in I$

(vi) $\tan^2\theta = \tan^2\alpha \Rightarrow \tan\theta = \pm\tan\alpha = \tan(\pm\alpha)$

$\Rightarrow \theta = n\pi \pm \alpha$, where $n \in I$

TRAIN YOUR BRAIN

Example 2: Solve $4\cos^2\theta - 4\sin\theta - 1 = 0, 0 \le \theta \le 2\pi$.

Sol. $4(1-\sin^2\theta) - 4\sin\theta - 1 = 0$

$\Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0$

$\Rightarrow \sin\theta = \frac{1}{2}, -\frac{3}{2}$ (Not possible)

$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Thus we have two principal solutions.

Example 3: Solve : $\sqrt{3}\sec 2\theta = 2$.

Sol. $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

$\theta = n\pi \pm \frac{\pi}{12}, n \in I$

TYPES OF TRIGONOMETRIC EQUATION

Type I: Solutions of Equations by Factorising

Solution of trigonometric equation by factorization or equation which are expressed in quadratic form or which can be expressed in quadratic form.

TRAIN YOUR BRAIN

Example 4: Solve $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$ in $[0, 2\pi]$.

Sol. $(1+\cos x)(2\sin x - \cos x) - (1-\cos^2 x) = 0$

$(1+\cos x)[(2\sin x - \cos x) - (1 - \cos x)]$

$= 0, (1+\cos x)(2\sin x - 1) = 0$

$\sin x = \frac{1}{2}, \cos x = -1$.

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi$.

Example 5: Find the general solutions of equation $1 - \cos x \cot x = \cot x - \cos x$.

Sol. $(1+\cos x)(1-\cot x) = 0$

$\Rightarrow \quad \cos x = -1$ or $\cot x = 1$

$\Rightarrow \quad (2n+1)\pi, \quad n\pi + \pi/4, \ n \in I$

Example 6: Find the general solutions of equation $3\cos^2 x - 10\cos x + 3 = 0$.

Sol. $(3\cos x - 1)(\cos x - 3) = 0 \Rightarrow \cos x = \frac{1}{3}, 3$ but $\cos x \ne 3$

$\therefore \quad \cos x = \frac{1}{3} \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

Example 7: Find the solutions of the equation,

$\log_{\sqrt{2}\sin x}(1+\cos x) = 2$ in the interval $x \in [0, 2\pi]$.

Sol. $2\sin^2 x = 1 + \cos x$; $2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad \cos x = \frac{1}{2}$ or -1

$\Rightarrow \quad x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ but $x = \pi$ and $\frac{5\pi}{3}$ are rejected

$\Rightarrow \quad x = \frac{\pi}{3}$

Type-II: Solutions of Equations Reducible to Quadratic Equations

Rules:

1. Transform the terms to be a only one trigonometric ratio involving angles in same form.
2. Factorize the equation and express it in $f(x) \times g(x) = 0$
   $\Rightarrow f(x) = 0$ or $g(x) = 0$
3. Solve both the equations one by one to get the general value of the variables.

TRAIN YOUR BRAIN

Example 8: Solve the equation

$$\sin^2 x(\tan x + 1) = 3 \sin x (\cos x - \sin x) + 3.$$

Sol. The given equation makes no sense when $\cos x = 0$; therefore we can suppose that $\cos x \neq 0$. Nothing that the right-hand member of the equation is equal to $3 \sin x \cos x + 3 \cos^2 x$, and dividing both members by $\cos^2 x$, we obtain $\tan^2 x (\tan x + 1) = 3 (\tan x + 1)$,

$$\Rightarrow (\tan^2 x - 3)(\tan x + 1) = 0$$

$$\Rightarrow x_1 = -\frac{\pi}{4} + k\pi, x_2 = \frac{\pi}{3} + k\pi, x_3 = -\frac{\pi}{3} + k\pi.$$

Example 9: Find the general solution set of the equation

$$\log_{\tan x}(2 + 4\cos^2 x) = 2.$$

Sol. We have $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

$$\Rightarrow 2 + 4\cos^2 x = \tan^2 x$$

$$\Rightarrow 3 + 4\cos^2 x = \sec^2 x$$

$$\Rightarrow 4\cos^4 x + 3\cos^2 x - 1 = 0$$

let $\cos^2 x = t \Rightarrow 4t^2 + 3t - 1 = 0 \Rightarrow (4t - 1)(t + 1) = 0$

$$\Rightarrow t = 1/4 \text{ or } t = -1$$

$$\Rightarrow \cos^2 x = \frac{1}{4} \text{ or } \cos^2 x = -1 \text{ (not possible)}$$

$$\Rightarrow \cos^2 x = \cos^2 \frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3},\ n \in I$$

TYPE-III:

Solving equations by introducing an auxiliary argument

A Trigonometric Equation is of the form

$$a \cos\theta \pm b \sin\theta = c$$

Rule: 1. Divide by $\sqrt{a^2 + b^2}$ on both the sides

2. Reduce the given equation into either $\sin(\theta \pm a)$ or $\cos(\theta \pm a)$
3. Simplify the given equation.

TRAIN YOUR BRAIN

Example 10: Solve $\sin x + \cos x = \sqrt{2}$

Sol. $\because$ $\sin x + \cos x = \sqrt{2}$ ....(i)

$\therefore$ Divide both sides of equation (i) by $\sqrt{2}$,

We get $\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 1$

$$\Rightarrow \sin x \cdot \sin\frac{\pi}{4} + \cos x \cdot \cos\frac{\pi}{4} = 1$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1$$

$$\Rightarrow x - \frac{\pi}{4} = 2n\pi, n \in I$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{4}, n \in I$$

$\therefore$ Solution of given equation is $2n\pi + \frac{\pi}{4}, n \in I$

Example 11: Solve the equation

$$\cos 7x - \sin 5x = \sqrt{3}\ (\cos 5x - \sin 7x).$$

Sol. Rewrite the equation in the form

$$\frac{1}{2}\cos 7x + \frac{\sqrt{3}}{2}\sin 7x = \frac{\sqrt{3}}{2}\cos 5x + \frac{1}{2}\sin 5x$$

$$\text{or } \sin\frac{\pi}{6}\cos 7x + \cos\frac{\pi}{6}\sin 7x = \sin\frac{\pi}{3}\cos 5x + \cos\frac{\pi}{3}\sin 5x, \text{ i.e. } \sin\left(\frac{\pi}{6} + 7x\right) = \sin\left(\frac{\pi}{3} + 5x\right).$$

But $\sin\alpha = \sin\beta$ if and only if either $\alpha - \beta = 2k\pi$ or $\alpha + \beta = (2m + 1)\pi$ $(k, m = 0, \pm 1, \pm 2, ...)$.

Hence $\frac{\pi}{6} + 7x - \frac{\pi}{3} - 5x = 2k\pi$

$$\text{or } \frac{\pi}{6} + 7x + \frac{\pi}{3} + 5x = (2m + 1)\pi.$$

Thus, the roots for the equation are

$$\left.\begin{aligned} x &= \frac{\pi}{12}(12k + 1), \\ x &= \frac{\pi}{24}(4m + 1) \end{aligned}\right\} (k, m = 0, \pm 1, \pm 2, ...).$$

TYPE-IV:

Solving equations by Transforming a sum of Trigonometric functions into a product

TRAIN YOUR BRAIN

Example 12: General solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is

(a) $\frac{n\pi}{3}$ (b) $\frac{2n\pi}{3}$

(c) $2n\pi$ (d) $n\pi$

Sol. $2\sin 3x \cos 2x = 2\sin 3x \cos x$

$$\Rightarrow \sin 3x = 0 \text{ or } \cos 2x = \cos x$$

$$\Rightarrow 3x = n\pi \text{ or } 2x = 2n\pi \pm x$$

$$\therefore x = \frac{n\pi}{3}, 2n\pi, \frac{2n\pi}{3} \Rightarrow x = \frac{n\pi}{3}$$

Example 13: Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Sol. $\cos 3x + \sin 2x - \sin 4x = 0$

$$\Rightarrow \cos 3x + 2\cos 3x \cdot \sin(-x) = 0$$

$$\Rightarrow \cos 3x - 2\cos 3x \cdot \sin x = 0$$

$$\Rightarrow \cos 3x (1 - 2\sin x) = 0$$

$$\Rightarrow \cos 3x = 0$$

$$\text{or } 1 - 2\sin x = 0$$

$\Rightarrow \quad 3x = (2n+1)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is $(2n+1)\frac{\pi}{6}, n \in I$

or $n\pi + (-1)^n \frac{\pi}{6}, n \in I$

TYPE-V:

Solving equations by transforming a product of trigonometric functions into a sum

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Example 14: Number of solutions of the trigonometric equation in $[0, \pi]$, $\sin 3\theta = 4\sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$.

(*a*) 4 (*b*) 6

(*c*) 8 (*d*) 10

Sol. $\sin 3\theta = 4 \sin\theta \sin(3\theta - \theta)\sin(3\theta + \theta)$

$= 4 \sin\theta (\sin^2 3\theta - \sin^2\theta)$

$\Rightarrow \quad \sin 3\theta + 4\sin^3\theta = 4 \sin\theta \sin^2 3\theta$

$\Rightarrow \quad 3 \sin\theta = 4 \sin\theta \sin^2 3\theta$

$\Rightarrow \quad \sin\theta = 0$ or $\sin^2 3\theta = \frac{3}{4}$

$\Rightarrow \quad \theta = n\pi$ or $3\theta = n\pi \pm \frac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi$ or $\theta = \frac{n\pi}{3} \pm \frac{\pi}{9}, n \in I$

$\Rightarrow \quad \theta = 0, \pi, \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

Example 15: Find the number of solutions of the equations, $\sin x + 2\sin 2x = 3 + \sin 3x$ in $[0, \pi]$

(*a*) No solution (*b*) Infinite solution

(*c*) Exactly one solution (*d*) Two solutions

Sol. We have,

$\Rightarrow \quad \sin 3x - \sin x - 2\sin 2x + 3 = 0$

$\Rightarrow \quad 2\cos 2x \cdot \sin x - 4 \sin x \cdot \cos x + 3 = 0$

$\Rightarrow \quad \sin x (2\cos 2x - 4\cos x) + 3 = 0$

$\Rightarrow \quad \sin x (4\cos^2 x - 4\cos x - 2) + 3 = 0$

$\Rightarrow \quad \sin x (4\cos^2 x - 4\cos x + 1) + 3 - 3\sin x = 0$

$\Rightarrow \quad \sin x (2\cos x - 1)^2 + 3(1 - \sin x) = 0 \quad \ldots(i)$

since $x \in [0, \pi]$,

$\therefore \quad \sin x \geq 0$ and $1 - \sin x \geq 0$

$\therefore$ each part in equation (i) must be zero.

i.e. $\sin x (2\cos x - 1)^2 = 0$ and

$3(1 - \sin x) = 0$

from the second equation of system we have

$\sin x = 1$

$\Rightarrow \quad \cos x = 0$ hence $\sin x (2\cos^2 x - 1)^2 \neq 0$

$\therefore$ not a single solution of the second equation is a solution of the first.

Hence the original equation has no real solution.

TYPE-VI:

SOLVING EQUATIONS BY A CHANGE OF VARIABLE

(i) Equations of the form

$P(\sin x \pm \cos x, \sin x \cdot \cos x) = 0$,

where $P(y, z)$ is a polynomial, can be solved by the change $\cos x \pm \sin x = t$

$\Rightarrow \quad 1 \pm 2\sin x \cdot \cos x = t^2$.

(ii) Equations of the form $a \cdot \sin x + b \cdot \cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle.

(iii) Many equations can be solved by introducing a new variable. eg. the equation

$\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to

$2(y+1)\left(y - \frac{1}{2}\right) = 0$ by substituting,

$\sin 2x \cdot \cos 2x = y$.

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Example 16: Solve the equation $\sin 2x - 12(\sin x - \cos x) + 12 = 0$

Sol. Putting $\sin x - \cos x = t$ and using the identity $(\sin x - \cos x)^2 = 1 - 2\sin x \cos x$, we rewrite the original equation in the form $t^2 + 12t - 13 = 0$.

This equation has the roots $t_1 = -13$ and $t_2 = 1$. But $t = \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$, and thus, $|t| \leq \sqrt{2}$. Consequently, the root $t_1 = -13$ must be discarded. Therefore, the original equation is reduced to the equation $\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. $\Rightarrow \quad x_1 = \pi + 2k\pi$,

$x_2 = \frac{\pi}{2} + 2k\pi$.

Example 17: Solve the equation $1 + 2\csc x = -\frac{\sec^2 \frac{x}{2}}{2}$.

Sol. Transform the given equation to the form

$2\cos^2 \frac{x}{2}(2 + \sin x) + \sin x = 0$.

Using the formula $2\cos^2\frac{x}{2} = 1 + \cos x$ and opening the brackets, we obtain

$$2 + 2(\sin x + \cos x) + \sin x \cdot \cos x = 0. \quad ...(1)$$

By the substitution $\sin x + \cos x = t$ equation (1) is reduced to the quadratic equation $t^2 + 4t + 3 = 0$ whose roots are $t_1 = -1$ and $t_2 = -3$. Since $|\sin x + \cos x| \le \sqrt{2}$, the original equation can only be satisfied by the roots of the equation

$$\sin x + \cos x = -1. \quad ...(2)$$

Solving equation (2), we obtain $x_1 = -\frac{\pi}{2} + 2k\pi$ and $x_2 = (2k+1)\pi$.

here x_2 should be discarded because $\sin x_2 = 0$, and therefore the original equation makes no sense for $x = x_2 \Rightarrow x = -\frac{\pi}{2} + 2k\pi$

TYPE-VII:

Solving equations with the use of the Boundness of the functions sin x & cos x

Remember:

$-1 \le \sin x \le 1, -1 \le \cos x \le 1, \tan x \in R, \cot x \in R.$

$$|\operatorname{cosec} x| \ge 1, \quad |\sec x| \ge 1.$$

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Example 18: Solve for x : $\cos x + \cos 2x + \cos 3x = 3$.

Sol. $\cos x = 1$ and $\cos 2x = 1$ and $\cos 3x = 1$

when $\cos x = 1$

$\Rightarrow x = 2n\pi, n \in I$

when $\cos 2x = 1$

$\Rightarrow x = \frac{2n\pi}{2} = n\pi, n \in I$

when $\cos 3x = 1$

$\Rightarrow 3x = 2n\pi$

$\Rightarrow x = \frac{2n\pi}{3}, n \in I$

$2n\pi, n \in I$ **Ans.**

Example 19: Solve for x : $\sin^3 x - \cos^3 x = 1 + \sin x \cos x$.

Sol. $(\sin x - \cos x)(\sin^2 x + \cos^2 x + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow (\sin x - \cos x)(1 + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow \sin x - \cos x = 1$

$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

$\Rightarrow x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$

$\Rightarrow x = 2n\pi \pm \frac{3\pi}{4} - \frac{\pi}{4}$

$\Rightarrow x = (2n-1)\pi, 2n\pi + \frac{\pi}{2}$

or $1 + \sin x \cos x = 0$

$2 \sin x \cos x = -2$

$\sin 2x = -2$

$\Rightarrow$ No solution.

Example 20: Solve for

$$x: \sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$$

Sol. $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow \sin\left(\frac{5x}{4}\right) = 1$

$\Rightarrow \frac{5x}{4} = (4n+1)\frac{\pi}{2}$

$\Rightarrow x = (4n+1)\frac{2\pi}{5}$

and $\cos x = 1$

$x = 2m\pi$

$x = 0, \pm 2\pi, \pm 4\pi$

Period of the given equation is 8π.

$\therefore$ Consider $x \in [0, 8\pi)$

$\Rightarrow x = \frac{2\pi}{5}, 2\pi, \frac{18\pi}{5}, \frac{26\pi}{5}, \frac{34\pi}{5}$

Common solution $= 2\pi$

$\therefore 8n\pi + 2\pi = 2\pi(4n+1)$ **Ans.**

Example 21: If $x, y \in [0, 2\pi]$ then find the total number of order pairs (x, y) satisfying the equation $\sin x \cos y = 1$.

Sol. We have $\sin x \cos y = 1$

$\Rightarrow \sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

If $\sin x = 1, \cos y = 1 \Rightarrow x = \frac{\pi}{2}, y = 0, 2\pi$

If $\sin x = -1, \cos y = -1$

$\Rightarrow x = \frac{3\pi}{2}, y = \pi$

Then the possible ordered pairs are

$$\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, \pi\right).$$

Example 22: Solve the equation $\frac{1 - \tan x}{1 + \tan x} = 1 + \sin 2x$.

Sol. The equation makes no sense for $x = \frac{\pi}{2} + k\pi$ and for $x = -\frac{\pi}{4} + k\pi$. For all the other values of x it is equivalent to the equation $\frac{\cos x - \sin x}{\cos x + \sin x} = 1 + \sin 2x$.

After simple transformations we obtain $\sin x (3 + \sin 2x + \cos 2x) = 0$.

It is obvious that the equation $\sin 2x + \cos 2x + 3 = 0$ has no solution, and therefore, the original equation is reduced to the equation $\sin x = 0 \Rightarrow x = k\pi$

TYPE-VIII:

Solution of trigonometric equation of the form $f(x) = \sqrt{\phi(x)}$.

(i) $f(x) \geq 0, \phi(x) \geq 0$ (ii) $f^2(x) = \phi(x)$

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Example 23: Solve for x, $\sqrt{(1-\cos x)} = \sin x$.

Sol. $\sin x \geq 0$ and $1 - \cos x \geq 0$ ...(i)

$\therefore \quad 1 - \cos x = \sin^2 x = 1 - \cos^2 x$

$\Rightarrow \quad (1 - \cos x)[1 - (1 + \cos x)] = 0$

$\Rightarrow \quad \cos x = 0, 1$

when $\cos x = 0, x = 2n\pi \pm \frac{\pi}{2}$ but $\sin x \geq 0$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2}$

when $\cos x = 1, \quad x = 2n\pi$

Both are satisfying in equality (i)

MISCELLANEOUS QUESTIONS

A trigonometric equation is of the form

$b_0 \sin^n x + b_1 \sin^{n-1} x . \cos x +$

$b_2 \sin^{n-2} x . \cos^2 x + \ldots\ldots + b_n \cos^n x = 0$

where, $b_0, b_1, b_2, \ldots\ldots b_n \in R$ is a homogenous equation of sin x and cos x, where cos x is nonzero.

Rule : 1. Divide both the sides by highest power of cos x.

2. The given equation can be reduced to

$b_0 \tan^n x + b_1 \tan^{n-1} x + \ldots\ldots + b_n = 0$

3. and then use the factorization method.

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Example 24: Solve : $2\sin^2 x - 5 \sin x \cos x - 8\cos^2 x = -2$

Sol. The given equation can be written as

$2\sin^2 x - 5 \sin x \cos x - 8 \cos^2 x$

$= -2(\sin^2 x + \cos^2 x)$

$\Rightarrow 2 \tan^2 x - 5 \tan x - 8 = -2(\tan^2 x + 1)$

$\Rightarrow 4 \tan^2 x - 5 \tan x - 6 = 0$

$\Rightarrow (\tan x - 2)(4\tan x + 3) = 0$

$\Rightarrow \tan x = 2, \ \tan x = -\frac{3}{4}$

$\Rightarrow x = n\pi + \alpha, x = m\pi + \beta,$

where $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), n \in Z$.

TRIGONOMETRIC INEQUALITIES AND SYSTEM OF INEQUALITY

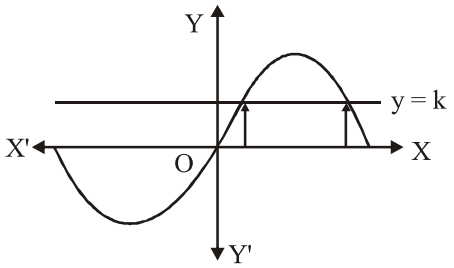
To solve the trigonometric inequalities of the type $f(x) \leq a$ or $f(x) \geq a$, where f(x) is some trigonometric ratio we take following steps

1. Draw the graph of f(x) in an interval length equal to the fundamental period of f(x).
2. Draw the line y = a.
3. Take the portion of the graph for which the inequality is satisfied.
4. To generalise, add $p \cdot n, n \in I$ and in the final solution where p is the fundamental period of f(x).

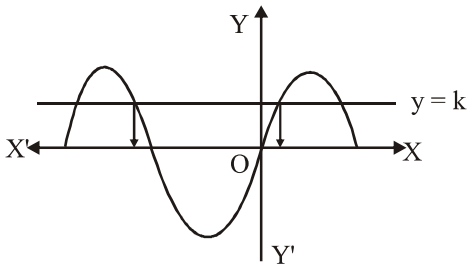
Suppose we have to solve $f(x) > k$ or $f(x) < k$.

When we solve the inequation, we often use the graphs of the functions y = f(x) and y = k.

Then, the solution of the inequality $f(x) > k$ is the values of x for which the point (x, f(x)) of the graph of y = f(x) lies above the striaght line y = k.

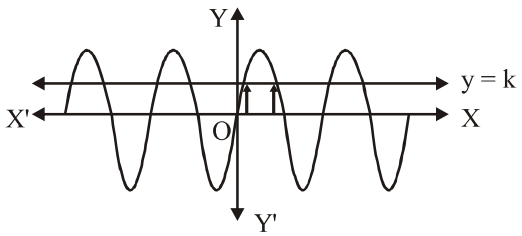


Similarly, when we solve $f(x) < k$, then the solution of the inequalition $f(x) < k$ is the values of x for which the point (x, f(x)) of the graph of y = f(x) lies below the striaght line y = k.



- An inequation is of the form $\sin x > k, \cos x < k, \cos x > k, \tan x > k, \tan x < k$.

Rule: Find the smallest values of x that satisfies the given inequation and then add $2n\pi$ with that values of x.



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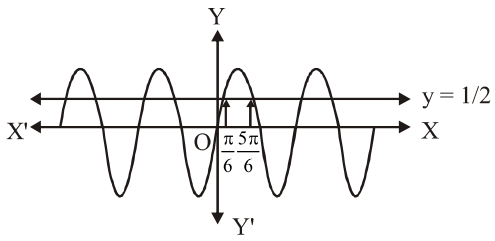
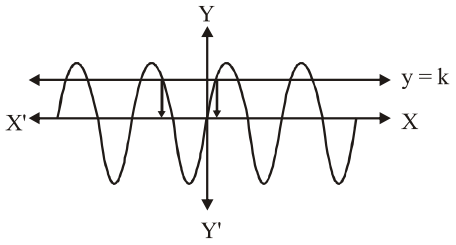
Example 25: Solve $\sin x > 1/2$.

Sol. Here, we should construct the graph of $y = \sin x$ and $y = 1/2$.



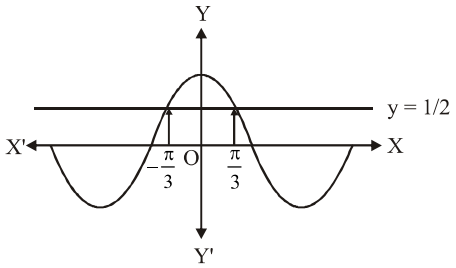
Hence, the solution set is

$$x = \bigcup_{n\in I}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right).$$



Example 26: Solve : $\cos x \geq \frac{1}{2}$.

Sol. Here, we should draw the graph of $y = \cos x$ and $y = 1/2$



Hence, the solution set is

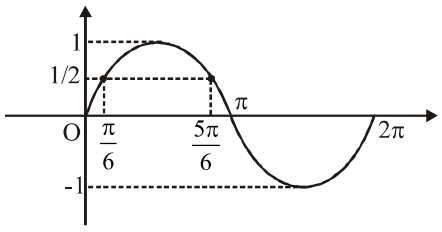
$$x = \bigcup_{n\in I}\left(2n\pi-\frac{\pi}{3}, 2n\pi+\frac{\pi}{3}\right).$$

Example 27: Solve : $2\sin^2\theta - \sin\theta \geq 0$, where $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

Sol. $\sin\theta\,(2\sin\theta - 1) \geq 0$ which is possible. only where

$$\sin\theta \geq \frac{1}{2} \quad \text{or} \quad \sin\theta \leq 0$$

$$\sin\theta \geq \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$$



$$\sin\theta \leq 0 \quad \Rightarrow \quad \pi \leq \theta \leq \frac{3\pi}{2}$$

$$\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$$

Example 28: Solve: $\sin\theta + \sqrt{3}\cos\theta \geq 1$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

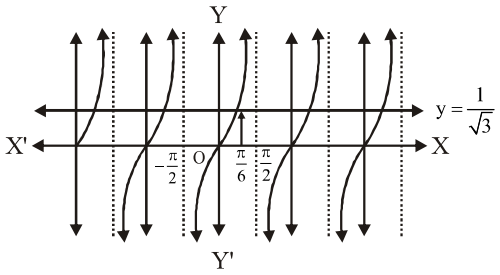
Sol. $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \geq \frac{1}{2}$

$$\sin\left(\theta+\frac{\pi}{3}\right) \geq \frac{1}{2}$$

$$\frac{\pi}{6} \leq \theta+\frac{\pi}{3} \leq \frac{5\pi}{6} \quad \Rightarrow \quad -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

Example 29: Solve : $\tan x \geq \frac{1}{\sqrt{3}}$

Sol. Here, we should draw the graphs of $y \tan x$ and $y = \frac{1}{\sqrt{3}}$.



Hence, the solution set is

$$x = \bigcup_{n\in I}\left[n\pi+\frac{\pi}{6}, n\pi+\frac{\pi}{2}\right)$$

IMPORTANT POINTS

1. Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

   To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of n = ...–2, –1, 0, 1, 2, 3..... etc. and then to find the angles in [0, 2π]. If all the angles in both solutions are same, the solutions are equivalent.

2. While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation

tanx = 2 sinx. Here by dividing both sides by sinx, we get cosx = $\frac{1}{2}$. This is not equivalent to the original equation. Here the roots obtained by sinx = 0, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

3. While equating one of the factors to zero, take care of the other factor that it should not become infinite. For example, if we have the equation sinx = 0, which can be written as cos x tan x = 0. Here we cannot put cosx = 0, since for cos x = 0, tanx = sinx/ cosx is infinite.

4. **Avoid squaring:** When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.
For example : Consider the equation,
$\sin\theta + \cos\theta = 1$ ...(1)
Squaring we get
$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ ...(2)
i.e. $2\theta = n\pi$ or $\theta = n\pi/2$,
This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$
Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$
and $\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1, \neq 1.$
The reason for this is simple.
The equation (2) is not equivalent to (1) and (2) contains two equations : $\sin\theta + \cos\theta = 1$
and $\sin\theta + \cos\theta = -1$. Therefore we get extra solutions. Thus if squaring is must, verify each of the solution.

5. **Some necessary restrictions:** If the equation involves tanx, secx, take $\cos x \neq 0$. If cot x or cosec x appear, take $\sin x \neq 0$.
If log appear in the equation, i.e. log [f(θ)] appear in the equation, use $f(\theta) > 0$ and base of log $> 0, \neq 1$.
Also note that $\sqrt{[f(\theta)]}$ is always positive, for example $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm\sin\theta$.

6. **Verification :** Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.
$a = OA - OB = l\cos\beta - l\cos\alpha$
$b = OP - OQ = l\sin\alpha - l\sin\beta$

$$\Rightarrow \frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}$$

$$\Rightarrow a = b\tan\frac{\alpha+\beta}{2}$$

Topicwise Questions

SOLUTION BY FACTORIZATION

1. If $2\tan^2\theta = \sec^2\theta$, then the general value of θ is [where $n \in I$]

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$

(c) $n\pi \pm \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{4}$

2. If $\sqrt{3}\tan 2\theta + \sqrt{3}\tan 3\theta + \tan 2\theta \tan 3\theta = 1$, then the general value of θ is [where $n \in I$]

(a) $n\pi + \frac{\pi}{5}$ (b) $\left(n + \frac{1}{6}\right)\frac{\pi}{5}$

(c) $\left(2n \pm \frac{1}{6}\right)\frac{\pi}{5}$ (d) $\left(n + \frac{1}{3}\right)\frac{\pi}{5}$

3. If $\tan 2\theta \tan\theta = 1$, then the general value of θ is [where $n \in I$]

(a) $\left(n + \frac{1}{2}\right)\frac{\pi}{3}$ (b) $\left(n + \frac{1}{2}\right)\pi$

(c) $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{3}$ (d) None of these

4. The solution of the equation $4\cos^2 x + 6\sin^2 x = 5$ [where $n \in I$]

(a) $x = n\pi \pm \frac{\pi}{2}$ (b) $x = n\pi \pm \frac{\pi}{4}$

(c) $x = n\pi \pm \frac{3\pi}{2}$ (d) Both (a) and (c)

5. If $\sin\left(\frac{\pi}{4}\cot\theta\right) = \cos\left(\frac{\pi}{4}\tan\theta\right)$, [where $n \in I$] then θ may be equal to

(a) $n\pi + \frac{\pi}{4}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $n\pi - \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{6}$

6. The general value θ is obtained from the equation $\cos 2\theta = \sin\alpha$, is[where $n \in I$]

(a) $2\theta = \frac{\pi}{2} - \alpha$ (b) $\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$

(c) $\theta = \frac{n\pi + (-1)^n\alpha}{2}$ (d) $\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

7. General solution of $\tan 5\theta = \cot 2\theta$ is [where $n \in I$]

(a) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$ (b) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}$

(c) $\theta = \frac{n\pi}{7} + \frac{\pi}{2}$ (d) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$

8. The general value of θ in the equation $2\sqrt{3}\cos\theta = \tan\theta$, is [where $n \in I$]

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $n\pi + (-1)^n\frac{\pi}{3}$ (d) $n\pi + (-1)^n\frac{\pi}{4}$

9. If $(2\cos x - 1)(3 + 2\cos x) = 0, 0 \le x \le 2\pi$, [where $n \in I$] then $x =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{2}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)$ (d) $\frac{5\pi}{3}$

10. If $\tan(\cot x) = \cot(\tan x)$, [where $n \in I$] then $\sin 2x =$

(a) $(2n+1)\frac{\pi}{4}$ (b) $\frac{4}{(2n+1)\pi}$

(c) $4\pi(2n+1)$ (d) $\frac{\pi}{2}(2n-1)$

11. If $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$, [where $n \in I$] then the general value of θ is

(a) $2n\pi + \frac{\pi}{4}$ (b) $(2n+1)\pi + \frac{\pi}{4}$

(c) $n\pi + \frac{\pi}{4}$ (d) $n\pi \pm \frac{\pi}{4}$

12. If $\tan\theta = -\frac{1}{\sqrt{3}}$ and $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, [where $n \in I$] then the principal value of θ will be

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ (d) $-\frac{\pi}{6}$

13. The solution set of the equation $4\sin\theta.\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$ in the interval $(0, 2\pi)$ is

(a) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ (b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

(c) $\left\{\frac{3\pi}{4}, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (d) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

14. All solutions of the equation $2\sin\theta + \tan\theta = 0$ are obtained by taking all integral values of m and n in:

(a) $2n\pi + \frac{2\pi}{3}, n \in I$

(b) $n\pi$ or $2m\pi \pm \frac{2\pi}{3}$ where $n, m \in I$

(c) $n\pi$ or $m\pi \pm \frac{\pi}{3}$ where $n, m \in I$

(d) $n\pi$ or $2m\pi \pm \frac{\pi}{3}$ where $n, m \in I$

15. The most general solution of $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is

(a) $n\pi + \frac{7\pi}{4}, n \in I$ (b) $n\pi + (-1)^n\frac{7\pi}{4}, n \in I$

(c) $2n\pi + \frac{7\pi}{4}, n \in I$ (d) None of these

16. If $2\cos^2(\pi + x) + 3\sin(\pi + x)$ vanishes then the values of x lying in the interval from 0 to 2π are

(a) $x = \pi/6$ or $5\pi/6$ (b) $x = \pi/3$ or $5\pi/3$

(c) $x = \pi/4$ or $5\pi/4$ (d) $x = \pi/2$ or $5\pi/2$

17. If $\cos 2\theta + 3\cos\theta = 0$, then

(a) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{\sqrt{17}-3}{4}\right)$

(b) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{-\sqrt{17}-3}{4}\right)$

(c) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{\pm\sqrt{17}-3}{4}\right)$

(d) none of these

18. The general solution of the equation, $2\cos 2x = 3.2\cos^2 x - 4$ is

(a) $x = 2n\pi, n \in I$ (b) $x = n\pi, n \in I$

(c) $x = n\pi/4, n \in I$ (d) $x = n\pi/2, n \in I$

19. If $x \in (0, 1)$, then greatest root of the equation $\sin 2\pi x = \sqrt{2}\cos\pi x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) $\frac{3}{4}$ (d) $\frac{1}{3}$

20. The general solution of $\sec\theta + \tan\theta = \sqrt{3}$ is

(a) $\theta = 2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}, n \in I$

(b) $\theta = 2n\pi + \frac{\pi}{6}, n \in I$

(c) $2n\pi - \frac{\pi}{6}, n \in I$

(d) $2n\pi - \frac{\pi}{2}, n \neq 2k+1, n, k \in I$

21. The general solution of $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0$ is

(a) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, \theta = n\pi, n \in I$

(b) $\theta = n\pi, n \in I$

(c) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$

SOLVING SIMULTANEOUS EQUATION

22. The most general value of θ which will satisfy both the equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is [where $n \in I$]

(a) $n\pi + (-1)^n\frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{6}$ (d) $2n\pi + \frac{7\pi}{6}$

23. If $\sin A = \sin B$ and $\cos A = \cos B$, then

(a) $A = B + n\pi, n \in I$ (b) $A = B - n\pi, n \in I$

(c) $A = 2n\pi + B, n \in I$ (d) $A = n\pi - B, n \in I$

SOLUTION OF EQUATION REDUCIBLE TO QUADRATIC EQUATION

24. If $3(\sec^2\theta + \tan^2\theta) = 5$, [where $n \in I$] then the general value of θ is

(a) $2n\pi + \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{6}$

(c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi \pm \frac{\pi}{3}$

25. If $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$, [where $n \in I$] the general value of θ is

(a) $n\pi \pm \frac{\pi}{3}$ (b) $n\pi \pm \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{6}$

26. If $\tan^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=0$, [where $n\in I$] then the general value of θ is

(a) $n\pi+\frac{\pi}{4},n\pi+\frac{\pi}{3}$ (b) $n\pi-\frac{\pi}{4},n\pi+\frac{\pi}{3}$

(c) $n\pi+\frac{\pi}{4},n\pi-\frac{\pi}{3}$ (d) $n\pi-\frac{\pi}{4},n\pi-\frac{\pi}{3}$

27. If $4\sin^2\theta+2(\sqrt{3}+1)\cos\theta=4+\sqrt{3}$, [where $n\in I$] then the general value of θ is

(a) $2n\pi\pm\frac{\pi}{3}$ (b) $2n\pi+\frac{\pi}{4}$

(c) $n\pi\pm\frac{\pi}{3}$ (d) $n\pi-\frac{\pi}{3}$

28. The equation $3\sin^2 x+10\cos x-6=0$ is satisfied, if [where $n\in I$]

(a) $x=n\pi\pm\cos^{-1}(1/3)$

(b) $x=2n\pi\pm\cos^{-1}(1/3)$

(c) $x=n\pi\pm\cos^{-1}(1/6)$

(d) $x=2n\pi\pm\cos^{-1}(1/6)$

29. The solution of equation $\cos^2\theta+\sin\theta+1=0$ lies in the interval [where $n\in I$]

(a) $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$

(c) $\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$ (d) $\left(\frac{5\pi}{4},\frac{7\pi}{4}\right)$

30. The smallest positive angle which satisfies the equation $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$, is

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

31. If $12\cot^2\theta-31\operatorname{cosec}\theta+32=0$, then the value of $\sin\theta$ is

(a) $\frac{3}{5}$ or 1 (b) $\frac{2}{3}$ or $\frac{-2}{3}$

(c) $\frac{4}{5}$ or $\frac{3}{4}$ (d) $\pm\frac{1}{2}$

32. If $5\cos^2\theta+7\sin^2\theta-6=0$, then the general value of θ is

(a) $2n\pi\pm\frac{\pi}{4}$ (b) $n\pi\pm\frac{\pi}{4}$

(c) $n\pi+(-1)^n\frac{\pi}{4}$ (d) $n\pi+(-1)^n\frac{\pi}{2}$

SUM TO PRODUCT AND PRODUCT TO SUM

33. If $\cos 7\theta=\cos\theta-\sin 4\theta$, then the general value of θ is [where $n\in I$]

(a) $\frac{n\pi}{4},\frac{n\pi}{3}+\frac{\pi}{18}$ (b) $\frac{n\pi}{3},\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$

(c) $\frac{n\pi}{4},\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$ (d) $\frac{n\pi}{6},\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$

34. If $\cos\theta+\cos 2\theta+\cos 3\theta=0$, then the general value of θ is [where $m\in I$]

(a) $\theta=2m\pi\pm\frac{2\pi}{3}$ (b) $\theta=m\pi\pm\frac{\pi}{4}$

(c) $\theta=m\pi\pm(-1)^m\frac{2\pi}{3}$ (d) Both (a)&(b)

Equation of the type a sinx + b cos x = C

35. If $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$, then the most general value of θ is [where $n\in I$]

(a) $n\pi+(-1)^n\frac{\pi}{4}$ (b) $(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$

(c) $n\pi+\frac{\pi}{4}-\frac{\pi}{3}$ (d) $n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$

36. The equation $\sqrt{3}\sin x+\cos x=4$ has

(a) Only one solution
(b) Two solutions
(c) Infinitely many solutions
(d) No solution

37. The equation $3\cos x+4\sin x=6$ has

(a) Finite solution (b) Infinite solution
(c) One solution (d) No solution

38. The general solution of equation sinx + sin5x = sin2x + sin4x is :

(a) $\frac{n\pi}{2}$; $n\in I$ (b) $\frac{n\pi}{5}$; $n\in I$

(c) $\frac{n\pi}{3}$; $n\in I$ (d) $\frac{2n\pi}{3}$; $n\in I$

SOLUTION USING THE BOUNDNESS

39. One root of the equation $\cos x-x+\frac{1}{2}=0$ lies in the interval

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{2},0\right]$

(c) $\left[\frac{\pi}{2},\pi\right]$ (d) $\left[\pi,\frac{3\pi}{2}\right]$

40. If $\sec x \cos 5x + 1 = 0$, where $0 < x < 2\pi$, then $x =$

(*a*) $\frac{\pi}{5}, \frac{\pi}{5}$ (*b*) $\frac{\pi}{5}$

(*c*) $\frac{\pi}{4}$ (*d*) $\frac{\pi}{3}$

41. If the equation $1 + \sin^2 x\theta = \cos\theta$ has a non-zero solution in θ, then x must be

(*a*) An integer (*b*) A rational number

(*c*) An irrational number (*d*) None of these

42. The number of the solution of the equation $3 \sin x + 4 \cos x - x^2 - 16 = 0$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

43. The solution set of $\sin^4 x - \tan^8 x = 1$ is given by

(*a*) $x = 2n\pi + \frac{\pi}{8}, n \in I$ (*b*) $x = n\pi + \frac{\pi}{12}, n \in I$

(*c*) $x = 2n\pi + \frac{\pi}{24}, n \in I$ (*d*) None of these

Learning Plus

1. $\sin 3\theta = 4 \sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$ in $0 \le \theta \le \pi$ has

(*a*) 2 real solutions (*b*) 4 real solutions

(*c*) 6 real solutions (*d*) 8 real solutions

2. General solution of the equation, $\cot 3\theta - \cot\theta = 0$ is

(*a*) $\theta = (2n-1)\frac{\pi}{2}, n \in I$ (*b*) $\theta = (2n-1)\frac{\pi}{4}, n \in I$

(*c*) $\theta = (2n-1)\frac{\pi}{3}, n \in I$ (*d*) None of these

3. The arithmetic mean of the roots of the equation $4\cos^3 x - 4\cos^2 x - \cos(\pi + x) - 1 = 0$ in the interval $[0, 315]$ is equal to

(*a*) 49π (*b*) 50π

(*c*) 51π (*d*) 100π

4. $\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$ if

(*a*) $\theta = n\pi + \frac{\pi}{3}, n \in I$ (*b*) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

(*c*) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in I$ (*d*) $\theta = n\pi + \frac{\pi}{6}, n \in I$

5. If $\sin\theta + 7\cos\theta = 5$, then $\tan(\theta/2)$ is a root of the equation

(*a*) $x^2 - 6x + 1 = 0$ (*b*) $6x^2 - x - 1 = 0$

(*c*) $6x^2 + x + 1 = 0$ (*d*) $x^2 - x + 6 = 0$

6. $\frac{\sin 3\theta}{2\cos 2\theta + 1} = \frac{1}{2}$ if

(*a*) $\theta = n\pi + \frac{\pi}{6}, n \in I$ (*b*) $\theta = 2n\pi - \frac{\pi}{6}, n \in I$

(*c*) $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$ (*d*) $\theta = n\pi - \frac{\pi}{6}, n \in I$

7. The general solution of the equation $\tan x + \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x + \frac{2\pi}{3}\right) = 3$ is

(*a*) $\frac{n\pi}{4} + \frac{\pi}{12}, n \in I$

(*b*) $\frac{n\pi}{3} + \frac{\pi}{6}, n \in I$

(*c*) $\frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

(*d*) None

8. A triangle ABC is such that $\sin(2A + B) = \frac{1}{2}$. If A, B, C are in A.P. then the angle A, B, C are respectively

(*a*) $\frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{3}$ (*b*) $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$

(*c*) $\frac{\pi}{3}, \frac{\pi}{4}, \frac{5\pi}{12}$ (*d*) $\frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{4}$

9. The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is

(*a*) ϕ

(*b*) $(\pi/4)$

(*c*) $\{n\pi + \pi/4 \mid n = 1, 2, 3.....\}$

(*d*) $\{2n + \pi/4 \mid n = 1, 2, 3 .....\}$

10. The number of solutions of $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + 4\sin 4\theta = 10$ in $(0, \pi)$ is

(*a*) 1 (*b*) 2

(*c*) 4 (*d*) 0

11. The number of all possible triplets (a_1, a_2, a_3) such that : $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) infinite

12. The value 'a' for which the equation $4\text{cosec}^2(\pi(a+x)) + a^2 - 4a = 0$ has a real solution is :

(*a*) $a = 1$ (*b*) $a = 2$

(*c*) $a = 3$ (*d*) None of these

13. The solution of $|\cos x| = \cos x - 2\sin x$ is

(*a*) $x = n\pi, n \in I$

(*b*) $x = n\pi + \frac{\pi}{4}, n \in I$

(*c*) $x = n\pi + (-1)^n \frac{\pi}{4}, n \in I$

(*d*) $(2n+1)\pi + \frac{\pi}{4}, n \in I$

14. If m, $n \in N$ $(n > m)$, then number of solutions of the equation $n|\sin x| = m|\sin x|$ in $[0, 2\pi]$ is

(*a*) m (*b*) n

(*c*) mn (*d*) 3

15. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $x \in (0, \pi)$, then

(*a*) $x = \frac{\pi}{4}, y = 1$

(*b*) $y = 0$

(*c*) $y = 2$

(*d*) $x = \frac{3\pi}{4}$

16. The number of values of x for which $\sin 2x + \cos 4x = 2$ is

(*a*) Zero (*b*) 1

(*c*) 2 (*d*) Infinite

17. The number of solutions of the equation $\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1$ is

(*a*) Zero (*b*) 1

(*c*) 2 (*d*) Infinite

18. The number of values of $\theta \in [0, 2\pi]$ satisfying $r \sin\theta = \sqrt{3}$ and $r + 4\sin\theta = 2(\sqrt{3}+1)$ is

(*a*) 4 (*b*) 5

(*c*) 6 (*d*) 7

19. The solution set of the equation $4\sin^4 x + \cos^4 x = 1$ is

(*a*) $x = n\pi + \frac{\pi}{6}$

(*b*) $x = n\pi \pm \theta$ where $\cos^2\theta = \frac{3}{5}$ and $x = n\pi, n \in I$

(*c*) $x = (2n+1)\frac{\pi}{2}$

(*d*) $x = 2n\pi \pm \frac{\pi}{3}$

20. The number of solutions of the equation $\sin x \cdot \cos x (\cos x - \sin x)^2 \cdot (\sin x + \cos x) = \lambda$, where $\lambda > \frac{1}{2\sqrt{2}}$ in the interval $[0, 4\pi]$, is

(*a*) 0 (*b*) 2

(*c*) 4 (*d*) 8

21. The complete set of values of x, $x \in \left(-\frac{\pi}{2}, \pi\right)$ satisfying the inequality $\cos 2x > |\sin x|$ is:

(*a*) $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

(*b*) $\left(-\frac{\pi}{2}, \frac{\pi}{6}\right) \cup \left(-\frac{\pi}{6}, \frac{5\pi}{6}\right)$

(*c*) $\left(-\frac{\pi}{2}, -\frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

(*d*) $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

22. Values of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x} = 1$ are in A.P. whose common difference is:

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{2}$ (*d*) $\frac{2\pi}{3}$

23. If α and β are distinct roots of the equation $a\cos\theta + b\sin\theta = c$, then $\cos(\alpha + \beta) =$

(*a*) $\frac{2ab}{a^2 + b^2}$ (*b*) $\frac{2ab}{a^2 - b^2}$

(*c*) $\frac{a^2 - b^2}{a^2 + b^2}$ (*d*) $\frac{a^2 + b^2}{a^2 - b^2}$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. If $\sin(x - y) = \cos(x + y) = 1/2$ then the values of x & y lying between 0 and π are given by:

(*a*) $x = \pi/4, y = 3\pi/4$

(*b*) $x = \pi/4, y = \pi/12$

(*c*) $x = 5\pi/4, y = 5\pi/12$

(*d*) $x = 11\pi/12, y = 3\pi/4$

2. $\cos 15x = \sin 5x$ if

(*a*) $x = -\frac{\pi}{20} + \frac{n\pi}{5}, n \in I$

(*b*) $x = \frac{\pi}{40} + \frac{n\pi}{10}, n \in I$

(*c*) $x = \frac{3\pi}{20} + \frac{n\pi}{5}, n \in I$

(*d*) $x = -\frac{3\pi}{40} + \frac{n\pi}{10}, n \in I$

3. $\sin x - \cos^2 x - 1$ assumes the least value for the set of values of x given by:

(*a*) $x = n\pi + (-1)^{n+1}(\pi/6), n \in I$

(*b*) $x = n\pi + (-1)^n(\pi/6), n \in I$

(*c*) $x = n\pi + (-1)^n(\pi/3), n \in I$

(*d*) $x = n\pi - (-1)^n(\pi/6), n \in I$

4. The equation $2\sin\frac{x}{2} \cdot \cos^2 x + \sin^2 x = 2\sin\frac{x}{2} \cdot \sin^2 x + \cos^2 x$ has a root for which

(*a*) $\sin 2x = 1$ (*b*) $\sin 2x = -1$

(*c*) $\cos x = \frac{1}{2}$ (*d*) $\cos 2x = -\frac{1}{2}$

5. $5\sin^2 x + \sqrt{3}\sin x \cos x + 6\cos^2 x = 5$ if

(*a*) $\tan x = -1/\sqrt{3}$

(*b*) $\sin x = 0$

(*c*) $x = n\pi + \pi/2, n \in I$

(*d*) $x = n\pi + \pi/6, n \in I$

6. $\sin^2 x + 2\sin x \cos x - 3\cos^2 x = 0$ if

(*a*) $\tan x = 3$

(*b*) $\tan x = -1$

(*c*) $x = n\pi + \pi/4, n \in I$

(*d*) $x = n\pi + \tan^{-1}(-3), n \in I$

7. $\sin^2 x - \cos 2x = 2 - \sin 2x$ if

(*a*) $x = n\pi/2, n \in I$

(*b*) $\tan x = 3/2$

(*c*) $x = (2n + 1)\pi/2, n \in I$

(*d*) $x = n\pi + (-1)^n \sin^{-1}(2/3), n \in I$

8. $\sin x, \sin 2x, \sin 3x$ are in A.P if

(*a*) $x = n\pi/2, n \in I$ (*b*) $x = n\pi, n \in I$

(*c*) $x = 2n\pi, n \in I$ (*d*) $x = (2n + 1)\pi, n \in I$

9. $\sin x + \sin 2x + \sin 3x = 0$ if

(*a*) $\sin x = 1/2$ (*b*) $\sin 2x = 0$

(*c*) $\sin 3x = \sqrt{3}/2$ (*d*) $\cos x = -1/2$

10. $\cos 4x \cos 8x - \cos 5x \cos 9x = 0$ if

(*a*) $\cos 12x = \cos 14x$ (*b*) $\sin 13x = 0$

(*c*) $\sin x = 0$ (*d*) $\cos x = 0$

11. The general solution of the equation $\cos x \cdot \cos 6x = -1$, is

(*a*) $x = (2n + 1)\pi, n \in I$ (*b*) $x = 2n\pi, n \in I$

(*c*) $x = (2n - 1)\pi, n \in I$ (*d*) None of these

12. The general solution of the equation $\cos x \cdot \cos 6x = -1$, is :

(*a*) $x = (2n + 1)\pi, n \in I$

(*b*) $x = 2n\pi, n \in I$

(*c*) $x = (2n - 1)\pi, n \in I$

(*d*) none of these

Comprehension – 1 (No. 13 to 15)

Let a, b, c, d $\in$ R. Then the cubic equation of the type $ax^3 + bx^2 + cx + d = 0$ has either one root real or all three roots are real. But in case of trigonometric equations of the type $a\sin^3 x + b\sin^2 x + c\sin x + d = 0$ can possess several solutions depending upon the domain of x.

To solve an equation of the type $a\cos\theta + b\sin\theta = c$. The equation can be written as $\cos(\theta - \alpha) = c/\sqrt{(a^2 + b^2)}$.

The solution is $\theta = 2n\pi + \alpha \pm \beta$, where $\tan\alpha = b/a$, $\cos\beta = c/\sqrt{(a^2 + b^2)}$.

13. On the domain $[-\pi, \pi]$ the equation $4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$ possess

(*a*) only one real root (*b*) three real roots

(*c*) four real roots (*d*) six real roots

14. In the interval $[-\pi/4, \pi/2]$, the equation, $\cos 4x + \frac{10\tan x}{1+\tan^2 x} = 3$ has

(*a*) no solution (*b*) one solution

(*c*) two solutions (*d*) three solutions

15. $|\tan x| = \tan x + \frac{1}{\cos x}$ $(0 \le x \le 2\pi)$ has

(*a*) no solution (*b*) one solution

(*c*) two solutions (*d*) three solutions

Comprehension – 2 (No. 16 to 18)

Equation of the form P (sinx ± cosx, sinx cosx)=0 where P(y, z) is a polynomial, can be solved by the change : cos x ± sin x = t ; 1 ± 2 sin x cos x = t^2. Reduce the given equation into $P\left(t, \frac{t^2-1}{2}\right) = 0$

16. General solutin of sin x + cos x = 1 + sin x cos x is

(*a*) $\frac{\pi}{2} + 2\pi n$ & $2n\pi$ (*b*) $\frac{\pi}{4} + 2\pi n$ & $(2n+1)\pi$

(*c*) $2n\pi$ (*d*) None of these

17. If $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$, then x is

(*a*) $n\pi \pm \frac{\pi}{3}$ (*b*) $2n\pi \pm \frac{\pi}{3}$

(*c*) $n\pi \pm \frac{\pi}{6}$ (*d*) $2n\pi \pm \frac{\pi}{6}$

18. $\sin^4 x + \cos^4 x = \sin x \cos x$ then x is–

(*a*) $(6n+1)\frac{\pi}{6}$ (*b*) $n\pi$

(*c*) $(4n+1)\frac{\pi}{4}$ (*d*) None of these

19. Match the following for number of solutions in $[0, 2\pi]$

Column - I	**Column - II**
(*a*) $\sin^2\theta - \tan^2\theta = 1$	(p) 2
(*b*) $\sin\theta + \cos\theta = 1$	(q) 0
(*c*) $\tan\theta + \sec\theta = 2\cos\theta$	(r) 3
(*d*) $3\sin^2\theta - 4\sin\theta + 1 = 0$	(s) 1

20.

Column - I	**Column - II**
(*a*) $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$	(p) $\sin x = 1/2$
(*b*) $1 + \sin 2x = \cos x + \sin x$	(q) $\tan x = -1$
(*c*) $4x^4 + x^6 + \sin^2 5x = 0$	(r) $x = 0$
(*d*) $\tan x = 1/\sqrt{3}$	(s) $x = 19\pi/6$

NUMERICAL BASED QUESTIONS

21. Number of values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

22. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan\alpha$ is:

23. The number of solution of the equation $1 + \sin x \sin^2 x/2 = 0$ in $[-\pi, \pi]$ is

24. Number of roots of the equation $\cos^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$ which lie in the interval $[-\pi, \pi]$ is

25. The number of solutions of

$\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in $[0, 2p]$ is

26. The number of values of x between 0 and 2π that satisfies the equation

$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ must be

27. $\sin 3\theta = 4\sin\theta \sin 2\theta \sin 4\theta$ in $0 \le \theta \le \pi$ has how many real solutions :

28. If $2\tan^2 x - 5\sec x - 1 = 0$ has 7 different roots in $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, Find the greatest value of n.

29. If $x \in \left[0, \frac{\pi}{2}\right]$, then find the number of solutions of the equation $\sin 7x + \sin 4x + \sin x = 0$.

30. Find the number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is: [*JEE Main-2016*]

(*a*) 3 (*b*) 5
(*c*) 7 (*d*) 9

2. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, the the value of $\cos 4x$ is: [*JEE Main-2017*]

(*a*) $-\frac{7}{9}$ (*b*) $-\frac{3}{5}$

(*c*) $\frac{1}{3}$ (*d*) $\frac{2}{9}$

3. If sum of all the solutions of the equation $8\cos x \cdot \left(\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0, \pi]$ is $k\pi$, then k is equal to : [*JEE Main-2018*]

(*a*) $\frac{13}{9}$ (*b*) $\frac{8}{9}$

(*c*) $\frac{20}{9}$ (*d*) $\frac{2}{3}$

4. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is [*JEE Main-2019 (January)*]

(*a*) 2 (*b*) 1
(*c*) 3 (*d*) 4

5. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is: [*JEE Main-2019 (January)*]

(*a*) π (*b*) $\frac{5\pi}{4}$

(*c*) $\frac{\pi}{2}$ (*d*) $\frac{3\pi}{8}$

6. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ldots\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is [*JEE Main-2019 (January)*]

(*a*) $\frac{1}{512}$ (*b*) $\frac{1}{1024}$

(*c*) $\frac{1}{256}$ (*d*) $\frac{1}{2}$

7. Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is [*JEE Main-2019 (April)*]

(*a*) $\frac{13\pi}{6}$ (*b*) π

(*c*) 2π (*d*) $\frac{5\pi}{3}$

8. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x$, $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : [*JEE Main-2019 (April)*]

(*a*) 5 (*b*) 4
(*c*) 7 (*d*) 3

9. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to : [*JEE Main-2019 (April)*]

(*a*) [2, 6] (*b*) [3, 7]
(*c*) R (*d*) [1, 4]

10. If the equation $\cos^4\theta + \sin^4\theta + \lambda = 0$ has real solutions for θ, then λ lies in the interval [*JEE Main-2020 (September)*]

(*a*) $\left[-1, -\frac{1}{2}\right]$ (*b*) $\left[-\frac{3}{2}, -\frac{5}{4}\right]$

(*c*) $\left(-\frac{1}{2}, -\frac{1}{4}\right]$ (*d*) $\left(-\frac{5}{4}, -1\right)$

11. If $\sqrt{3}(\cos^2 x) = (\sqrt{3}-1)\cos x + 1$, the number of solutions of the given equation when $x \in \left[0, \frac{x}{2}\right]$ is [*JEE Main-2021 (February)*]

12. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in: [*JEE Main-2021 (February)*]

(*a*) $\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

(*b*) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

(*c*) $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(*d*) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

13. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10} \sin x + \log_{10} \cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$ then the value of n is equal to:

[*JEE Main-2021 (March)*]

(*a*) 20 (*b*) 12
(*c*) 9 (*d*) 16

14. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is:

[*JEE Main-2021 (March)*]

(*a*) 3 (*b*) 4
(*c*) 2 (*d*) 5

15. The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$ in the interval $[0, 2\pi]$ is:

[*JEE Main-2021 (March)*]

16. Sum of all values of x in $[0, 2\pi]$ for which $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$, is equal to:

[*JEE Main-2021 (July)*]

(*a*) 8π (*b*) 11π
(*c*) 9π (*d*) 12π

17. The sum of the solution of the equation $\frac{\cos x}{1+\sin x} = |\tan 2x|$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}, -\frac{\pi}{4}\right\}$ is:

[*JEE Main-2021 (August)*]

(*a*) $\frac{\pi}{10}$ (*b*) $-\frac{11\pi}{30}$
(*c*) $-\frac{7\pi}{30}$ (*d*) $-\frac{\pi}{15}$

18. Let S be sum of all solution (in radian) of the equation $\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$ in $[0, 4\pi]$. Then $8s/\pi =$

[*JEE Main-2021 (August)*]

19. Number of solutions of the equation $32^{\tan^2 x} + 32^{\sec^2 x} = 81$, $0 \le x \le \frac{\pi}{4}$ is

[*JEE Main-2021 (August)*]

(*a*) 3 (*b*) 1
(*c*) 0 (*d*) 2

20. If n is the number of solutions of the equation $2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right) = 1, x \in [0, \pi]$ and S is the sum of all these solutions, then the ordered pair (n, S) is

[*JEE Main-2021 (September)*]

(*a*) $\left(3, \frac{13\pi}{19}\right)$ (*b*) $\left(2, \frac{2\pi}{3}\right)$
(*c*) $\left(2, \frac{8\pi}{9}\right)$ (*d*) $\left(3, \frac{5\pi}{3}\right)$

JEE-ADVANCED PREVIOUS YEAR'S

1. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has [*JEE Advanced-2014*]

(*a*) infinitely many solutions
(*b*) three solutions
(*c*) one solution
(*d*) no solution

2. The number of distinct solutions of the equation [*JEE Advanced-2015*]

$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is

3. Let $S = \left\{x \in (-\pi, \pi) : x \ne 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to

[*JEE Advanced-2016*]

(*a*) $-\frac{7\pi}{9}$ (*b*) $-\frac{2\pi}{9}$
(*c*) 0 (*d*) $\frac{5\pi}{9}$

4. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

[*JEE Advanced-2016*]

(*a*) $2(\sec\theta - \tan\theta)$ (*b*) $2\sec\theta$
(*c*) $-2\tan\theta$ (*d*) 0

5. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to [*JEE Advanced-2016*]

(*a*) $3 - \sqrt{3}$ (*b*) $2\left(3 - \sqrt{3}\right)$
(*c*) $2\left(\sqrt{3} - 1\right)$ (*d*) $2\left(2 + \sqrt{3}\right)$

6. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a\cos x + 2b\sin x = c$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is _____ [*JEE Advanced-2018*]

Comprehension for 7 and 8

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

[*JEE Advanced-2019*]

$X = \{x : f(x) = 0\}, Y = \{x : f'(x) = 0\}$

$Z = \{x : g(x) = 0\}, W = \{x : g'(x) = 0\},$

List-I contains the sets X, Y, Z and W. List - II contains some information regarding these sets.

List-I		List-II
(I)	X	(P) $\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q) an arithmetic progression
(III)	Z	(R) NOT an arithmetic progression
(IV)	W	(S) $\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T) $\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

7. Which of the following is the only CORRECT combination ?

(*a*) (II), (R), (S)

(*b*) (I), (P), (R)

(*c*) (II), (Q), (T)

(*d*) (I), (Q), (U)

8. Which of the following is the only CORRECT combination ?

(*a*) (IV), (Q), (T)

(*b*) (IV), (P), (R), (S)

(*c*) (III), (R), (U)

(*d*) (III), (P), (Q), (U)

9. Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is ....

[*JEE(Advanced)-2020*]

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

ANSWER KEY

Topicwise Questions

1. (*c*)	**2.** (*b*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)	**6.** (*d*)	**7.** (*a*)	**8.** (*c*)	**9.** (*b*)	**10.** (*b*)
11. (*b*)	**12.** (*b*)	**13.** (*d*)	**14.** (*b*)	**15.** (*c*)	**16.** (*a*)	**17.** (*a*)	**18.** (*b*)	**19.** (*c*)	**20.** (*b*)
21. (*b*)	**22.** (*d*)	**23.** (*c*)	**24.** (*c*)	**25.** (*c*)	**26.** (*a*)	**27.** (*a*)	**28.** (*b*)	**29.** (*d*)	**30.** (*a*)
31. (*c*)	**32.** (*b*)	**33.** (*c*)	**34.** (*d*)	**35.** (*d*)	**36.** (*d*)	**37.** (*d*)	**38.** (*c*)	**39.** (*a*)	**40.** (*c*)
41. (*b*)	**42.** (*a*)	**43.** (*d*)							

Learning Plus

1. (*d*)	**2.** (*a*)	**3.** (*b*)	**4.** (*b*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*b*)	**9.** (*a*)	**10.** (*d*)
11. (*d*)	**12.** (*b*)	**13.** (*d*)	**14.** (*d*)	**15.** (*a*)	**16.** (*a*)	**17.** (*b*)	**18.** (*a*)	**19.** (*b*)	**20.** (*a*)
21. (*d*)	**22.** (*c*)	**23.** (*c*)							

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*b*, *d*)	**2.** (*a*, *b*, *c*, *d*)	**3.** (*a*, *d*)	**4.** (*a*, *b*, *c*, *d*)	**5.** (*a*, *c*)	**6.** (*c*, *d*)	**7.** (*b*, *c*)			
8. (*b*, *c*, *d*)	**9.** (*b*, *d*)	**10.** (*a*, *b*, *c*)	**11.** (*a*,*c*)	**12.** (*a*, *c*)	**13.** (*d*)	**14.** (*c*)	**15.** (*b*)	**16.** (*a*)	**17.** (*b*)
18. (*c*)	**19.** (*a*) $\rightarrow$ (q), (*b*) $\rightarrow$ (r), (*c*) $\rightarrow$ (p), (*d*) $\rightarrow$ (r)			**20.** (*a*) $\rightarrow$ (p), (*b*) $\rightarrow$ (q), (*c*) $\rightarrow$ (r), (*d*) $\rightarrow$ (s)					
21. [0006]	**22.** [0000]	**23.** [0000]	**24.** [0004]	**25.** [0000]					
26. [0004]	**27.** [0008]	**28.** [15]	**29.** [5]	**30.** [2]					

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*)	**2.** (*a*)	**3.** (*a*)	**4.** (*a*)	**5.** (*c*)	**6.** (*a*)	**7.** (*c*)	**8.** (*a*)	**9.** (*a*)	**10.** (*a*)
11. (*a*)	**12.** (*b*)	**13.** (*b*)	**14.** (*a*)	**15.** (*a*)	**16.** (*c*)	**17.** (*a*)	**18.** [56]	**19.** (*b*)	**20.** (*a*)

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*d*)	**2.** [8]	**3.** (*c*)	**4.** (*c*)	**5.** (*c*)	**6.** [0.5]	**7.** (*c*)	**8.** (*b*)	**9.** (1.00)

PRAYAS1.0: t.me/realprayasians
PR[illegible]sians2

4 Solution of Triangle

INTRODUCTION

In any Δ, the three sides and the three angles are generally called the elements of the triangle.

A triangle which does not contain a right angle is called an oblique triangle.

In any ΔABC, the measures of the angles ∠BAC, ∠CBA and ∠ACB are denoted by the letters.

A, B and C respectively, and the sides BC, AC and AB opposite to the angles A, B and C are respectively denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations.

(*i*) $A+B+C=\pi$

(*ii*) $a+b>c;\ b+c>a;\ c+a>b$

SINE FORMULA

In any triangle the sides are proportional to the sines of the opposite angles i.e.

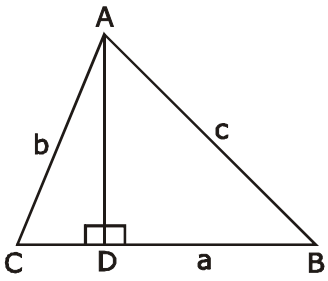
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

(1) Let the triangle ABC be acute-angled.

From A draw AD perpendicular to the opposite side; then

AD = AB sin (∠ABD) = c sin B and AD = AC sin (∠ABCD) = b sin C

$\therefore \quad b\sin C=c\sin B$ i.e. $\frac{b}{\sin B}=\frac{c}{\sin C}$



(2) Let the triangle ABC have an obtuse angle at B

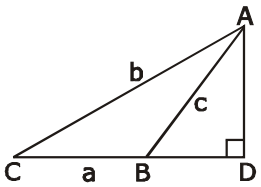
Draw AD perpendicular to CB produced; then

AD = AC sin ∠ACD = b sin C and

AD = AB sin ∠ABD

= c sin (180° – B) = c sin B;

$\therefore b\sin C=c\sin B$ i.e. $\frac{b}{\sin B}=\frac{c}{\sin C}$



In a similar manner it may be proved that either of these ratios is equal to $\frac{a}{\sin A}$

Thus $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

TRAIN YOUR BRAIN

Example 1: If the angles of a ΔABC are $\frac{\pi}{7}, \frac{2\pi}{7}$ and $\frac{4\pi}{7}$ and R is the radius of the circumcircle then $a^2+b^2+c^2$ has the value equal to

Sol. $a^2+b^2+c^2=4R^2(\sin^2A+\sin^2B+\sin^2C)=2R^2\left[1-\cos\frac{2\pi}{7}+1-\cos\frac{4\pi}{7}+1-\cos\frac{8\pi}{7}\right]$

$=2R^2[3-(\cos\theta+\cos2\theta+\cos4\theta)]$ where $\theta=2\pi/7$

now let $S=\cos\theta+\cos2\theta+\cos3\theta$ ($\cos4\theta=\cos3\theta$)

$$2\sin\frac{\theta}{2}S=\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}+\sin\frac{7\theta}{2}-\sin\frac{5\theta}{2}=\sin\frac{7\theta}{2}-\sin\frac{\theta}{2}$$

$$=\sin\pi-\sin\frac{\theta}{2}$$

$$=-\sin\frac{\theta}{2},\ S=-\frac{1}{2}$$

$\Rightarrow a^2+b^2+c^2=2R^2(3+1/2)=7R^2$

Example 2: In a triangle ABC, a cos A + b cos B + c cos C = s. Prove that the triangle is equilateral.

Sol. The given result can be written as 2a cos A + 2b cos B + 2c cos C = a + b + c

Using sine rule we get 2 sin A cos A + 2 sin B cos B + 2 sin C cos C = sin A + sin B + sin C

$\Rightarrow \sin2A+\sin2B+\sin2C=\sin A+\sin B+\sin C$

$$\Rightarrow 4\sin A\sin B\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$\Rightarrow \quad 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1$

$\Rightarrow \quad 4 \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] \sin \frac{C}{2} = 1$

$\Rightarrow \quad 4 \sin^2 \frac{C}{2} - 4 \cos \frac{A-B}{2} \sin \frac{C}{2} + 1 = 0$. This is a quadratic equation in $\sin \frac{C}{2}$ which must have real roots.

Hence $16 \cos^2 \frac{A-B}{2} \le 1$

$\Rightarrow \cos^2 \frac{A-B}{2} \le 1$. But $\cos^2 \frac{A-B}{2} \le 1$

$\Rightarrow \cos^2 \frac{A-B}{2} = 1 \Rightarrow A = B$

Similarly it can be prove that $B = C \Rightarrow A = B = C$

Example 3: In any ΔABC, prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}}$.

Sol. Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (let)

$\Rightarrow \quad a = k \sin A, b = k \sin B$ and $c = k \sin C$

$\therefore \quad$ L.H.S. $= \frac{a+b}{c}$

$= \frac{k(\sin A + \sin B)}{k \sin C}$

$= \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin \frac{C}{2} \cos \frac{C}{2}} = \frac{\cos \frac{C}{2} \cos\left(\frac{A-B}{2}\right)}{\sin \frac{C}{2} \cos \frac{C}{2}}$

$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}} =$ R.H.S.

Hence L.H.S. = R.H.S. Proved

COSINE FORMULA

To find an expression for one side (c) of a triangle in terms of other to sides and the included angle (C).

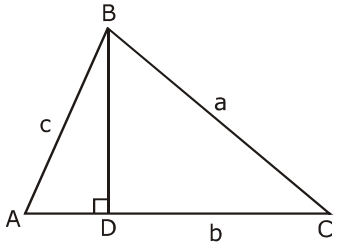
(1) Let C be an acute angle.

Draw BD perpendicular to AC ;

$AB^2 = BC^2 + CA^2 - 2AC \cdot CD$;

$\therefore \quad c^2 = a^2 + b^2 - 2ba \cos C$

$= a^2 + b^2 - 2ab \cos C$.



(2) Let C be an obtuse angle.

Draw BD perpendicular to AC produced ;

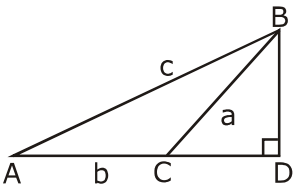
$AB^2 = BC^2 + CA^2 + 2AC \cdot CD$;

$\therefore \quad c^2 = a^2 + b^2 + 2ba \cos BCD$

$= a^2 + b^2 + 2ab \cos (180^\circ - C)$

$= a^2 + b^2 - 2ab \cos C$

Hence in each case, $c^2 = a^2 + b^2 - 2ab \cos C$



Similarly it may be shown that

$a^2 = b^2 + c^2 - 2bc \cos A$ and b^2

$= c^2 + a^2 - 2ac \cos B$

From the above result we obtain

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$;

$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$;

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

These results enable us to find the cosines of the angles when the numerical values of the sides are given.

TRAIN YOUR BRAIN

Example 4: In a ΔABC, prove that $a (b \cos C - c \cos B) = b^2 - c^2$

Sol. $\because$ We have to prove $a (b \cos C - c \cos B) = b^2 - c^2$

$\because$ from cosine rule we know that

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\therefore$ L.H.S.

$= a \left\{ b\left(\frac{a^2 + b^2 - c^2}{2ab} \right) - c\left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right\}$

$= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2}$

$= (b^2 - c^2) =$ R.H.S.

Example 5: If in ΔABC, $\angle A = 60^\circ$ then find the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$.

Sol. $\because \quad \angle A = 60^\circ$

$$\because \left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = \left(\frac{c+a+b}{c}\right)\left(\frac{b+c-a}{b}\right)$$

$$= \frac{(b+c)^2-a^2}{bc} = \frac{(b^2+c^2-a^2)+2bc}{bc}$$

$$= \frac{b^2+c^2-a^2}{bc}+2 = 2\left(\frac{b^2+c^2-a^2}{2bc}\right)+2$$

$$= 2\cos A + 2 = 3$$

$$\left(\because \angle A = 60^\circ \Rightarrow \cos A = \frac{1}{2}\right)$$

$$\therefore \quad \left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = 3$$

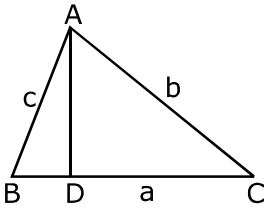
PROJECTION FORMULA

To express one side of a triangle in terms of the adjacent angles and the other two sides.

(1) Let ABC be an acute-angled triangle

Draw AD perpendicular to BC;

then $BC = BD + CD = AB\cos\angle ABD + AC\cos\angle ACD$;



i.e. $a = c\cos B + b\cos C$

(2) Let the triangle ABC have an obtuse angle C.

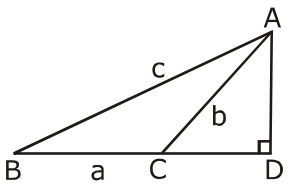
Draw AD perpendicular to BC produced; then

$BC = BD - CD = AB\cos$

$\angle ABD - AC\cos\angle ACD$;

$\therefore \quad a = \cos B - b\cos(180^\circ - C)$

$= c\cos B + b\cos C$



Thus in each each case $a = b\cos C + c\cos B$.

Similarly it may be shown that

$b = c\cos A + a\cos C$, and $c = a\cos B + b\cos A$

TRAIN YOUR BRAIN

Example 6: In a ΔABC prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$

Sol. $\because$ L.H.S. $= (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$

$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$= (b\cos A + a\cos B) + (c\cos A + a\cos C)$

$+ (c\cos B + b\cos C) = a+b+c =$ R.H.S.

Example 7: In any triangle ABC, find.

$$(a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}$$

Sol. $a^2\left(\sin^2\frac{C}{2}+\cos^2\frac{C}{2}\right) + b^2\left(\sin^2\frac{C}{2}+\cos^2\frac{C}{2}\right)$

$$-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$

$= a^2 + b^2 - 2ab\cos C = c^2$

NAPIER'S ANALOGY - TANGENT RULE

(*i*) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(*ii*) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(*iii*) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

TRAIN YOUR BRAIN

Example 8: Find the unknown elements of the ΔABC in which $a = \sqrt{3}+1, b = \sqrt{3}-1, C = 60^\circ$.

Sol. $\because \quad a = \sqrt{3}+1, b = \sqrt{3}-1, C = 60^\circ$

$\because \quad A + B + C = 180^\circ$

$\therefore \quad A + B = 120^\circ$ ...(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right)$

$$= \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$= \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30^\circ$$

$$= \frac{2}{2\sqrt{3}}\cot 30^\circ$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = 1$$

$$\therefore \quad \frac{A-B}{2} = \frac{\pi}{4} = 45^\circ \Rightarrow A - B = 90^\circ \qquad \text{...(ii)}$$

From equation (i) and (ii), we get

$A = 105°$ and $B = 15°$

Now,

$\because$ From **sine-rule,**

we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \quad c = \frac{a \sin C}{\sin A} = \frac{(\sqrt{3}+1)\sin 60°}{\sin 105°} = \frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$\because \quad \sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow c = \sqrt{6}$$

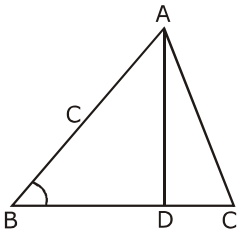
$\therefore \quad c = \sqrt{6}$, $A = 105°$, $B = 15°$ Ans.

AREA OF A TRIANGLE

To find the area of a triangle. Let Δ denote the area of the triangle ABC. Draw AD perpendicular to BC.

$$\therefore \quad \Delta = \frac{1}{2} (\text{base} \times \text{altitude})$$

$$= \frac{1}{2} BC \,.\, AD = \frac{1}{2} BC \,.\, AB \sin B = \frac{1}{2} ca \sin B$$



TRAIN YOUR BRAIN

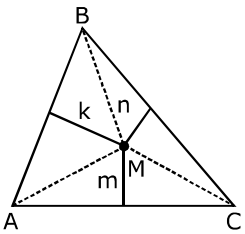
Example 9: Find the area of a triangle with angles α, β and γ knowing that the distances from an arbitrary point M taken inside the triangle to its sides are equal to m,nand k. (fig.)

Sol. The area S of the triangle ABC can be found by the formula

$S = \frac{1}{2}$ AC. BC. $\sin\gamma$, by sin rule

$$\frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha} = \frac{AB}{\sin\gamma},$$

whence we find that $AC = \frac{a\sin\beta}{\sin\alpha}$ and $AB = \frac{a\sin\gamma}{\sin\alpha}$.



We have :

$$S = \frac{1}{2} AC \,.\, BC \,.\, \sin\gamma = \frac{1}{2} \frac{a\sin\beta}{\sin\alpha} \,.\, a\sin\gamma$$

$= \frac{a^2 \sin\beta\sin\gamma}{2\sin\alpha}$. On the other hand,

$S = S_{AMB} + S_{BMC} + S_{AMC}$

$$= \frac{1}{2} AB \,.\, k + \frac{1}{2} BC \,.\, n + \frac{1}{2} AC \,.\, m$$

$$= \frac{1}{2} \cdot \frac{a\sin\gamma}{\sin\alpha} \,.\, k + \frac{1}{2} an + \frac{1}{2} \cdot \frac{a\sin\beta}{\sin\alpha} \,.m$$

$$= \frac{a(k\sin\gamma + n\sin\alpha + m\sin\beta)}{2\sin\alpha}$$

Hence, $\frac{a^2 \sin\beta\sin\gamma}{2\sin\alpha} = \frac{a(k\sin\gamma + n\sin\alpha + m\sin\beta)}{2\sin\alpha}$, we get

$: x = \frac{k\sin\gamma + n\sin\alpha + m\sin\beta}{\sin\beta\sin\gamma}$.

Substituting this value of a into the first of the above formulas for the area of the triangle ABC, we obtain : $S \Rightarrow$

$$\frac{a^2 \sin\beta\sin\gamma}{2\sin\alpha} = \frac{(k\sin\gamma + n\sin\alpha + m\sin\beta)^2}{2\sin\alpha\sin\beta\sin\gamma}.$$

TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

(*i*) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$;

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$;

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(*ii*) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$;

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$;

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(*iii*) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

where $s = \frac{a+b+c}{2}$ & Δ = area of triangle

(*iv*) Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

TRAIN YOUR BRAIN

Example 10: If a, b, c are in A.P., then the numerical value of $\tan\frac{A}{2}\tan\frac{C}{2}$ is

Sol. Given $2b = a + c$

$\Rightarrow 3b = 2s = a + b + c$

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{\Delta}{s(s-a)}\cdot\frac{\Delta}{s(s-c)}\cdot\frac{s-b}{s-b}$$

$$= \frac{2s-2b}{2s} = \frac{b}{3b} = \frac{1}{3}$$

Example 11: With usual notions, prove that in a triangle ABC,

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}.$$

Sol. Using $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}$ etc.

$$LHS = \frac{s(s-a)+s(s-b)+s(s-c)}{\Delta}$$

$$= \frac{3s^2 - s(a+b+c)}{\Delta} = \frac{3s^2-2s^2}{\Delta} = \frac{s^2}{\Delta} = \frac{s}{r}$$

Example 12: In a ΔABC if $b\sin C(b\cos C + c\cos B) = 42$, then find the area of the ΔABC.

Sol. $\because$ $b\sin C(b\cos C + c\cos B) = 42$ ...(i) given

$\because$ From **projection rule**, we know that

$a = b\cos C + c\cos B$ put in (i), we get

$ab\sin C = 42$ ...(ii)

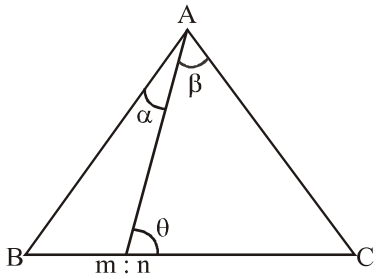
$\because$ $\Delta = \frac{1}{2}ab\sin C$

$\therefore$ from equation (ii), we get

$\therefore$ $\Delta = 21$ sq. unit

m-n Rule

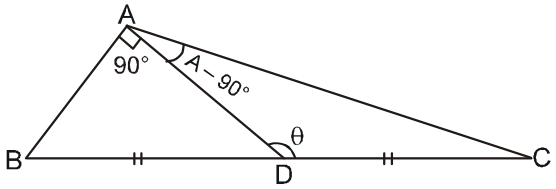
In any triangle, $(m+n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C$



TRAIN YOUR BRAIN

Example 13: If the median AD of a triangle ABC is perpendicular to AB, prove that $\tan A + 2\tan B = 0$.

Sol. From the figure, we see that $\theta = 90° + B$ (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

$(1+1)\cot(90° + B) = 1.\cot 90° - 1.\cot(A - 90°)$

$\Rightarrow -2\tan B = \cot(90° - A)$

$\Rightarrow -2\tan B = \tan A$

$\Rightarrow \tan A + 2\tan B = 0$ **Hence proved.**

RADIUS OF CIRCUMCIRCLE

Circumcircle is the circle passing thorugh vertices of triangle ABC. Its radius is known as circumradius (R) & centre is circumcentre (C).

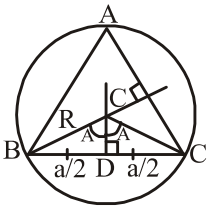
Circumcentre is the point of intersection of perpendicular bisectors of sides.

In ΔBCD

$$\frac{a/2}{R} = \sin A$$

$\Rightarrow a = 2R\sin A$

or $R = \frac{a}{2\sin A}$



Area of triangle and circumradius (R)

we know that

$$\Delta = \frac{1}{2}a.b.\sin c \qquad \left(\because \sin c = \frac{c}{2R}\right)$$

$$\Rightarrow \Delta = \frac{1}{2}a.b.\frac{c}{2R}$$

or $R = \frac{abc}{4\Delta}$

TRAIN YOUR BRAIN

Example 14: In a ΔABC, if $a = 18$ cm., $b = 24$ cm. and $c = 30$ cm., then find its circum-radius

Sol. Clearly, the triangle is right angled.

($\because 18^2 + 24^2 = 30^2$)

Thus, the area of the triangle

$$= \frac{1}{2}\times 24\times 18 = 12\times 18$$

Therefore, the circum-radius = R

$$= \frac{abc}{4\Delta} = \frac{18\times 24\times 30}{4\times 12\times 18} = 15 \text{ cm.}$$

Example 15: In an euqilateral triangle of side $2\sqrt{3}$ cm., then find the circum - radius

Sol. As we know that,

$$\frac{a}{\sin A} = 2R$$

$$\Rightarrow 2R = \frac{2\sqrt{3}}{\sin(60^\circ)}$$

$$\Rightarrow R = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$$

Hence, the circum-radius is 2cm.

Example 16: If the length of the sides of a triangle are 3, 4 and 5 units, then find its circum-radius R.

Sol. Let a = 3, b = 4 and c = 5. Clearly, it is a right angled triangle

Thus, $\Delta = \frac{1}{2} \times 4 \times 3 = 6$ sq. u

Hence, the circum radius R

$$= \frac{abc}{4\Delta}$$

$$= \frac{3\times 4\times 5}{6} = 10 \text{ unit}$$

RADIUS OF THE INCIRCLE

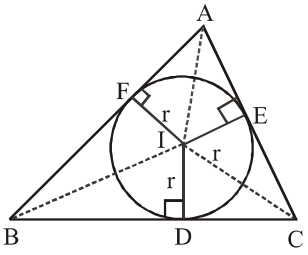
To find the radius of the circle inscribed in a triangle. Let ℓ be the circle inscribed in the triangle ABC, and D, E, F the points of contact; then ID, IE, IF are perpendicular to the sides.

Now Δ = sum of the areas of the triangles BIC, CIA, AIB

$$= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr$$

$$= \frac{1}{2} (a+b+c)r = sr$$

$$\Rightarrow r = \frac{\Delta}{s}$$



(*a*) $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$

(*b*) $r = (s-a)\tan\frac{A}{2}$

$= (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(*c*) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(*d*) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

RADIUS OF THE EX-CIRCLES

A circle which touches one side of a triangle and the produced of other two sides is said to be an escribed circle of the triangle. Thus the triangle ABC has three escribed circles, one touching BC, and produced AB and AC, a second touching CA, and produced BC and AB ; a third touching AB, and produced CA, CB.

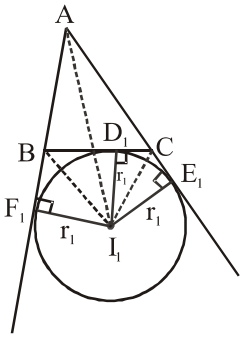
To find the radius of an escribed circle of a triangle. Let I_1 be the centre of the circle touching the side BC and the two sides AB and AC produced. Let D_1, E_1, F_1 be the points of contact; then the lines joining I_1 to these points are perependicular to the sides.

Let r_1 be the radius ; then

Δ = area ABC

= area ABI_1C – area BI_1C

= area BI_1A + area CI_1A – area BI_1C



$$= \frac{1}{2} cr_1 + \frac{1}{2} br_1 - \frac{1}{2} ar_1$$

$$= \frac{1}{2} (c+b+a) r_1 = (s-a) r_1$$

$$\therefore r_1 = \frac{\Delta}{s-a}$$

Similarly, if r_2, r_3 be the radii of the escribed circles opposite to the angles

B and C respectively

$$r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}.$$

Many important relations connecting a triangle and its circles may be established by elementary geometry.

With the notation of previous articles, since tangents to a circle from the same point are equal.

we have, AF = AE, BD = BF, CD = CE;

$\therefore$ AF + (BD + CD) = half the sum of the sides ;

$\therefore$ AF + a = s

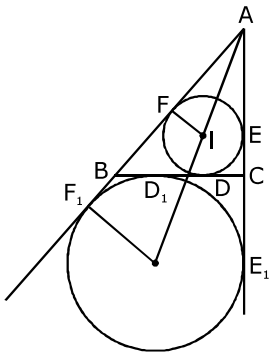
$\therefore$ AF = s – a = AE

Similarly, BD = BF = s – b, CD = CE = s – c. Also

$r_1 = AF \tan\frac{A}{2} = (s-a)\tan\frac{A}{2}$

Similarly, $r_2 = (s-b)\tan\frac{B}{2}$,

$r_3 = (s-c)\tan\frac{C}{2}$.



Again, $AF_1 = AE_1$, $BF_1 = BD_1$, $CE_1 = CD_1$

$\therefore$ $2AF_1 = AF_1 + AE_1 = (AB + BD_1) + (AC + CD_1)$

= sum of the sides

$\therefore$ $AF_1 = s = AE_1$

$\therefore$ $BD_1 = BF_1 = s - c$, $CD_1 = CE_1 = s - b$

Also $r_1 = AF_1 \tan\frac{A}{2} = s\tan\frac{A}{2}$

Similarly, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$.

Note : $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

$= 4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$ & so on

TRAIN YOUR BRAIN

Example 17: With usual notation in a triangle ABC, prove that $r^2 + s^2 + 4Rr = ab + bc + ca$.

Sol. $r^2 + s^2 + 4Rr$

$= \frac{\Delta^2}{s^2} + s^2 + \frac{abc}{\Delta}\cdot\frac{\Delta}{s}$ $\left(r = \frac{\Delta}{s}, R = \frac{abc}{4\Delta}\right)$

$= \frac{s(s-a)(s-b)(s-c)}{s.s} + s^2 + \frac{abc}{s}$

$= \frac{(s-a)(s-b)(s-c) + s^2 + abc}{s}$

$= \frac{2s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc + abc}{s}$

= ab + bc + ca (proved)

Example 18: In a ΔABC, prove that $\sin A + \sin B + \sin C = \frac{s}{R}$

Sol. In a ΔABC, we know that

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore$ $\sin A = \frac{a}{2R}$,

$\sin B = \frac{b}{2R}$ and $\sin C = \frac{c}{2R}$.

$\therefore$ $\sin A + \sin B + \sin C = \frac{a+b+c}{2R} = \frac{2s}{2R}$

$\because$ a + b + c = 2s

$\Rightarrow$ $\sin A + \sin B + \sin C = \frac{s}{R}$.

Example 19: In a ΔABC if a = 13 cm, b = 14 cm and c = 15 cm, then find its circumradius.

Sol. $\because$ $R = \frac{abc}{4\Delta}$ ...(i)

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because$ $s = \frac{a+b+c}{2} = 21$ cm

$\therefore$ $\Delta = \sqrt{21\times8\times7\times6} = \sqrt{7^2\times4^2\times3^2}$

$\Rightarrow$ $\Delta = 84\ cm^2$

$\therefore$ $R = \frac{13\times14\times15}{4\times84} = \frac{65}{8}$ cm $\therefore$ $R = \frac{65}{8}$ cm.

Example 20: In a ΔABC, prove that $s = 4R\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$.

Sol. In a ΔABC,

$\because$ $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

and $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ and $R = \frac{abc}{4\Delta}$

$\because$ $R.H.S. = 4R\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$.

$= \frac{abc}{\Delta}\cdot s\sqrt{\frac{s(s-a)(s-b)(s-c)}{(abc)^2}} = s$

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = L.H.S.$

Hence R.H.S = L.H.S. proved.

Example 21: The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

Sol. $\frac{s-a}{\Delta}$, $\frac{s-b}{\Delta}$, $\frac{s-c}{\Delta}$ are in A.P.

$\Rightarrow$ a, b, c are in A.P.

$\Rightarrow$ 2b = a + c

$\Rightarrow$ 2s = 24 $\Rightarrow$ s = 12

$\Rightarrow$ $\sqrt{12(12-a)4(12-16+a)} = 24$

$\Rightarrow$ $12\times4(12-a)(a-4) = 24\times24$

$\Rightarrow$ $-a^2 + 16a - 48 = 12$

$\Rightarrow$ $a^2 - 16a + 60 = 0$

$\Rightarrow$ $(a-10)(a-6) = 0$

$\Rightarrow$ $a = 10$ or $a = 6$

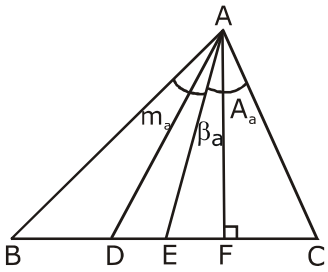
Hence, the lengths of side are 6, 8, 10 cm.

LENGTH OF ANGLE BISECTORS, MEDIANS & ALTITUDE

(*i*) Length of an angle bisector from the angle $A = \beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$.

(*ii*) Length of median from the angle $A = m_a$

$= \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$



(*iii*) Length of altitude from the angle $A = A_a = \dfrac{2\Delta}{a}$.

Note: $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

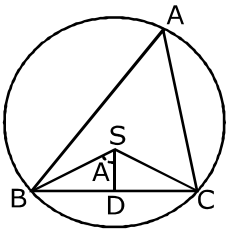
To find the radius of the circle circumscribing a triangle. Let S be the centre of the circle circumscribing the triangle ABC, and R its radius. Bisect $\angle BSC$ by SD, which will also bisect BC at right angles.

$\angle BSC = 2A$

and $\dfrac{a}{2} = BD = BS \sin BSD = R \sin A$

$\therefore \quad R = \dfrac{a}{2\sin A}$

Thus $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ or $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$



The circum-radius may be expressed in a form not involving the angles, as

$R = \dfrac{a}{2\sin A} = \dfrac{abc}{2bc\sin A} = \dfrac{abc}{4\Delta}$

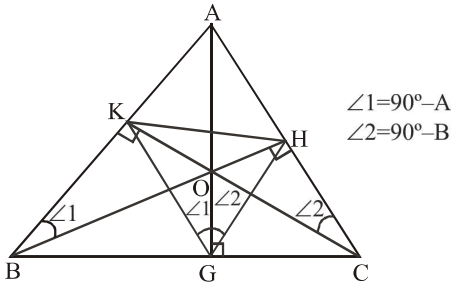
TRAIN YOUR BRAIN

Example 22: Show that $2R^2 \sin A \sin B \sin C = \Delta$.

Sol. L.H.S. $= \dfrac{1}{2} \cdot 2R\sin A \cdot 2R \sin B \cdot \sin C = \dfrac{1}{2} ab \sin C = \Delta$

ORTHOCENTRE AND PEDAL TRIANGLE

Let G, H, K be the feet of the perpendiculars from the angular points on the opposite sides of the triangle ABC, then GHK is called the Pedal triangle of ABC. The three perpendiculars AG, BH, CK meet in a point O which is called the orthocentre of the triangle ABC.



To find the sides and angles of the pedal triangle.

In the figure, the points K, O, G, B are concyclic :

$\therefore \quad \angle OGK = OBK = 90° - A$

Also the points H, O, G, C are concyclic :

$\therefore \quad \angle OGH = \angle OCH = 90° - A$

$\therefore \quad \angle KGH = 180° - 2A$

Thus the angles of the pedal triangle are

$180° - 2A, 180° - 2B, 180° - 2C$

Again, the triangles AKH, ABC are similar :

$\therefore \quad \dfrac{HK}{BC} = \dfrac{AK}{AC} = \cos A$

$\therefore \quad HK = a \cos A$

Thus the sides of the pedal triangle are a cos A, b cos B, c cos C.

In terms of R, the equivalent forms become R sin 2A, R sin 2B, R sin 2C.

If the angle ACB of the given triangle is obtuse, the expression $180° - 2C$, and c cos C are both negative, and the values we have obtained required some modification. In this case the angles are 2A, 2B, $2C - 180°$, and the sides a cos A, b cos B, – c cos C.

Remarks:

(*i*) The distances of the orthocentre from the angular points of the ΔABC are 2 R cos A, 2R cos B and 2R cos C.

(*ii*) The distances of orthocentre from sides are 2R cos B cos C, 2R cos C cos A & 2R cos B cos C.

(*iii*) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

To find the area and circum-radius of the pedal triangle.

Area $= \frac{1}{2}$ (product of two sides) × (sine of included angle) $= \frac{1}{2} R \sin 2B \,.\, R \sin 2C \,.\, \sin(180^\circ - 2A)$

$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$. The circum-radius

$= \frac{HK}{2\sin HGK} = \frac{2\sin 2A}{2\sin(180^\circ - 2A)} = \frac{R}{2}$.

TRAIN YOUR BRAIN

Example 23: If f, g, h denote sides, the pedal triangle of a ΔABC, then show that

$$f.\frac{(b^2-c^2)}{a^2} + g.\frac{(c^2-a^2)}{b^2} + h.\frac{(a^2-g^2)}{c^2} = 0$$

Sol. Sides are a cos A, b cos B, c cos C. Hence LHS $\left(\frac{b^2-c^2}{a}\right)$

$\cos A + \left(\frac{c^2-a^2}{b}\right) \cos B + \left(\frac{a^2-b^2}{c}\right) \cos C$

Put the values of cos A etc get the result.

Example 24: Vertex A of a variable triangle ABC, inscribed in a circle of radius R, is a fixed point. If the angles subtended by the side BC at orthocentre (H), circumcentre (O) and incentre (I) are equal than identify the locus of orthocentre of triangle ABC.

Sol. The angles subtended by the side BC at points H, O and I are $B + C$, $2A$ and $\pi - \left(\frac{B+C}{2}\right)$ respectively.

$\Rightarrow \quad B + C = 2A = 180 - \left(\frac{B+C}{2}\right)$

$\Rightarrow A = \frac{\pi}{3}$ and $B + C = \frac{2\pi}{3}$.

Also in triangle ABC, $HA = 2R \cos A = R$

$\Rightarrow$ HA is contant

$\Rightarrow$ locus of orthocentre is a circle having centre at the vertex A.

EXCENTRAL TRIANGLE

Let ABC be a triangle and I_1, I_2, I_3 its ex-centres ; then triangle $I_1I_2I_3$ is called the Ex-central triangle of ABC. Let I be the in-centre ; then from the construction for finding the positions of the in-centre and ex-centres, it follows that :

(*i*) The points I, I_1 lie on the line bisecting the angle BAC; the points I, I_2 lie on the line bisecting the angle ABC; the points I, I_3 lie on the line bisecting the angle ACB.

(*ii*) The points I_2, I_3 lie on the line bisecting the angle BAC externally; the points I_3,I_1 lie on the line bisecting the angle ABC externally ; the points I_1, I_2 lie on the line bisecting the angle ABC externally.

(*iii*) The line A_1 is perpendicular to I_2I_3; the line BI_2 is perpendicular to I_3I_1 ; the line CI_3 is perpendicular to I_1I_2. Thus the triangle ABC is the Pedal triangle of its ex-central triangle $I_1I_2I_3$.

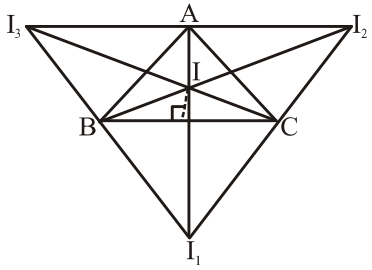
(*iv*) The angles IBI_1 and ICI_1 are right angles; hence the points B, I, C, I_1 are concyclic . Similarly, the points C, I, A, I_2, and the points A, I, B, I_3 are concyclic.

(*v*) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(*vi*) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(*vii*) Each of the four points I, I_1,I_2,I_3 is the orthocentre of the triangle formed by joining the other three points.

To find the sides and angles of the ex-central triangle.



With the figure of the-last article.

$\angle BI_1C = \angle BI_1I + \angle CI_1I$

$= \angle BCI + \angle CBI = \frac{C}{2} + \frac{B}{2}$

$= 90^\circ - \frac{A}{2}$. Thus the angles are

$90^\circ - \frac{A}{2}$ $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$.

Again, the points B, I_3, I_2, C are concyclic.

$\therefore \quad \angle I_1I_2I_3 = I_3BC = \angle I_1BC$

$\therefore$ the triangles $I_1I_2I_3$

= supplement of $\angle I_1BC$ are similar

$\therefore \quad \frac{I_2I_3}{BC} = \frac{I_3I_1}{I_1C} = \sec\left(90^\circ - \frac{A}{2}\right) = \text{cosec}\ \frac{A}{2}$

$\therefore \quad I_2I_3 = a\ \text{cosec}\ \frac{A}{2} = 4R \cos \frac{A}{2}$

Thus the sides are $4R \cos \frac{A}{2}$, $4R \cos \frac{B}{2}$, $4R \cos \frac{C}{2}$

To find the area and circum-radius of the ex-central triangle.

AREA:

$$= \frac{1}{2} \text{ (product of two sides)} \times \text{(sine of included angle)}$$

$$= \frac{1}{2} \times 4R\cos\frac{B}{2} \times 4R\cos\frac{C}{2} \times \sin\left(90^\circ - \frac{A}{2}\right)$$

$$= 8R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

THE CIRCUM-RADIUS

$$= \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^\circ - \frac{A}{2}\right)} = 2R$$

To find the distance between the in-centre and ex-centres.

The angles $\angle IBI_1$, $\angle ICI_1$ are right angles $\therefore$ II_1 is the diameter of the circum-circle of the triangle BCI_1

$$\therefore II_1 = \frac{BC}{\sin BI_1C} = \frac{a}{\cos\frac{A}{2}} = 4R\sin\frac{A}{2}.$$

Thus the distances are

$$4R\sin\frac{A}{2}, 4R\sin\frac{B}{2}, 4R\sin\frac{C}{2}$$

We have proved that OG, OH, OK bisect the angles HGK, KHG, GHK respectively, so that O is the in-centre of the triangle GHK. Thus the orthocentre of a triangle is the in-centre of the pedal triangle.

Again, the line CGB which is at right angles to OG bisect $\angle HGK$ externally. Similarly the lines AHC and BKA bisect $\angle KHG$ and $\angle GKH$ externally, so that ABC is the ex-central triangle of its pedal triangle GHK.

TRAIN YOUR BRAIN

Example 25: If I is the incentre and I_1, I_2, I_3 are the centre of escribed circles of the ΔABC, Prove that

(*i*) $II_1.II_2.II_3 = 16R^2r$

(*ii*) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = I_1I_2^2 + II_3^2$

Sol.

(*i*) $\because$ We know that

$$II_1 = a\sec\frac{A}{2}, II_2 = b\sec\frac{B}{2} \text{ and } II_3 = c\sec\frac{C}{2}$$

$\because$ $I_1I_2 = c$, $\text{cosec}\frac{C}{2}$,

$$I_2I_3 = a\,\text{cosec}\frac{A}{2} \text{ and } I_3I_1 = b\,\text{cosec}\frac{B}{2}$$

$\because$ $II_1.II_2.II_3 = abc\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$ ....(i)

$\because$ $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$

$\therefore$ equation (i) becomes

$\because$ $II_1.II_2.II_3 = (2R\sin A)(2R\sin B)$

$$(2R\sin C)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$$

$$= 8R^3 . \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$$= 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$\because$ $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $II_1.II_2.II_3 = 16R^2r$

(*ii*) $II_1^2 + I_2I_3^2 = II_2^2 I_3I_1^2 = II_3^2 + I_1I_2^2$

$\because$ $II_1^2 + I_2I_3^2$

$$= a^2\sec^2\frac{A}{2} + a^2 . \text{cosec}^2\frac{A}{2} = \frac{a^2}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$$

$\because$ $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$

$$\therefore II_1^2 + I_2I_3^2 = \frac{16R^2\sin^2\frac{A}{2}.\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}.\cos^2\frac{A}{2}} = 16R^2$$

Hence $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

N. DISTANCE OF SPECIAL POINTS FROM VERTICES AND SIDES OF A TRIANGLE

(*i*) Circumcentre (O) : $OA = R$ & $O_a = R\cos A$

(*ii*) Incentre (I) : $IA = r\,\text{cosec}\frac{A}{2}$ & $I_a = r$

(*iii*) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\frac{A}{2}$ & $I_{1a} = r_1$

(*iv*) Orthocentre (H) : HA
$= 2R\cos A$ & $H_a = 2R\cos B\cos C$

(*v*) Centrod (G) : GA

$$= \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2} \text{ & } G_a = \frac{2\Delta}{3a}$$

DISTANCE BETWEEN SPECIAL POINTS

(*a*) The distance between circumcentre and orthocentre is

$$= R\sqrt{1 - 8\cos A\cos B\cos C}$$

(*b*) The distance between circumcentre and incentre is

$$= \sqrt{R^2 - 2Rr} = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

(*c*) Distance between circumcentre and centroid

$$OG = \sqrt{R^2 - \frac{1}{9}(a^2 + b^2 + c^2)}$$

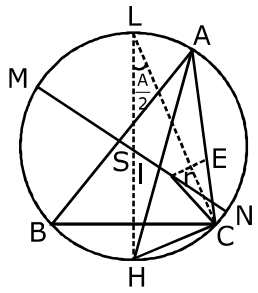
(*d*) The distance between incentre and orthocentre is $\sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$

To find the distance between the in-centre and circum-centre.

Let S be the circum-centre and I the in-centre.Produce AI to meet the circum-circle in H; join CH and CI. Draw IE perpendicular to AC. Produce HS to meet the circumference in L, and join CL. Then

$$\angle HIC = \angle IAC + \angle ICA = \frac{A}{2} + \frac{C}{2};$$

$$\angle HCI = \angle ICB + \angle BCH = \frac{C}{2} + \angle BAH = \frac{C}{2} + \frac{A}{2};$$



$\therefore \quad \angle HCI = HIC;$

$\therefore \quad HI = HC = 2R \sin \frac{A}{2}.$

Also $AI = IE \text{ cosec } \frac{A}{2} = r \text{ cosec } \frac{A}{2}$

$\therefore \quad AI \,.\, IH = 2Rr$

Produce SI to meet the circumference in M and N. Since MIN, AIH are chords of the circle.

$AI.IH = MI.\ IN = (R + SI)(R - SI);$

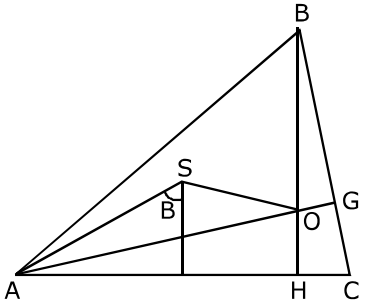
$\therefore \quad 2Rr = R^2 - SI^2;\ SI^2 = R^2 - 2Rr$

To find the distance of the orthocentre from the circum-centre.

With the usual notation, we have

$SO^2 = SA^2 + AO^2 - 2SA \,.\, AO \cos SAO.$

Now AS = R; AO = AH cosec C = c cos A cosec C



$= 2R \sin C \cos A \text{ cosec } C = 2R \cos A;\ \angle SAO$

$= \angle SAC - \angle OAC = (90^\circ - B) - (90^\circ - C) = C - B$

$\therefore \ SO^2 = R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B)$

$= R^2 - 4R^2 \cos^2 A \ \{\cos (B + C) + \cos (C - B)\}$

$= R^2 - 8R^2 \cos A \cos B \cos C.$

TRAIN YOUR BRAIN

Example 26: If r and R are radii of the incircle and circumcircle of ΔABC, prove that

$$8rR\left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}\right)$$

$$= 2\,bc + 2\,ca + 2ab - a^2 - b^2 - c^2$$

Sol. $LHS = \frac{8\Delta}{s} \frac{abc}{4\Delta} \Sigma \cos^2 \frac{A}{2} = \frac{abc}{s}\left[\Sigma 2\cos^2 \frac{A}{2}\right]$

$$= \frac{abc}{s}[(1 + \cos A) + (1 + \cos B) + (1 + \cos C)]$$

$$= \frac{abc}{s}\left[\frac{(b+c)^2 - a^2}{2bc} + \frac{(c+a)^2 - b^2}{2ca} + \frac{(a+b)^2 - c^2}{2ab}\right]$$

$$= \frac{abc}{s} \frac{(2s)}{2abc}[a(b + c - a) + b(c + a - b) + c(a + b - c)]$$

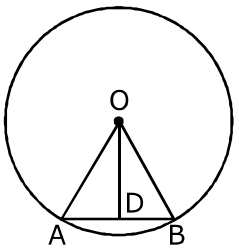
INSCRIBED AND CIRCUMSCRIBED POLYGONS

To find the perimeter and area of a regular polygon of n sides inscribed a circle.

Let r be the radius of the circle, and AB a side of the polygon.

Join OA, OB, and draw OD bisecting $\angle AOB$; then AB is bisected at right angles in D.

And $\angle AOB = \frac{1}{n}$ (four right angles) $= \frac{2\pi}{n}$



Perimeter of polygon

$= nAB = 2nAD = 2nAO \sin \Delta AOD = 2nr \sin \frac{\pi}{n}$

Area of polygon = n (area of triangle AOB)

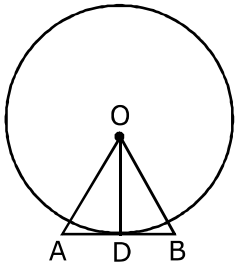
$= \frac{1}{2} nr^2 \sin \frac{2\pi}{n}.$

To find the perimeter and area of a regular polygon of n sides circumscribed about a given circle.

Let r be the radius of the circle, and AB a side of the polygon. Let AB touch the circle at D. Join OA, OB, OD; then OD bisects AB at right angles, and also bisects ∠AOB.

Perimeter= nAB = 2nAD=2nOD tan ∠AOD=2nr tan $\frac{\pi}{n}$.

Area of polygon = n (area of triangle AOB)



$= nOD \cdot AD = nr^2 \tan \frac{\pi}{n}$

TRAIN YOUR BRAIN

Example 27: If a, b, c denote the sides of a ΔABC, show that the value of the expression,

$a^3 (p - q) (p - r) + b^2 (q - r) + b^2 (q - r) (q - p) + c^2 (r - p) (r - q)$ cannot be negative where p, q, r ∈ R.

Sol. Let p > q > r and $\left.\begin{matrix} p-q=y>0 \\ q-r=z>0 \end{matrix}\right\} \Rightarrow p-r=y+z>0$

Consider $E = a^2y(y+z) - b^2zy + c^2z(y+z)$

$= a^2y^2 + c^2z^2 + yz(a^2 + c^2 - b^2)$

Now $b < a + c$

$\Rightarrow \quad b_{max} = a + c$

$E_{least} = a^2y^2 + c^2z^2 + yz[a^2 + c^2 - (a+c)^2]$

$= a^2y^2 + c^2z^2 - 2acyz = (ay - cz)^2 \geq 0$

Example 28: In a triangle ABC if $2 \cos \frac{B}{2} \cos \frac{C}{2} = \frac{1}{2} + \left(\frac{b+c}{a}\right) \sin \frac{A}{2}$ then find the measure of angle A.

Sol. Given $2 \cos \frac{B}{2} \cos \frac{C}{2} = \frac{1}{2} + \left(\frac{b+c}{a}\right) \sin \frac{A}{2}$

$$\text{or } \cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)$$

$$= \frac{1}{2} + \left(\frac{\sin B + \sin C}{\sin A}\right) \sin \frac{A}{2}$$

$$\text{or } \sin \frac{A}{2} + \cos\left(\frac{B-C}{2}\right)$$

$$= \frac{1}{2} + \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\text{or } \sin \frac{A}{2} + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \frac{\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$$

$$\text{or } \sin \frac{A}{2} + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \sin \frac{A}{2} = \frac{1}{2}$$

$\therefore \quad \angle A/2 = 30^\circ \Rightarrow \angle A = 60^\circ$

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PRAYAS2.0: t.me/realprayasians2

Topicwise Questions

SINE & COSINE RULE

1. If the angles of a triangle ABC be in A.P., then

(a) $c^2 = a^2 + b^2 - ab$ (b) $b^2 = a^2 + c^2 - ac$

(c) $a^2 = b^2 + c^2 - ac$ (d) $b^2 = a^2 + c^2$

2. In ΔABC. $a \sin(B-C) + b \sin(C-A) + c \sin(A-B) =$

(a) 0 (b) $a+b+c$

(c) $a^2 + b^2 + c^2$ (d) $2(a^2 + b^2 + c^2)$

3. In ΔABC, $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C =$

(a) 0 (b) $a^2 + b^2 + c^2$

(c) $2(a^2 + b^2 + c^2)$ (d) $\frac{1}{2abc}$

4. If the sides of a triangle are p, q and $\sqrt{p^2 + pq + q^2}$, then the biggest angle is

(a) $\pi/2$ (b) $2\pi/3$

(c) $5\pi/4$ (d) $7\pi/4$

5. In a triangle ABC, if $B = 3C$, then the values of $\sqrt{\left(\frac{b+c}{4c}\right)}$ and $\left(\frac{b-c}{2c}\right)$ are

(a) $\sin C, \sin\frac{A}{2}$ (b) $\cos C, \sin\frac{A}{2}$

(c) $\sin C, \cos\frac{A}{2}$ (d) $\cos C, \cos\frac{A}{2}$

6. In a ΔABC, if $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then $\cos C =$

(a) $\frac{7}{5}$ (b) $\frac{5}{7}$

(c) $\frac{17}{36}$ (d) $\frac{16}{17}$

7. In a triangle ABC, $b = \sqrt{3}$, $c = 1$ and $\angle A = 30^o$, then the largest angle of the triangle is

(a) 135° (b) 90°

(c) 60° (d) 120°

8. In a ΔABC, $2ac\sin\left(\frac{A-B+C}{2}\right)$ is equal to

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

9. If in triangle ABC, $\cos A = \frac{\sin B}{2\sin C}$, then the triangle is

(a) Equilateral (b) Isosceles

(c) Right angled (d) Scalene

10. If a, b and c are the sides of a triangle such that $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$ then the angles opposite to the side c is

(a) 45° or 135° (b) 30° or 100°

(c) 50° or 100° (d) 60° or 120°

Projection formula, Napier's Analogy & area of triangle

11. If in a triangle ABC, $b = \sqrt{3}$, $c = 1$ and $B - C = 90^o$ then $\angle A$ is

(a) 30° (b) 45°

(c) 75° (d) 15°

12. If a^2, b^2, c^2 are in A. P. then which of the following are also in A.P.

(a) $\sin A, \sin B, \sin C$ (b) $\tan A, \tan B, \tan C$

(c) $\cot A, \cot B, \cot C$ (d) cosec A, cosec B, cosec C

13. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the sides are proportional to

(a) $1:1:\sqrt{2}$ (b) $1:\sqrt{2}:1$

(c) $\sqrt{2}:1:1$ (d) $1:1:1$

14. If in the ΔABC, $AB = 2BC$, then $\tan\frac{B}{2} : \cot\left(\frac{C-A}{2}\right)$

(a) 3:1 (b) 2:1

(c) 1:2 (d) 1:3

15. If in a triangle ABC, $a = 5, b = 4, A = \frac{\pi}{2} + B$, then $C =$

(a) $\tan^{-1}\left(\frac{1}{9}\right)$ (b) $\tan^{-1}\frac{1}{40}$

(c) Cannot be evaluated (d) $2\tan^{-1}(1/9)$

16. If A is the area and $2s$ the sum of 3 sides of triangle, then

(a) $A \le \frac{s^2}{3\sqrt{3}}$ (b) $A \le \frac{s^2}{2}$

(c) $A > \frac{s^2}{\sqrt{3}}$ (d) $A \ge \frac{s^2}{3\sqrt{3}}$

17. If in a triangle ABC right angled at $B, s-a=3$, $s-c=2$, then the values of a and c are respectively

(a) 2, 3 (b) 3, 4

(c) 4, 3 (d) 6, 8

18. In a ΔABC a, c, A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then $\sin A =$

(a) $\sqrt{\frac{9a^2-c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2-c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2+c^2}{8a^2}}$ (d) $\sqrt{\frac{9a^2+c^2}{8c^2}}$

19. In a ΔABC, $a^2 \sin 2C + c^2 \sin 2A =$

(a) Δ (b) 2Δ

(c) 3Δ (d) 4Δ

20. The product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle is equal to

(a) Δ (b) 2Δ

(c) 3Δ (d) 4Δ

21. In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex A from the side BC is (in cm)

(a) $\frac{25}{13}$ (b) $\frac{60}{13}$

(c) $\frac{65}{12}$ (d) $\frac{144}{13}$

22. In a ΔABC, if $b = 2$, $c = \sqrt{3}$ and $\angle A = \frac{\pi}{6}$, then value of R is equal to

(a) $\frac{1}{2}$ (b) 1

(d) 2 (d) $\frac{1}{4}$

Half-Angled Formulae

23. In ΔABC, if $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ be in H. P. then a, b, c will be in

(a) A. P. (b) G.P.

(c) H. P. (d) A.G..P.

24. In any triangle ABC, $\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}} =$

(a) $\frac{a-b}{a+b}$ (b) $\frac{a-b}{c}$

(c) $\frac{a-b}{a+b+c}$ (d) $\frac{c}{a+b}$

25. In any triangle ABC, the value of $a(b^2 + c^2)\cos A + b(c^2 + a^2)\cos B + c(a^2 + b^2)\cos C$ is

(a) $3abc^2$ (b) $3a^2bc$

(c) $3abc$ (d) $3ab^2c$

26. In a triangle ABC, AD is altitude from A. Given $b > c$, $\angle C = 23^\circ$ and $AD = \frac{abc}{b^2-c^2}$, then $\angle B =$

(a) 67° (b) 44°

(c) 113° (d) 157°

27. In a ΔABC if the sides are $a = 3, b = 5$ and $c = 4$, then $\sin\frac{B}{2} + \cos\frac{B}{2}$ is equal to

(a) $\sqrt{2}$ (b) $\frac{\sqrt{3}+1}{2}$

(c) $\frac{\sqrt{3}-1}{2}$ (d) 1

28. Which of the following is true in a triangle ABC

(a) $(b+c)\sin\frac{B-C}{2} = 2a\cos\frac{A}{2}$

(b) $(b+c)\cos\frac{A}{2} = 2a\sin\frac{B-C}{2}$

(c) $(b-c)\cos\frac{A}{2} = a\sin\frac{B-C}{2}$

(d) $(b-c)\sin\frac{B-C}{2} = 2a\cos\frac{A}{2}$

29. In a ΔABC, if $3a = b + c$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2}$ is

(a) 1 (b) 2

(c) $\sqrt{3}$ (d) $\sqrt{2}$

30. In ΔABC, $2R^2 \sin A \sin B \sin C =$

(a) s^2 (b) ab + bc + ca

(c) Δ (d) zero

31. In an equilateral triangle the inradius and the circum-radius are connected by

(a) $r = 4R$ (b) $r = R/2$

(c) $r = R/3$ (d) $r = R/4$

32. If the sides of the triangle are $5K, 6K, 5K$ and radius of incircle is 6 then value of K is equal to

(a) 4 (b) 5

(c) 6 (d) 7

33. The circum-radius of the triangle whose sides are 13, 12 and 5 is

(a) 15 (b) 13/2

(c) 15/2 (d) 6

Learning Plus

1. In a ΔABC, A : B : C = 3 : 5 : 4. Then $a + b + c\sqrt{2}$ is equal to
(*a*) 2b (*b*) 2c
(*c*) 3b (*d*) 3a

2. If in a Δ ABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :
(*a*) right angled (*b*) isosceles
(*c*) equilateral (*d*) obtuse angled

3. In a ΔABC $\frac{bc\sin^2 A}{\cos A + \cos B \cos C}$ is equal to
(*a*) $b^2 + c^2$ (*b*) bc
(*c*) a^2 (*d*) $a^2 + bc$

4. In a triangle ABC, $(a + b + c)(b + c - a) = kbc$, if
(*a*) $k < 0$ (*b*) $k > 6$
(*c*) $0 < k < 4$ (*d*) $k > 4$

5. In a ΔABC, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area (ΔABC) $= \frac{9\sqrt{3}}{2}$ cm^2. Then 'a' is
(*a*) $6\sqrt{3}$ cm (*b*) 9 cm
(*c*) 18 cm (*d*) None of these

6. If AD, BE and CF are the medians of ΔABC, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to
(*a*) 4 : 3 (*b*) 3 : 2
(*c*) 3 : 4 (*d*) 2 : 3

7. The distance between the middle point of BC and the foot of the perpendicular form A is
(*a*) $\frac{-a^2 + b^2 + c^2}{2a}$ (*b*) $\frac{b^2 - c^2}{2a}$
(*c*) $\frac{b^2 + c^2}{\sqrt{bc}}$ (*d*) None of these

8. In a acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to
(*a*) $\frac{\Delta}{2R}$ (*b*) $\frac{\Delta}{3R}$
(*c*) $\frac{\Delta}{4R}$ (*d*) $\frac{\Delta}{R}$

9. If in a ΔABC, $\Delta = a^2 - (b - c)^2$, then tan A =
(*a*) 15/16 (*b*) 8/15
(*c*) 8/17 (*d*) 1/2

10. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the ΔABC on the sides a, b and c respectively. If $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh}$ then the value of λ is
(*a*) 1/4 (*b*) 1/2
(*c*) 1 (*d*) 2

11. If in a triangle ABC, right angle at B, s – a = 3 and s – c = 2, then
(*a*) a = 2, c = 3 (*b*) a = 3, c = 4
(*c*) a = 4, c = 3 (*d*) a = 6, c = 8

12. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are :
(*a*) in A.P. (*b*) in G.P.
(*c*) in H.P. (*d*) None

13. In a Δ ABC if b + c = 3a, then $\cot\frac{B}{2} \cdot \cot\frac{C}{2}$ has the value equal to:
(*a*) 4 (*b*) 3
(*c*) 2 (*d*) 1

14. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to
(*a*) $\frac{a - c}{2b}$ (*b*) $\frac{a - b}{2c}$
(*c*) $\frac{a - c}{b}$ (*d*) $\frac{a - b}{c}$

15. If R denotes circumradius, then in ΔABC, $\frac{b^2 - c^2}{2aR}$ is equal to
(*a*) cos (B – C) (*b*) sin (B – C)
(*c*) cos B – cos C (*d*) None of these

16. In a Δ ABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a + b + c}$ is equal to:
(*a*) $\frac{r}{R}$ (*b*) $\frac{R}{2r}$
(*c*) $\frac{R}{r}$ (*d*) $\frac{2r}{R}$

17. If the sides of a triangle are 3 : 7 : 8, then R : r is equal to
(*a*) 2 : 7 (*b*) 7 : 2
(*c*) 3 : 7 (*d*) 7 : 3

18. In a right angled triangle R is equal to
(*a*) $\frac{s+r}{2}$ (*b*) $\frac{s-r}{2}$
(*c*) $s-r$ (*d*) $\frac{s+r}{a}$

19. If the area of triangle is 100 sq. cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then the value of (b – a) is equal to
(*a*) 20 (*b*) 16
(*c*) 8 (*d*) 4

20. In a ΔABC, the inradius and three exradii are r, r_1, r_2 and r_3 respectively. In usual notations the value of $r \cdot r_1 \cdot r_2 \cdot r_3$ is equal to
(*a*) 2Δ (*b*) Δ^2
(*c*) $\frac{abc}{4R}$ (*d*) None of these

21. In a ΔABC if $r_1 > r_2 > r_3$, then
(*a*) a > b > c
(*b*) a < b < c
(*c*) a > b and b < c
(*d*) a < b and b > c

22. In a triangle ABC, right angled at B, the inradius is
(*a*) $\frac{AB+BC-AC}{2}$
(*b*) $\frac{AB+AC-BC}{2}$
(*c*) $\frac{AB+BC+AC}{2}$
(*d*) None of these

23. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to
(*a*) R, R, R
(*b*) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$
(*c*) 2R, 2R, 2R
(*d*) $\frac{2}{R}, \frac{2}{R}, \frac{2}{R}$

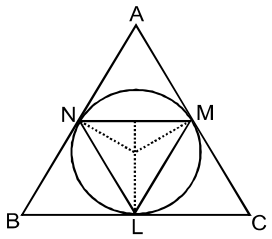
24. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in
(*a*) A.P.
(*b*) G.P.
(*c*) H.P.
(*d*) None of these

25. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to
(*a*) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$
(*b*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$
(*c*) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$
(*d*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$

26. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ is equal to
(*a*) 2
(*b*) $\frac{1}{2}$
(*c*) 1
(*d*) None of these

27. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to
(*a*) 0 (*b*) 1
(*c*) 2 (*d*) None of these

28. If the incircle of the Δ ABC touches its sides respectively at L, M and N and if x, y, z be the circumradii of the triangles MIN, NIL and LIM where I is the incentre then the product xyz is equal to



(*a*) Rr^2 (*b*) rR^2
(*c*) $\frac{1}{2}Rr^2$ (*d*) $\frac{1}{2}rR^2$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. Which of the following holds good for any triangle ABC?

(*a*) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(*b*) $\frac{\sin A}{a}+\frac{\sin B}{b}+\frac{\sin C}{c}=\frac{3}{2R}$

(*c*) $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$

(*d*) $\frac{\sin 2A}{a^2}=\frac{\sin 2B}{b^2}=\frac{\sin 2C}{c^2}$

2. If in a ΔABC, a = 5, b = 4 and $\cos(A-B)=\frac{31}{32}$, then

(*a*) c = 6

(*b*) $\sin A=\left(\frac{5\sqrt{7}}{16}\right)$

(*c*) area of $\Delta ABC=\frac{15\sqrt{7}}{4}$

(*d*) None of these

3. In a triangle ABC, with usual notations the length of the bisector of internal angle A is :

(*a*) $\frac{2bc\ \cos\frac{A}{2}}{b+c}$

(*b*) $\frac{2bc\ \sin\frac{A}{2}}{b+c}$

(*c*) $\frac{abc\ \operatorname{cosec}\frac{A}{2}}{2R(b+c)}$

(*d*) $\frac{2\Delta}{b+c}\cdot\operatorname{cosec}\frac{A}{2}$

4. If $r_1=2r_2=3r_3$, then

(*a*) $\frac{a}{b}=\frac{4}{5}$ (*b*) $\frac{a}{b}=\frac{5}{4}$

(*c*) $\frac{a}{c}=\frac{3}{5}$ (*d*) $\frac{a}{c}=\frac{5}{3}$

5. In a ΔABC, following relations holds good. In which case(s) the triangle is a right angled triangle?

(*a*) $r_2+r_3=r_1-r$ (*b*) $a^2+b^2+c^2=8R^2$

(*c*) $r_1=s$ (*d*) $2R=r_1-r$

6. If in a triangle ABC, cos A cos B + sin A sin B sin C = 1, then the triangle is

(*a*) isosceles

(*b*) right angled

(*c*) equilateral

(*d*) None of these

7. AD, BE and CF are the perpendiculars from the angular points of an acute angled Δ ABC upon the opposite sides, then :

(*a*) $\frac{\text{Perimeter of }\Delta DEF}{\text{Perimeter of }\Delta ABC}=\frac{r}{R}$

(*b*) Area of $\Delta DEF = 2\Delta\cos A\cos B\cos C$

(*c*) Area of $\Delta AEF=\Delta\cos^2 A$

(*d*) Circum-radius of $\Delta DEF=\frac{R}{2}$

8. The product of the distances of the incentre from the angular points of a Δ ABC is:

(*a*) $4R^2r$ (*b*) $4Rr^2$

(*c*) $\frac{(abc)\,R}{s}$ (*d*) $\frac{(abc)\,r}{s}$

9. In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:

(*a*) $\tan\theta = 3\tan B$

(*b*) $3\tan\theta = \tan C$

(*c*) $\frac{6\tan\theta}{\tan^2\theta-9}=\tan A$

(*d*) angle B = angle C

10. With usual notations, in a Δ ABC the value of $\Pi(r_1-r)$ can be simplified as:

(*a*) $abc\,\Pi\tan\frac{A}{2}$ (*b*) $4rR^2$

(*c*) $\frac{(abc)^2}{R(a+b+c)^2}$ (*d*) $4Rr^2$

Comprehension – 1 (No. 11 to 13)

Consider a triangle ABC, where x, y, z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a, b, c respectively and let the letters R, r, S, Δ denote the circumradius, inradius semi-perimeter and area of the triangle respectively.

11. If $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2+b^2+c^2}{k}$, then the value of k is :–

(*a*) R (*b*) S

(*c*) 2R (*d*) $\frac{3}{2}$ R

12. If cot A + cot B + cot C = k $\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$, then the value of k is

(*a*) R^2 (*b*) rR

(*c*) Δ (*d*) $a^2 + b^2 + c^2$

13. The value of $\frac{c\sin B + b\sin C}{x} + \frac{a\sin C + c\sin A}{y} + \frac{b\sin A + a\sin B}{z}$ is equal to

(*a*) $\frac{R}{r}$ (*b*) $\frac{S}{R}$

(*c*) 2 (*d*) 6

Comprehension – 2 (No. 14 to 16)

G is the centroid of triangle ABC. Perpendiculars from vertices A, B, C meet the sides BC, CA, AB at D, E, F respectively. P, Q, R are feet of the perpendiculars from G on sides BC, CA, AB respectively. L, M, N are the mid points of sides BC, CA, AB respectively, then

14. Length of the side PG is

(*a*) $\frac{1}{2}$ b sin C (*b*) $\frac{1}{2}$ c sin C

(*c*) $\frac{2}{3}$ b sin C (*d*) $\frac{1}{3}$ c sin B

15. (Area of ΔGPL) to (Area of ΔALD) is equal to

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{9}$

(*c*) $\frac{2}{3}$ (*d*) $\frac{4}{9}$

16. Area of ΔPQR is

(*a*) $\frac{1}{9}(a^2 + b^2 + c^2)\sin A \sin B \sin C$

(*b*) $\frac{1}{18}(a^2 + b^2 + c^2)\sin A \sin B \sin C$

(*c*) $\frac{2}{9}(a^2 + b^2 + c^2)\sin A \sin B \sin C$

(*d*) $\frac{1}{3}(a^2 + b^2 + c^2)\sin A \sin B \sin C$

17. Match the column :

Column–I	**Column–II**
(*a*) In a ΔABC, 2B = A + C and $b^2 = ac$. Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to	(p) 8
(*b*) In any right angled triangle ABC, the value of $\frac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)	(q) 1
(*c*) In a ΔABC if a = 2, bc = 9, then the value of 2RΔ is equal to	(r) 5
(*d*) In a ΔABC, a = 5, b = 3 and c = 7, then the value of 3 cos C + 7 cos B is equal to	(s) 9

18. Match the column :

Column – I	**Column – II**
(*a*) In a ΔABC, a = 4, b = 3 and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	(p) 3
(*b*) If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then value of tan A.tan C is equal to	(q) 7
(*c*) In a ΔABC, a = 5, b = 4 and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to	(r) 6
(*d*) In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of (8 cos B) is equal to	(s) 11

NUMERICAL BASED QUESTIONS

19. The value of

$a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=$

20. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 2, b = 3, c = 4 and H be the orthocentre. Find $\frac{\sqrt{15}}{2}(HA)$.

21. A triangle has base 6 cm and an area of 12 sq. cm. The difference of the base angles is 60°. The opposite angle is given by the equation $8\sin A - 6\cos A =$

22. The ratio of the circum-radius and the in-radius of ΔABC, whose sides are in the ratio 4 : 5 : 6, is expressed as $\left(\frac{m}{n}\right)$; $m, n \in N$, L.C.M. $(m, n) = 1$. Then find $(m-n)$.

23. Let ABC and ABC' be two non-congruent triangles with sides AB = 4, AC = AC' = $2\sqrt{2}$ and angle B = 30°. Find the absolute value of the difference between the areas of these triangles.

24. The sides of a ΔABC are in A.P. as well as in G.P., then the value of $\left(\frac{r_1}{r_2}-\frac{r_2}{r_3}\right)$ is (where r_1, r_2, r_3 are radii of excircles)

25. In a triangle ABC, if $\frac{\sin A}{4}=\frac{\sin B}{5}=\frac{\sin C}{6}$, then the value of

$\cos A+\cos B+\cos C=\frac{K}{M}$ then $\left(\frac{K}{M}\right)=$ .....

(where $(\cdot)$ denotes least integer function)

26. If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them 60^0. If the third side is 3, the remaining fourth side is

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to:

[*JEE Main-2019 (January)*]

(*a*) $\frac{3}{4}$ (*b*) $\frac{5}{4}$

(*c*) $\frac{7}{4}$ (*d*) $\frac{3}{2}$

2. With the usual notation, in ΔABC, If $\angle A + \angle B = 120^o$, $a = \sqrt{3}+1$ and $b = \sqrt{3}-1$, then the ratio $\angle A : \angle B$, is:

[*JEE Main - 2019 (January)*]

(*a*) 7 : 1 (*b*) 5 : 3

(*c*) 9 : 7 (*d*) 3 : 1

3. Given $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ for ΔABC with usual notation. If $\frac{\cos A}{\alpha}=\frac{\cos B}{\beta}=\frac{\cos C}{\gamma}$, then the ordered tried (α, β, γ) has a value:

[*JEE Main-2019 (January)*]

(*a*) (7, 19, 25) (*b*) (3, 4, 5)

(*c*) (5, 12 ,13) (*d*) (19, 7, 25)

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is :-

[*JEE Main-2019 (January)*]

(*a*) $\frac{3}{2}y$ (*b*) $\frac{c}{\sqrt{3}}$

(*c*) $\frac{c}{3}$ (*d*) $\frac{y}{\sqrt{3}}$

5. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is :

[*JEE Main-2019 (April)*]

(*a*) 5 : 9 : 13 (*b*) 5 : 6 : 7

(*c*) 4 : 5 : 6 (*d*) 3 : 4 : 5

6. The angles A, B and C of a triangle ABC are in A.P. and a: b = 1 : $\sqrt{3}$. If c = 4cm, then the area (in sq. cm) of this triangle is: [*JEE Main-2019 (April)*]

(*a*) $4\sqrt{3}$ (*b*) $\frac{2}{\sqrt{3}}$

(*c*) $2\sqrt{3}$ (*d*) $\frac{4}{\sqrt{3}}$

7. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is :

[*JEE Main-2021 (February)*]

(a) An equilateral triangle having each of its side of length $\sqrt{3}$.

(b) An equilateral triangle of height $\frac{2r}{3}$.

(c) A right angle triangle having two of its sides of length $2r$ and r.

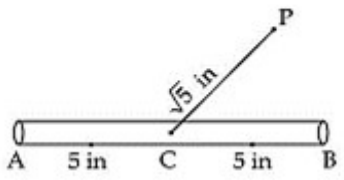
(d) An isosceles triangle with base equal to $2r$.

8. If in a triangle $ABC, AB = 5$ units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of ΔABC is 5 units, then the area (in sq. units) of ΔABC is: [*JEE Main-2021 (February)*]

(a) $10 + 6\sqrt{2}$ (b) $8 + 2\sqrt{2}$

(c) $6 + 8\sqrt{3}$ (d) $4 + 2\sqrt{3}$

9. 10 inches long pencil AB with mid point C and a small eraser P are placed on the horizontal top of a table such that $PC = \sqrt{5}$ inches and $\angle PCB = \tan^{-1}(2)$. The acute angle through which the pencil must be rotated about C so that the perpendicular distance between eraser and pencil becomes exactly 1 inch is: [*JEE Main-2021 (August)*]



(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $6\tan^{-1}(1)$

(c) $\tan^{-1}\left(\frac{1}{3}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

10. Let $\frac{\sin A}{\sin B} = \frac{\sin(A-C)}{\sin(C-B)}$, where A, B, C are angles of triangle ABC. If the lengths of the sides opposite these angles are a, b, c respectively, then:

[*JEE Main-2021 (August)*]

(a) $b^2 - a^2 = a^2 + c^2$ (b) b^2, c^2, a^2 Are in A.P.

(c) c^2, a^2, b^2 Are in A.P. (d) a^2, b^2, c^2 Are in A.P.

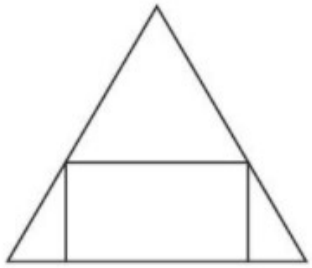
11. Let A B C D be a square of side of unit length. Let a circle C_1 cantered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to

[*JEE Main-2021 (March)*]

12. In ΔABC, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of $2R + r$ (in cm) is equal to

[*JEE Main-2021 (March)*]

13. If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is [*JEE Main-2021 (March)*]



JEE-ADVANCED PREVIOUS YEAR'S

1. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and c = $\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively.

Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals [*IIT JEE-2012*]

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$

(c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

2. In a triangle PQR, P is the largest angle and cosP = $\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[*JEE Advanced-2013*]

(a) 16 (b) 18

(c) 24 (d) 22

3. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

[*JEE Advanced- 2014*]

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

Paragraph for 4 and 5: Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$ respectively, of a triangle PQR.

4. If the triangle PQR varies, then the minimum value of cos (P + Q) + (Q + R) + cos(R + P) is

[*JEE Advanced-2017*]

(*a*) $-\frac{3}{2}$ (*b*) $\frac{3}{2}$

(*c*) $\frac{5}{3}$ (*d*) $-\frac{5}{3}$

5. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$ [*JEE Advanced-2017*]

(*a*) sin (P + Q) (*b*) sin (P + R)

(*c*) sin (Q + R) (*d*) $\sin^2 R$

6. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE ?

[*JEE Advanced-2018*]

(*a*) $\angle QPR = 45^\circ$

(*b*) The area of the traingel PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$

(*c*) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(*d*) The area of the circumcircle of the triangle PQR is 100π.

7. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct?

[*JEE Advanced-2019*]

(*a*) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(*b*) Radius of incircle of $\Delta PQR \frac{\sqrt{3}}{2}(2-\sqrt{3})$

(*c*) Length of $RS = \frac{\sqrt{7}}{2}$

(*d*) Length of $OE = \frac{1}{6}$

8. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2Y}{x+y+z},$$

then which of the following statements is/are TRUE?

[*JEE(Advanced)-2020*]

(*a*) $2Y = X + Z$ (*b*) $Y = X + Z$

(*c*) $\tan\frac{X}{2} = \frac{x}{y+z}$ (*d*) $x^2+z^2-y^2=xz$

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ANSWER KEY

Topicwise Questions

1. (*b*) **2.** (*a*) **3.** (*a*) **4.** (*b*) **5.** (*b*) **6.** (*b*) **7.** (*d*) **8.** (*b*) **9.** (*b*) **10.** (*a*)
11. (*a*) **12.** (*c*) **13.** (*a*) **14.** (*d*) **15.** (*d*) **16.** (*a*) **17.** (*b*) **18.** (*b*) **19.** (*d*) **20.** (*b*)
21. (*b*) **22.** (*b*) **23.** (*c*) **24.** (*b*) **25.** (*c*) **26.** (*c*) **27.** (*a*) **28.** (*c*) **29.** (*b*) **30.** (*c*)
31. (*b*) **32.** (*a*) **33.** (*b*)

Learning Plus

1. (*c*) **2.** (*c*) **3.** (*c*) **4.** (*c*) **5.** (*b*) **6.** (*c*) **7.** (*b*) **8.** (*d*) **9.** (*b*) **10.** (*a*)
11. (*b*) **12.** (*a*) **13.** (*c*) **14.** (*d*) **15.** (*b*) **16.** (*a*) **17.** (*b*) **18.** (*b*) **19.** (*c*) **20.** (*b*)
21. (*a*) **22.** (*a*) **23.** (*a*) **24.** (*a*) **25.** (*a*) **26.** (*b*) **27.** (*b*) **28.** (3)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *b*) **2.** (*a*, *b*, *c*) **3.** (*a*, *c*, *d*) **4.** (*b*,*d*) **5.** (*a*, *b*, *c*, *d*) **6.** (*a*, *b*) **7.** (*a*, *b*, *c*, *d*) **8.** (*b*,*d*)
9. (*a*, *c*, *d*) **10.** (*a*, *c*, *d*) **11.** (*c*) **12.** (*c*) **13.** (*d*) **14.** (*d*) **15.** (*b*) **16.** (*b*)
17. (*a*) → (q), (*b*) → (p), (*c*) → (s), (*d*) → (r) **18.** (*a*) → (s), (*b*) → (p), (*c*) → (r),(*d*) → (q) **19.** [0000] **20.** [7]
21. [0003] **22.** [0009] **23.** [0004] **24.** [0000] **25.** [0002] **26.** [0002]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*) **2.** (*a*) **3.** (*a*) **4.** (*b*) **5.** (*c*) **6.** (*c*) **7.** (*a*) **8.** (*c*) **9.** (*a*) **10.** (*b*)
11. (1) **12.** (15) **13.** (3)

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*c*) **2.** (*b*, *d*) **3.** (*b*) **4.** (*a*) **5.** (*a*) **6.** (*b*,*c*,*d*) **7.** (*b*,*c*,*d*) **8.** (*b*,*c*)

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PREFACE

In this era of cut throat competition among **JEE** aspirants, only few are lucky enough to make it through. During the preparation, a student needs the most relevant study material to cover all the topics with proper planning to crack **JEE**.

We know there is abundance of study materials floating in the market but what We felt missing, is a self-driven and well-planned Study material. We have tried to bring clarity and Quality in the students by introducing this **Aspire for JEE**

The book covers all the important concepts and theories with a rich amount of practice questions as well. This book is designed for the perfect guidance you need.

The study material is consistent with the syllabus offered in NCERT Books as well and has been curated by industry experts under the careful observation of **Physics Wallah Team**.

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PRAYAS2.0: t.me/realprayasians2

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1 Basic Mathematics and Logarithm

Topicwise Questions

1. (*b*) It is a property

2. (*b*) x and y are rational and

$(x+y)+(x-2y)\sqrt{2}=2x-y+(x-y-1)\sqrt{6}$

Since real part of both sides are equal and coefficient of irrational parts should be zero in both sides

$\therefore \quad x-2y=0 \Rightarrow x=2y$

and $x-y-1=0 \Rightarrow y=1 \therefore x=2$

$x=2, y=1$

3. (*d*) It is a property

4. (*a*) Here $(x-1)^2+(x-2)^2+(x-3)^2=0$

Here sum of three + ve number can not be equal to zero then all three number independently be zero.

$\therefore$ There are no any real root.

5. (*c*) Let $P(x)=x^3-a^2x+x+2$ be the given polynomial

Then by factor theorem, $(x-a)$ is a factor of $P(x)$ iff $P(A)=0$

$\Rightarrow \quad a^3-a^2 . a+a+2=0$

$\Rightarrow \quad a+2=0 \Rightarrow a=-2$

6. (*a*) Every irrational number can be expressed on the number line. This statement is always true.

7. (*d*) $\underset{x}{(\text{rational})} \times \underset{y}{(\text{irrational})} = \text{irrational except when } x=0$

8. (*b*) $2x^2-4xy+xy-2y^2=7$

$2x(x-2y)+y(x-2y)=7$

$(x-2y)(2x+y)=7$

x, y are integers $\Rightarrow x-2y, 2x+y$ are also integers

Four cases are possible

Case I: $x-2y=1, 2x+y=7 \Rightarrow x=3, y=1$

$|x+y|=4$

Case II: $x-2y=7, 2x+y=1 \Rightarrow x=\frac{9}{5}$ rejected

Case III: $x-2y=-1, 2x+y=-7$

$\Rightarrow \quad x=-3, y=-1$

$|x+y|=4$

Case IV: $x-2y=-7, 2x+y=-1 \Rightarrow x=-\frac{9}{5}$ rejected

Hence $|x+y|=4$

9. (*b*) $a(a-b)+b(b-c)+c(c-a)=0$

$\Rightarrow a^2+b^2+c^2-ab-bc-ca=0$

Multiplying & deviding by 2,

$$\frac{1}{2}[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(a^2+c^2-2ac)]=0$$

$$\Rightarrow \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$$

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0 \Rightarrow a=b=c$

10. (*a*) $2x^3-5x^2+x+2$

$=ax^3+(-b-2a)x^2+(a-1+2b)x+2$

On comparing coefficient of x^3 & x, we get

$a=2$ & $b=1$

11. (*b*) $[e]-[-\pi]$

here $e \approx 2.7$

$\pi \approx 3.14$

$\therefore \quad [2.7]-[-3.14]$

$=2-(-4)=2+4=6$

12. (*b*) Taking log on both sides,

$(3x^2-10x+3)\log|x-3|=0$

$\log|x-3|=0$ or $3x^2-10x+3=0$

$x-3 \neq 0; |x-3|=1$ or $(x-3)(3x-1)=0$

$x \neq 3; (x-3)=\pm 1$ or $x=3; x=\frac{1}{3}$

$\therefore x=2, 4, 1$ by 3 $\quad [\because x \neq 3]$

Hence, three real solutions

13. (*b*) Applying base change theorem,

$= \log_{abc}\sqrt{bc} + \log_{abc}\sqrt{ca} + \log_{abc}\sqrt{ab}$

$= \log_{abc}\sqrt{bc}\,.\,\sqrt{ca}\,.\sqrt{ab} = \log_{abc} abc = 1$

14. (*d*) Given, $\log_x \log_{18}(\sqrt{2}+\sqrt{8}) = \frac{1}{3}$

$\Rightarrow \log_{18}(\sqrt{2}+2\sqrt{2}) = x^{1/3} \Rightarrow \log_{18} 3\sqrt{2} = x^{1/3}$

$\Rightarrow (3\sqrt{2})^2 = \sqrt{bc} \;\Rightarrow 18 = 18^{2x^{1/3}}$

$2x^{1/3} = 1 \Rightarrow x^{1/3} = \frac{1}{2} \Rightarrow x = \frac{1}{8} \Rightarrow 1000x = 125$

15. (*c*) $\sqrt{\log_{10}(-x)} = \log_{10}\sqrt{x^2}$

$\Rightarrow x < 0$

$\sqrt{\log_{10}(-x)} = \log_{10}(-x)$

$\Rightarrow \log^2{}_{10}(-x) - \log_{10}(-x) = 0$

$\Rightarrow \log_{10}(-x) = 0, 1 \Rightarrow x = -1, -10$

two solution

16. (*c*) $\log_2 15\,.\,\log_{1/6} 2\,.\,\log_3 1/6$

$= \frac{\log_e 15}{\log_e 2} \times \frac{\log_e 2}{\log_e 1/6} \times \frac{\log_e 1/6}{\log_e 3} = \frac{\log_e 15}{\log_e 3}$

$= \frac{\log_e (3\times 5)}{\log_e 3} = 1 + \log_3 5$

$\therefore \; [1 + \log_3 5] = 2$

17. (*d*) $= \frac{2^{\log_2(a^4)} - 3^{\log_3(a^2+1)} - 2a}{7^{\log_7(a^2)} - a - 1} = \frac{a^4 - (a^2+1) - 2a}{a^2 - a - 1}$

$= \frac{(a^2)^2 - (a+1)^2}{(a^2 - a - 1)} = a^2 + a + 1$

18. (*b*) $3^{\log_3(x^2)} - 2x - 3 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = -1, 3, (x = -1$ reject $\because x > 0)$

number of values of x is one

19. (*c*) $\log_2 7 \Rightarrow \log_2 4 < \log_2 7 < \log_2 8$

$\Rightarrow 2 < \log_2 7 < 3$ i.e. not integer

Let $\log_2 7 = \frac{p}{q}$ (where p and q are coprime)

$\Rightarrow 2^{p/q} = 7 \;\Rightarrow 2^p = 7^q$

which is not possible so $\log_2 7$ is an irratoinal number

20. (*c*) $\Rightarrow \text{antilog}_{16} 0.75 = (16)^{0.75}$

$= (16)^{3/4} = (2^4)^{3/4} = 2^3 = 8$

21. (*b*) $\log_{\sqrt{0.9}} \log_5 (\sqrt{x^2+5+x}) > 0$

$\log_5 (\sqrt{x^2+5+x}) < 1$

$(x^2+5+x)^{1/2} < 5$ and $x^2+x+5 > 0$

$\Rightarrow \quad x^2+5+x < 25$

$\Rightarrow \quad x^2+x-20 < 0$

$\Rightarrow \quad (x+5)(x-4) < 0$

$\Rightarrow \quad x \in (-5, 4)$

$\therefore \quad n = 8$

22. (*a*) $x^2 - 4 > 0 \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

$\log_5(x^2-4) > 0 \Rightarrow x^2 - 4 > 1 \Rightarrow x^2 - 5 > 0$

$\Rightarrow x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

Now

$\log_{0.5} \log_5 (x^2-4) > \log_{0.5} 1 \Rightarrow \log_5(x^2-4) < 1$

$x^2 - 4 < 5 \Rightarrow x^2 - 9 < 0, x \in (-3, 3)$

$\therefore$ **Ans. :** $\left(-3, -\sqrt{5}\right) \cup \left(\sqrt{5}, 3\right)$

23. (*d*) $\log_{1-x}(x-2) \geq -1$

$1 - x > 0 \Rightarrow 1 > x \Rightarrow x \in (-\infty, 1) - \{0\}$

$x - 2 > 0 \Rightarrow x > 2$ No solution.

24. (*b*) $2 - \log_2 (x^2+3x) \geq 0$

$\log_2(x^2+3x) \leq 2$

$\Rightarrow x^2 + 3x \leq 4$

$\Rightarrow x^2 + 3x - 4 \leq 0$

$\Rightarrow (x+4)(x-1) \leq 0$

$\Rightarrow x \in [-4, 1]$

and $x^2 + 3x > 0 \Rightarrow x \in (-\infty, -3) \cup (0, \infty)$

Ans. : $[-4, -3) \cup (0, 1]$



25. (*c*) When $x \leq -\frac{3}{4}$ $-4x - 3 - 3x + 4 = 12 - 11 = 7x$

$\Rightarrow x = -\frac{11}{7}$

When $-\frac{3}{4} < x \leq \frac{4}{3}$; $4x + 3 - 3x + 4 = 12$

$x = 5$ will not satisfy

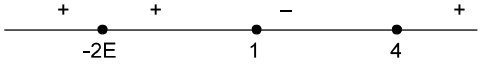
When $x > \frac{4}{3}$ $4x + 3 + 3x - 4 = 12$

$7x = 13 \Rightarrow x = \frac{13}{7}$

26. (*a*) Since $(x^2+x-2) - (x^2-2x-8) = 3x + 6 = 3(x+2)$

$\therefore \quad (x^2-2x-8)(x^2+x-2) \leq 0$

i.e. $\quad (x-4)(x+2)(x+2)(x-1) \leq 0$



$\therefore$ Solution set is $[1,4] \cup \{-2\}$

27. (*b*) $||x-1|-1| \le 1$

$\Rightarrow -1 \le |x-1|-1 \le 1$

$\Rightarrow 0 \le |x-1| \le 2$

$\Rightarrow -2 \le x-1 \le 2$

$\Rightarrow -1 \le x \le 3$

$\therefore$ **Ans. :** $x \in [-1, 3]$

28. (*d*) $|x|^2 - 3|x| + 2 = 0$

$\Rightarrow (|x|-2)(|x|-1) = 0$

$\Rightarrow |x| = 1, 2$

$\Rightarrow x = \pm 1, \pm 2$

$\therefore$ number of real roots is 4.

Learning Plus

1. (*c*) A & B are two rational number then $\frac{A}{B}$ is

Also rational number if $B \neq 0$.

2. (*d*) $\frac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)} = ?$

Let $a-b=A, (b-c)=B, (c-a)=C$

$[\because A+B+C=0 \Rightarrow A^3+B^3+C^3=3ABC]$

$\therefore \frac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)} = \frac{3ABC}{ABC} = 3$

3. (*b*) $P(4) = k4^3 + 3\cdot4^2 - 3$ & $Q(4) = 2\cdot4^3 - 5\cdot4 + k$

remainder is same

$P(4) = Q(4) \Rightarrow 64k + 48 - 3 = 128 - 20 +$

$\Rightarrow 63k = 108 - 45 \Rightarrow k = \frac{63}{63} = 1$

4. (*c*) Putting $x = 1$, remainder $= 7$

5. (*a*) $x^{x.x^{1/3}} = \left(x.x^{1/3}\right)^x \Rightarrow x^{x^{1+\frac{1}{3}}} = \left(x^{1+\frac{1}{3}}\right)^x$

$\Rightarrow x^{x^{4/3}} = \left(x^{4/3}\right)^x = x^{x^{4/3}} = x^{\frac{4}{3}x} \Rightarrow x^{4/3} = \frac{4}{3}x;$

Also is an obvious solution.

6. (*a*) $4^{\left(x^2+2\right)} - 9.2^{\left(x^2+2\right)} + 8 = 0$

$\Rightarrow \left(2^{\left(x^2+2\right)}\right)^2 - 9.2^{\left(x^2+2\right)} + 8 = 0$

Put $2^{\left(x^2+2\right)} = y$. Then $y^2 - 9y + 8 = 0$, which gives $y = 8, y = 1$.

when $y = 8 \Rightarrow 2^{x^2+2} = 8 \Rightarrow 2^{x^2+2} = 2^3 \Rightarrow x^2 + 2 = 3$

$\Rightarrow x^2 = 1 \Rightarrow x = 1, -1$.

when $y = 1 \Rightarrow 2^{x^2+2} = 1 \Rightarrow 2^{x^2+2} = 2^\circ$

$\Rightarrow x^2 + 2 = 0 \Rightarrow x^2 = -2$, which is not

7. (*a*) Let x be the required logarithm , then by definition

$\log_{2\sqrt{2}} 32\sqrt[5]{4} = x$

$\left(2\sqrt{2}\right)^x = 32\sqrt[5]{4} \Rightarrow \left(2.2^{1/2}\right)^x = 2^5.2^{2/5};$

$\therefore 2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$

Here, by equating the indices, $\frac{3}{2}x = \frac{27}{5}$

$\therefore x = \frac{18}{5} = 3.6.$

8. (*b*) $\log_3 4.\log_4 5.\log_5 6.\log_6 7.\log_7 8.\log_8 9$

$= \frac{\log 4}{\log 3}.\frac{\log 5}{\log 4}.\frac{\log 6}{\log 5}.\frac{\log 7}{\log 6}.\frac{\log 8}{\log 7}.\frac{\log 9}{\log 8} = \frac{\log 9}{\log 3}$

$= \log_3 9 = \log_3 3^2 = 2.$

9. (*d*) $\frac{1}{\log_b b + \log_b a + \log_b c} + \frac{1}{\log_c c + \log_c a + \log_c b} + \frac{1}{\log_a a + \log_a b + \log_a c}$

$= \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$

$= \log_{abc} b + \log_{abc} c + \log_{abc} a = \log_{abc} abc = 1$

10. (*a*) $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}\left(\log_e a + \log_e b\right)$

$= \frac{1}{2}\log_e(ab) = \log_e\sqrt{ab}$

$\Rightarrow \frac{a+b}{2} = \sqrt{ab} \Rightarrow a + b = 2\sqrt{ab}$

$\Rightarrow \left(\sqrt{a} - \sqrt{b}\right)^2 = 0 \Rightarrow \sqrt{a} - \sqrt{b} = 0 \Rightarrow a = b$

11. (*c*) $\log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}} = \log_7 \log_7 7^{7/8} = \log_7 (7/8)$

$= \log_7 7 - \log_7 8 = 1 - \log_7 2^3 = 1 - 3\log_7 2.$

12. (*d*) $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$= 3^{4\log_3 5} + 3^{3.\frac{1}{2}\log_3 36} + 3^{4\log_9 7}$

$= 3^{\log_3 5^4} + 3^{\log_3 36^{3/2}} + 3^{\log_3 7^{4/2}}$

$= 5^4 + 36^{3/2} + 7^2 = 890$

13. (*d*) $\log_{1000} x^2 = \log_{10^3} x^2 = 2\log_{10^3} x = \frac{2}{3}\log_{10} x = \frac{2}{3}.y$

14. (*b*) $x = \log_a bc \Rightarrow 1 + x = \log_a a + \log_a bc = \log_a abc$

$\therefore (1+x)^{-1} = \log_{abc} a$

$\therefore (1+x)^{-1} + (1+y)^{-1} + (1+z)^{-1} = \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1.$

15. (*c*) $a = \log_{24} 12 = \frac{\log 12}{\log 24} = \frac{2\log 2 + \log 3}{3\log 2 + \log 3}$

$b = \log_{36} 24 = \frac{3\log 2 + \log 3}{2(\log 2 + \log 3)}$

$c = \log_{48} 36 = \frac{2(\log 2 + \log 3)}{4\log 2 + \log 3}$

$\therefore abc = \frac{2\log 2 + \log 3}{4\log 2 + \log 3}$

$\Rightarrow 1 + abc = \frac{6\log 2 + 2\log 3}{4\log 2 + \log 3} = 2.\frac{3\log 2 + \log 3}{4\log 2 + \log 3} = 2bc.$

16. (*c*) $\log_7 \log_5 \left(\sqrt{x^2+5+x}\right) = 0 = \log_7 1$

$\Rightarrow \log_5 \left(x^2+5+x\right)^{1/2} = 1 = \log_5 5$

$\Rightarrow (x^2+5+x)^{1/2} = 5$

$\Rightarrow (x^2+x+5) = 25 \Rightarrow x^2 + x - 20 = 0$

$\Rightarrow (x-4)(x+5) = 0 \Rightarrow x = 4, -5 \Rightarrow x = 4$

17. (*a*) $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+.....)} = \left(\frac{1}{20}\right)^{2\log_{20}\left(\frac{0.1}{1-0.1}\right)}$

$= 20^{-2\log_{20}(1/9)} = 20^{2\log_{20} 9} = 20^{\log_{20} 9^2} = 9^2 = 81$

18. (*b*) $\log_2 . \log_3 .... \log_{99} \log_{100} 100^{99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$

$= \log_2 \log_3 .... \log_{99}^{99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$

$[\log_{100} 100 = 1]$

$= \log_2 . \log_3 .... \log_{98}^{98^{97^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$

$= \log_2 . \log_3 .... \log_{97}^{97^{96^{\cdot^{\cdot^{\cdot^{2^1}}}}}} = \log_2 \log_3 3^{2^1}$

$= \log_2 2' \log_3 3 = \log_2 2 = 1.$

19. (*d*) $\log_2(x+5) = 6 - x \Rightarrow x + 5 = 2^{6-x}$

$\Rightarrow x + 5 = 64.2^{-x}$

Let $y = x + 5, y = 64.2^{-x}$ will intersect at one point.

Number of solutions = 1.

20. (*c*) $y = 3^{12} \times 2^8 \Rightarrow \log_{10} y = 12\log_{10} 3 + 8\log_{10} 2$

$= 12 \times 0.47712 + 8 \times 0.30103$

$= 5.72544 + 2.40824 = 8.13368$

$\therefore$ Number of digits in $y = 8 + 1 = 9$.

21. (*d*) $\log_{|x|}(x^2+x+1) \geq 0$

$D: |x| \neq 0, 1$

Case-I: if $|x| < 1$

$x^2 + x + 1 \leq 1 \Rightarrow x(x+1) \leq 0$

$x \in (-1, 0)$

Case-II: if $|x| > 1$

$x^2 + x + 1 > 1 \Rightarrow x(x+1) \geq 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (0, \infty)$

$\therefore \quad x \in (-\infty, -1) \cup (1, \infty)$

22. (*b*) $\log_{1/2}\left(x^2 - 6x + 12\right) \geq -2$ ...(*i*)

For log to be defined, $x^2 - 6x + 12 > 0$

$\Rightarrow (x-3)^2 + 3 > 0$, which is true $\forall\, x \in R$.

From (i), $x^2 - 6x + 12 \leq \left(\frac{1}{2}\right)^{-2}$

$\Rightarrow x^2 - 6x + 12 \leq 4 \Rightarrow x^2 - 6x + 8 \leq 0$

$\Rightarrow (x-2)(x-4) \leq 0 \Rightarrow 2 \leq x \leq 4$

$\therefore x \in [2, 4]$.

23. (*c*) $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$ ...(*i*)

For log to be defined $x - 1 > 0 \Rightarrow x > 1$

From (*i*), $\log_{(0.2)^2}(x-1) \geq \log_{0.2}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.2}(x-1) \geq \log_{0.2}(x-1) \Rightarrow \sqrt{x-1} \leq (x-1)$

$\Rightarrow \sqrt{x-1}(1-\sqrt{x-1}) \leq 0 \Rightarrow 1 - \sqrt{x-1} \leq 0$

$\Rightarrow \sqrt{x-1} \geq 1 \Rightarrow x \geq 2 \therefore x \in [2, \infty)$

24. (*a*) $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\log_{0.3}(x-1) < \frac{1}{2}\log_{(0.3)}(x-1)$

$\log_{0.3}(x-1) < \log_{(0.3)}(x-1)^{1/2}$

Here base is less than 1, therefore the inequality is reversed

$(x-1) > (x-1)^{1/2}$

$(x-1)^2 > (x-1)$

$x^2\ 1-2x-x+1>0$

$x^2-3x+2>0$

$x^2-2x-x+2>0$

$x(x-2)-1(x-2)>0$

$(x-1)(x-2)>0$

$\underset{1}{+}\ \underset{2}{-}\ +$

$x\in(-\infty,1)\cup(2,\infty)$

$\because x-1>0\Rightarrow x>1$

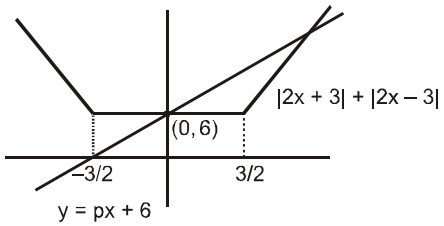
than $x\in(2,\infty)$

25. (*b*) $f(x)=|x-1|+|x-2|+|x-3|$

$$=\begin{cases}-x+1-x+2-x+3=6-3x, & x\le 1\\ x-1-x+2-x+3=4-x & 1<x\le 2\\ x-1+x-2-x+3=x & 2<x\le 3\\ x-1+x-2+x-3=3x-6 & x>3\end{cases}$$

min $\quad f(x)=2.$

26. (*d*)



For more than 2 solution $p=0$

27. (*c*) $[2x]-3\{2x\}=1$

$\{2x\}=\dfrac{[2x]-1}{3}$

$0\le\{2x\}<1$

$\Rightarrow\quad 1\le[2x]<4$

$\Rightarrow\quad [2x]=1,2,3$

$[2x]=1,\ \{2x\}=0$

$2x=1+0$

$x=\dfrac{1}{2}$

$[2x]=2\ ;\ \{2x\}=\dfrac{1}{3}$

$2x=2+\dfrac{1}{3}$

$2x=\dfrac{7}{3}$

$x=\dfrac{7}{6}$

$[2x]=3,\quad \{2x\}=\dfrac{2}{3},\quad 2x=3+\dfrac{2}{3}$

we get three value of x

Advanced Level Multiconcept Questions

1. (a,b)

$\dfrac{y}{x}=x$

$\Rightarrow y=x^2\quad[\because x\ne 0]$

$\Rightarrow y\ne 0$

$\because x^2>0\Rightarrow y>0\quad\therefore y\ne -1$

2. (a,b,c)

$x^{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$

$\left[\dfrac{3}{4}(\log_2 x)^2+\log_2 x-\dfrac{5}{4}\right]\log_2 x=\log_2\sqrt{2}=\dfrac{1}{2}$

Let $\log_2 x=t$

$(3t^2+4t-5)\,t=2\Rightarrow 3t^3+4t^2-5t-2=0$

$t=1,-2,-\dfrac{1}{3}=\log_2 x$

$x=2,\ \dfrac{1}{4},\ 2^{-\frac{1}{3}}$

3. (a,b,c)

$(\log_5 x)^2+\log_{5x}\dfrac{5}{x}=1$

$\Rightarrow(\log_5 x)^2+\log_{5x}5-\log_{5x}x=1$

$\Rightarrow(\log_5 x)^2+\dfrac{\log_5 5}{\log_5 5+\log_5 x}-\dfrac{\log_5 x}{\log_5 5+\log_5 x}=1$

$\Rightarrow(\log_5 x)^2+\dfrac{1}{1+\log_5 x}-\dfrac{\log_5 x}{1+\log_5 x}=1$

Let $\log_5 x = t$

$\therefore \; t^2 + \dfrac{1}{1+t} - \dfrac{t}{1+t} = 1$

$\Rightarrow \; \dfrac{t^2(1+t)+1-t}{1+t} = 1$

$\Rightarrow \; t^3 + t^2 + 1 - t = 1 + t$

$t^3 + t^2 - 2t = 0$

$t(t^2 + t - 2) = 0$

$t(t-1)(t+2) = 0$

$t = 0, 1, -2$

$\therefore \; \log_5 x = 0, 1, -2$

$\therefore \; x = 1, 5, \dfrac{1}{25}$

4. (a, b, c, d)

$= \log_3 135 \log_3 15 - \log_3 5 \log_3 405$

$= \log_3(5 \times 3^3) \cdot \log(5 \times 3) - \log_3 5 . \log_3 (5 \times 3^4)$

$= (\log_3 5 + \log_3 3^3)(\log_3 5 + \log_3 3) - \log_3 5 (\log_3 5 + \log_3 3^4)$

$= (x+3)(x+1) - x(x+4)$

{Let $\log_3 5 = x$}

$= x^2 + 4x + 3 - x^2 - 4x = 3$

which is Prime, rational Integer and natural number

5. (a, b)

$\Rightarrow \log_3 xy = 2\log_3 3 + \log_3 2$

$\Rightarrow \log_3 xy = \log_3(2 \times 9) \Rightarrow xy = 18$ ...(*i*)

and $\log_{27}(x+y) = \dfrac{2}{3} \Rightarrow x + y = 27^{2/3}$

$\Rightarrow x + y = 3^2 \Rightarrow x + y = 9$ ...(*ii*)

from equation (i) & (ii)

$\therefore x^2 - 9x + 18 = 0 \Rightarrow (x-6)(x-3) = 0$

$\Rightarrow x = 6, 3$ so $(x, y) \equiv (6, 3) \equiv (3, 6)$

6. (a, b, c, d)

$\dfrac{4}{2} \log_x 2 + \dfrac{\log_x 64}{\log_x 2x}$

$\Rightarrow \dfrac{2x \log_x 2}{x} + \dfrac{6\log_x 2}{1 + \log_x 2} = 3$

Let $\alpha = \log_x 2$

$2\alpha + \dfrac{6\alpha}{1+\alpha} = 3$

$2\alpha + 2\alpha^2 + 6\alpha - 3 - 3\alpha = 0$

$\Rightarrow 2\alpha^2 + 5\alpha - 3 = 0$

$\Rightarrow (\alpha + 3)(2\alpha - 1) = 0 \Rightarrow a = -3, 1/2$

$\therefore \; \log_x 2 = -3 \Rightarrow x = 2^{-1/3}$ (Irrational)

or $\log_x 2 = \dfrac{1}{2} \Rightarrow x = 4$ (Integer)

7. (a, b, c, d)

Taking $\log_3$ on both sides

$[(\log_3 x)^2 - \dfrac{9}{2}\log_3 x + 5](\log_3 x) = \dfrac{3}{2}$

Let $\log_3 x = \alpha$

$\Rightarrow \; \dfrac{(2\alpha^3 - 9\alpha^2 + 10\alpha)}{2} = \dfrac{3}{2}$

$\Rightarrow 2\alpha^3 - 9\alpha^2 + 10\alpha - 3 = 0$

$\Rightarrow (\alpha - 1)(2\alpha^2 - \alpha + 3) = 0$

$\Rightarrow (\alpha - 1)(\alpha - 3)(2\alpha - 1) = 0 \Rightarrow \alpha = 1, 3, \dfrac{1}{2}$

$\therefore \; \log_3 x = 1; \log_3 x = 3 ; \log_3 x = \dfrac{1}{2}$

$\Rightarrow x = 3 ; \; x = 3^3 = 27 ; x = \sqrt{3}$

Exactly three solution, one is irrational solution and every real number is also complex.

8. (b, c)

$(\log_2 x)^4 - \left(\log_2 \left(\dfrac{x}{2}\right)^3\right)^2 + 9\left[\log_2 32 - \log_2 x^2\right] < 4(\log_2 x)^2$

(note $x > 0$)

$(\log_2 x)^4 - (3\log_2 x - 3)^2 + 45 - 18\log_2 x < 4(\log_2 x)^2$

Let $\log_2 x = t$

$t^4 - (3t-3)^2 + 45 - 18t < 4t^2$

$\Rightarrow \; t^4 - (9t^2 + 9 - 18t) - 18t + 45 < 4t^2$

$\Rightarrow t^4 - 13t^2 + 36 < 0$

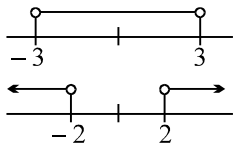
$\Rightarrow (t^2 - 4)(t^2 - 9) < 0$

$\Rightarrow \; 4 < t^2 < 9$

$t^2 < 9 \; \Rightarrow \; -3 < t < 3$

and $t^2 > 4 \; \Rightarrow \; t > 2$ or $t < -2$

hence, $t \in (-3, -2) \cup (2, 3)$



$x \in \left(\dfrac{1}{8}, \dfrac{1}{4}\right) \cup (4, 8) \; \Rightarrow \; B, C$

9. (a, b, d)

$\dfrac{1}{2} \le \log_{0.1} x \le 2 \Rightarrow \left(\dfrac{1}{10}\right)^{1/2} \ge x \ge \left(\dfrac{1}{10}\right)^2$

10. (b, c)

$|x-5|+|x+5|=10$

Case-I: $x \geq 5$, the equation becomes

$(x-5)+(x+5)=10$

$\Rightarrow 2x=10$

$\Rightarrow x=5$ which satisfies the case, therefore accepted.

Case-II: $-5 < x < 5$ The above equation becomes

$-(x-5)+(x+5)=10$

$\Rightarrow -x+5+x+5=10$

$\Rightarrow 10=10$ which is true.

So, the solution is $x \in (-5, 5)$

Case-III: $x \leq -5$, The above equation becomes

$-(x-5)-(x+5)=10$

$\Rightarrow -x+5-x-5=10$

$\Rightarrow -2x=10$

$\Rightarrow x=-5$ which satisfies the above case so, accepted.

$\therefore$ final answer is $x \in [-5, 5]$

11. (b)

12. (a)

13. (d) $x = 3^{\log 5 - \log 7}$

$y = 5^{\log 7 - \log 3}$

$z = 7^{\log 3 - \log 5}$

$\therefore \quad x \cdot y \cdot z = 1$

$\therefore \quad A = 1$

$\log_2 (6 \log_2 |x| - 3) - \log_2 (4 \log |x| - 5) = \log_2 3$

$\dfrac{6\log_2 |x| - 3}{4\log_2 |x| - 5} = 3$

let $\log_2 |x| = 1$

$\therefore \dfrac{6t-3}{4t-5} = 3$

$6t-3 = 12t-15,\ 6t = 12$

$\therefore t = 2,\ \log_2 |x| = 2, |x| = 4$

$\therefore x = \pm 4$

$B = 16 + 16 = 32$

$\log_2 (\log_2 3) + \log_2 (\log_3 4) + \log_2 (\log_4 5) + \log_2 (\log_5 6)$
$+ \log_2 (\log_6 7) + \log_2 (\log_7 8) = \log_2 (\log_2 8) = \log_2 3$

$\therefore C = 1$

14. (b)

15. (d)

16. (a) $\left(\sqrt{x}\right)^{\log_5 x - 1} = 5 \Rightarrow (\log_5 x - 1)\dfrac{1}{2}\log_5 x = 1$

$\Rightarrow \log_5 x (\log_5 x - 1) = 2$

Put $\log_5 x = y$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2$ or $-1 \Rightarrow \log_5 x = 2$ or -1

$x = 25$ or $x = \dfrac{1}{5}$

$\alpha = 25$ and $\beta = \dfrac{1}{5}$

(a) Now, let $N = (25)^{10}$

$\log_{10} N = 10 \cdot 2\log_{10} 2 = 20\,[1 - 0.3010] = 20\,[0.6990] = 13.980 = 13.98$

$\Rightarrow N$ is a 14 digit number.

(b) Let $x = \left(\dfrac{1}{5}\right)^{10}$

$\log x = 10(-\log 5) = -10(1 - \log 2) = -10\,[0.699]$
$= -6.99$

Hence, there are 6 zeroes after decimal before a significant figure.

(c) $\left(\dfrac{1}{5}\right)^{\log_{25} 9} = \left(\dfrac{1}{5}\right)^{\log_5 3} = \dfrac{1}{5^{\log_5 3}} = \dfrac{1}{3}$. **Ans.**

17. (a) Q, R, S, T; (b) P; (c) Q, R, S, T; (d) P, R, S

(a) Let $\log_2 (x-1) = A$ & $\log_2 x = B$ then

$\Rightarrow (A+B)^2 - 4AB = 1$

$\Rightarrow (A-B)^2 = 1$

$\Rightarrow (A-B) = \pm 1$

$\therefore \quad \log_2 \left(\dfrac{x-1}{x}\right) = 1$ or $2^{-1} \therefore x = -1$ or 2

Only 2 is solution.

(b) $(8x-3) = 13 \quad [a^{\log_a b} = b]$

$(8x = 16)$

$\therefore x = 2$

(c) $N = \left(\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_6 \pi}\right) \Rightarrow (\log_{\pi^2} + \log_{\pi 6})$

$\Rightarrow \log_\pi 12 > \qquad \therefore \pi^2 = 10$

$\Rightarrow \log_\pi \pi^2 < \log_\pi 12 < \log_\pi \pi^3 \qquad \pi^3 < 28$

$\Rightarrow 2 < \log_\pi 12 < 3$

(d) $l = (\log_3 4 + \log_2 9)^2 - (\log_3 4 - \log_2 9)^2$

$m = 0.8\ (1 + 9^{\log_3 8})^{\log_{65} 5}$

$\therefore \quad l = (A+B)^2 - (A-B)^2 = 4AB = 4\log_3 4 \cdot$

$\log_2 9 \Rightarrow 4 \cdot \dfrac{\log 4}{\log 3} \times \dfrac{\log 9}{\log 2} \Rightarrow 16$

$l = 16$

$m = 0.8\ (1 + 3^{\log_3 8^2})^{\log_{65} 5}$

$\Rightarrow (0.8)\ (1+64)^{\log_{65} 15} \Rightarrow (0.8)\ (65)^{\log_{65}(5)}$

$\Rightarrow (0.8)(5) = 4$

$\therefore\ m = 4$

so, $l + m = 16 + 4 = 20$

18. $(a)\ R;\ (b)\ S;\ (c)\ P;\ (d)\ Q]$

(a) Antilog $_{27}(2/3) = x \Rightarrow (2/3) = \log_{27} x$

$\Rightarrow x = (27)^{2/3} = 9$ **Ans**

(b) We have to find characteristic of $\log_2 2008$

we know , $\log_2 1024 = 10$ and $\log_2 2048 = 11$

$\therefore\ 10 < \log_2 2008 < 11$

$\therefore$ It has characteristic = 10 **Ans.**

$(c)\ \dfrac{\log 2}{\log e} \times \dfrac{\log 625}{\log b} = \dfrac{4\log 2}{\log 10} \times \dfrac{\log 10}{\log e}$

$\Rightarrow \log 5 = \log b \Rightarrow\ b = 5$ **Ans.**

$(d)\ y = \left(\dfrac{5}{6}\right)^{100} \Rightarrow \log_{10} y = 100(\log_{10} 5 - \log_{10} 6)$

$= 100(1 - \log_{10} 2 - \log_{10} 3 - \log_{10} 2)$

$\Rightarrow \qquad -7.91 \Rightarrow -8 + 0.09$

since char is –8. Hence number of zeros after decimal

= 7 **Ans.**

INTEGER TYPE

19. **[1]** Let $x + |x - 2| = y$

$\therefore$ Equation becomes

$\log_x y^2 = \log_x (5y - 6)$

$\Rightarrow\ y^2 = 5y - 6$

$\Rightarrow\ y^2 - 5y + 6 = 0$

$\Rightarrow\ y = 2$ or 3

If $y = 2$

then $x + |x - 2| = 2 \Rightarrow 0 < x < 1 \cup 1 < x \le 2$

If $y = 3$

then $x + |x - 2| = 3 \Rightarrow x = \dfrac{5}{2}$ only

Hence number of integral solutions is 1.

20. **[2]** Let $\log_{3x} 45 = \log_{4x} 40\sqrt{3} = k$

$45 = (3x)^k,\ 40\sqrt{3} = (4k)^k$

$\dfrac{45}{40\sqrt{3}} = \left(\dfrac{3k}{4x}\right)^k \Rightarrow \dfrac{3\sqrt{3}}{8} = \left(\dfrac{3}{4}\right)^k$

$\Rightarrow \left(\dfrac{3}{4}\right)^{\frac{3}{2}} = \left(\dfrac{3}{4}\right)^k \Rightarrow k = \dfrac{3}{2}$

$\log_{3x} 45 = \dfrac{3}{2}$

$45 = (3x)^{\frac{3}{2}}$

squaring both sides

$45 \times 45 = (3x)^3$

$x^3 = 75$

$\log_7 x^3 = \log_7 75$

$\Rightarrow \log_7 49 < \log_7 75 < \log_7 343$

$\Rightarrow\ 2 < \log_7 75 < 3$;

Characteristic = 2.

21. **[0006]**

$y = \sqrt{\log_2 3 \cdot \log_2 12 \cdot \log_2 48 \cdot \log_2 192 + 16} - \log_2 12 \cdot \log_2 48 + 10$

$= \sqrt{\log_2 3 \cdot (2 + \log_2 3)(4 + \log_2 3)(6 + \log_2 3) + 16} - (2 + \log_2 3)(4 + \log_2 3) + 10$

Let us put $\log_2 3 = x$

$= \sqrt{x(2 + x)(4 + x)(6 + x) + 16} - (2 + x)(4 + x) + 10$

$= \sqrt{(x^2 + 6x)(x^2 + 6x + 8) + 16} - (x^2 + 6x + 8) + 10$

Put again $x^2 + 6x = \alpha$

$= \sqrt{\alpha(\alpha + 8) + 16} - (\alpha + 8) + 10$

$= \sqrt{\alpha^2 + 8\alpha + 16} - (\alpha + 8) + 10$

$= \sqrt{(\alpha + 4)^2} - (\alpha + 8) + 10$

$= (\alpha + 4) - (\alpha + 8) + 10 = y = 6.$

22. **[0007]**

As, $\dfrac{1}{\log_a (2 - \sqrt{3})} + \dfrac{1}{\log_b \left(\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}\right)}$

$= \log_{2-\sqrt{3}} a + \log_{\frac{\sqrt{3}-1}{\sqrt{3}+1}} b$

$= \log_{2-\sqrt{3}} a + \log_{2-\sqrt{3}} b = \log_{2-\sqrt{3}} (ab)$

Now, $(2 + \sqrt{3})^{\log_{2-\sqrt{3}}(ab)} = \dfrac{1}{12}$

$\Rightarrow (2 - \sqrt{3})^{\log_{2-\sqrt{3}}\left(\frac{1}{ab}\right)} = \dfrac{1}{12}$

$\Rightarrow \dfrac{1}{ab} = \dfrac{1}{12} \Rightarrow ab = 12$

As a, b are co-prime numbers, so either $a = 4$, $b = 3$ or $a = 3, b = 4$.

Hence , $(a + b) = 7$. **Ans.**

23. [0003]

$N = 10^p ; p = \log_{10}8 - \log_{10}9 + 2\log_{10}6$

$p = \log\left(\frac{8 \cdot 36}{9}\right) = \log_{10}32$

$\therefore$ N $= 10^{\log_{10}32} = 32$

Hence characteristic of $\log_3 32$ is 3.

24. [0009]

We have,

$2^{2x} - 8 \cdot 2^x + 15 = 0$

$\Rightarrow (2^x - 3)(2^x - 5) = 0$

$\Rightarrow 2^x = 3$ or $2^x = 5$

Hence smallest x is obtained by equating $2^x = 3$

$\Rightarrow x = \log_2 3$

So, $p = \log_2 3$

Hence, $4^p = 2^{2\log_2 3} = 2^{\log_2 9} = 9.$

25. [5625]

Let $\log_{10}x = a$; $\log_{10}y = b$ and $\log_{10}z = c$

Here $xyz = 10^{81}$

$\Rightarrow \log_{10}x + \log_{10}y + \log_{10}z = 81$

i.e. $a + b + c = 81$ ....(1)

Also $a(b + c) + bc = 468$

$ab + bc + ca = 468$ ....(2)

Now $a^2 + b^2 + c^2 = (a + b + c)^2 - 2\sum ab = (81)^2 - (2)(468)$

$= 6561 - 936 = 5625$ **Ans.**

26. [2.25]

$\log(2y - 3x)^2 = \log xy$

$(2y - 3x)^2 = xy$

$4y^2 - 12xy + 9x^2 + xy$

Dividing the equation by y^2

$9\left(\frac{x}{y}\right)^2 - 13\frac{x}{y} + 4 = 0$

$\left(\frac{x}{y} - 1\right)\left(\frac{9x}{y} - 4\right) = 0$

$\frac{x}{y} = 1, \frac{x}{y} = \frac{4}{9}.$

$z = y$ disregarded as for $x = y$, $2y - 3x$ is negative.

Hence $\frac{y}{x} = \frac{9}{4}.$

27. [12]

$\log_{225}x + \log_{64}y = 4$

$\log_x 225 - \log_y 64 = 1$

Let us put $\log_{225}x = a$, $\log_{64}y = b$

$a + b = 4$

$\frac{1}{a} - \frac{1}{b} = 1$

$b - (4 - b) = (4 - b)\, b$

$2b - 4 = 4b - b^2$

$b^2 - 2b - 4 = 0$, Let b_1, b_2 be the roots

$b_1 + b_2 = 2$

$\log_{64}y_1 + \log_{64}y_2 = 2$

$\log_{64}y_1y_2 = 64^2 = 2^{12}$

Similarly $a_1 + a_2 = (4 - b_1) + (4 - b_2)$

$= 8 - (b_1 + b_2) = 8 - 2$

$a_1 + a_2 = 6$

$\log_{225}x_1 + \log_{225}x_2 = 6$

$x_1x_2 = 225^6 = 15^{12}.$

Now, $\log_{30}x_1x_2y_1y_2 = \log_{30}2^{12} \times 15^{12}$

$= 12 \log_{30}30 = 12.$ **Ans.**

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*) $\left|\sqrt{x}-2\right|+\sqrt{x}\left(\sqrt{x}-4\right)+2=0$

$|\sqrt{x}-2|+\left(\sqrt{x}\right)^2-4\sqrt{x}+2=0$

$|\sqrt{x}-2|^2+\left|\sqrt{x}-2\right|-2=0$

$|\sqrt{x}-2|=-2$ (not possible) or $|\sqrt{x}-2|=1$

$\sqrt{x}-2=1,-1$

$\sqrt{x}=3,1$

$x=9,\ 1$

Sum = 10

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*d*)

Let $\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}\ldots\ldots}=t\Rightarrow\sqrt{4-\frac{1}{3\sqrt{2}}t}=t$

$\Rightarrow 4-\frac{1}{3\sqrt{2}}t=t^2$

$\Rightarrow t^2+\frac{1}{3\sqrt{2}}t-4=0\Rightarrow 3\sqrt{2}\,t^2+t-12\sqrt{2}=0$

$\Rightarrow t=\frac{-1\pm\sqrt{1+4\times3\sqrt{2}\times12\sqrt{2}}}{2\times3\sqrt{2}}=\frac{-1\pm17}{2\times3\sqrt{2}}$

$t=\frac{16}{6\sqrt{2}},\frac{-18}{6\sqrt{2}}\Rightarrow t=\frac{8}{3\sqrt{2}},\frac{-3}{\sqrt{2}}$ and $\frac{-3}{\sqrt{2}}$ is rejected

so $6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\times\frac{8}{3\sqrt{2}}\right)=6+\log_{3/2}\left(\frac{4}{9}\right)$

$=6+\log_{3/2}\left(\left(\frac{2}{3}\right)^2\right)=6-2=4$

2. (*a*, *b*, *c*)

$3^x=4^{x-1}\Rightarrow x=(x-1)\log_3 4$

$\Rightarrow x(1-2\log_3 2)=-2\log_3 2$

$x=\frac{2\log_3 2}{2\log_3 2-1}$ **Ans.** (*a*)

Again $x\log_2 3=(x-1)\cdot 2$

$\Rightarrow x(\log_2 3-2)=-2\Rightarrow x=\frac{2}{2-\log_2 3}$ **Ans.** (*b*)

$x=\frac{1}{1-\frac{1}{2}\log_2 3}=\frac{1}{1-\log_4 3}$ **Ans.**(*c*)

3. (8)

$\log_2 9^{\frac{2}{\log_2(\log_2 9)}}\times 7^{\frac{1/2}{\log_4 7}}$

$=(\log_2 9)^{2\log_{\log_2 9}2}\times 7^{\frac{1}{2}\log_7 4}$

$=4\times 2=8$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

2 Trigonometric Ratio and Identities

Topicwise Questions

1. (*b*) $\tan\alpha + \cot\alpha = a$

By squaring both side, we get

$\Rightarrow \tan^2\alpha + \cot^2\alpha = a^2 - 2$

By squaring both sides again, we get

$\Rightarrow \tan^4 a + \cot^4 a + 2 = a^4 - 4a^2 + 4$

$\Rightarrow \tan^4\alpha + \cot^4\alpha = a^4 - 4a^2 + 2$

2. (*b*) Let angles of triangle be $a-d, a, a+d$ (as angles in Δ are in A.P.)

Sum of angles = 180°

$(a-d)+(a)+(a+d) = 180° \Rightarrow 3a = 180° \Rightarrow a = 60°$

$$\frac{\text{Smallest angle (in rod.)}}{\text{largest angle (in degree)}} = \frac{60}{\pi}$$

$$\frac{a-d}{a+d}\times\frac{180}{\pi} = \frac{60}{\pi} \Rightarrow \frac{a-d}{a+d} = \frac{1}{3}$$

$3a - 3d = a + d$

$2a = 4d$

$$d = \frac{a}{2} = \frac{60}{2} = 30°$$

Angles are 30°, 60°, 90°

3. (*a*) Let radius of circle be *r*.

length of sector be *l*.

Then, $l = r\theta$

Also, according to question,

$$2r + l = \frac{1}{2}.2\pi r$$

$2r + r\theta = \pi r$

$$\theta = \frac{(\pi - 2)}{r}$$

$\theta = (\pi - 2)$ radian

4. (*b*) Given expression

$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha]$

$= 3[1 + 2\cos^2\alpha \sin^2\alpha] - 2[1 + 3\cos^2\alpha \sin^2\alpha]$

$= 3 + 6\cos^2\alpha \sin^2\alpha - 2 - 6\cos^2\alpha \sin^2\alpha = 1$

5. (*a*) $\cos(540° - \theta) - \sin(630° - \theta)$

$= -\cos\theta - (-\cos\theta) = 0$

6. (*a*) $\sin(\pi + \theta).\sin(\pi - \theta).\operatorname{cosec}^2\theta$

$= (-\sin\theta)(\sin\theta)\operatorname{cosec}^2\theta$

$= -\sin^2\theta \operatorname{cosec}^2\theta = -1$

7. (*d*) $\cos 0 + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

$= 1 + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right) + \cos\left(\pi - \frac{2\pi}{7}\right) + \cos\left(\pi - \frac{\pi}{7}\right)$

$= 1 + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = 1$

8. (*a*) $\cos 10° - \sin 10°$

$\because \cos 10° > 0$ & $\sin 10° > 0$

$\therefore \cos 10° + \sin 10° > 0$

$\Rightarrow (\cos 10° - \sin 10°)(\cos 10° + \sin 10°)$

$\Rightarrow \cos^2 10° - \sin^2 10°$

$\Rightarrow \cos 20° > 0$

$\Rightarrow \cos 10° - \sin 10° > 0$

9. (*b*) The true relation is $\sin 1 > \sin 1°$

Since value of $\sin\theta$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

10. (*d*) We have,

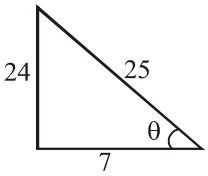
$\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin^2\theta + 1 = 2\sin\theta$

$\Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow (\sin\theta - 1)^2 = 0 \Rightarrow \sin\theta = 1$

Required value of $\sin^{10}\theta + \operatorname{cosec}^{10}\theta = (1)^{10} + \frac{1}{(1)^{10}} = 2$.

11. (*c*)



$\text{Sec}\theta = \frac{25}{7}$

$\tan\theta = \frac{24}{7}$

$\text{Sec}\theta + \tan\theta = \frac{25}{7} + \frac{24}{7} = \frac{25+24}{7} = \frac{49}{7}$

Lies in second Quadrent than $\sec\theta + \tan\theta = -7$

12. (*d*) $3\tan A + 4 = 0 \Rightarrow \tan A = -\frac{4}{3}$

$\Rightarrow \sin A = \pm\frac{\tan A}{\sqrt{1+\tan^2 A}} = \pm\frac{-4/3}{\sqrt{1+16/9}} = \frac{4}{5}$

(Q A is in 2nd quadrant)

and $\cos A = -\frac{3}{5}$, Thus, $2\cot A - 5\cos A + \sin A$

$= 2\left(-\frac{3}{4}\right) - 5\left(-\frac{3}{5}\right) + \frac{4}{5} = \frac{23}{10}$

13. (*d*) $e^{\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \log_{10}\tan 3^\circ + \ldots\ldots\ldots + \log_{10}\tan 89^\circ}$

$= e^{\log_{10}(\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots\ldots \tan 89^\circ)} = e^{\log_{10} 1} = e^\circ = 1$

14. (*d*) $\tan A + \cot A = 4$

$\Rightarrow \tan^2 A + \cot^2 A + 2\tan A \cot A = 16$

$\Rightarrow \tan^2 A + \cot^2 A = 14 \Rightarrow \tan^4 A + \cot^4 A + 2 = 196$

$\Rightarrow \tan^4 A + \cot^4 A = 194.$

15. (*b*) $\cos A + \sin(270^\circ + A) - \sin(270^\circ - A) + \cos(180^\circ + A)$

$\cos A - \cos A + \cos A - \cos A = 0$

16. (*d*) $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8}$

$= \sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{\pi}{8}$

$= 2\left(\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right) = 2 \times 1 = 2.$

17. (*a*) We know that A+C = 180°

$\Rightarrow$ A = 180°–C

$\Rightarrow$ cosA + cosC = 0 ...(i)

$\Rightarrow$ cosB + cosD = 0 ...(ii)

From (i) and (ii) cosA – cosB + cosC – cosD = 0

18. (*a*) $\frac{\sin 24^\circ\cos 6^\circ - \sin 6^\circ\cos 4^\circ}{\sin 21^\circ\cos 39^\circ - \sin 39^\circ\cos 21^\circ} = \frac{\sin(24^\circ - 6^\circ)}{\sin(21^\circ - 39^\circ)}$

$= \frac{\sin 18^\circ}{\sin(-18^\circ)} = -1$

19. (*d*) $\because 3\sin\alpha = 5\sin\beta$

$\Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{5}{3} \Rightarrow \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{8}{2}$

$\Rightarrow \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)} = 4$

20. (*a*) $\tan A + \tan B = a$

$\tan A \tan B = b$

$\Rightarrow \tan(A+B) = \frac{a}{1-b}$

$\therefore \sin^2(A+B) = \left[\frac{|a|}{\sqrt{a^2+(1-b)^2}}\right]^2 = \frac{a^2}{a^2+(1-b)^2}$

21. (*b*) $\because \tan A < 0$ and $A + B + C = 180^\circ$

$\Rightarrow A > 90^\circ$

$\Rightarrow B + C < 90^\circ$

$\Rightarrow \tan(B+C) > 0 \quad \Rightarrow \frac{\tan B + \tan C}{1 - \tan B\tan C} > 0$

$\Rightarrow 1 - \tan B\tan C > 0 \quad \Rightarrow \tan B\tan C < 1$

22. (*c*) $\tan A - \tan B = x$

$\cot B - \cot A = y$

$\frac{\tan A - \tan B}{\tan A \tan B} = y$

$\Rightarrow \tan A \tan B = \frac{x}{y}$

Now $\cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A\tan B}{\tan A - \tan B}$

$= \frac{1+\frac{x}{y}}{x} = \frac{1}{x} + \frac{1}{y}$

23. (*a*) $\frac{\tan(180^\circ - 25^\circ) - \tan(90^\circ + 25^\circ)}{1 + (\tan(180^\circ - 25^\circ)\tan(90^\circ + 25^\circ)}$

$= \frac{-\tan 25^\circ + \frac{1}{\tan 25^\circ}}{2} = \frac{1-x^2}{2x}$

24. (*b*) $\because \cot(A+B) = \cot 225^\circ = 1$

$\Rightarrow \frac{\cot A\cot B - 1}{\cot A + \cot B} = 1$

$\Rightarrow \cot A\cot B = 1 + \cot A + \cot B$

Now $\frac{\cot A.\cot B}{1 + \cot A + \cot B + \cot A\cot B}$

$= \frac{1 + \cot A + \cot B}{2(1 + \cot A + \cot B)} = \frac{1}{2}$

25. (*d*) $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

$\Rightarrow -\cos(\alpha+\beta) + 1 = 0$

$\Rightarrow \cos(\alpha+\beta) = 1$

$\Rightarrow \sin(\alpha+\beta)=0$

then $1+\cot\alpha\tan\beta=1+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$

$=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}=\dfrac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}=0$

26. (*c*) $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}=\cos(2\times 15^\circ)=\cos 30^\circ$

27. (*d*) $\tan^2\theta=2\tan^2\phi+1$ ...(*i*)

$\cos 2\theta+\sin^2\phi=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}+\sin^2\phi$

$=\dfrac{1-2\tan^2\varphi-1}{1+2\tan^2\varphi+1}+\sin^2\phi=\dfrac{-2\tan^2\varphi}{2(1+\tan^2\varphi)}+\sin^2\phi$

$=-\sin^2\phi+\sin^2\phi=0$

which is independent of ϕ

28. (*b*) $\left(1+\cos\dfrac{\pi}{10}\right)\left(1+\cos\dfrac{3\pi}{10}\right)$

$\left(1-\cos\dfrac{3\pi}{10}\right)\left(1-\cos\dfrac{\pi}{10}\right)=\left(1-\cos^2\dfrac{\pi}{10}\right)$

$\left(1-\cos^2\dfrac{3\pi}{10}\right)$

$\sin^2\dfrac{\pi}{10}\cdot\sin^2\dfrac{3\pi}{10}$

$=\left(\dfrac{\sqrt5-1}{4}\cdot\dfrac{\sqrt5+1}{4}\right)^2=\left(\dfrac{4}{16}\right)^2=\dfrac{1}{16}$

29. (*b*)

$=\dfrac{(\cos 6x+\cos 4x)+5(\cos 4x+\cos 2x)+10(\cos 2x+1)}{\cos 5x+5\cos 3x+10\cos x}$

$=\dfrac{2\cos 5x\cos x+10\cos 3x\cos x+20\cos^2 x}{\cos 5x+5\cos 3x+10\cos x}$

$=\dfrac{2\cos x[\cos 5x+5\cos 3x+10\cos x]}{[\cos 5x+5\cos 3x+10\cos x]}$

$=2\cos x$

30. (*c*) Given, $\cos(A-B)=\dfrac{3}{5}$

$\Rightarrow \cos A\cos B+\sin A\sin B=\dfrac{3}{5}$ ...(*i*)

Also, $\tan A\tan B=2$

$\Rightarrow 2\cos A\cos B-\sin A\sin B=0$ ...(*ii*)

From (*i*) and (*ii*), we get

$\cos A\cos B=\dfrac{1}{5}$ & $\sin A\sin B=\dfrac{2}{5}$

$\therefore \cos A\cos B-\sin A\sin B=\dfrac{1}{5}-\dfrac{2}{5}$

$\Rightarrow \cos(A+B)=-\dfrac{1}{5}$

31. (*d*) We know that $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$=\dfrac{1}{\sqrt{10}}\sqrt{1-\dfrac{1}{5}}+\dfrac{1}{\sqrt5}\sqrt{1-\dfrac{1}{10}}$

$=\dfrac{1}{\sqrt{10}}\sqrt{\dfrac{4}{5}}+\dfrac{1}{\sqrt5}\sqrt{\dfrac{9}{10}}=\dfrac{1}{\sqrt{50}}(2+3)=\dfrac{5}{\sqrt{50}}=\dfrac{1}{\sqrt2}$

$\sin(A+B)=\sin\dfrac{\pi}{4}$

Hence, $A+B=\dfrac{\pi}{4}$.

32. (*c*) $2\tan(A-B)=2\left(\dfrac{\tan A-\tan B}{1+\tan A\tan B}\right)$

$=2\dfrac{(2\tan B+\cot B-\tan B)}{1+(2\tan B+\cot B)\tan B}=2\dfrac{\tan B+\cot B}{2(1+\tan^2 B)}$

$=\dfrac{\cot B(\tan^2 B+1)}{(1+\tan^2 B)}=\cot B$

33. (*b*) We have $\tan A=-\dfrac{1}{2}$ and $\tan B=-\dfrac{1}{3}$

Now, $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{-\frac{1}{2}-\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=-1$

$\Rightarrow \tan(A+B)=\tan\dfrac{3\pi}{4}$. Hence, $A+B=\dfrac{3\pi}{4}$.

34. (*b*) $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\dfrac{2\sin(A+B)\sin(A-B)}{\sin 2A-\sin 2B}$

$=\dfrac{2\sin(A+B)\sin(A-B)}{2\cos(A+B)\sin(A-B)}=\tan(A+B)$.

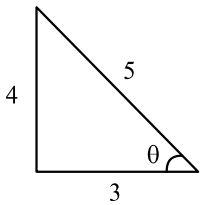
35. (*a*) Since $\tan 3A=\dfrac{\tan A+\tan 2A}{1-\tan A\tan 2A}$

$\Rightarrow \tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A$

36. (*b*) $\tan\theta=\dfrac{-4}{3}$

As $\tan\theta$ is (–ve), $\theta\in 2^{nd}$ Quad.

or $\theta\in 4^{th}$ Quad.



$\sin\theta=\dfrac{4}{5}$ if $\theta\in 2^{nd}$ Quad.

$\sin\theta=\dfrac{-4}{5}$ if $\theta\in 4^{th}$ Quad.

37. (*d*) $\sin\theta_1+\sin\theta_2+\sin\theta_3=3$

As max value of $\sin\theta=2$

$\sin\theta_1=\sin\theta_2=\sin\theta_3=1$

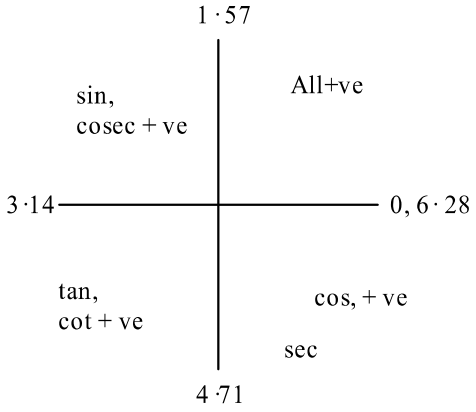
$\cos\theta_1=\cos\theta_2=\cos\theta_3=0$

$\cos\theta_1+\cos\theta_2+\cos\theta_3=0$

38. (*b*) As $\cos 90° = 0$

$\cos 1° \cdot \cos 2° \cdot \cos 3°..... \cos 90° ..... \cos 179° = 0$

39. (*b*)



$\sin 1 \cdot \cos 2 \cdot \tan 3 \cdot \cot 4 \cdot \sec 5 \cdot \operatorname{cosec} 6$

sign of above expression

$= (+)(-)(-)(+)(+)(-)$

$= (-)$ ve

Negative

40. (*c*) $\sin\left(\frac{\pi}{10}\right)\cdot\sin\left(\frac{13\pi}{10}\right) = \sin\left(\frac{\pi}{10}\right)\sin\left(\pi+\frac{3\pi}{10}\right)$

$= \sin\left(\frac{\pi}{10}\right)\left(-\sin\left(\frac{3\pi}{10}\right)\right) = -\sin 18° \sin 54°$

$= -\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{-4}{16} = \frac{-1}{4}$

41. (*b*) $\frac{2\sin\alpha}{1+\sin\alpha+\cos\alpha} = \lambda$

$\frac{2\sin\alpha}{1+\sin\alpha+\cos\alpha}\cdot\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha-\cos\alpha} = \lambda$

$\frac{2\sin\alpha+2\sin^2\alpha-2\sin\alpha\cos\alpha}{1+\sin^2\alpha+2\sin\alpha-\cos^2\alpha} = \lambda$

$\frac{2\sin\alpha+2\sin^2\alpha-2\sin\alpha\cos\alpha}{2\sin^2\alpha+2\sin\alpha} = \lambda$

$\frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{2\sin(\sin\alpha+1)} = \lambda$

$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha} = \lambda$

42. (*a*) $8x^2-26x+15=0$

$\longrightarrow$ roots $= \tan\frac{\alpha}{2}, \tan\frac{\beta}{2}$

$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2} = \frac{26}{8} = \frac{13}{4}; \quad \tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} = \frac{15}{8}$

$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{\frac{26}{8}}{1-\frac{15}{8}} = \frac{26}{8}\left(\frac{-8}{7}\right)$

$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{-26}{7}$

$\cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{1-\frac{676}{49}}{1+\frac{676}{49}}$

$\cos(\alpha+\beta) = \frac{-627}{725}$

43. (*a*) $\sin A, \cos A, \tan A$ are in *G.P.*

$\cos^2 A = \sin A \cdot \tan A$

$\cos^2 A = \frac{\sin^2 A}{\cos A} \Rightarrow \cos^3 A = 1-\cos^2 A$

$\Rightarrow \cos^3 A + \cos^2 A = 1$

44. (*c*) $\operatorname{cosec} A + \cot A = \frac{11}{2}$ ...(1)

$\operatorname{cosec} A - \cot A = \frac{2}{11}$ (as $\operatorname{cosec}^2 A - \cot^2 A = 1$) ...(2)

Subtracting (2) from (1),

$2\cot A = \frac{11}{2}-\frac{2}{11} = \frac{121-4}{22} = \frac{117}{22}$

$\cot A = \frac{117}{44}$

$\tan A = \frac{44}{117}$

45. (*c*) $\cos A = \frac{3}{4}$

$16\cos^2\frac{A}{2} - 32\sin\frac{A}{2}\sin\frac{5A}{2}$

$= \frac{16(1+\cos A)}{2} - 16(\cos 2A - \cos 3A)$

$= \frac{16(1+\cos A)}{2} - 16\{(2\cos^2 A - 1) - (4\cos^3 A - 3\cos A)\}$

$= 8\left(1+\frac{3}{4}\right) - 16\left\{2\times\frac{9}{16} - 1 - 4\times\frac{27}{64} + 3\times\frac{3}{4}\right\} = 3$

46. (*b*) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \cos\theta(4\cos^2\theta - 3)$

$= \frac{1}{2}\left(a+\frac{1}{a}\right)\left\{4\times\frac{1}{4}\left(a+\frac{1}{a}\right)^2 - 3\right\}$

$= \frac{1}{2}\left(a+\frac{1}{a}\right)\left\{a^2+\frac{1}{a^2}-1\right\}$

$= \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$

47. (*d*) $\sin 12^\circ . \sin 48^\circ \sin 54^\circ$

Multiplying & dividing by 2, we get

$= \frac{1}{2}(2\sin 12^\circ \sin 48^\circ)\sin 54^\circ$

$= \frac{1}{2}[\cos 36^\circ - \cos 60^\circ]\sin 54^\circ$

$= \frac{1}{2}\left[\cos 36^\circ - \frac{1}{2}\right]\sin 94^\circ$

$= \frac{1}{4}[2\cos 36^\circ \sin 54^\circ - \sin 54^\circ]$

$= \frac{1}{4}[\sin 90^\circ + \sin 18^\circ - \sin 54^\circ]$

$= \frac{1}{4}\left[1 + \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8}$

48. (*c*) $A = \tan 6^\circ \tan 42^\circ$

$B = \cot 66^\circ \cot 78^\circ$

$\frac{A}{B} = \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$

$\Rightarrow \frac{A}{B} = \frac{\tan 6^\circ \tan(60^\circ - 6^\circ)\tan(60^\circ + 6^\circ)}{\tan 54^\circ} . \tan 78^\circ \tan 42^\circ$

$\Rightarrow \frac{A}{B} = \frac{\tan 18^\circ . \tan(60^\circ - 18^\circ)\tan(60^\circ + 18^\circ)}{\tan 54^\circ}$

$= \frac{\tan 54^\circ}{\tan 54^\circ}$

$\Rightarrow \frac{A}{B} = 1 \Rightarrow A = B$

49. (*d*) $\cos\frac{\pi}{10} \cdot \cos\frac{2\pi}{10} \cdot \cos\frac{4\pi}{10} \cdot \cos\frac{8\pi}{10} \cdot \cos\frac{16\pi}{10}$

$= \frac{\sin 2^5 \frac{\pi}{10}}{2^5 \sin\frac{\pi}{10}} = \frac{1}{32}\frac{\sin\frac{32\pi}{10}}{\sin\frac{\pi}{10}} = \frac{1}{32}\frac{\sin\left(3\pi + \frac{2\pi}{10}\right)}{\sin\left(\frac{\pi}{10}\right)}$

$= -\frac{1}{32} \cdot \frac{2\sin\frac{\pi}{10}\cos\frac{\pi}{10}}{\sin\frac{\pi}{10}} = -\frac{1}{16}\cos\frac{\pi}{10}$

$= -\frac{1}{64}\sqrt{10 + 2\sqrt{5}}$

50. (*d*) $\cot x + \cot(60^\circ + x) + \cot(120^\circ + x)$

$= \frac{1}{\tan x} + \tan(30^\circ - x) - \tan(30^\circ + x)$

$= \frac{1}{\tan x} + \left(\frac{1 - \sqrt{3}\tan x}{\sqrt{3} + \tan x}\right) - \left(\frac{1 + \sqrt{3}\tan x}{\sqrt{3} - \tan x}\right)$

$3 - \tan^2 x + \sqrt{3}\tan x - 3\tan^2 x - \tan^2 x$

$= \frac{+\sqrt{3}\tan^2 x - \sqrt{3}\tan^2 x - 3\tan^2 x - \tan^2 x}{\tan x(3 - \tan^2 x)}$

$= \frac{3 - 9\tan^2 x}{3\tan x - \tan^3 x}$

51. (*a*) We have, $\sin\alpha + \sin\beta = \frac{-21}{65}$

& $\cos\alpha + \cos\beta = \frac{-27}{65}$

Squaring and adding both sides, we get

$\Rightarrow 2 + 2\cos(\alpha - \beta) = \frac{21^2 + 27^2}{65^2}$

$\Rightarrow 2(1 + \cos(\alpha - \beta)) = \frac{21^2 + 27^2}{65^2}$

$\Rightarrow 2 . 2\cos^2\left(\frac{\alpha - \beta}{2}\right) = \frac{3^2(7^2 + 9^2)}{65^2}$

$\Rightarrow \cos\left(\frac{\alpha - \beta}{2}\right) = \pm\frac{3\sqrt{130}}{130} = \pm\frac{3}{\sqrt{130}}$

$\because \frac{\pi}{2} < \left(\frac{\alpha - \beta}{2}\right) < \frac{3\pi}{2} \therefore \cos\left(\frac{\alpha - \beta}{2}\right) < 0$

Hence $\cos\left(\frac{\alpha - \beta}{2}\right) = \frac{-3}{\sqrt{130}}$

52. (*c*) $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$

$= \tan x + \frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} + \frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x}$

$= \tan x + \frac{8\tan x}{1 - 3\tan^2 x} = \frac{3(3\tan x - \tan^3 x)}{1 - 3\tan^2 x} = 3\tan 3x$

Therefore, the given equation is $3\tan 3x = 3$

$\Rightarrow \tan 3x = 1.$

53. (*a*) $\sin 4\theta = 2\sin 2\theta \cos 2\theta$

$= 2.2\sin\theta\cos\theta(1 - 2\sin^2\theta)$

$= 4\sin\theta(1 - 2\sin^2\theta)\sqrt{1 - \sin^2\theta}$

54. (*c*) $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = \sqrt{2 + \sqrt{2.2\cos^2 2\theta}}$

$= \sqrt{2 + 2\cos 2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta .$

55. (*b*) We have $\tan A = \frac{1}{2}$

$\Rightarrow \tan 3A = \frac{3\tan A - \tan^3 A}{1 - \tan^2 A} = \frac{3.\frac{1}{2} - \frac{1}{8}}{1 - 3.\frac{1}{4}} = \frac{12 - 1}{2} = \frac{11}{2}$

56. (*b*) We have $x + \frac{1}{x} = 2\cos\theta ,$

Now $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3x\frac{1}{x}\left(x + \frac{1}{x}\right)$

$= (2\cos\theta)^3 - 3(2\cos\theta) = 8\cos^3\theta - 6\cos\theta$

$= 2(4\cos^3\theta - 3\cos\theta) = 2\cos 3\theta$.

Trick : Put $x = 1 \Rightarrow \theta = 0^\circ$

Then $x^3 + \frac{1}{x^3} = 2 = 2\cos 3\theta$

57. (*a*) $\because \alpha + \beta + \gamma = 2\pi \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$

$\Rightarrow \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

58. (*d*) If $A + B + C = \frac{3\pi}{2}$ then

$2\cos(A+B)\cos(A-B) + \cos 2C$
$= -2\sin C\cos(A-B) + \cos 2C$
$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C$
$= 1 - 2\sin C[\cos(A-B) + \sin C]$
$= -2\sin C[\cos(A-B) + \sin[3\pi/2 - (A+B)]]$
$= 1 - 2\sin C[\cos(A-B) - \cos(A+B)]$
$= 1 - 2\sin C \cdot 2\sin A\sin B$
$= 1 - 4\sin A\sin B\sin C$

59. (*c*) $A + B + C = \pi \Rightarrow A = \pi - (B + C)$

$\cos A = \cos(\pi - (B+C)) = -\cos(B+C)$ ...(1)

Also, $\cos A = \cos B \cdot \cos C$ ...(2)

$-\cos(B + C) = \cos B \cdot \cos C$ (from (1) & (2))

$\sin B \sin C - \cos B \cos C = \cos B \cos C$

$\sin B \sin C = 2\cos B \cos C$

$\tan B \cdot \tan C = 2$.

60. (*c*) $f(\theta) = \sin^4\theta + \cos^2\theta = \sin^2\theta(1 - \cos^2\theta) + \cos^2\theta$
$= \sin^2\theta + \cos^2\theta - \sin^2\theta\cos^2\theta$

$f(\theta) = 1 - \frac{1}{4}\sin^2 2\theta$

$\because \ 0 \le \sin^2 2\theta \le 1$

$f(\theta)_{max} = 1$

$f(\theta)_{min} = 1 - \frac{1}{4} = 3/4$

$\therefore$ Range is $\left[\frac{3}{4}, 1\right]$

61. (*d*) Maximum value of $f(x) = \sqrt{1^2 + 1^2} = \sqrt{2}$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

1. (*b*) $\sin x + \sin^2 x = 1 \Rightarrow \sin x = 1 - \sin^2 x$

$\Rightarrow \sin x = \cos^2 x$ ...(1)

$\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 1$

$= \sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x - 1$ (from ...(1))

$= \sin^3 x(\sin^3 x + 3\sin^2 x + 3\sin x + 1) - 1$

$= \sin^3 x((\sin x + 1)^3) - 1$

$= \sin x(1+\sin x))^3 - 1 = (\sin x + \sin^2 x)^3 - 1$

$= (\cos^2 x + \sin^2 x)^3 - 1$ (from ...(1))

$= 1^3 - 1 = 1$

2. (*a*) $\tan 75° - \cot 75°$

$$= \frac{\sin 75°}{\cos 75°} - \frac{\cos 75°}{\sin 75°} = \frac{\sin^2 75° - \cos^2 75°}{\cos 75° \sin 75°}$$

$$= \frac{-2\cos(2\times 75°)}{\sin(2\times 75°)} \quad \begin{pmatrix} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \& \sin 2\theta = 2\sin\theta\cos\theta \end{pmatrix}$$

$$= \frac{-2\cos 150°}{\sin 150°} = \frac{2\cos 30°}{\sin 30°} = 2\cot 30° = 2\sqrt{3}$$

From (*i*), (*ii*) and (*iii*)

3. (*c*) $\tan\alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{3+1+2\sqrt{3}}{3-1}$

$$= \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3} = \tan 75°$$

$\alpha = 75°$

Now,

$\cos 2d + (2+\sqrt{3})\sin 2\alpha = \cos(2\times 75°) + (2+\sqrt{3})$ $\sin(2\times 75°)$

$= \cos 150° + (2+\sqrt{3})\sin(150°)$

$$= -\frac{\sqrt{3}}{2} + (2+\sqrt{3})\cdot\frac{1}{2} = \frac{1}{2}(-\sqrt{3}+2+\sqrt{3})$$

$$= \frac{1}{2}\cdot 2 = 1$$

4. (*a*) $\tan\left(7\frac{1}{2}°\right) = \tan 75° = \frac{\sin(7.5°)}{\cos(7.5°)}$

$$= \frac{2\sin^2(7.5°)}{2\sin(7.5°)\cos(7.5°)} = \frac{1-\cos(2\times 7.5°)}{\sin(2\times 7.5°)}$$

$$\left(\begin{matrix} 2\sin^2\theta = 1-\cos 2\theta \\ 2\sin\theta\cos\theta = \sin 2\theta \end{matrix}\right)$$

$$= \frac{1-\cos 15°}{\sin 15°} \quad ...(i)$$

$\sin 15° = \sin(45° - 30°)$

$= \sin 45°\cos 30° - \sin 30°\cos 45°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad ...(ii)$$

$\cos 15° = \cos(45° - 30°)$

$= \cos 45°\cos 30° + \sin 45°\sin 30°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad ...(iii)$$

From (*i*), (*ii*) and (*iii*)

$$\tan\left(7\frac{1}{2}°\right) = \frac{1-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}, \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}-3-\sqrt{3}-\sqrt{3}-1}{3-1}$$

$$= \frac{2\sqrt{6}-2\sqrt{3}+2\sqrt{2}-4}{2} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}$$

5. (*d*) $a\sin x + b\cos(c+x) + b\cos(c-x) = \alpha$

$a\sin x + b(\cos(c+x) + \cos(c-x) = \alpha$

$$a\sin x + b.\,2\cos\frac{(c+x+c-x)}{2}\cos\left(\frac{c+x-c+x}{2}\right) = \alpha$$

$a\sin x + 2b\cos c\cos x = \alpha$

$$\cos c = \frac{\alpha - a\sin x}{2b\cos x} = \frac{1}{2b}(\alpha\sec x - a\tan x)$$

Let $f(x) = |\cos c|$

for minimum value of $f(x) = \frac{1}{2b}(\alpha\sec x - a\tan x)$

Differentiating.

$$f^1(x)\frac{1}{2b}(\alpha\sec x\tan x - a\sec^2 x)$$

$f^1(x) = 0 \Rightarrow \alpha \sec x \tan x - a \sec^2 x = 0$

$\sec x\,(\alpha \tan x - a \sec x) = 0 \Rightarrow \sec x = 0$

or $\alpha \tan x - a \sec x = 0$

$$\alpha \cdot \frac{\sin x}{\cos x} = \frac{a}{\cos x}$$

$$\Rightarrow \sin x = \frac{a}{\alpha}$$

$$\sin x = \frac{a}{\alpha} i \qquad \cos x = \sqrt{1 - \frac{a^2}{\alpha^2}} = \sqrt{\frac{\alpha^2 - a^2}{\alpha}}$$

$$\cos c = \frac{\alpha - a \sin x}{2b \cos x} = \frac{\alpha - a.\frac{a}{\alpha}}{2b\frac{\sqrt{\alpha^2 - a^2}}{\alpha}} = \frac{\alpha^2 - a^2}{2b\sqrt{\alpha^2 - a^2}}$$

$$= \frac{\sqrt{\alpha^2 - a^2}}{2b} = \sqrt{\frac{\alpha^2 - a^2}{4b^2}}$$

6. (*c*) $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ ...(i)

$\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$ ...(ii)

$0 < A.B < \frac{\pi}{2}$

$(i) \div (ii)$

$$\frac{\tan A}{\tan B} = \frac{\sqrt{3}}{\sqrt{3}} \Rightarrow \frac{\tan A}{\sqrt{3}} = \frac{\tan B}{\sqrt{5}} = K(\text{Let}); \; k > 0$$

$\tan A = \sqrt{3}k; \qquad \tan B = \sqrt{5}k$

Also, $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2} \Rightarrow 2 \sin A = \sqrt{3} \sin B$

$$\Rightarrow \frac{2 \sin A}{\cos A}.\cos A = \frac{\sqrt{3} \sin B}{\cos B}, \cos B$$

$$\frac{2 \tan A}{\sec A} = \frac{\sqrt{3} \tan B}{\sec B}$$

$$\Rightarrow \frac{2 \tan A}{\sqrt{1 + \tan^2 A}} = \frac{\sqrt{3} \tan B}{\sqrt{1 + \tan^2 B}}$$

$$\Rightarrow \frac{2\sqrt{3k}}{\sqrt{1 + 3k^2}} = \frac{\sqrt{3}.\sqrt{5k}}{\sqrt{1 + 5k^2}}$$

$$\Rightarrow \frac{2k}{\sqrt{1 + 3k^2}} = \frac{\sqrt{5k}}{\sqrt{1 + 5k^2}}$$

$$\Rightarrow \frac{4k^2}{1 + 3k^2} = \frac{5k^2}{1 + 5k^2}$$

$4k^2 + 20k^4 = 5k^2 + 15k^4$

$5k^4 - k^2 = 0 \qquad \Rightarrow \qquad k^2(5k^2 - 1) = 0$

$\Rightarrow k^2 = 0 \qquad or \qquad k^2 = \frac{1}{5}$

But $k > 0$, so $\qquad k = \frac{+1}{\sqrt{5}}$

$$\tan A + \tan B = \sqrt{3k} + \sqrt{5k} = \frac{\sqrt{3} + \sqrt{5}}{\sqrt{5}}$$

7. (*b*) $f(\theta) = 3\cos^2\theta - 8\sqrt{3}\sin\theta\cos\theta + 5\sin^2\theta - 7$

$= 3(\cos^2\theta + \sin^2\theta) + 2\sin^2\theta - 4\sqrt{3}(2\sin\theta\cos\theta) - 7$

$= 3 + (1 - \cos 2\theta) - 4\sqrt{3}\sin 2\theta - 7$

$= -1\cos 2\theta - 4\sqrt{3}\sin 2\theta - 3$

Range of $a\cos 2\theta + b\sin 2\theta$: $\left[-\sqrt{a^2 + b^2} + \sqrt{a^2 + b^2}\right]$

Range of

$$f(\theta): \left[-\sqrt{(-1)^2 + \left(4\sqrt{3}^2\right)} - 3, +\sqrt{(-1)^2 + \left(4\sqrt{3}\right)^2} - 3\right]$$

$= \left[-\sqrt{49} - 3, +\sqrt{49} - 3\right]$

$= \left[-7 - 3, \quad +7 - 3\right]$

$= \left[-10, +4\right]$

8. (*c*) $f(\theta) = 1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta + 2$

$\left(\cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta\right)$

$$= 1 + \frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}} + \frac{2\cos\theta}{\sqrt{2}} + \frac{2\sin\theta}{\sqrt{2}}$$

$$= 1 + \frac{3}{\sqrt{2}}(\sin\theta + \cos\theta)$$

$\sin\theta + \cos\theta \in \left[-\sqrt{2}, +\sqrt{2}\right]$

$$\frac{3}{\sqrt{2}}(\sin\theta + \cos\theta) \in \left[-\sqrt{2}.\frac{3}{2}, +\sqrt{2}.\frac{3}{\sqrt{2}}\right]$$

$1 + \frac{3}{\sqrt{2}}(\sin\theta + \cos\theta) \in [1 - 3, \; 1 + 3]$

$f\theta \in [-2, \; +4]$

$f(\theta)\max = 4$

9. (*c*) $\frac{\sin\alpha - \sin\gamma}{\cos\gamma - \sin\alpha} = \frac{2\sin\left(\frac{\alpha - \gamma}{2}\right)\cos\left(\frac{\alpha + \gamma}{2}\right)}{2\sin\left(\frac{\alpha + \gamma}{2}\right)\sin\left(\frac{\alpha - \gamma}{2}\right)}$

$$= \frac{\cos\left(\frac{\alpha + \gamma}{2}\right)}{\sin\left(\frac{\alpha + \gamma}{2}\right)} = \cot\left(\frac{\alpha + \gamma}{2}\right)$$

$= \cot(\beta)$

10. (*c*) $\frac{\tan^3 A}{1+\tan^2 A}+\frac{\cot^3 A}{1+\cot^2 A}=\frac{\tan^3 A}{\sec^2 A}+\frac{\cot^3 A}{\operatorname{cosec}^2 A}$

$=\frac{\sin^3 A}{\cos^3 A}.\cos^2 A+\frac{\cos^3 A}{\sin^3 A}.\sin^2 A=\frac{\sin^3 A}{\cos A}+\frac{\cos^3 A}{\sin A}$

$=\frac{\sin^4 A+\cos^4 A}{\sin A\cos A}$

$=\frac{(\sin^2 A)^2+(\cos^2 A)^2+2\sin^2 A\cos^2 A-2\sin 2A\cos 2A}{\sin A\cos A}$

$=\frac{(\sin^2 A+\cos^2 A)^2-2\sin^2 A\cos^2 A}{\sin A\cos A}$

$=\frac{1-2\sin^2 A\cos^2 A}{\sin A\cos A}$

$=\sec A\operatorname{cosec} A-2\sin A\cos A$

Comparing it with $p\sec A\operatorname{cosec} A+q\sin A\cos A$

$p=1,\ q=-2$

11. (*c*) $\log_3(\sin x)-\log_3(\cos x)-\log_3(1-\tan x)-\log_3(1+\tan x)=-1$

$\log_3\left(\frac{\sin x}{\cos x(1-\tan x)(1+\tan x)}\right)=-1$

$\log_3\left(\frac{\tan x}{1-\tan^2 x}\right)=-1$

$\frac{\tan x}{1-\tan^2 x}=3^{-1}\Rightarrow\frac{2\tan x}{1-\tan^2 x}=\frac{2}{3}$

$\Rightarrow\tan(2x)=\frac{2}{3}$

12. (*c*) $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}=\cos^2\frac{\pi}{n}+\cos^2\frac{2\pi}{n}+\cos^2\frac{3\pi}{n}+\ldots+\cos^2\frac{(n-1)\pi}{n}$

$=\frac{1}{2}\left(1+\cos\frac{2\pi}{n}\right)+\frac{1}{2}\left(1+\cos\frac{4\pi}{n}\right)+\frac{1}{2}\left(1+\cos\frac{6\pi}{n}\right)+\ldots..+\frac{1}{2}\left(1+\cos 2\frac{(n-1)\pi}{n}\right)$

$=\frac{1}{2}\left\{(n-1)+\left(\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots.(n-1)\text{ term}\right)\right\}$

$=\frac{n-1}{2}+\frac{1}{2}\frac{\sin\left(\frac{n-1\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}.\cos\left(\frac{2\pi}{n}+(n-2)\frac{\pi}{n}\right)$

$=\frac{n-1}{2}+\frac{1}{2}\frac{\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}.\cos\pi=\frac{n-1}{2}-\frac{1}{2}=\frac{n-2}{2}=\frac{n}{2}-1$

13. (*b*) $2\sin 2^\circ+4\sin 4^\circ+6\sin 6^\circ+\ldots\ldots+178\sin 178^\circ+180\sin 180^\circ$

$=2\sin 2^\circ+4\sin 4^\circ+6\sin 6^\circ+\ldots\ldots+178\sin 2^\circ+180.0$

$=(2+178)\sin 2^\circ+(4+176)\sin 4^\circ+(6+174)\sin 6^\circ+\ldots\ldots+90\sin 90^\circ$

$=180(\sin 2^\circ+\sin 4^\circ+\sin 6^\circ+\ldots\ldots\sin 88^\circ)+90.1$

$=180\times\sin\frac{\left(44.\frac{2}{2}\right)}{\sin\left(\frac{2}{2}\right)}\times\sin\left(2+(44-1)\frac{2}{2}\right)+90$

$=180\times\frac{\sin 44^\circ}{\sin 1^\circ}\times\sin 45^\circ+90$

$=180\times\frac{\sin(45^\circ-1)}{\sin 1^\circ}\times\frac{1}{\sqrt{2}}+90$

$=\frac{180}{\sqrt{2}}\frac{(\sin 45^\circ\cos 1^\circ-\sin 1^\circ\cos 45^\circ)}{\sin 1^\circ}+90$

$=\frac{180}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}.\cot 1^\circ-\frac{1}{\sqrt{2}}\right)+90$

$=\frac{180}{\sqrt{2}}\cot 1^\circ-\frac{180}{2}+90=90\cot 1^\circ$

14. (*a*) $\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta).....(1+\sec 2^n\theta)$

Let us solve it for $n=1$

$\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)$

$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(1+\frac{1}{\cos\theta}\right)\left(1+\frac{1}{\cos 2\theta}\right)$

$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(\frac{1+\cos\theta}{\cos\theta}\right)\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)$

$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.\frac{2\cos^2\frac{\theta}{2}}{\cos\theta}.\frac{2\cos^2\theta}{\cos 2\theta}$

$$= \frac{\sin\frac{\theta}{2}.\,2\cos\frac{\theta}{2}.2\cos\theta}{\cos 2\theta} = \frac{\sin\theta\,.\,2\cos\theta}{\cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

for n = n,

$\tan^n 2\theta$

15. (*c*) $f(x) = 27^{\cos 3x}\,.\,81^{\sin 3x} = 33^{\cos 3x}\,.\,34^{\sin 3x} = 33^{\cos 3x + 4\sin 3x}$

$$-\sqrt{3^2+4^2} \le 3\cos 3x + 4\sin 3x \le +\sqrt{3^2+4^2}$$

$$-5 \le 3\cos 3x + 4\sin 3x \le +5$$

$$3^{-5} \le 3^{3\cos 3x + 4\sin 3x} \le +3^5$$

$$\frac{1}{243} \le f(x) \le 243$$

Hence $f(x)|_{\text{min.}} = \frac{1}{243}$

16. (*b*) $\cos(\alpha+\beta) + \sin(\alpha+\beta) = 0$

$\Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta + \sin\alpha\cos\beta - \sin\beta\cos\alpha = 0$

$\cos\alpha\cos\beta - \sin\beta\cos\alpha = \sin\alpha\sin\beta - \sin\alpha\cos\beta$

$\cos\alpha(\cos\beta - \sin\beta) = \sin\alpha(\sin\beta - \cos\beta)$

$\Rightarrow \cos\alpha = -\sin\alpha$

$\tan\alpha = -1$

$(\cos\beta \ne \sin\beta)$ as $\tan\beta = \frac{1}{2009}$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \frac{3.(-1) - (-1)^3}{1 - 3(-1)^2} = \frac{-3+1}{1-3}$$

$$= \frac{-2}{-2} = 1$$

17. (*d*) $\sin x(\sin x + \cos x) = a$

$\sin^2 x + \sin x\cos x = 2a$

$2\sin^2 x + 2\sin x\cos x = 2a$

$1 - \cos 2x + \sin 2x = 2a$

$\sin 2x - \cos 2x = 2a - 1$

$$\sqrt{1^2+(-1)^2} \le \sin 2x - \cos 2x \le \sqrt{1^2+(-1)^2}$$

$$-\sqrt{2} \le \sin 2x - \cos 2x \le \sqrt{2}$$

$$-\sqrt{2} \le 2a - 1 \le \sqrt{2}$$

$$1 - \sqrt{2} \le 2a \le 1 + \sqrt{2}$$

$$\frac{1-\sqrt{2}}{2} \le a \le \frac{1+\sqrt{2}}{2}$$

18. (*a*) $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \frac{\sin\theta + 2\sin\theta\cos\theta}{(1+\cos 2\theta) + \cos\theta}$

$$= \frac{\sin\theta(1+2\cos\theta)}{2\cos^2\theta + \cos\theta}$$

$$= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

As $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \tan\theta \in (-\infty, \infty)$

19. (*a*) $f(\alpha, \beta) = \cos^2\alpha + \sin^2\alpha \cdot \cos 2\beta$

$= 1 - \sin^2\alpha + \sin^2\alpha \cdot \cos 2\beta$

$= 1 - \sin^2\alpha(1 - \cos 2\beta)$

$= 1 - \sin^2\alpha \cdot 2\sin^2\beta$

$= 1 - 2\sin^2\alpha \cdot \sin^2\beta$

$f(\beta, \alpha) = 1 - 2\sin^2\beta \cdot \sin^2\alpha$

$f(\alpha, \beta) = f(\beta, \alpha)$

20. (*a*) $f(x) = \cos 10x + \cos 8x + 3\cos 4x + 3\cos 2x$

$g(x) = 8\cos x\cos^3 3x$

$f(x) = \cos 10x + \cos 2x + \cos 8x + \cos 4x + 2\cos 4x + 2\cos 2x$

$$= 2\cos\left(\frac{10x+2x}{2}\right)\cos\left(\frac{10x-2x}{2}\right)$$

$$+ 2\cos\left(\frac{8x+4x}{2}\right)\cos\left(\frac{8x-4x}{2}\right) + 2\cos 4x + 2\cos 2x$$

$= 2\cos 6x\cos 4x + 2\cos 6x\cos 2x + 2\cos 4x + 2\cos 2x$

$= 2\cos 6x(\cos 4x + \cos 2x) + 2(\cos 4x + \cos 2x)$

$= (\cos 4x + \cos 2x)(2\cos 6x + 2)$

$$= 2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)2 \cdot (1 + \cos(2 \cdot 3x))$$

$= 2\cos 3x\cos x \cdot 2 \cdot 2\cos^2 3x$

$= 8\cos x\cos^3 3x = g(x)$

Hence, $f(x) = g(x)$

Advanced Level Multiconcept Questions

1. (b, d) $\dfrac{\sin x + \cos x}{\cos^3 x} = \tan x \sec^2 x + \sec^2 x$

$= \sec^2 x (1 + \tan x) = (1 + \tan^2 x)(1 + \tan x)$

2. (a, b)

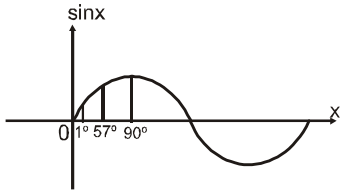
$1 = [(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)]^2$

$\Rightarrow (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = \pm 1$

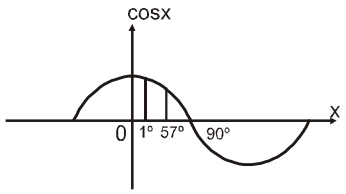
So each side is equal to ± 1

3. (b, c)

1 radian $\simeq$ 57° (approx.)



$\therefore \quad \sin 1 > \sin 1°$



$\therefore \quad \cos 1° > \cos 1$

4. (a, c)

$f(x) = \cos 9x + \cos(-10)x = \cos 9x + \cos 10x$

$f\left(\dfrac{\pi}{2}\right) = \cos \dfrac{9\pi}{2} + \cos 5\pi = -1$

$f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$

$f(-\pi) = 0$

$f\left(\dfrac{\pi}{4}\right) = \cos \dfrac{9\pi}{4} + \cos \dfrac{5\pi}{2} = \cos \dfrac{9\pi}{4}$

$= \cos\left(2\pi + \dfrac{\pi}{4}\right) = \cos \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

5. (a, b, c, d)

$\dfrac{1 + \tan 11°}{1 - \tan 11°} = \tan(45° + 11°) = \tan 56°$

6. (c, d)

$0 < \theta < \pi/2$

$\tan 3\theta = \tan(2\theta + \theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$

$\Rightarrow \tan 3\theta - \tan\theta \tan 2\theta \tan 3\theta = \tan 2\theta + \tan\theta$

$\Rightarrow \tan 3\theta - \tan 2\theta - \tan\theta = \tan\theta \tan 2\theta \tan 3\theta$

$\Rightarrow \tan 3\theta + \tan 3\theta = \tan\theta \cdot \tan 2\theta \tan 3\theta$

($\because$ given that $\tan\theta + \tan 2\theta + \tan 3\theta = 0$)

$\Rightarrow \tan 3\theta (2 - \tan\theta \tan 2\theta) = 0$

$\Rightarrow \tan 3\theta = 0$ or $\tan\theta \tan 2\theta = 2$.

7. (b, c)

$y = a\cos^2 x + 2b \sin x \cos x + c \sin^2 x$ & $\tan x = \dfrac{2b}{a - c}$

$z = a \sin^2 x - 2b \sin x \cos x + c \cos^2 x$

$\Rightarrow y + z = a + c$

and $y - z = (a - c)\left(\cos^2 x - \sin^2 x\right) + 4b \sin x \cos x$

$= (a - c)\cos 2x + 2b \sin 2x$ ($\because 2b = (a - c)\tan x$)

$= (a - c)[\cos 2x + \tan x . \sin 2x] = (a - c)$

$\left[2\cos 2x + \dfrac{\sin x}{\cos x}\sin 2x\right] = \dfrac{(a - c)\cos(2x - x)}{\cos x}$

$= (a - c)$.

8. (b, d)

$$\left\{\frac{2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)}\right\}^n + \left\{\frac{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)}\right\}^n$$

$= \cot^n\left(\dfrac{A-B}{2}\right) + (-1)^n \cot^n\left(\dfrac{A-B}{2}\right)$

$0 \; ; n \in$ odd

$2\cot^n\left(\dfrac{A-B}{2}\right) ; n \in$ even

9. (b, d)

Let $\tan\dfrac{\alpha}{2} = t$

$(a + 2)\, 2t + (2a - 1)(1 - t^2) = (2a + 1)(t^2 + 1)$

$\Rightarrow 2at + 4t + 2a - 2at^2 - 1 + t^2 = 2a + 1 + 2at^2 + t^2$

$\Rightarrow 4at^2 - 2t(2 + a) + 2 = 0 \Rightarrow 2at^2 - 2t - at + 1 = 0$

$\Rightarrow 2t(at - 1) - 1(at - 1) = 0 \Rightarrow t = 1/2 , t = 1/a$

$\Rightarrow \tan\alpha = \dfrac{2\tan\alpha/2}{1 - \tan^2\alpha/2}$

$\Rightarrow \tan\alpha = \dfrac{2 \times 1/2}{1 - 1/4} = 4/3$

or $\tan\alpha = \dfrac{2/a}{1 - 1/a^2} = \dfrac{2a}{a^2 - 1}$

10. (a, b)

$3\sin\beta = \sin(2\alpha + \beta)$

$\Rightarrow \dfrac{\sin(2\alpha + \beta)}{\sin\beta} = \dfrac{3}{1}$

Applying componendo & dividendo

$\Rightarrow \frac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \frac{3+1}{3-1} = \frac{2}{1}$

$\Rightarrow \frac{2\sin(\alpha+\beta).\cos\alpha}{2\cos(\alpha+\beta)\sin\alpha} = 2$

$\Rightarrow \tan(\alpha+\beta) = 2\tan\alpha$

$\Rightarrow \tan(\alpha+\beta) - 2\tan\alpha = 0$

Hence, independent of α & β both.

11. (*a*, *b*)

$\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$

$\Rightarrow \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C) = 0$

$\Rightarrow \sin(A+B+C) = 0 \Rightarrow A+B+C = n\pi, n \in I$

12. (*a*, *b*)

$\tan A + \tan B + \tan C = 6$, $\tan A \tan B = 2$

In any ΔABC,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow 6 = 2\tan C \Rightarrow \tan C = 3$

$\therefore \tan A + \tan B + 3 = 6$

$\Rightarrow \tan A + \tan B = 3$ & $\tan A \tan B = 2$

Now $(\tan A - \tan B)^2 = (\tan A + \tan B)^2 - 4\tan A \tan B$

$= 9 - 8 = 1$

$\Rightarrow \tan A - \tan B = \pm 1$

$\therefore \tan A - \tan B = 1$ or $\tan A - \tan B = -1$

$\tan A + \tan B = 3$ $\tan A + \tan B = 3$

on solving on solving

$\tan A = 2$ $\tan A = 1$

$\tan B = 1$ $\tan B = 2$

13. (*a*, *c*)

$h = \sqrt{\{\cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta)\}^2 + \{\sin 2\alpha + \sin 2\beta + 2\sin(\alpha+\beta)\}^2}$

$\Rightarrow h = [4\cos^2(\alpha+\beta)(\cos(\alpha-\beta)+1)^2 + 4\sin^2(\alpha+\beta)(\cos(\alpha-\beta)+1)^2]^{1/2}$

$\Rightarrow h = [4\{\cos(\alpha-\beta)+1\}^2 \{\cos^2(\alpha+\beta) + \sin^2(\alpha+\beta)\}]^{1/2}$

$\Rightarrow h = 2(1+\cos(\alpha-\beta)) \Rightarrow h = 2 \times 2\cos^2\left(\frac{\alpha-\beta}{2}\right)$

$\Rightarrow h = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

14. (*b*, *d*)

Let $Y = 1 + 4\sin\theta + 3\cos\theta$

$\because$ Max value of $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}$

Max value of $Y = 1 + \sqrt{4^2+3^2} = 6$

$\because$ Min value of $a\sin\theta + b\cos\theta = -\sqrt{a^2+b^2}$

Min value of $Y = 1 - \sqrt{4^2+3^2} = 1-5 = -4$

15. (*c*, *d*)

$\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x \cos y \cos z$

(Given $x + y = z$)

$= 1 + \cos(x+y)\cos(x-y) + \cos^2 z - 2\cos x \cos y \cos z$

$= 1 + \cos z[\cos(x-y) + \cos(x+y)] - 2\cos x \cos y \cos z$

$= 1 + \cos z \cdot 2\cos x \cos y - 2\cos x \cos y \cos z$

$= 1$

$= \cos(x+y-z)$

16. (*b*, *d*)

$\sin^6 x + \cos^6 x = a^2$

$\Rightarrow (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x) = a^2$

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x = a^2 \Rightarrow 1 - 3\sin^2 x \cos^2 x = a^2$

$\Rightarrow 1 - \frac{3}{4}\sin^2 2x = a^2 \Rightarrow \frac{4(1-a^2)}{3} = \sin^2 2x$

$\Rightarrow 0 \le \frac{4}{3}(1-a^2) \le 1$

$1 - a^2 \ge 0$ and $4 - 4a^2 \le 3$

$a^2 \le 1$ and $\frac{1}{4} \le a^2$

$-1 \le a \le 1$ and $a \ge \frac{1}{2}$ or $a \le -\frac{1}{2}$

$a \in \left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$

17. (*c*) $\tan A \tan B \tan C = \tan A + \tan B + \tan C = \frac{p}{q}$

18. (*b*) $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$= \frac{\sin A \sin B \cos C + \cos A \sin B \sin C + \cos B \sin C \sin A}{\cos A \cos B \cos C}$

$= \frac{\sin B.\sin(A+C) + \cos B \sin C \sin A}{q}$

$= \frac{1 - \cos^2 B + \cos B \sin C \sin A}{q}$

$= \frac{1 + \cos B(\sin C \sin A - \cos B)}{q}$

$= \frac{1 + \cos B \cos C \cos A}{q} = \frac{1+q}{q}$

19. (*d*) $\tan^3 A + \tan^3 B + \tan^3 C - 3\tan A \tan B \tan C$

$= (\tan A + \tan B + \tan C)\{(\tan A + \tan B + \tan C)^2 - 3(\tan A \tan B + \tan B \tan C + \tan C \tan A)\}$

$= \frac{p}{q}\left[\left(\frac{p}{q}\right)^2 - 3\left(\frac{1+q}{q}\right)\right]$

$\therefore \tan^3 A + \tan^3 B + \tan^3 C$

$= \frac{3p}{q} + \frac{p}{q^3}(p^2 - 3q - 3q^2)$

$= \frac{3pq^2 + p^3 - 3pq - 3pq^2}{q^3} = \frac{p^3 - 3pq}{q^3}$

20. (*c*) Given $\cos\alpha + \cos\beta = a$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = a \qquad .....(i)$$

and $\sin\alpha + \sin\beta = b$

$$\Rightarrow 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = b \qquad .....(ii)$$

by (*i*) & (*ii*)

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a} \quad \Rightarrow \quad \tan\theta = \frac{b}{a}$$

$\therefore \sin2\theta + \cos2\theta$

$$= \frac{2\left(\frac{b}{a}\right)}{1+\frac{b^2}{a^2}} + \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2} + \frac{a^2-b^2}{a^2+b^2}$$

$$= \frac{a^2+b^2-2b^2+2ab}{a^2+b^2} \quad = 1 + \frac{2b(a-b)}{a^2+b^2}$$

$\therefore n = 2$

21. (*a*) $\sin^n A = x$

$\Rightarrow \sin^2 A = x$

$\therefore \sin A \sin 2A \sin 3A \sin 4A$

$= \sin A (2 \sin A \cos A) (3 \sin A - 4 \sin^3 A) (4 \sin A \cos A (1 - 2 \sin^2 A))$

$= 8 \sin^4 A (1 - \sin^2 A) (1 - 2 \sin^2 A) (3 - 4 \sin^2 A)$

If we put $\sin^2 A = x$, then given expression is a polynomial of degree 5 in x.

22. (*b*) If $p = 5$

$\therefore \sin x + (p-5), \cos x, \tan x$

$\Rightarrow \sin x, \cos x, \tan x$ are in G.P.

$\therefore \cos^2 x = \sin x \tan x$

$\cos^3 x = \sin^2 x$

$\therefore \cos^3 x = 1 - \cos^2 x$

$\therefore \cos^3 x + \cos^2 x = 1$

taking cube both sides

$\therefore \cos^9 x + \cos^6 x + 3\cos^5 x = 1$

$\Rightarrow \cos^9 x + \cos^6 x + 3\cos^5 x - 1 = 0$

23. (*a*) $\rightarrow$ (q), (*b*) $\rightarrow$ (r),(*c*) $\rightarrow$ (q), (*d*) $\rightarrow$ (p)

(*a*) $\sin 420° \cos 390° + \cos(-660°) \sin(-330°)$

$= \sin(360° + 60°) \cos(360° + 30°) + \cos 660° (-\sin 330°)$

$= \sin 60° \cos 30° - \cos(720° - 60°) \sin(360° - 30°)$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \cos 60° (-\sin 30°) = \frac{3}{4} + \frac{1}{2}\cdot\frac{1}{2} = 1$$

(*b*) $\tan 315° \cot(-405°) + \cot(495°) \tan(-585°)$

$= \tan(360° - 45°)(-\cot(360° + 45°)) + \cot(360° + 135°)(-\tan(720° - 135°))$

$= (-\tan 45°)(-\cot 45°) + \cot 135° \tan 135° = 1 + 1 = 2$

(*c*) $37° = 45° - 8°$

$$\Rightarrow \tan 37° = \frac{1-\tan 8°}{1+\tan 8°} \Rightarrow 1 + \tan 37°$$

$$= \frac{2}{1+\tan 8°}$$

Similarly $1 + \tan 23° = \dfrac{2}{1+\tan 22°}$

$$\therefore \frac{(1+\tan 8°)(1+\tan 37°)}{(1+\tan 22°)(1+\tan 23°)} = 1$$

(*d*) $\left[\frac{\pi}{4}\right] + \left[\frac{1}{3}\sin^2 x\right] = 0 + 0 = 0$

24. (*a*) $\rightarrow$ (q, s), (*b*) $\rightarrow$ (p), (*c*) $\rightarrow$ (q), (*d*) $\rightarrow$ (p)

(*a*) $x + \dfrac{1}{x} = 2\cos\theta \geq 2$ or ≤ -2

$\Rightarrow \cos\theta = 1$ or -1

(*b*) $\sin\theta + \operatorname{cosec}\theta = 2$

$\because \sin\theta + \dfrac{1}{\sin\theta} \geq 2$ or ≤ -2

but given that $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta + \dfrac{1}{\sin\theta}$ $= 2$ which is possible only when $\sin\theta = 1$

$$\therefore \sin^{2008}\theta + \operatorname{cosec}^{2008}\theta = \sin^{2008} + \frac{1}{\sin^{2008}}$$

$= 1 + 1 = 2$

(*c*) $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$$= 1 - \frac{1}{2}\sin^2 2\theta$$

$\because 0 \leq \sin^2 2\theta \leq 1$

$$\therefore \frac{1}{2} \leq 1 - \frac{1}{2}\sin^2 2\theta \leq 1$$

$\therefore$ maximum value $= 1$

(*d*) $2\sin^2\theta + 3\cos^2\theta = 2\sin^2\theta + 3 - 3\sin^2\theta = 3 - \sin^2\theta$

$\because 0 \leq \sin^2\theta \leq 1$

$\therefore 2 \leq 3 - \sin^2\theta \leq 3$

$\therefore$ least value $= 2$

25. [6]

$$\frac{\cos 5A}{\cos A} + \frac{\sin 5A}{\sin A}$$

$$= \frac{\sin A\cos 5A + \cos A\sin 5A}{\sin A\cos A}$$

$$= \frac{2\sin 6A}{\sin 2A} = \frac{2[3\sin 2A - 4\sin^3 2A]}{\sin 2A}$$

$$= 6 - 8\sin^2 2A = 6 - 8\cdot\left[\frac{1-\cos 4A}{2}\right]$$

$= 2 + 4\cos 4A = a + b\cos 4A$

$a + b = 6$ **Ans.**

26. [2]

$\cos\theta$

$= \dfrac{2\cos(\theta-\phi).\cos(\theta+\phi)}{\cos(\theta-\phi)+\cos(\theta+\phi)} = \dfrac{2(\cos^2\theta-\sin^2\phi)}{2\cos\theta.\cos\phi}$

$\Rightarrow \cos^2\theta \,.\, \cos\phi = \cos^2\theta - \sin^2\phi$

$\Rightarrow \sin^2\phi = \cos^2\theta\,(1-\cos\phi)$

$\Rightarrow 4\sin^2\dfrac{\phi}{2}\,.\,\cos^2\dfrac{\phi}{2} = 2\cos^2\theta\,.\,\sin^2\dfrac{\phi}{2}$

$\Rightarrow \cos\theta\sec\dfrac{\phi}{2} = \sqrt{2}$

27. [1]

Since sin A. cos A and tan A are G.P.

we have $\cos^2 A = \sin A \tan A$

$\Rightarrow \cot^2 A = \sec A$

$\Rightarrow \cot^4 A = 1 + \tan^2 A \quad \Rightarrow \cot^6 A - \cot^2 A = 1$

28. [1]

We have $\operatorname{cosec}\theta - \sin\theta = a^3$

$\Rightarrow \cos^2\theta = a^3 \sin\theta$ & $\sec\theta - \cos\theta = b^3$

$\Rightarrow \sin^2\theta = b^3\cos\theta$ on simplifying

$\Rightarrow a^2 b^2 (a^2 + b^2) = 1$

29. [4]

$\sin\theta + \sin^2\theta + \sin^2\theta = 1$

$\sin\theta\,(1+\sin^2\theta) = 1 - \sin^2\theta$

$\Rightarrow \sin\theta\,(2-\cos^2\theta) = \cos^2\theta \Rightarrow$ on squaring

$\Rightarrow \sin^2\theta\,(2-\cos^2\theta)^2 = \cos^4\theta$

$\Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

30. [2]

Explanation:

The value of $(1-\cot 23^\circ)(1-\cot 22^\circ)$

$\Rightarrow \dfrac{(\sin 23^\circ - \cos 23^\circ)(\sin 22^\circ - \cos 22^\circ)}{\sin 23^\circ \sin 22^\circ}$

$\Rightarrow \dfrac{\sqrt{2}\sin(23^\circ-45^\circ).\sqrt{2}\sin(22^\circ-45^\circ)}{\sin 23^\circ \sin 22^\circ} \Rightarrow 2$

31. $[\sqrt{3}]$

Explanation:

$60^\circ = 40^\circ + 20^\circ$

$\tan(60^\circ) = \tan(40^\circ + 20^\circ)$

$\sqrt{3} = \dfrac{\tan 40^\circ + \tan 20^\circ}{1 - \tan 20^\circ \tan 40^\circ}$

$\sqrt{3} = \sqrt{3}\tan 20^\circ \tan 40^\circ = \tan 40^\circ + \tan 20^\circ$

$\tan 20^\circ + \tan 40^\circ + \sqrt{3}\tan 20^\circ \tan 40^\circ = \sqrt{3}$

32. [1.5]

$\cos^4\dfrac{\pi}{8} + \cos^4\dfrac{7\pi}{8} + \cos^4\dfrac{3\pi}{8} + \cos^4\dfrac{5\pi}{8}$

$= \cos^4\dfrac{\pi}{8} + \cos^4\dfrac{\pi}{8} + \cos^4\left(\dfrac{\pi}{2} - \dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{\pi}{2} + \dfrac{\pi}{8}\right)$

$= 2\left[\cos^4\dfrac{\pi}{8} + \sin^4\dfrac{\pi}{8}\right]$

$= 2\left[(\cos^2\dfrac{\pi}{8} + \sin^2\dfrac{\pi}{8})^2 - 2\sin^2\dfrac{\pi}{8}\cos^2\dfrac{\pi}{8}\right]$

$= 2\left[1 - \dfrac{1}{2}\left(\sin\dfrac{\pi}{4}\right)^2\right] = 2\left[1 - \dfrac{1}{4}\right] = \dfrac{3}{2}$

33. [0.0625]

$\cos\dfrac{\pi}{15}.\cos\dfrac{2\pi}{15}.\cos\dfrac{4\pi}{15}.\cos\dfrac{8\pi}{15}$

$= \dfrac{1}{4}\left(2\cos\dfrac{4\pi}{15}\cos\dfrac{\pi}{15}\right)\left(2\cos\dfrac{8\pi}{15}\cos\dfrac{2\pi}{15}\right)$

$= \dfrac{1}{4}(\cos 60^\circ + \cos 36^\circ)(\cos 120^\circ + \cos 72^\circ)$

$= \dfrac{1}{4}\left(\dfrac{1}{2} + \dfrac{\sqrt{5}+1}{4}\right)\left(-\dfrac{1}{2} + \dfrac{\sqrt{5}-1}{4}\right)$

$= \dfrac{1}{4}\left[-\dfrac{1}{4} + \dfrac{1}{2}\left(\dfrac{\sqrt{5}-1}{4} - \dfrac{\sqrt{5}+1}{4}\right) + \dfrac{5-1}{16}\right] = -\dfrac{1}{16}$

34. [10]

$5\cos\theta + 3\cos\left(\theta + \dfrac{\pi}{3}\right) + 3$

$= 5\cos\theta + 3(\cos\theta\cos\dfrac{\pi}{3} - \sin\theta\sin\dfrac{\pi}{3}) + 3$

$= 5\cos\theta + \dfrac{3}{2}\cos\theta - \dfrac{3\sqrt{3}}{2}\sin\theta + 3$

$= \dfrac{13}{2}\cos\theta - \dfrac{3\sqrt{3}}{2}\sin\theta + 3$

$\therefore$ maximum value $= 3 + \sqrt{\left(\dfrac{13}{2}\right)^2 + \left(-\dfrac{3\sqrt{3}}{2}\right)^2}$

$= 3 + \sqrt{\dfrac{196}{4}} = 3 + 7 = 10$

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*b*) $3(1-\sin 2\theta)^2+6(1+\sin 2\theta)+4\sin^6\theta$

$=3(1-2\sin 2\theta+\sin^2 2\theta)+6+6\sin 2\theta+4\sin^6\theta$

$=9+3\sin^2 2\theta+4\sin^6\theta$

$=9+12\sin^2\theta\cos^2\theta+4(1-\cos^2\theta)^3$

$=9+12(1-\cos^2\theta)\cos^2\theta+4$

$(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$=13+12\cos^2\theta-12\cos^4\theta-12\cos^2\theta+12\cos^4\theta-4\cos^6\theta$

$=13-4\cos^6\theta$

2. (*c*) Using quadratic formula,

$x=\dfrac{(\cos\theta\sin\theta+1)\pm\sqrt{(\cos\theta\sin\theta+1)^2-4\sin\theta\cos\theta}}{2\sin\theta}$

$=\dfrac{(\cos\theta\sin\theta+1)^2\pm(\cos\theta\sin\theta-1)}{2\sin\theta}$

$=\cos\theta,\ \text{cosec}\,\theta$

$\alpha=\cos\theta,\ \beta=\text{cosec}\,\theta$

$\therefore\ \sum_{n=0}^{\infty}\alpha^n+\dfrac{(-1)^n}{\beta^n}=\sum_{n=0}^{\infty}(\cos\theta)^n+\sum_{n=0}^{\infty}(-\sin\theta)^n$

$=\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\sin\theta}$

$\therefore$ (*c*) is the correct answer.

3. (*a*) $5\sin\left(\theta-\dfrac{\pi}{6}\right)+3\cos\theta$

$5\left(\sin\theta\cos\dfrac{\pi}{6}-\cos\theta\sin\dfrac{\pi}{6}\right)+3\cos\theta$

$=\dfrac{5\sqrt{3}}{2}\sin\theta+\dfrac{1}{2}\cos\theta$

Maximum value is $\sqrt{\left(\dfrac{5\sqrt{3}}{2}\right)^2+\left(\dfrac{1}{2}\right)^2}=\sqrt{\dfrac{76}{4}}=\sqrt{19}$

4. (*d*) A.M. $\geq$ G.M.

$\dfrac{\sin^4\alpha+4\cos^4\beta+1+1}{4}\geq\left(\sin^4\alpha.4\cos^4\beta.1.1\right)^{\frac{1}{4}}$

$\sin^4\alpha+4\cos^2\beta+2\geq 4\sqrt{2}\sin\alpha\cos\beta$ given that

$\sin^4\alpha+4\cos^4\beta+2=4\sqrt{2}\sin\alpha\cos\beta$

$\Rightarrow$ A.M. = G.M. $\Rightarrow \sin^4\alpha=1=4\cos^4\beta$

$\sin\alpha=\pm 1,\cos\beta=\pm\dfrac{1}{\sqrt{2}},\ \text{As}\ \alpha\beta\in[0,\pi]$

$\Rightarrow \sin\alpha=1,\ \cos\beta=\pm\dfrac{1}{\sqrt{2}}$

$\Rightarrow \sin\beta=\dfrac{1}{\sqrt{2}}$ as $\beta\in[0,\pi]$

$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta=-\sqrt{2}$

5. (*d*) $0<\alpha+\beta=\dfrac{\pi}{2}$ and $\dfrac{-\pi}{4}-\alpha-\beta<\dfrac{\pi}{4}$

if $\cos(\alpha+\beta)=\dfrac{3}{5}$ then $\tan(\alpha-\beta)=\dfrac{4}{3}$

and it $\sin(\alpha-\beta)=\dfrac{5}{13}$ then $\tan(\alpha-\beta)=\dfrac{5}{12}$

(since $\alpha-\beta$ here lies in the first quadrant)

Now $\tan(2a)=\tan\{(\alpha+\beta)+(\alpha-\beta)\}$

$=\dfrac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta).\tan(\alpha-\beta)}=\dfrac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}=\dfrac{63}{16}$

6. (*b*) $\dfrac{1}{2}(2\cos^2 10^\circ-2\cos 10^\circ\cos 50^\circ+2\cos^2 50^\circ)$

$\Rightarrow \dfrac{1}{2}(1+\cos 20^\circ-(\cos 60^\circ+\cos 40^\circ)+1+\cos 100^\circ)$

$\Rightarrow \dfrac{1}{2}\left(\dfrac{3}{2}+\cos 20^\circ+2\sin 70^\circ\sin(-30^\circ)\right)$

$\Rightarrow \dfrac{1}{2}\left(\dfrac{3}{2}+\cos 20^\circ-\sin 70^\circ\right)\Rightarrow\dfrac{3}{4}$

7. (*d*) $(\sin 10^\circ\sin 30^\circ\sin 70^\circ)\sin 30^\circ$

$\dfrac{1}{4}(\sin 30^\circ)^2=\dfrac{1}{4}\cdot\dfrac{1}{4}=\dfrac{1}{16}$

8. (*a*) $2^{\sqrt{\sin^2 x-2\sin x+5}}\cdot 4^{-\sin^2 y}\leq 1$

$\Rightarrow 2^{\sqrt{(\sin x-1)^2+4}}\leq 2^{2\sin^2 y}$

$\Rightarrow \sqrt{(\sin x-1)^2+4}\leq 2\sin^2 y$

$\Rightarrow \sin x=1$ and $|\sin y|=1$

9. (*b*) $2y=2\sin x\sin(x+2)-2\sin^2(x+1)$

$2y=\cos 2-\cos(2x+2)-(1-\cos(2x+2))$

$=\cos 2-1$

$2y=-2\sin^2 1$

$y=-\sin^2 1\leq 0$

10. (*c*) $(k+1)\tan^2 x-\sqrt{2}\cdot\lambda\tan x+(k-1)=0$

$\tan\alpha+\tan\beta=\dfrac{\sqrt{2}\lambda}{k+1}$

$\tan\alpha\tan\beta=\dfrac{k-1}{k+1}$

$$\tan(\alpha+\beta)=\frac{\frac{\sqrt{2}\lambda}{k+1}}{1-\frac{k-1}{k+1}}=\frac{\sqrt{2}\lambda}{2}=\frac{\lambda}{\sqrt{2}}$$

$$\tan^2(\alpha+\beta)=\frac{\lambda^2}{2}=50$$

$\lambda=10.$

11. (*c*) $2[{}^6C_0x^6+{}^6C_2x^4(x^2-1)+{}^6C_4x^2(x^2-1)^2+{}^6C_6(x^2-1)^3]$
$=2[x^6+15(x^6-x^4)+15x^2(x^4-2x^2+1)$
$+(-1+3x^2-3x^4+x^6)]$
$=2(32x^6-48x^4+18x^2-1)$
$\alpha=-96$ and $\beta=36 \quad \therefore \alpha-\beta=-132$

12. (*a*) $\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha}=\frac{1}{7}$ and $\frac{\sqrt{2}\sin\beta}{\sqrt{2}}=\frac{1}{\sqrt{10}}$

$$\tan\alpha=\frac{1}{7}$$

$$\sin\beta=\frac{1}{\sqrt{10}}$$

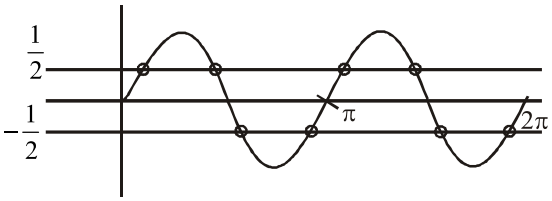
$$\tan\beta=\frac{1}{3}$$

$$\tan 2\beta=\frac{2.\frac{1}{3}}{1-\frac{1}{9}}=\frac{\frac{2}{3}}{\frac{8}{9}}=\frac{3}{4}$$

$$\tan(\alpha+2\beta)=\frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}}=\frac{\frac{4+21}{28}}{\frac{25}{28}}=1$$

13. (*a*) $\cos^3\frac{\pi}{8}\left[4\cos^3\frac{\pi}{8}-3\cos\frac{\pi}{8}\right]$
$+\sin^3\frac{\pi}{8}\left[3\sin\frac{\pi}{8}-4\sin^3\frac{\pi}{8}\right]$
$=4\cos^6\frac{\pi}{8}-4\sin^6\frac{\pi}{8}-3\cos^4\frac{\pi}{8}+3\sin^4\frac{\pi}{8}$
$=4\left[\left(\cos^2\frac{\pi}{8}-\sin^2\frac{\pi}{8}\right)\right]$
$\left[\left(\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)\right]$
$-3\left[\left(\cos^2\frac{\pi}{8}-\sin^2\frac{\pi}{8}\right)\right]$
$=\cos\frac{\pi}{4}\left[4\left(1-\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)-3\right]$
$=\frac{1}{\sqrt{2}}\left[1-\frac{1}{2}\right]=\frac{1}{2\sqrt{2}}$

14. (8) $\log_{1/2}|\sin x|=2-\log_{1/2}|\cos x|$
$\log_{1/2}|\sin x\cos x|=2$
$|\sin x\cos x|=\frac{1}{4}$
$\sin 2x=\pm\frac{1}{2}$



Number of solution = 8

15. (*a*) $y=1+\cos^2\theta+\cos^4\theta+.....$
$\Rightarrow y=\frac{1}{1-\cos^2\theta}\Rightarrow\frac{1}{y}=\sin^2\theta$
$x=\frac{1}{1-(-\tan^2\theta)}=\frac{1}{\sec^2\theta}$
$\Rightarrow \text{cso}^2\theta=x$
$\Rightarrow \frac{1}{y}+x=1$
$\Rightarrow y(1-x)=1$

16. (*d*) $\because a\cos\theta=b\cos\left(\theta+\frac{2\pi}{3}\right)=c\cos\left(\theta+\frac{4\pi}{3}\right)=k$
$ab+bc+ca=k^2$
$$\left[\frac{1}{\cos\theta.\cos\left(\theta+\frac{2\pi}{3}\right)}+\frac{1}{\cos\left(\theta+\frac{2\pi}{3}\right).\cos\left(\theta+\frac{4\pi}{3}\right)}+\frac{1}{\cos\left(\theta+\frac{4\pi}{3}\right).\cos\theta}\right]$$
$$=K^2\left[\frac{\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)}{\cos\theta.\cos\left(\theta+\frac{2\pi}{3}\right).\cos\left(\theta+\frac{4\pi}{3}\right)}\right]=0$$
So, angle between the given vectors will be $\frac{\pi}{2}$.

17. (*c*) $\frac{2^{\sin x}+2^{\cos x}}{2}\geq(2^{(\sin x+\cos x)})^{\frac{1}{2}}$
$\Rightarrow 2^{\sin x}+2^{\cos x}\geq 2.2^{\frac{\sin x+\cos x}{2}}$
$\Rightarrow 2^{\sin x}+2^{\cos x}\geq 2^{1-\frac{1}{\sqrt{2}}}$
(as $\sin x+\cos x\geq=-\sqrt{2}$)

18. (*b*) $L+M=1-2\sin^2\frac{\pi}{8}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ ...(*i*)
and $L-M=-\cos\frac{\pi}{8}$ ...(*ii*)
By (*i*) and (*ii*)
$L=\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\cos\frac{\pi}{8}\right)=\frac{1}{2\sqrt{2}}-\frac{1}{2}\cos\frac{\pi}{8}$
& $M=\frac{1}{2}\left(\frac{1}{\sqrt{2}}+\cos\frac{\pi}{8}\right)=\frac{1}{2\sqrt{2}}+\frac{1}{2}\cos\frac{\pi}{8}$

19. (*c*) $\cos^2x+\cos^4x+\cos^6x+\ .....\infty$
↳ G.P. $a=\cos^2x$
$r=\cos^2x$

$$\text{Sum} = \frac{a}{1-r} = \frac{\cos^2 x}{1-\cos^2 x} = \frac{\cos^2 x}{\sin^2 x} = \cot^2 x$$

$$e^{(\cos^2 x + \cos^4 x + \ldots\infty)\ln 2} = e^{\cot^2 x \times \ln 2} = e^{\ln 2^{\cot^2 x}} = 2^{\cot^2 x}$$

$$t^2 - 9t + 8 = 0 \Rightarrow (t-1)(t-8) = 0 \Rightarrow t = 1 \text{ or } 8$$

$\longrightarrow$ Root = $2^{\cot^2 x}$

$\Rightarrow 2^{\cot^2 x} = 1$ or $2^{\cot^2 x} = 8 = 2^3$

$\cot^2 x = 0$ $\quad$ $\cot^2 x = 3$

$\cot x = 0$ $\quad$ $\cot x = \pm\sqrt{3}$

But $0 < x < \frac{\pi}{2} \Rightarrow \cot x$ is tve $\Rightarrow$ $\cot z = \sqrt{3}$

$$\frac{2\sin x}{\sin x + \sqrt{3}\cos x} = \frac{2}{1 + \sqrt{3}\cot x} = \frac{2}{1 + \sqrt{3}.\sqrt{3}} = \frac{2}{4} = \frac{1}{2}$$

20. [11] $3\sin x + 4\cos x = k + 1$

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le \sqrt{3^2+4^2}$

$-5 \le k + 1 \le 5$

$-6 \le k \le 4$

Integer value of $k = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$

No. of integers = 11

21. (*d*) $15\sin^4\alpha + 10\cos^4\alpha = 6$

$15\sin^4\alpha + 10(\cos^2\alpha)^2 = 6$

$15\sin^4\alpha + 10(1 - \sin^2\alpha)^2 = 6$

$15\sin^4\alpha + 10(1 + \sin^4\alpha - 2\sin^2\alpha)^2 = 6$

$25\sin^4\alpha - 20\sin^2\alpha + 10 = 6$

$(5\sin^2\alpha)^2 + (2)^2 - 2\cdot 5\sin\alpha\cdot 2 = 0$

$(5\sin^2\alpha - 2)^2 = 0 \Rightarrow \sin^2\alpha = \frac{2}{5}$;

$\cos^2\alpha = \frac{3}{5}$

$27\sec^6\alpha + 8\,\text{cosec}^6\alpha = 27\left(\frac{5}{3}\right)^3 + 8\left(\frac{5}{2}\right)^3$

$= 125 + 125 = 250$

22. (*a*) $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - \left(2\cos^2\left(\frac{x+y}{2}\right) - 1\right)$$

$$= \frac{3}{2}$$

$$2\cos\left(\frac{x+y}{2}\right)\left\{\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right\} = \frac{3}{2} - 1$$

$$2\cos\left(\frac{x+y}{2}\right)\cdot 2\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) = \frac{1}{2}$$

$$\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) = \frac{1}{8}$$

As $0 < x, y < \pi$, possibility is $\frac{x}{2} = 30°$; $\frac{y}{2} = 30°$

$\Rightarrow x = 60°$; $y = 60°$

Then, $\sin x + \cos y = \sin 60° + \cos 60° = \frac{1+\sqrt{3}}{2}$

23. (*c*) $2x = \tan\left(\frac{\pi}{9}\right) + \tan\left(\frac{7\pi}{18}\right)$

$$\Rightarrow \quad x = \frac{\tan\left(\frac{\pi}{9}\right) + \tan\left(\frac{7\pi}{18}\right)}{2}$$

Also, $2y = \tan\left(\frac{\pi}{9}\right) + \tan\left(\frac{5\pi}{18}\right)$

$$\tan\left(\frac{7\pi}{18}\right) = \cot\left(\frac{\pi}{2} - \frac{7\pi}{18}\right) = \cot\left(\frac{2\pi}{18}\right) = \cot\frac{\pi}{9}$$

$$\tan\left(\frac{5\pi}{18}\right) = \cot\left(\frac{\pi}{2} - \frac{5\pi}{18}\right) = \cot\left(\frac{4\pi}{18}\right) = \cot\left(\frac{2\pi}{9}\right)$$

$$|x - 2y| = \left|\frac{\tan\left(\frac{\pi}{9}\right) + \tan\left(\frac{7\pi}{18}\right)}{2} - \tan\left(\frac{\pi}{9}\right) - \tan\left(\frac{5\pi}{18}\right)\right|$$

$$= \left|\frac{\cot\left(\frac{\pi}{9}\right) - \tan\left(\frac{\pi}{9}\right)}{2} - \cot\left(\frac{2\pi}{9}\right)\right|$$

$$= \left|\cot\left(\frac{2\pi}{9}\right) - \cot\left(\frac{2\pi}{9}\right)\right| = 0$$

24. (*b*) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{1+\cos 2\theta}{\sin 2\theta}$

Putting $\theta = \frac{\pi}{24}$

$$\cot\left(\frac{\pi}{24}\right) = \frac{1 + \cos\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)} = \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1} \cdot \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{2\sqrt{6} + 3 + \sqrt{3} + 2\sqrt{2} + \sqrt{3} + 1}{3 - 1}$$

$$= \frac{2\sqrt{6} + 2\sqrt{3} + 2\sqrt{2} + 4}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$$

25. (*c*) $\sin\theta+\cos\theta=\frac{1}{2} \xrightarrow{()^2} \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$

$=\frac{1}{2}$

$\Rightarrow \sin 2\theta=\frac{1}{4}-1=\frac{-3}{4}$

$\cos 4\theta=1-2\sin^2 2\theta=1-2\cdot\frac{9}{16}=1-\frac{9}{8}=\frac{-1}{8}$

$\sin 6\theta=\sin 3(2\theta)=3\sin 2\theta-4\sin^3 2\theta$

$=(3-4\sin^2 2\theta)\sin 2\theta$

$=\left(3-4\cdot\frac{9}{4.4}\right)\left(\frac{-3}{4}\right)=\left(3-\frac{9}{4}\right)\left(\frac{-3}{4}\right)=\frac{-9}{16}$

$16(\sin(2\theta)+\cos(4\theta)+\sin(6\theta))=16\left(\frac{-3}{4}-\frac{1}{8}-\frac{9}{16}\right)$

$=-12-2-9=-23$

26. (*c*) $2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{5\pi}{8}\right)$

$\sin\left(\frac{6\pi}{8}\right)\sin\left(\frac{7\pi}{8}\right)$

$=2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{8}\right)$

$\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{8}\right)$

$=2\sin^2\left(\frac{\pi}{4}\right)\cdot\sin^2\left(\frac{\pi}{8}\right)\cdot\sin^2\left(\frac{3\pi}{8}\right)$

$=2\cdot\frac{1}{2}\cdot\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)$

$=\frac{1}{4}\left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2$

$=\frac{1}{4}\cdot\left(\sin\left(\frac{2\pi}{8}\right)\right)^2=\frac{1}{4}\sin^2\left(\frac{\pi}{4}\right)=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*a, c, d*)

$2\cos\theta(1-\sin\varphi)=2\sin\theta\cos\phi-1$

$2\cos\theta+1=2\sin(\theta+\phi)$

$\frac{1}{2}=\sin(\theta+\phi)-\cos\theta$

$\tan(2\pi-\theta)>0$, $\tan\theta<0$

$\Rightarrow -1<\sin\theta<-\frac{\sqrt{3}}{2}$

$270^\circ<\theta<300^\circ$

2. (*b,c*)

$\cos\alpha=\left(\frac{1-a}{1+a}\right);\qquad a=\tan^2\frac{\alpha}{2}$

$\cos\beta=\left(\frac{1-b}{1+b}\right);\qquad b=\tan^2\frac{\beta}{2}$

$2\left(\left(\frac{1-b}{1+b}\right)-\left(\frac{1-b}{1+a}\right)\right)+\left(\left(\frac{1-a}{1+a}\right)\left(\frac{1-b}{1+b}\right)\right)=1$

$\Rightarrow 2((1-b)(1+a)-(1-a)(1+a))+(1-a)(1-b)$

$=(1+a)(1+b)$

$2(1+a-b-ab-(1+b-a-ab))+1-a-b+ab$

$=1+a+b+ab$

$4(a-b)=2(a+b)$

$2a-2b=a+b$

3. (*a, b, d*)

$$f(n)=\frac{\sum_{k=0}^{n}\left(\cos\left(\frac{\pi}{n+2}\right)-\cos\left(\frac{2k+3}{n+3}\right)\pi\right)}{\sum_{k=0}^{n}\left(1-\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n)=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)-\left(\sum_{k=0}^{n}\cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{(n+1)-\left(\sum_{k=0}^{n}\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n)=\frac{(n+1)\cos\frac{\pi}{n+2}-\left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}\cdot\cos\left(\frac{n+3}{n+3}\right)\pi\right)}{(n+1)-\left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}\cdot\cos\left(\frac{2(n+2)\pi}{2(n+2)}\right)\right)}$$

$$f(n)=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)+\cos\left(\frac{\pi}{n+2}\right)}{(n+1)+1}$$

$\Rightarrow g(x)=\cos\left(\frac{\pi}{n+2}\right)$

(*a*) $\sin\left(7\cos^{-1}\cos\frac{\pi}{7}\right)=\sin\pi=0$

(*b*) $f(4)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

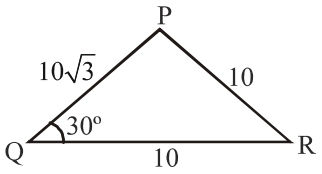
(*c*) $\lim_{x\to\infty}\cos\left(\frac{\pi}{n+2}\right)=1$

(*d*) $\alpha=\tan\left(\cos^{-1}\cos\frac{\pi}{8}\right)=\sqrt{2}-1 \Rightarrow \alpha+1=\sqrt{2}$

$\alpha^2+2\alpha-1=0$

4. (b, c, d)

$$\cos 30^\circ = \frac{(10\sqrt{3})^2 + (10)^2 - (PR)^2}{2\times 10\sqrt{3}\times 10}$$



$\Rightarrow$ $PR = 10$

$\because$ $QR = PR \Rightarrow \angle PQR = \angle QPR$

$\angle QPR = 30^\circ$

(b) area of $\Delta PQR = \frac{1}{2}\times 10\sqrt{3} \times 10 \times \sin 30^\circ$

$$= \frac{1}{2} \times 10 \times 10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3}$$

$\angle QRP = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$

$$(c)\ r = \frac{\Delta}{S} = \frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}}$$

$$= 5\sqrt{3}.(2-\sqrt{3}) = 10\sqrt{3} - 15$$

$$(d)\ R = \frac{a}{2\sin A} = \frac{10}{2\sin 30^\circ} = 10$$

$\therefore$ Area $= \pi R^2 = 100\pi$

5. (b, c, d)

$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$$

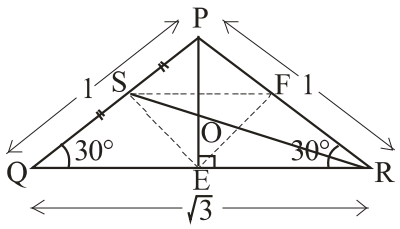
$$\Rightarrow P = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ and } Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since $p > q \Rightarrow P > Q$

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$ (not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$$



Now, area of $\Delta SEF = \frac{1}{4}$ area of ΔPQR

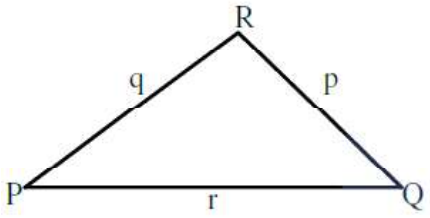
$\Rightarrow$ area of $\Delta SOE = \frac{1}{3}$ area of $\Delta SEF = \frac{1}{12}$ area of ΔPQR

$$= \frac{1}{12}.\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$$

$$RS = \frac{1}{2}\sqrt{2(3)+2(1)-1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3}PE = \frac{1}{3}.\frac{1}{2}\sqrt{2(1)^2+2(1)^2-3} = \frac{1}{6}$$

6. (a, b)



$$(a)\ \text{Cos } P = \frac{q^2+r^2-p^2}{2qr} = \frac{q^2+r^2}{2qr} - \frac{p^2}{2qr} \geq 1 - \frac{p^2}{2qr}$$

(as $p^2 + q^2 \geq 2qr$ (AM $\geq$ GM), so (a) is correct }

(b) $(p+q)\cos R \geq (q-r)\cos P + (p-r)\cos Q$

$\Rightarrow (p\cos R + r\cos P) + (q\cos R + r\cos Q)$

$\geq q\cos P + p\cos Q$

$\Rightarrow q + p \geq r$

So (b) is correct

$$(c)\ \frac{q+r}{p} = \frac{\sin Q + \sin R}{\sin P} \geq \frac{2\sqrt{\sin Q \times \sin R}}{\sin P}$$

so (c) is incorrect

(d) $\cos Q > \frac{p}{r} \Rightarrow \sin R \cos Q > \sin P$

$\Rightarrow \sin P + \sin(R-Q) > 2\sin P$

$\Rightarrow \sin(R-Q) > \sin P$

need not necessarily hold true if $R < Q$

Hence (a), (b)

3 Trigonometric Equation

Topicwise Questions

1. (*c*) $2\tan^2\theta = \sec^2\theta \Rightarrow 2\tan^2\theta = \tan^2\theta + 1$

$\Rightarrow \tan^2\theta = 1 = \tan^2\left(\frac{\pi}{4}\right) \Rightarrow \theta = n\pi \pm \frac{\pi}{4}$.

2. (*b*) $\sqrt{3}\tan 2\theta + \sqrt{3}\tan 3\theta + \tan 2\theta \tan 3\theta = 1$

$\Rightarrow \frac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta} = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan 5\theta = \tan\frac{\pi}{6}$

$\Rightarrow 5\theta = n\pi + \frac{\pi}{6} \Rightarrow \theta = \left(n + \frac{1}{6}\right)\frac{\pi}{5}$

3. (*a*) $\tan 2\theta = \cot\theta \Rightarrow \tan 2\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow 2\theta = n\pi + \frac{\pi}{2} - \theta \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{6}$

4. (*b*) $4 + 2\sin^2 x = 5$

$\Rightarrow \sin^2 x = \frac{1}{2} = \sin^2\frac{\pi}{4} \Rightarrow x = n\pi \pm \frac{\pi}{4}$

5. (*a*) We have $\frac{\pi}{4}\cot\theta = \frac{\pi}{2} - \frac{\pi}{4}\tan\theta \Rightarrow \tan\theta + \cot\theta = 2$

$\Rightarrow \sin 2\theta = 1 = \sin\frac{\pi}{2} \Rightarrow \theta = n\pi + \frac{\pi}{4}$

6. (*d*) $\cos 2\theta = \cos\left(\frac{\pi}{2} - \alpha\right) \Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

7. (*a*) $\tan 5\theta = \tan\left(\frac{\pi}{2} - 2\theta\right) \Rightarrow 5\theta = n\pi + \frac{\pi}{2} - 2\theta$

$\Rightarrow 7\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{7} + \frac{\pi}{14}$

8. (*c*) $2\sqrt{3}\cos^2\theta = \sin\theta \Rightarrow 2\sqrt{3}\sin^2\theta + \sin\theta - 2\sqrt{3} = 0$

$\Rightarrow \sin\theta = \frac{-1 \pm 7}{4\sqrt{3}} \Rightarrow \sin\theta = \frac{-8}{4\sqrt{3}}$, (Impossible)

and $\sin\theta = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{3}$

9. (*b*) $(2\cos x - 1)(3 + 2\cos x) = 0$

Then, $\cos x = \frac{1}{2}$ as $\cos x \neq \frac{-3}{2}$

$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}; \begin{cases} \text{for } n = 0,\ x = \frac{\pi}{3}, \frac{5\pi}{3} \\ \text{for } n = 1,\ x = \frac{5\pi}{3} \end{cases}$

10. (*b*) $\tan(\cot x) = \cot(\tan x)$

$\Rightarrow \tan(\cot x) = \tan\left(\frac{\pi}{2} - \tan x\right)$

$\Rightarrow \cot x = n\pi + \frac{\pi}{2} - \tan x \Rightarrow \cot x + \tan x = n\pi + \frac{\pi}{2}$

$\Rightarrow \frac{2}{\sin 2x} = n\pi + \frac{\pi}{2} \Rightarrow \sin 2x = \frac{2}{n\pi + \frac{\pi}{2}} = \frac{4}{(2n+1)\pi}$

11. (*b*) $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$; $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore$ The general value is $2n\pi + \frac{5\pi}{4}$ or $(2n+1)\pi + \frac{\pi}{4}$

12. (*b*) $\tan\theta = \frac{-1}{\sqrt{3}} = \tan\left(\pi - \frac{\pi}{6}\right)$, $\sin\theta = \frac{1}{2} = \sin\left(\pi - \frac{\pi}{6}\right)$

and $\cos\theta = \frac{-\sqrt{3}}{2} = \cos\left(\pi - \frac{\pi}{6}\right)$

Hence principal value is $\theta = \frac{5\pi}{6}$

13. (*d*) $4\sin\theta.\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$

$\Rightarrow 2\cos\theta(2\sin\theta - 1) - \sqrt{3}(2\sin\theta - 1) = 0$

$\Rightarrow (2\sin\theta - 1)(2\cos\theta - \sqrt{3}) = 0$

$\Rightarrow \sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

14. (*b*) $2\sin\theta + \tan\theta = 0 \Rightarrow \sin\theta = 0$ or $2 + \frac{1}{\cos\theta} = 0$

$\Rightarrow \theta = n\pi$ or $2 = -\frac{1}{\cos\theta} \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = 2m\pi \pm \frac{2\pi}{3}$.

15. (*c*) Given, $\tan\theta = -1$ & $\cos\theta = \frac{1}{\sqrt{2}}$

principal solution

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ & $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$

common principal solutoin is $\frac{7\pi}{4}$

then general solution is $\theta = 2n\pi + \frac{7\pi}{4}, n \in I$

16. (*a*) $2\cos^2(\pi + x) + 3\sin(\pi + x) = 0$

$\Rightarrow 2\cos^2 x - 3\sin x = 0 \Rightarrow 2 - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow 2\sin^2 x + 3\sin x - 2 = 0 \Rightarrow \sin x = -2, \frac{1}{2}$

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

17. (*a*) $\cos 2\theta + 3\cos\theta = 0 \Rightarrow 2\cos^2\theta + 3\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{3 \pm \sqrt{9+8}}{4} = \frac{-3 \pm \sqrt{17}}{4}$

As $-1 \le \cos\theta \le 1$ *a* $\therefore \cos\theta = \frac{-3 + \sqrt{17}}{4}$ only

$\Rightarrow \theta = 2n\pi \pm \alpha$ where $\cos\alpha = \frac{\sqrt{17} - 3}{4}$

18. (*b*) Given, $2\cos 2x = 3.2\cos^2 x - 4$

$\Rightarrow 2\cos 2x = 3(\cos 2x + 1) - 4$

$\Rightarrow \cos 2x = 1 \Rightarrow 2x = 2n\pi ; n \in I$

$\Rightarrow x = n\pi ; n \in I$

19. (*c*) $\sin 2\pi x = \sqrt{2}\cos\pi x$

$2\sin\pi x\cos\pi x = \sqrt{2}\cos\pi x$

$\cos\pi x(2\sin\pi x - \sqrt{2}) = 0$

$\cos\pi x = 0$

or $\sin\pi x = \frac{1}{\sqrt{2}}$

[As $x \in (0, 1), px \in (0, \pi)$]

$\pi x = \frac{\pi}{2}$ or $\pi x = \frac{\pi}{4}, \frac{3\pi}{4}$

$x = \frac{1}{2}$ $x = \frac{1}{4}, \frac{3}{4}$

$x = \frac{1}{2}, \frac{1}{4}, \frac{3}{9}$

Greatest root = $\frac{3}{4}$

20. (*b*) $\sec\theta + \tan\theta = \sqrt{3}$ ...(*i*)

$\sec\theta - \tan\theta = \frac{1}{\sqrt{3}}$ ...(*ii*)

(i) + (ii),

$2\sec\theta = \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

$\sec\theta = \frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \cos\frac{\pi}{6} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{6}$ $n \in z$

21. (*b*) $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0$

$\frac{\sin^2\theta}{\cos\theta} + \frac{\sqrt{3}\sin\theta}{\cos\theta} = 0$

$\Rightarrow \frac{\sin\theta}{\cos\theta}(\sin\theta + \sqrt{3}) = 0$

$\Rightarrow \sin\theta = 0$ or $\sin\theta = -\sqrt{3}$ (Not possible)

$\sin\theta = 0$

$\theta = n\pi, n \in z$

22. (*d*) $\sin\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$

$\tan\theta = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right) \Rightarrow \theta = \left(\pi + \frac{\pi}{6}\right)$

Hence general value of θ is $2n\pi + \frac{7\pi}{6}$

23. (*c*) $\sin A = \sin B$

$A = n\pi + (-1)^n B, n \in Z$

$A = B, \pi - B, 2\pi + B, 3\pi - B, 4\pi + B \ldots$ ...(*i*)

Also, $\cos A = \cos B$

$A = 2n\pi \pm B, n \in Z$

$A = B, 2\pi + B, 4\pi + B, 6\pi + B \ldots$ ...(*ii*)

From (*i*) and (*ii*), common solution is

$A = 2n\pi + B, n \in Z$

24. (*c*) $\sec^2\theta + \tan^2\theta = \frac{5}{3}$, also $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \tan^2\theta = \frac{1}{3} = \tan^2\left(\frac{\pi}{6}\right) \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$.

25. (*c*) $\cot\theta+\tan\theta=2\,\text{cosec}\theta \Rightarrow \dfrac{2}{\sin\theta}=\dfrac{1}{\sin\theta\cos\theta}$

$\Rightarrow \cos\theta=\dfrac{1}{2}$ or $\sin\theta=0 \Rightarrow \theta=2n\pi\pm\dfrac{\pi}{3} \Rightarrow \theta=n\pi$

26. (*a*) $\tan^2\theta-\tan\theta-\sqrt{3}\tan\theta+\sqrt{3}=0$

$\Rightarrow \tan\theta(\tan\theta-1)-\sqrt{3}(\tan\theta-1)=0$

$\Rightarrow (\tan\theta-\sqrt{3})(\tan\theta-1)=0 \Rightarrow \theta=n\pi+\dfrac{\pi}{3},\ n\pi+\dfrac{\pi}{4}$

27. (*a*) $4-4\cos^2\theta+2(\sqrt{3}+1)\cos\theta=4+\sqrt{3}$

$\Rightarrow 4\cos^2\theta-2(\sqrt{3}+1)\cos\theta+\sqrt{3}=0$

$\Rightarrow \cos\theta=\dfrac{2(\sqrt{3}+1)\pm\sqrt{4(\sqrt{3}+1)^2-16\sqrt{3}}}{8}$

$\Rightarrow \cos\theta=\dfrac{\sqrt{3}}{2}$ or $1/2 \Rightarrow \theta=2n\pi\pm\dfrac{\pi}{6}$ or $2n\pi\pm\pi/3$

28. (*b*) $3\sin^2 x+10\cos x-6=0$

$3(1-\cos^2 x)+10\cos x-6=0$

On solving, $(\cos x-3)(3\cos x-1)=0$

Either $\cos x=3$, (which is not possible) or $\cos x=\dfrac{1}{3}$

29. (*d*) We have, $\cos^2\theta+\sin\theta+1=0$

$\Rightarrow 1-\sin^2\theta+\sin\theta+1=0$

$\Rightarrow \sin^2\theta-\sin\theta-2=0$

$\Rightarrow (\sin\theta+1)(\sin\theta-2)=0$

$\sin\theta=2$, which is not possible and $\sin\theta=-1$.

Therefore, solution of given equation lies in the interval $\left(\dfrac{5\pi}{4},\dfrac{7\pi}{4}\right)$.

30. (*a*) $2-2\cos^2\theta+\sqrt{3}\cos\theta+1=0$

$\Rightarrow 2\cos^2\theta-\sqrt{3}\cos\theta-3=0$

$\Rightarrow \cos\theta=\dfrac{\sqrt{3}\pm\sqrt{3+24}}{4}=\dfrac{\sqrt{3}(1\pm3)}{4}=\sqrt{3}\left(-\dfrac{1}{2}\right)$

$\Rightarrow \theta=\dfrac{5\pi}{6}$

31. (*c*) $12\cot^2\theta-31\,\text{cosec}\theta+32=0$

$12(\text{cosec}^2\theta-1)-31\text{cosec}\,\theta+32=0$

$12\text{cosec}^2\theta-31\text{cosec}\,\theta+20=0$

$12\,\text{cosec}^2\theta-16\,\text{cosec}\,\theta-15\text{cosec}\theta+20=0$

$(4\,\text{cosec}\theta-5)(3\,\text{cosec}\theta-4)=0$

$\text{cosec}\theta=\dfrac{5}{4},\dfrac{4}{3}$; $\therefore\ \sin\theta=\dfrac{4}{5},\dfrac{3}{4}$

32. (*b*) $5-5\sin^2\theta+7\sin^2\theta=6 \Rightarrow 2\sin^2\theta=1$

$\Rightarrow \sin^2\theta=\dfrac{1}{2}=\sin^2\left(\dfrac{\pi}{4}\right) \Rightarrow \theta=n\pi\pm\dfrac{\pi}{4}$

33. (*c*) $\sin4\theta=\cos\theta-\cos7\theta \Rightarrow \sin4\theta=2\sin(4\theta)\sin(3\theta)$

$\Rightarrow \sin4\theta=0 \Rightarrow 4\theta=n\pi$ or $\sin3\theta=\dfrac{1}{2}=\sin\left(\dfrac{\pi}{6}\right)$

$\Rightarrow 3\theta=n\pi+(-1)^n\dfrac{\pi}{6} \Rightarrow \theta=\dfrac{n\pi}{4},\dfrac{n\pi}{3}+(-1)^n\dfrac{\pi}{18}$

34. (*d*) $\cos\theta+\cos2\theta+\cos3\theta=0$

$\Rightarrow (\cos\theta+\cos3\theta)+\cos2\theta=0$

$\Rightarrow 2\cos2\theta\cos\theta+\cos2\theta=0$

$\Rightarrow \cos2\theta(2\cos\theta+1)=0$

$\Rightarrow \cos2\theta=0=\cos\dfrac{\pi}{2} \Rightarrow \theta=\dfrac{\pi}{4} \Rightarrow \theta=2m\pi\pm\dfrac{\pi}{4}$

or $\cos\theta=\dfrac{-1}{2}=\cos\dfrac{2\pi}{3} \Rightarrow \theta=2m\pi\pm\dfrac{2\pi}{3}$.

35. (*d*) $\dfrac{\sqrt{3}}{2}\cos\theta+\dfrac{1}{2}\sin\theta=\dfrac{\sqrt{2}}{2}$

{dividing by $\sqrt{(\sqrt{3})^2+1^2}=2$}

$\Rightarrow \sin\left(\theta+\dfrac{\pi}{3}\right)=\dfrac{1}{\sqrt{2}}=\sin\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \theta=n\pi+(-1)^n\dfrac{\pi}{4}-\dfrac{\pi}{3}$

36. (*d*) Given equation is $\sqrt{3}\sin x+\cos x=4$

which is of the form $a\sin x+b\cos x=c$ with $a=\sqrt{3},b=1,c=4$.

Here $a^2+b^2=3+1=4<c^2$, therefore the given equation has no solution.

37. (*d*) $3\cos x+4\sin x=6$

$\Rightarrow \dfrac{3}{5}\cos x+\dfrac{4}{5}\sin x=\dfrac{6}{5} \Rightarrow \cos(x-\theta)=\dfrac{6}{5}$

[where $\theta=\cos^{-1}(3/5)$]

So, that equation has no solution.

38. (*c*) $\sin x+\sin5x=\sin2x+\sin4x$

$\Rightarrow 2\sin3x\cos2x=2\sin3x\cos x$

$\Rightarrow \sin3x=0$ or $\cos2x=\cos x$

$\Rightarrow 3x=n\pi$ or $2x=2n\pi\pm x$

$\Rightarrow x=\dfrac{n\pi}{3},\ 2n\pi,\ \dfrac{2n\pi}{3}$

$\Rightarrow x=\dfrac{n\pi}{3}$ (It includes all three possible)

39. (*a*) $f(x) = \cos x - x + \frac{1}{2}$, $f(0) = \frac{3}{2} > 0$

$$f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} + \frac{1}{2} = \frac{1-\pi}{2} < 0 \left(\because \pi = \frac{22}{7} \text{ nearly}\right)$$

$\therefore$ One root lies in the interval $\left[0, \frac{\pi}{2}\right]$.

40. (*c*) $\sec x \cos 5x = -1 \Rightarrow \cos 5x = -\cos x$

$$\Rightarrow 5x = 2n\pi \pm (\pi - x) \Rightarrow x = \frac{(2n+1)\pi}{6} \text{ or } \frac{(2n-1)\pi}{4}$$

Hence, $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{9\pi}{6}, \frac{11\pi}{6}$

41. (*b*) $1 + \sin^2 x\theta = \cos\theta$ …(*i*)

$0 \le \sin^2 x\theta \le 1$

$1 \le 1 + \sin^2 x\theta \le 2$

$1 \le \cos\theta \le 2$ (From (*i*)) …(*ii*)

But $\cos\theta \ne 1$; $-1 \le \cos\theta \le 1$ …(*iii*)

From (*ii*) and (*iii*)

$\cos\theta = 1 \Rightarrow \theta = 2m\pi, m \in z$

$1 + \sin^2 x\theta = 1$

$\sin^2 x\theta = 0$

$x\theta = n\pi, n \in z$

$$x = \frac{n\pi}{\theta}$$

$x = \frac{n\pi}{2m\pi} = \frac{n}{2m}$; x is rational number.

42. (*a*) $3\sin x + 4\cos x = x^2 + 16$…(*i*)

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le \sqrt{3^2+4^2}$

$-5 \le 3\sin x + 4\cos x \le 5$

Also, $x^2 \ge 0$

$x^2 + 16 \ge 16$

In equation (*i*),

LHS $\in [-5, 5]$

RHS $\in [16, \infty]$

So, no solution.

No. of solution = 0.

43. (*d*) $\sin^4 x = 1 + \tan^8 x$ …(*i*)

$0 \le \sin^4 x \le 1$

Also, $\tan^8 x \ge 0$

$1 + \tan^8 x \ge 1$

LHS $\in [0, 1]$; RHS $= [1, \infty)$

Only possibility

$\sin^4 x = 1 = 1 + \tan^8 x$

$\sin^4 x = 1$ and $1 = 1 + \tan^8 x$

$\sin^2 x = 1$ and $\tan^8 x = 0$

$\sin^2 x = \sin^2 \frac{\pi}{2}$

$\tan x = 0; x = n\pi, n \in \text{I}$

$$x = n\pi \pm \frac{\pi}{2}, n \in I$$

There will be no common solution.

Learning Plus

1. (*d*) Given

$\sin 3\theta = 4 \sin\theta \,.\, \sin 2\theta \sin 4\theta$, where $0 \le \theta \le \pi$

$\Rightarrow 3\sin\theta - 4\sin^3\theta - 4\sin\theta \sin 2\theta \sin 4\theta = 0$

$\Rightarrow \sin\theta [3 - 4\sin^2\theta - 4\sin 2\theta \sin 4\theta]$

$\Rightarrow \sin\theta [3 - 2 + 2\cos 2\theta - 2\cos 2\theta + 2\cos 6\theta]$

$\Rightarrow \sin\theta [1 + 2\cos 6\theta] = 0$

$\Rightarrow \sin\theta = 0$ or $\cos 6\theta = -\frac{1}{2}$

$\theta = 0, \pi$ or $0 \le \theta \le \pi$

$0, \pi \in [0, \pi]$ or $0 \le 6\theta \le 6\pi$ (3 rounds)

Number of solutions in 1 round $= 2$

total number of solutions in $\theta \in [0, \pi]$

$= 2 + 6 = 8$

2. (*a*) Given $\cot 3\theta - \cot\theta = 0$

$\Rightarrow \cot 3\theta = \cot\theta \Rightarrow 3\theta = n\pi + \theta$; $n \in I$

$\Rightarrow 2\theta = n\pi$; $n \in I$

$$\Rightarrow \theta = \frac{n\pi}{2} \because n \in I \begin{cases} 3\theta \ne n\pi \text{ or } \theta \ne m\pi \\ \theta \ne \frac{n\pi}{3} \end{cases}$$

$\Rightarrow \theta = (2n-1) \,.\, \frac{\pi}{2}$; $n \in \text{I}$

{$\cot\theta = 0$ & $\cot\theta \ne \infty$, so n = odd int}

θ shout be odd integral multiple of $\frac{\pi}{2}$

$\Rightarrow \quad \theta = (2n-1) \cdot \frac{\pi}{2}$; $n \in \text{I}$

3. (*b*) Given $4\cos^3 x - 4\cos^2 x - \cos(\pi + x) - 1 = 0$

$\Rightarrow 4\cos^3 x - 4\cos^2 x + \cos x - 1 = 0$

A.M. of roots $x \in [0, 315]$

$\Rightarrow 4\cos^2 x(\cos x - 1) + (\cos x - 1) = 0$

$\Rightarrow (\cos x - 1)(4\cos^2 x + 1) = 0$

$\Rightarrow \cos x = 1$ or $\cos^2 x = -\frac{1}{4}$ (not possible)

$x = 2n\pi, n \in I$

$0 \le 2n \le 100 x \in [0, 315]$

$0 \le n \le 50$ or $[0, 100\pi]$

$x = 0, 2\pi, 4\pi, 6\pi, \ldots, 100\pi$

$$\text{A.M.} = \frac{2\pi[0+1+2+3+\ldots+50]}{51} = \frac{2\pi}{51}\ \frac{50\times 51}{2} = 50\pi$$

4. (*b*) $\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$

$\Rightarrow 2(4\cos^3\theta - 3\cos\theta) = 2(2\cos^2\theta - 1) - 1$

$\Rightarrow 8\cos^3\theta - 4\cos^2\theta - 5\cos\theta\ 3 = 0$

$\Rightarrow (4\cos^2\theta - 3)(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

But when $\cos\theta = \pm\frac{\sqrt{3}}{2}$

then $2\cos 2\theta - 1 = 0$

$\therefore$ rejecting this value,

$\cos\theta = \frac{1}{2}$ is valid only

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

5. (*b*) $\sin\theta + 7\cos\theta = 5$

$\Rightarrow \frac{2t}{1+t^2} + \frac{7(1-t^2)}{1+t^2} = 5$ where $t = \tan\left(\frac{\theta}{2}\right)$

$\Rightarrow 2t + 7 - 7t^2 = 5 + 5t^2$

$\Rightarrow \tan\frac{\theta}{2}$ is root of $12t^2 - 2t - 2 = 0$ or $6t^2 - t - 1 = 0$.

6. (*c*) Given, $\frac{\sin 3\theta}{2\cos 2\theta + 1} = \frac{1}{2}$

$\Rightarrow \quad \frac{3\sin\theta - 4\sin^3\theta}{2 - 4\sin^2\theta + 1} = \frac{1}{2}$

$\Rightarrow \quad 2\sin\theta[3 - 4\sin^2\theta] = (3 - 4\sin^2\theta)$

$\Rightarrow \quad (2\sin\theta - 1)(3 - 4\sin^2\theta) = 0$

$\therefore \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

or $\quad \sin^2\theta = \frac{3}{4} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, m \in I$

which does not satisfy the given equation

7. (*c*) $\tan x + \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x + \frac{2\pi}{3}\right) = 3$

$\Rightarrow \quad 3\tan 3x = 3$

$\Rightarrow \quad \tan 3x = 1$

$\Rightarrow \quad x = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

8. (*b*) Given, $\sin(2A + B) = \frac{1}{2}$

$\Rightarrow \sin(2A + B) = \sin\frac{\pi}{6}$ or $\sin\frac{5\pi}{6}$

But $2A + B = \frac{\pi}{6}$ is not possible

$\therefore \quad 2A + B = \frac{5\pi}{6}$ ....(*i*)

Given that

$2B = A + C$ ...(*ii*)

We know

$A + B + C = \pi$ .... (*iii*)

From (*ii*) & (*iii*) $3B = \pi \Rightarrow B = \frac{\pi}{3}$

From (*i*) $2A = \frac{5\pi}{6} - \frac{\pi}{3} \Rightarrow A = \frac{\pi}{4}$

From (*iii*) $C = \pi - (A + B)$

$= \pi - \left(\frac{\pi}{3} + \frac{\pi}{4}\right) \Rightarrow C = \frac{5\pi}{12}$

9. (*a*) Given $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \times \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1$

$\Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4}$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}\ n \in I$

But at this value of x, $\tan 2x = \infty$

which is not acceptable

$\therefore \quad x \in \phi$

10. (*d*) Given

$\sin\theta + 2\sin 2\theta + 3\sin\theta + 4\sin 4\theta = 10$ in $(0, \pi)$

Using boundary of *SM*

It's only possible if $\Rightarrow 1 + 2 + 3 + 4 = 10$

(Using boundary values)

$\Rightarrow \sin\theta = 1$ & $\sin 2\theta = 1$ & $\sin 3\theta = 1$ & $\sin 4\theta = 1$

$\theta = 2m\pi + \frac{\pi}{2}, 2\theta = 2n\pi + \frac{\pi}{2}, 3\theta = 2k\pi + \frac{\pi}{2}, 4\theta = 2p\pi + \frac{\pi}{2}$

$m \in I\ n \in I\ k \in I\ p \in I$

$\theta = \frac{\pi}{2}\ \theta = \frac{\pi}{4}\ \theta = \frac{\pi}{6}, \frac{5\pi}{6}\ \theta = \frac{\pi}{8}, \frac{5\pi}{8}$

$\therefore x \in \left\{\frac{\pi}{2}\right\} \cap \left\{\frac{\pi}{4}\right\} \cap \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\} \cap \left\{\frac{\pi}{8}, \frac{5\pi}{8}\right\}$

$x \in \{\ \}$ number of solutoins is zero.

11. (*d*) $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x

$$a_1 + a_2 \cos 2x + a_3 \left(\frac{1-\cos 2x}{2}\right) = 0$$

$$\left(a_1 + \frac{a_3}{2}\right) + \cos^2 x \left(a_2 - \frac{a_3}{2}\right) = 0$$

As it is valid for all x,

$$a_1 + \frac{a_3}{2} = 0,\ a_2 - \frac{a_3}{2} = 0,\ \text{Let } a_3 = k$$

$$a_1 = -\frac{a_3}{2} = \frac{-k}{2};\ a_2 = \frac{a_3}{2} = \frac{k}{2}$$

$$(a_1, a_2, a_3) \equiv \left(\frac{-k}{2}, \frac{k}{2}, k\right) \text{ where } k \in R$$

$\Rightarrow$ Infinite number of triplets.

12. (*b*) Given $4\,\text{cosec}^2(\pi(a+x)) + a^2 - 4a = 0$

$\Rightarrow$ $4\,\text{cosec}^2(\pi(a+x)) = 4a - a^2$

$\Rightarrow$ $\sin^2(\pi(a+x)) = \dfrac{4}{4a-a^2}$

$0 \le \sin^2 x \le 1$

$0 \le \dfrac{4}{4a-a^2} \le 1$

$0 \le 4 \le 4a - a^2$

$\therefore$ $4a - a^2 \ge 4$

$\Rightarrow$ $(a-2)^2 \ge 0$

$a \ge 2$

if $a = 1$

$4 \le (4-1)1$

$4 \not\le 3$

$\Rightarrow$ $a < 2$ does not satisfy the inequation

$\therefore$ $a = 2$

13. (*d*) Given $|\cos x| = \cos x - 2\sin x$

Case-I : if $\cos x \ge 0$, then $\cos x = \cos x - 2\sin x$

$\Rightarrow \sin x = 0$

$\Rightarrow x = n\pi$ but $\cos x \ge 0$

so only even integral multiple of π is acceptable

$\therefore\ x = 2n\pi\ ;\ n \in I$

Case-II :

if $\cos x < 0$, then $-\cos x = \cos x - 2\sin x - 2\cos x = -2\sin x$

$\Rightarrow$ $\tan x = 1$

$\Rightarrow$ $x = n\pi + \dfrac{\pi}{4}$ (x can be in I & III quadrant)

But $\cos x < 0$ So x will be in III quadrant

s.t. n should be odd integer

$$x = (2m+1)\pi + \frac{\pi}{4},\ m \in I$$

14. (*d*) $n|\sin x| = m|\sin x|$

$(n-m)|\sin x| = 0$

As $n > m$

$\Rightarrow$ $n - m \ne 0$

$\Rightarrow$ $|\sin x| = 0$

$\sin S = 0$

$x = n\pi,\ n \in Z$

$\ln [0, 2\pi],\ x = 0, \pi, 2\pi$

No. of solution = 3.

15. (*a*) $\sin x + \cos x = \sqrt{y + \dfrac{1}{y}}$

$-\sqrt{1^2+1^2} \le \sin x + \cos x \le \sqrt{1^2+1^2}$

$-\sqrt{2} \le \text{LHS} \le \sqrt{2}$ ...(*i*)

Also, $y + \dfrac{1}{y} \ge 2$

$\sqrt{y + \dfrac{1}{y}} \ge \sqrt{2}$

RHS $\ge \sqrt{2}$ ...(*ii*)

From (*i*) and (*ii*)

$\sin x + \cos x = \sqrt{2}$

$\Rightarrow$ $\left(\dfrac{1}{\sqrt{2}}\sin x + \cos x \dfrac{1}{\sqrt{2}} = 1\ \sin\left(\dfrac{\pi}{4} + x\right) = 1\right)$

$\dfrac{\pi}{4} + x = n\pi + (-1)^n \dfrac{\pi}{2}$; As $x \in (0, \pi)$

$\Rightarrow$ $x = \dfrac{\pi}{4}$ Also, $y + \dfrac{1}{y} = 2$ $\Rightarrow$ $y = 1$

16. (*a*) $\sin 2x + \cos 4x = 2$

$\sin 2x + 1 - 2\sin^2 2x = 2$ ($\cos 2\theta = 1 - 2\sin^2\theta$)

$2\sin^2 2x - \sin 2x + 1 = 0$

$D = (1)^2 - 4.2.1 = 1 - 8 = -7$

As discriminant of the quadratic equation is negative.

Hences No solution.

Number of solution = 0

17. (*b*) $\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1$

Only possibility is

$\cos(\pi\sqrt{x-4}) = 1$ *and* $\cos(\pi\sqrt{x}) = 1$

Also, $x - 4 \ge 0 \Rightarrow x \ge 4$

and from $\sqrt{x}$, $x \ge 0 \Rightarrow x \ge 4$

$\cos(\pi\sqrt{x-4}) = 1$ *and* $\cos(\pi\sqrt{x}) = 1$

$\pi\sqrt{x-4} = 0 \Rightarrow x = 4$

Also, $\cos(\pi\sqrt{x}) = \cos(2\pi) = 1$

Thus, $x = 4$ is only solution.

18. (*a*) $\theta \in [0, 2\pi]$

$r \sin\theta = \sqrt{3} \quad \Rightarrow \quad \sin\theta = \dfrac{\sqrt{3}}{r}$

$r + 4\sin\theta = 2(\sqrt{3}+1) \Rightarrow r + \dfrac{4\sqrt{3}}{r} = 2(\sqrt{3}+1)$

$r^2 - 2r - 2\sqrt{3}r + 4\sqrt{3} = 0$

$\Rightarrow \quad r = 2 \quad \text{or} \quad 2\sqrt{3}$

$\sin\theta = \dfrac{\sqrt{3}}{r} \quad \Rightarrow \quad \sin\theta = \dfrac{\sqrt{3}}{2} \text{ or } \dfrac{\sqrt{3}}{2\sqrt{3}}$

$= \dfrac{\sqrt{3}}{2} \text{ or } \dfrac{1}{2}$

As $\theta \in [0, 2\pi]$

No. of solution = 4.

19. (*b*) $4\sin^4 x + \cos^4 x = 1$

$4\sin^4 x + (1 - \sin^2 x)^2 = 1$

$4\sin^4 x + 1 + \sin^4 x - 2\sin^4 x = 1$

$5\sin^4 x - 2\sin^2 x = 0$

$\sin^2 x (5\sin^2 x - 2) = 0$

$\sin^2 x = 0 \quad \text{or} \quad \sin^2 x = \dfrac{2}{5}$

$x = n\pi \quad \text{or} \quad x = n\pi \pm \theta,$

where $\sin^2\theta = \dfrac{2}{5} \quad \text{and} \quad \cos^2\theta = \dfrac{3}{5}$

20. (*a*) $\sin x \cdot \cos x (\cos x - \sin x)^2 (\sin x + \cos x) = \lambda$

$(2\sin x \cos x)(\cos x - \sin x)(\cos^2 x - \sin^2 x) = 2\lambda$

$\sin 2x \cdot (\cos x - \sin x) \cdot \cos 2x = 2\lambda$

$(2\sin 2x \cos 2x)(\cos x - \sin x) = 4\lambda$

$(\sin 4x)\left(\dfrac{1}{\sqrt{2}}\cos - \dfrac{1}{\sqrt{2}}\sin x\right) = \dfrac{4\lambda}{\sqrt{2}}$

$(\sin 4x)\left(\sin\left(\dfrac{\pi}{4} - x\right)\right) = 2\sqrt{2}\lambda$

As $\lambda > \dfrac{1}{2\sqrt{2}} \Rightarrow 2\sqrt{2}\lambda > 1 \Rightarrow \text{RHS} > 1$

But LHS ≤ 1

Hences no solution.

Number of solution = 0

21. (*d*) As $x \in \left(\dfrac{-\pi}{2}, \pi\right)$

$\cos 2x > |\sin x|$

Case 1: $\sin x \ge 0$ i.e. $x \in [0, \pi)$

$\cos 2x > \sin x$

$1 - 2\sin^2 x > \sin x$

$2\sin^2 x + \sin x - 1 < 0$

$(\sin x + 1)(2\sin x - 1) < 0$

$-1 < \sin x < \dfrac{1}{2} \qquad \ldots(i)$

Case 2: $\sin x < 0$

$\cos 2x > -\sin x$

$1 - 2\sin^2 x > -\sin x$

$2\sin^2 x - \sin x - 1 < 0$

$(\sin x - 1)(2\sin x + 1) < 0$

$-\dfrac{1}{2} < \sin x < 1 \qquad \ldots(ii)$

From (*i*) and (*ii*), $-\dfrac{1}{2} < \sin x < \dfrac{1}{2}$

$\Rightarrow x \in \left(-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right) \cup \left(\dfrac{5\pi}{6}, \pi\right)$

22. (*c*) $x \in [0, 2\pi]$

$\sin x\sqrt{8\cos^2 x} = 1$

$\sin^2 x \cdot 8\cos^2 x = 1$

$2(2\sin x \cos x)^2 = 1$

$2(\sin 2x)^2 = 1$

$2\sin^2 2x = 1$

$-\cos 4x = 1$

$\cos 4x = 0$

$4x = (4n+1) \cdot \dfrac{\pi}{2}$

$x = (4n+1)\dfrac{\pi}{8}, n \in Z$

$x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8} \ldots$

Common diff. $= d = \dfrac{5\pi}{8} - \dfrac{\pi}{8} = \dfrac{4\pi}{8} = \dfrac{\pi}{2}$

23. (*c*) $a\cos\theta + b\sin\theta = c$

$a\cos\theta = c - b\sin\theta$

$a^2\cos^2\theta = (c - b\sin\theta)^2$

$a^2(1-\sin^2\theta) = c^2 + b^2\sin^2\theta - 2bc\sin\theta$

$(a^2+b^2)\sin^2\theta - 2bc\sin\theta + c^2 - a^2 = 0$

$$\sin\alpha\sin\beta = \frac{c^2-a^2}{a^2+b^2} \qquad \ldots(i)$$

Also, $b\sin\theta = c - a\cos\theta$

$b^2\sin^2\theta = c^2 + a^2\cos^2\theta - 2ac\cos\theta$

$b^2(1-\cos^2\theta) = c^2 + a^2\cos^2\theta - 2ac\cos\theta$

$\cos^2\theta(a^2+b^2) - 2ac\cos\theta + c^2 - b^2 = 0$

$$\cos\alpha\cos\beta = \frac{c^2-a^2}{a^2+b^2}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \frac{c^2-b^2-c^2+a^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

Advanced Level Multiconcept Questions

1. (*b*, *d*)

$\sin(x-y) = \frac{1}{2}$ and $\cos(x+y) = \frac{1}{2}$

$\Rightarrow x-y = \frac{\pi}{6}, \frac{5\pi}{6}$ and $x+y = \frac{\pi}{3}, \frac{5\pi}{3}$

Adding $2x = \frac{\pi}{2}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6} \Rightarrow x = \frac{\pi}{4}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$

when $x = \frac{\pi}{4}, y = \frac{\pi}{12}$

when $x = \frac{7\pi}{12}$ no value of y is possible.

when $x = \frac{11\pi}{12}, y = \frac{3\pi}{4}$.

2. (*a*, *b*, *c*, *d*)

$\cos 15x = \sin 5x$

$\cos 15x = \cos\left(\frac{\pi}{2} - 5x\right)$ or $\cos\left(\frac{3\pi}{2} + 5x\right)$

$15x = 2n\pi \pm \left(\frac{\pi}{2} - 5x\right)$ or $15x = 2n\pi \pm \left(\frac{3\pi}{2} + 5x\right)$

$\Rightarrow x = \frac{n\pi}{10} + \frac{\pi}{40}, n \in I, x = \frac{n\pi}{5} + \frac{3\pi}{20}, n \in I$

and $x = \frac{n\pi}{5} - \frac{\pi}{20}, n \in I$ and $x = \frac{n\pi}{10} - \frac{3\pi}{40}, n \in I$

3. (*a*, *b*)

Let $E = \sin x - \cos^2 x - 1$

$\Rightarrow E = \sin x - 1 + \sin^2 x - 1 = \sin^2 x + \sin x - 2$

$= \left(\sin x + \frac{1}{2}\right)^2 - \frac{9}{4}$ assumes least value

when $\sin x = -\frac{1}{2} \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$.

4. (*a*, *b*, *c*, *d*)

$2\sin\frac{x}{2}\cdot\cos^2 x + \sin^2 x = 2\sin\frac{x}{2}\cdot\sin^2 x + \cos^2 x$

$\Rightarrow \quad 2\sin\frac{x}{2}(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x$

$\Rightarrow \quad \sin 2x = 1$ or -1 and $\cos x = 1 - 2\sin^2\frac{x}{2}$

$= 1 - 2\left(\frac{1}{4}\right) = \frac{1}{2}$

$\therefore \quad \cos 2x = 2\cos^2 x - 1 = 2\times\frac{1}{4} - 1 = -\frac{1}{2}$

5. (*a*, *c*)

$5\sin^2 x + \sqrt{3}\sin x\cos x + 6\cos^2 x = 5$

Case-I $\cos x = 0 \Rightarrow 5 + 0 + 0 = 5$

$\therefore \quad x = n\pi + \frac{\pi}{2}$

Case -II $\cos x \neq 0$

$\therefore \quad 5\tan^2 x + \sqrt{3}\tan x + 6 = 5(1 + \tan^2 x)$

$\Rightarrow \quad \tan x = -\frac{1}{\sqrt{3}}$

6. (*c*, *d*)

$\sin^2 x + 2\sin x\cos x - 3\cos^2 x = 0$

Case-I : $\cos x \neq 0 \therefore \tan^2 x + 2\tan x - 3 = 0$

$\Rightarrow \tan x = 3, 1 \Rightarrow x = n\pi + \tan^{-1}(-3), n\pi + \frac{\pi}{4}$

Case-II : $\cos x = 0 \Rightarrow 1 + 0 - 0 = 0$ not true.

7. (*b*, *c*) $\sin^2 x - \cos 2x = 2 - \sin 2x$

$\Rightarrow \sin^2 x - (1 - 2\sin^2 x) = 2 - 2\sin x\cos x$

$\Rightarrow 3\sin^2 x + 2\sin x\cos x = 3$

Case-I : $\cos x \neq 0 \therefore\ 3\tan^2 x + 2\tan x = 3(1+\tan^2 x)$

$\Rightarrow \tan x = \frac{3}{2}$

Case-II : $\cos x = 0 \therefore\ 3(1) + 2(\pm 1)(0) = 3$ which is true

$\therefore\ x = (2n+1)\frac{\pi}{2}$

8. (b, c, d)

$2\sin 2x = \sin x + \sin 3x$

$\Rightarrow 2\sin 2x = 2\sin x 2x \cos x$

$\Rightarrow \sin 2x = 0$ or $\cos x = 1$

$\Rightarrow 2x = n\pi$ or $x = 2m\pi$

$\Rightarrow x = \frac{n\pi}{2}, 2m\pi$

options $(A), (B), (C), (D)$ are all a part of $x = \frac{n\pi}{2}$.

9. (b, d)

$\sin x + \sin 2x + \sin 3x = 0$

$\Rightarrow 2\sin 2x \cos x + \sin 2x = 0$

$\Rightarrow \sin 2x = 0$ or $\cos x = \frac{-1}{2}$.

10. (a, b, c)

$\cos 4x \cos 8x - \cos 5x \cos 9x = 0$

$\Rightarrow\ 2\cos 4x \cos 8x = 2\cos 5x \cos 9x$

$\Rightarrow\ \cos 12x + \cos 4x = \cos 14x + \cos 4x$

$\Rightarrow\ 14x = 2n\pi \pm (12x)$

$\Rightarrow\ 2x = 2n\pi$ or $26x = 2n\pi$

$\Rightarrow\ x = n\pi$ or $\frac{n\pi}{13}$ $\therefore\ \sin x = 0$ or $\sin 13x = 0$

11. (a, c)

$\cos x \cos 6x = -1 \Rightarrow 2\cos x \cdot \cos 6x = -2$

$\Rightarrow\ \cos x\ 7x + \cos 5x = -2$

It is possible only when

$\cos 7x = -1$ & $\cos 5x = -1$

$7x = (2n+1)\pi$ & $5x = (2m+1)\pi$; $n, m \in I$

$x = (2n+1)\frac{\pi}{7}$ & $x = (2m+1)\frac{\pi}{5}$

$x = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}$ & $x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

common solution in one round is π

In general $x = (2n \pm 1)\pi, n \in I$

12. (a, c)

$\cos x . \cos 6x = -1$

$\Rightarrow$ Either $\cos x = 1$ and $\cos 6x - 1$

or $\cos x = -1$ and $\cos 6x = 1$

$\Rightarrow x = 2n\pi$ and $\cos 6x = -1$

or $x = (2n+1)\pi$ and $\cos 6x = 1$

If $x = 2n\pi$ then $\cos 6x$ cannot be -1

However if $x = (2n+1)\pi$ then $\cos 6x = 1$

$\therefore\ x = (2n+1)\pi$

$x = (2n-1)\pi$ is also as above.

13. (d) $4\sin^3 x + 2\sin^2 x - 2\sin x - 1$

$= (2\sin x + 1)(2\sin^2 x - 1) = 0$

$\therefore\ \sin x = -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}$

$\therefore$ there are 6 solutions.

14. (c) $3 = \cos 4x + \frac{10\tan x}{1+\tan^2 x} = \cos 4x + 5\sin 2x$

i.e $\quad 3 = 1 - 2\sin^2 2x + 5\sin 2x$

i.e $\quad \sin 2x = \frac{1}{2}$

$\therefore\ \quad 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

Thus there are two solutions.

15. (b) (i) when $\tan x \geq 0$, then the equation becomes $\tan x = \tan x + \frac{1}{\cos x}$ i.e $\frac{1}{\cos x} = 0$ (not possible)

(ii) when $\tan x < 0$, then the equation becomes $-\tan x = \tan x + \frac{1}{\cos x}$ i.e $\sin x = -\frac{1}{2}$

$\therefore\ x = \frac{11\pi}{6}$ is the only solution.

16. (a) Given $\sin x + \cos x = 1 + \sin x \cos x$

$t = 1 + \frac{t^2 - 1}{2}$

$(t-1)^2 = 0$

$t = 1$

$\sin x + \cos x = 1$

so $x = 2n\pi$ and $2n\pi + \frac{\pi}{2}$

17. (b) $(\cos x - \sin x)(2\sin x + 1) + 2\cos x = 0$

$\sin x(2\cos x - 1) + 3\cos x - 2(1 - \cos^2 x) = 0$

$\sin x(2\cos x - 1) + 2\cos x(\cos x + 2)$

$-1(\cos x + 2) = 0$

$\Rightarrow\ (\sin x + \cos x + 2)(2\cos x - 1) = 0$

$\therefore\ \cos x = \frac{1}{2}$

$\Rightarrow\ x = 2n\pi \pm \frac{\pi}{3}$

18. (c) Given $\sin^4 x + \cos^4 x = \sin x \cos x$

$\Rightarrow\ 1 - \frac{(t^2-1)^2}{2} = t^2 - 1$

$\Rightarrow\ t^4 - t^2 - 2 = 0$

Let $t^2 = u$

$u^2 - u - 2 = 0$

$u = 2$ or $u \neq -1$

$\therefore \quad t = \pm\sqrt{2}$

$\Rightarrow \quad \sin x + \cos x = \pm\sqrt{2}$

$\Rightarrow \quad \sin(x - \pi/4) = 1$

$\Rightarrow \quad x = n\pi + \dfrac{\pi}{4}$

19. $(a) \to (q), (b) \to (r), (c) \to (p), (d) \to (r)$

(a) $\sin^2\theta = \sec^2\theta$

$\Rightarrow \sin^2\theta\cos^2\theta = 1$

$\Rightarrow \sin^2\theta(1 - \sin^2\theta) = 1$

$\Rightarrow \sin^4\theta - \sin^2\theta + 1 = 0$

Let $\sin\theta = t$

$\therefore \; t^2 - t + 1 = 0 \; ; D = 1 - 4 = -3$

Since $D < 0$, there is no real solution

(b) $\sin\theta + \cos\theta = 1$

If $\sin\theta = 0$, then $\cos\theta = 1$

$\therefore$ In $[0, 2\pi]$, Number of solution = 2

and if $\sin\theta = 1$ then $\cos\theta = 0$

$\therefore$ In $[0, 2\pi]$, Number of solution = 1

Total number of solution = 3

(c) $\tan\theta + \sec\theta = 2\cos\theta$

$\Rightarrow \quad \dfrac{1}{\cos\theta}[\sin\theta + 1] = 2\cos\theta$

$\Rightarrow \quad 2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm 3}{4}$

$\therefore$ 2 real solutions

(d) $3\sin^2\theta - 4\sin\theta + 1 = 0$

If $\sin\theta = 1$, In $[0, 2\pi]$ one solution

& $\sin\theta = \dfrac{1}{3}$, in $[0, 2\pi]$, there are 2 solutions

Total number of solution = 3

20. $(a) \to (p), (b) \to (q), (c) \to (r), (d) \to (s)$

(a) $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow \quad (2\sin x - \cos x)(1 + \cos x) = 1 - \cos^2 x$

$\Rightarrow \quad (2\sin x - \cos x)(1 + \cos x)$

$= (1 - \cos x)(1 + \cos x)$

$\Rightarrow \quad (1 + \cos x)(2\sin x - 1) = 0$

$\therefore \quad \sin x = \dfrac{1}{2}$ or $\cos x = -1$

(b) $1 + \sin 2x = (\cos x + \sin x)$

$\Rightarrow \quad (\sin x + \cos x)^2 = (\cos x + \sin x)$

$\Rightarrow \quad (\cos x + \sin x)(\cos x + \sin x - 1) = 0$

So, $\cos x + \sin x = 1$ & $\cos x + \sin x = 0$

$\therefore \tan x = -1$

(c) $4x^2 + x^6 + \sin^2 5x = 0$

$\sin^2 5x = -(4x^4 + x^6) \to$ always negative

$\downarrow$

always positive

So solution is $x = 0$ since LHS is always positive and RHS is always negative.

(d) Given that, $\tan x = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad x = n\pi \pm \dfrac{\pi}{6}$

So, solution is $\dfrac{19\pi}{6}$

INTEGER TYPE

21. [0006]

Apply $\sin C + \sin D$ and then solve

$\theta = 0, \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}$ & π

22. [0000]

Given $\sin\theta = 3\sin(\theta + 2\alpha)$

$\Rightarrow \sin(\theta + \alpha - \alpha) = 3\sin(\theta + \alpha + \alpha)$

$\Rightarrow \sin(\theta + \alpha)\cos\alpha - \cos(\theta + \alpha)\sin\alpha$

$= 3\sin(\theta + \alpha)\cos\alpha + 3\cos(\theta + \alpha)\sin\alpha$

$\Rightarrow -2\sin(\theta + \alpha)\cos\alpha = 4\cos(\theta + \alpha)\sin\alpha$

$\Rightarrow \dfrac{-\sin(\theta + \alpha)}{\cos(\theta + \alpha)} = \dfrac{2\sin\alpha}{\cos\alpha}$

$\Rightarrow \tan(\theta + \alpha) + 2\tan\alpha = 0$

23. [0000]

$1 + \sin x\left(\dfrac{1 - \cos x}{2}\right) = 0$

$\Rightarrow 4 + 2\sin x = \sin 2x \Rightarrow$ *No. sol*

24. [0004]

$1 - \sin^2 x + \dfrac{\sqrt{3}+1}{2}\sin x - \dfrac{\sqrt{3}}{4} - 1 = 0$

$\sin^2 x - \dfrac{\sqrt{3}+1}{2}\sin x + \dfrac{\sqrt{3}}{4} = 0;$

$4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} = 0$

On solving we get

$\sin x = 1/2 \; ; \dfrac{\sqrt{3}}{2} = (\pi/6 \, , 5\pi/6 \, ; \pi/3 \, , 2\pi/3 \,]$

25. [0000]

$\sin x \cos x\,[\sin^2 x + \sin x \cos x + \cos^2 x] = 1$

$\Rightarrow \sin x \cos x + (\sin x \cos x)^2 = 1$

$\sin^2 2x + 2 \sin 2x - 4 = 0$

$\Rightarrow \sin 2x = \dfrac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$,

which is not possible.

26. [0004]

apply $\sin C + \sin D$ & $\cos C + \cos D$ then solve

27. [0008]

given equation can be written as

$3 \sin\theta - 4\sin^3\theta = 4\sin\theta \sin 2\theta \sin 4\theta$

hence either $\sin\theta = 0 \Rightarrow \theta = n\pi$

or $3 - 4\sin^2\theta = 4 \sin 2\theta \sin 4\theta$

$3 - 2(1 - \cos 2\theta) = 2(\cos 2\theta - \cos 6\theta)$

or $1 = -2 \cos 6\theta$

$\cos 6\theta = -\dfrac{1}{2} = \cos \dfrac{2\pi}{3}$

$6\theta = 2n\pi \pm \dfrac{2\pi}{3}$

if $0 \le \theta \le \pi$ then total solution are

$0, \dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \dfrac{8\pi}{9}, \pi$

is 8 real solutions.

28. [15]

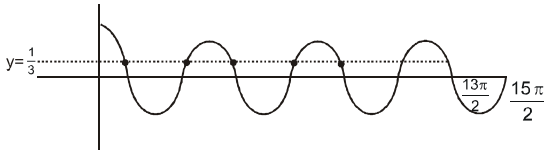
$2 \tan^2 x - 5 \sec x - 1 = 0$

$\Rightarrow 2(\sec^2 x - 1) - 5 \sec x - 1 = 0$

$\Rightarrow 2 \sec^2 x - 5 \sec x - 3 = 0$

$\Rightarrow \sec x = \dfrac{6}{2}, \dfrac{-1}{2} = 3, \dfrac{-1}{2}$

$\Rightarrow \sec x = 3 \left(\sec x \ne \dfrac{-1}{2}\right)$



$\Rightarrow \cos x = \dfrac{1}{3}$

$\Rightarrow$ 7 solutions in $\left[0, \dfrac{15\pi}{2}\right]$

$\therefore\ n = 15.$

29. [5]

$\sin 7x + \sin 4x + \sin x = 0$

$\Rightarrow 2 \sin 4x \cos 3x + \sin 4x = 0$

$\Rightarrow \sin 4x = 0$ or $\cos 3x = -\dfrac{1}{2}$

$\Rightarrow 4x = n\pi$ or $3x = 2n\pi \pm \dfrac{2\pi}{3}$

$\Rightarrow x = \dfrac{n\pi}{4}, \dfrac{2n\pi}{3} \pm \dfrac{2\pi}{9} = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}$.

30. [2]

Given $\tan x + \sec x = 2 \cos x$

or $\dfrac{\sin x + 1}{\cos x} = 2 \cos x$ or $2 \cos^2 x = 1 + \sin x$

or $2 - 2 \sin^2 x = 1 + \sin x$

or $2 \sin^2 x + \sin x - 1 = 0$

or $2 \sin^2 x + 2 \sin x - \sin x - 1 = 0$

or $(\sin x + 1)(2 \sin x - 1) = 0$

$\Rightarrow$ either $\sin x = -1$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow x = \dfrac{3\pi}{2} \Rightarrow x = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ $\quad [\because x \in (0, 2\pi)]$

but when $x = \dfrac{3\pi}{2}$, $\tan x$ and $\sec x$ are not defined.

$\therefore$ solutions of given equation are $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$ only

$\Rightarrow$ number of solutions $= 2$

JEE-MAIN PREVIOUS YEAR'S

1. (*c*) $\cos x+\cos 2x+\cos 3x+\cos 4x=0$

$(\cos x+\cos 4x)+(\cos 2x+\cos 3x)=0$

$2\cos\frac{5x}{2}\cos\frac{3x}{2}+2\cos\frac{5x}{2}\cos\frac{x}{2}=0$

$\Rightarrow 2\cos\frac{5x}{2}\left(\cos\frac{3x}{2}+\cos\frac{x}{2}\right)=0$

$\Rightarrow 4\cos\frac{5x}{2}\cos x\cos\frac{x}{2}=0$

$\Rightarrow \cos\frac{5x}{2}=0,\cos x=0,\cos\frac{x}{2}=0$

$\Rightarrow \frac{5x}{2}=(2n+1)\frac{\pi}{2},x=(2n+1)\frac{\pi}{2},\frac{x}{2}=(2n+1)\frac{\pi}{2}$

$\Rightarrow x=\frac{\pi}{5},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{9\pi}{5},\frac{\pi}{2},\frac{3\pi}{2}$

So, total 7 values.

2. (*a*) $5\left[\frac{1-t}{t}-t\right]=2(2t-1)+9$ {Let $\cos^2 x=t$}

$\Rightarrow 5(1-t-t^2)=t(4t+7)$

$\Rightarrow 9t^2+12t-5=0$

$\Rightarrow 9t^2+15t-3t-5=0$

$\Rightarrow (3t-1)(3t+5)=0$

$\Rightarrow t=\frac{1}{3}$ as $t\neq -\frac{5}{3}$

$\cos 2x=2\left(\frac{1}{3}\right)-1=-\frac{1}{3}$

$\cos 4x=2\left(-\frac{1}{3}\right)^2-1=-\frac{7}{9}$

3. (*a*) $8\cos x\left(\cos^2\frac{\pi}{6}-\sin^2 x-\frac{1}{2}\right)=1$

$\Rightarrow 8\cos x\left(\frac{1}{4}-\left(1-\cos^2 x\right)\right)=1$

$\Rightarrow 8\cos x\left(\cos^2 x-\frac{3}{4}\right)=1$

$\Rightarrow 2\cos 3x=1\Rightarrow \cos 3x=\frac{1}{2}$

$\therefore 3x+2n\pi\pm\frac{\pi}{3},n\in I$

$\Rightarrow x=\frac{2n\pi}{3}\pm\frac{\pi}{9}$

$x\in[0,\pi]: x=\frac{\pi}{9},\frac{2\pi}{3}+\frac{\pi}{9},\frac{2\pi}{3}-\frac{\pi}{9}$ only

sum $=\frac{13\pi}{9}$

4. (*a*) $\sin x-\sin 2x+\sin 3x=0$

$\Rightarrow (\sin x+\sin 3x)-\sin 2x=0$

$\Rightarrow 2\sin x.\cos x-\sin 2x=0$

$\Rightarrow \sin 2x\,(2\cos x-1)=0$

$\Rightarrow \sin 2x=0$ or $\cos x=\frac{1}{2}\Rightarrow x=0,\frac{\pi}{3}$

5. (*c*) $\sin^2 2\theta+\cos^4 2\theta=\frac{3}{4}$

Let $\cos^2 2\theta=t$

$\Rightarrow 1-\cos^2 2\theta+\cos^4 2\theta=\frac{3}{4}$

$\Rightarrow t=\frac{1}{2}\Rightarrow \cos^2 2\theta=\frac{1}{2}$

$\Rightarrow 2\cos^2 2\theta-1=0\Rightarrow \cos 4\theta=0$

$\Rightarrow 4\theta=(2n+1)\frac{\pi}{2}$

$\Rightarrow \theta=(2n+1)\frac{\pi}{8}\Rightarrow \theta=\frac{\pi}{8},\frac{3\pi}{8}\in\left[0,\frac{\pi}{2}\right]$

Sum of values of θ is $\frac{\pi}{2}$

6. (*a*) Using formula $\frac{\sin 2^n A}{2^n\sin A}=\cos A\cos 2\cos 2^2 A\ldots\cos 2^{n-1}A$

7. (*c*) $2(1-\sin^2\theta)+3\sin\theta=0$

$\Rightarrow 2\sin^2\theta-3\sin\theta-2=0$

$\Rightarrow (2\sin\theta+1)(\sin\theta-2)=0$

$\Rightarrow \sin\theta=-\frac{1}{2};\sin\theta=2$(reject)

root : $\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6},-\frac{\pi}{6},-\pi+\frac{\pi}{6}$

$\Rightarrow$ sum of values $=2\pi$

8. (*a*) $1+\sin^4 x=\cos^2 3x$

$\sin x=0$ and $\cos 3x=1$

$0,2\pi,-2\pi,-\pi,\pi$

9. (*a*) $\cos 2x + \alpha \sin x = 2\alpha - 7$

$\Rightarrow 2\sin^2 x - \alpha \sin x + 2\alpha - 8 = 0$

$\sin^2 x - \frac{\alpha}{2}\sin x + \alpha - 4 = 0$

$\Rightarrow \sin x = 2$ (rejected) or $\sin x = \frac{\alpha - 4}{2} \Rightarrow \left|\frac{\alpha - 4}{2}\right| \le 1$

$\Rightarrow \alpha \in [2, 6]$

10. (*a*) $\sin^4\theta + \cos^4\theta = -\lambda$

$\Rightarrow 1 - 2\sin^2\theta \cos^2\theta = -\lambda$

$\Rightarrow \lambda = \frac{(\sin 2\theta)^2}{2} - 1$

as $\sin^2 2\theta \in [0,1]$

$\Rightarrow \lambda \in \left[-1, -\frac{1}{2}\right]$

11. (*a*) $\sqrt{3}\cos^2 x = (\sqrt{3} - 1)\cos x + 1$

$\sqrt{3}\cos^2 x - (\sqrt{3} - 1)\cos x - 1 = 0$

$\sqrt{3}\cos^2 x - \sqrt{3}\cos x + \cos x - 1 = 0$

$\sqrt{3}\cos x(\cos x - 1) + 1(\cos x - 1) = 0$

$(\sqrt{3}\cos x + 1)(\cos x - 1) = 0$

$\cos x = \frac{-1}{\sqrt{3}}$ *or* 1

But $\cos x \ne \frac{-1}{\sqrt{3}}$ as $x \in \left[0, \frac{\pi}{2}\right]$, so $\cos x$ is $+ve$.

$\cos x = 1$

$x = 0$ in $\left[0, \frac{\pi}{2}\right]$

Only 1 solution.

12. (*b*) $\sin 2\theta + \tan 2\theta > 0$

$\sin 2\theta + \frac{\sin 2\theta}{\cos 2\theta} > 0$

$\sin 2\theta\left(\frac{1 + \cos 2\theta}{\cos 2\theta}\right) > 0$

$\tan 2\theta \cdot \underbrace{2\cos^2\theta}_{+ve} > 0$

$\tan 2\theta > 0$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(2\pi, \frac{5\pi}{2}\right) \cup \left(3\pi, \frac{7\pi}{2}\right)$

$\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

13. (*b*) $\log_{10}\sin x + \log_{10}\cos x = -1$

$\log_{10}(\sin x \cos x) = -1$

$\sin x \cos x = 10^{-1} = \frac{1}{10}$ …(*i*)

Also, $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$

$\log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$

$\sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{n}{10}$

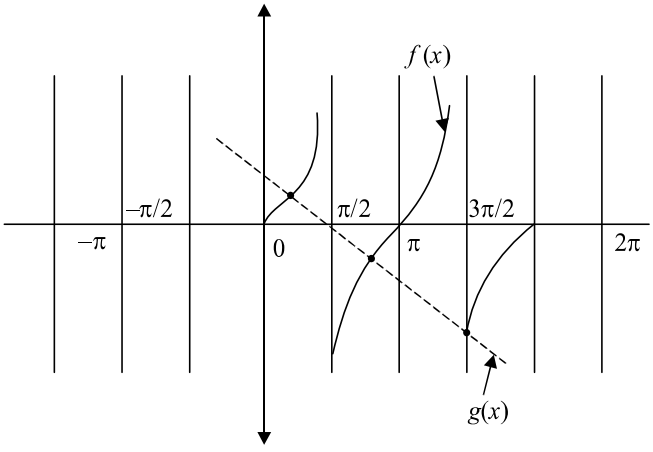
$1 + 2 \cdot \frac{1}{10} = \frac{n}{10} \Rightarrow \boxed{n = 12}$

14. (*a*) $x \in [0, 2\pi]$

$x + 2\tan x = \frac{\pi}{2}$

$\tan x = \frac{\pi}{4} - \frac{x}{2}$

Let us draw graph of $f(x) = \tan x$ and $g(x) = -\frac{x}{2} + \frac{\pi}{4}$



As both graph intersect at 3 points. Hence, number of solution = 3.

15. (*a*) $|\cot x| = \cot x + \frac{1}{\sin x}$

Case 1: $\cot > x\ 0$ i.e. $x \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

$\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow \frac{1}{\sin x} = 0 \Rightarrow$ Not possible

Case 2: $\cot x < 0$ i.e. $x \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

$-\cot x = \cot x + \frac{1}{\sin x} \Rightarrow -2\cot x = \frac{1}{\sin x}$

$-\frac{2\cos x}{\sin x} = \frac{1}{\sin x}$ $(\sin x \ne 0)$

$\Rightarrow \quad \cos x = \dfrac{-1}{2} \quad \Rightarrow \quad x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

But x can't be $\dfrac{4\pi}{3}$

$x = \dfrac{2\pi}{3}$

No. of solution : 1.

16. (*c*) $(\sin x + \sin 4x) + (\sin 2x + \sin 3x) = 0$

$2\sin\dfrac{5x}{2}\cos\dfrac{3x}{2} + 2\sin\dfrac{5x}{2}\cos\dfrac{x}{2} = 0$

$\Rightarrow 2\sin\dfrac{5x}{2}\left(\cos\dfrac{3x}{2} + \cos\dfrac{x}{2}\right) = 0$

$2\sin\dfrac{5x}{2}\left(2\cos x + \cos\dfrac{x}{2}\right) = 0$

$\Rightarrow \sin\dfrac{5x}{2} = 0 \Rightarrow \dfrac{5x}{2} = 0, 2\pi, 3\pi, 4\pi, 5\pi$

$\Rightarrow x = 0, \dfrac{2\pi}{5}, \dfrac{4\pi}{5}, \dfrac{6\pi}{5}, \dfrac{8\pi}{5}, 2\pi$

or $\cos\dfrac{x}{2} = 0 \Rightarrow \dfrac{x}{2} = \dfrac{\pi}{2} \Rightarrow x = \pi$

$\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Sum of values of x

$= \left(0 + \dfrac{2\pi}{5} + \dfrac{4\pi}{5} + \dfrac{4\pi}{5} + \dfrac{4\pi}{5} + 2\pi\right) + p + \left(\dfrac{\pi}{2} + \dfrac{3\pi}{2}\right)$

$= 9\pi$

17. (*b*) $-\dfrac{11\pi}{30}, \dfrac{\cos x}{1+\sin x} = |\tan 2x|$

$\Rightarrow \quad \dfrac{\cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2} + 2\sin\dfrac{x}{2} + \cos\dfrac{x}{2}} = |\tan 2x|$

$\Rightarrow \dfrac{\cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}}{\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right)} = |\tan 2x$

$\Rightarrow \quad \dfrac{\cos\dfrac{x}{2} - \sin\dfrac{x}{2}}{\cos\dfrac{x}{2} + \sin\dfrac{x}{2}} = |\tan 2x|$

$\Rightarrow \quad \dfrac{1 - \tan\dfrac{x}{2}}{1 + \tan\dfrac{x}{2}} = |\tan 2x|$

$\Rightarrow \dfrac{\tan\dfrac{\pi}{4} - \tan\dfrac{x}{2}}{1 + \tan\dfrac{\pi}{4}\tan\dfrac{x}{2}} = |\tan 2x|$

$\Rightarrow \quad \tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) = |\tan 2x|$

$\Rightarrow \quad \tan^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) = \tan^2 2x$

$2x = n\pi \pm \left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) \Rightarrow x = \dfrac{-3\pi}{10}, \dfrac{-\pi}{6}$

Sum of all values $= -\dfrac{3\pi}{10} + \left(\dfrac{-\pi}{6}\right) + \dfrac{\pi}{10} = -\dfrac{11\pi}{6}$

18. [56] $\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$

$(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta$

$- 2\sin^2\theta\cos^2\theta - \sin\theta\cos\theta = 0$

$(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta - \sin\theta \,.\, \cos\theta = 0$

$1 - 2\sin^2\theta\cos^2\theta - \sin\theta \,.\, \cos\theta = 0$

$2 - (2\sin\theta\cos\theta)^2 - (2\sin q\cos q) = 0$

$(\sin 2\theta)^2 + (\sin 2\theta) - 2 = 0$

$(\sin 2\theta - 1)(\sin 2\theta + 2) = 0$

$\sin 2\theta = 1$ or $\sin 2\theta = 2 =$ Not possible

$\sin 2\theta = 1 \quad \Rightarrow \quad 2\theta = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}, \dfrac{13\pi}{2}$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$

$S = \dfrac{\pi}{4} + \dfrac{5\pi}{4} + \dfrac{9\pi}{4} + \dfrac{13\pi}{4} = \dfrac{28\pi}{4} = 7\pi$

$\dfrac{8S}{\pi} = \dfrac{8\times 7\pi}{\pi} = 56$

19. (*b*) $(32)^{\tan^2 x} + (32)^{\sec^2 x} = 81$

$(32)^{\tan^2 x} + (32)^{1+\tan^2 x} = 81$

$\Rightarrow (32)^{\tan^2 x} + 32\cdot(32)^{\tan^2 x} = 81$

$(32)^{\tan^2 x}\cdot(1+32) = 81$

$\Rightarrow (32)^{\tan^2 x} = \dfrac{81}{33}$

$\Rightarrow$ Only 1 solution in $\left[0, \dfrac{\pi}{4}\right]$

20. (*a*) $2\cos x\left(4\sin\left(\dfrac{\pi}{4} + x\right)\sin\left(\dfrac{\pi}{4} - x\right) - 1\right) = 1$

$2\cos x\left(4\left(\sin^2\dfrac{\pi}{4} - \sin^2 x\right) - 1\right) = 1$

$2\cos x\left(4\left(\dfrac{1}{2} - \sin^2 x\right) - 1\right) = 1$

$2\cos x\,(2 - 4\sin^2 x - 1) = 1$

$2\cos x\,(1 - 4\sin^2 x) = 1$

$\Rightarrow \quad 2\cos x\,(1-4(1-\cos^2 x))=1$

$2\cos x\,(4\cos^2 x-3)=1$

$\Rightarrow \quad 2\,(4\cos^2 x-3\cos x)=1$

$\cos 3x=\frac{1}{2}$

$\Rightarrow \quad 3x=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3}$

$x=\frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9}$ (As $x\in[0,\pi]$, $3x\in[0,3\pi]$)

$n=3,\ S=\frac{\pi+5\pi+7\pi}{9}=\frac{13\pi}{9}$

$(n,S)\equiv\left(3,\frac{13\pi}{9}\right)$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*d*) $\sin x+2\sin 2x-\sin 3x=3.$

$\sin x\,(1+2\cos x-3+4\sin^2 x)=3.$

$(4\sin^2 x+2\cos x-2)=\frac{3}{\sin x}$

$2-4\cos^2 x+2\cos x=\frac{3}{\sin x}$

$\frac{9}{4}-\left(2\cos x-\frac{1}{2}\right)^2=\frac{3}{\sin x}.$

L.H.S. $\le\frac{9}{4}$ R.H.S. $\ge 3.$

No solution.

2. [8] $\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$

$\Rightarrow \frac{5}{4}\cos^2 2x+1-\frac{1}{2}\sin^2 2x+1-\frac{3}{4}\sin^2 2x=2$

$\Rightarrow \cos^2 2x=\sin^2 2x$

$\Rightarrow \tan^2 2x=1$

Now $2x\in[0,4\pi]$

$\Rightarrow x=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{9\pi}{8},\frac{11\pi}{8},\frac{13\pi}{8},\frac{15\pi}{8}$

so number of solution = 8

3. (*c*) $\sqrt{3}\sin x+\cos x=2\cos 2x$

$\Rightarrow \quad \cos 2x=\cos(x-\pi/3)$

$\Rightarrow x=2n\pi-\pi/3$ or $1/3(2n\pi+\pi/3)$

$\therefore x=-\pi/3,-5\pi/9,\pi/9,7\pi/9$

Hence, (*c*)

4. (*c*) $x=\sec\theta\pm\tan\theta$ $\qquad x=-\tan\theta\pm\sec\theta$

$\alpha_1=\sec\theta-\tan\theta$ $\qquad \beta_2=-\tan\theta-\sec\theta$

Hence, $\alpha_1+\beta_2=-2\tan\theta$

Hence, (*c*)

5. (*c*) $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}=\frac{1}{\sin\frac{\pi}{6}}\sum_{k=1}^{13}\frac{\sin\left[\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)-\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\right]}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$

$=2\sum_{k=1}^{13}\left[\cot\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)\right]$

$=\left[\left\{\cot\left(\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right\}+\left\{\cot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{2\pi}{6}\right)\right\}+.....+\left\{\cot\left(\frac{\pi}{4}+\frac{12\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right\}\right]$

$2\left[1-\cot\left(\frac{29\pi}{12}\right)\right]=2\left[1-\cot\left(\frac{5\pi}{12}\right)\right]=2\sqrt{3}-1$

6. [0.5]

$\sqrt{3}\cos x+\frac{2b}{a}\sin x=\frac{c}{a}$

Now, $\sqrt{3}\cos\alpha+\frac{2b}{a}\sin\alpha=\frac{c}{a}$ ...(1)

$\sqrt{3}\cos\beta+\frac{2b}{a}\sin\beta=\frac{c}{a}$ ...(2)

$\sqrt{3}[\cos\alpha-\cos\beta]+\frac{2b}{a}(\sin\alpha-\sin\beta)=0$

$$\sqrt{3}\left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]$$
$$+\frac{2b}{a}\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]=0$$

$$-\sqrt{3}+2\sqrt{3}.\frac{b}{a}=0 \quad \Rightarrow \quad \frac{b}{a}=\frac{1}{2}=0.5$$

7. (*c*)

8. (*b*)

Sol. $f(x)=\sin(\pi\cos x)$

$X: \{x: f(x)=0\}$

$f(x)=0 \Rightarrow \sin(\pi\cos x)\, x=n \Rightarrow \cos=1,-1.0 \Rightarrow x=\frac{n\pi}{2}$

$X=\left\{\frac{n\pi}{2}: n\in N\right\}=\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi.....\right\}$

g(x) = cos (2π sin x)

Z = {x : g(x) = 0}

$\cos(2\pi\sin x)=0 \Rightarrow 2\pi\sin x=(2n+1)\frac{\pi}{2} \Rightarrow \sin x=\frac{(2n+1)}{4}$

$\sin x=-\frac{1}{4},\frac{1}{4},\frac{-3}{4},\frac{3}{4}$

$Z=\left\{n\pi\pm\sin^{-1}\left(\frac{1}{4}\right),n\pi\pm\sin^{-1}\left(\frac{3}{4}\right),n\in I\right\}$

$Y=\{X: f'(x)=0\}$

$f(x)=\sin(\pi\cos x)=0$

$\Rightarrow \pi\cos x=(2n+1)\frac{\pi}{2} \Rightarrow \cos x=\frac{(2n+1)}{2}$

$\Rightarrow \cos x=-\frac{1}{2},\frac{1}{2}$

$Y=\left\{n\pi,n\pi\pm\frac{\pi}{3}\right\}=\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{5\pi}{3},2\pi.....\right\}$

$W=\{x: g'(x)=0\}$

$g(x)=\cos(2\pi\sin x)$

$\Rightarrow g'(x)=-\sin(2\pi\sin x).(2\pi\cos x)=0$

$\cos x=0 \Rightarrow x=(2n+1)\frac{\pi}{2}$

$\sin(2\pi\sin x)=0 \Rightarrow 2p\sin x=np \Rightarrow \sin x=\frac{n}{2}$

$=-1,-\frac{1}{2},0,\frac{1}{2},1$

$W=\left\{\frac{n\pi}{2},n\pi\pm n\in 1\right\}$

$=\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\pi,\frac{7\pi}{6},\frac{3\pi}{2},.....\right\}$

Now check the options

9. [1.00]

Let $\pi x-\frac{\pi}{4}=\theta\in\left[\frac{-\pi}{4},\frac{7\pi}{4}\right]$

So, $\left(3-\sin\left(\frac{\pi}{2}+2\theta\right)\right)\sin\theta\geq\sin(\pi+3\theta)$

$\Rightarrow (3-\cos 2\theta)\sin\theta\geq-\sin 3\theta$

$\Rightarrow \sin\theta\,[3-4\sin^2\theta+3-\cos 2\theta]\geq 0$

$\Rightarrow \sin\theta\,(6-2(1-\cos 2\theta)-\cos 2\theta)\geq 0$

$\Rightarrow \sin\theta\,(4+\cos 2\theta)\geq 0$

$\Rightarrow \sin\theta\geq 0$

$\Rightarrow \theta\in[0,\pi] \Rightarrow 0\leq\pi x-\frac{\pi}{4}\leq\pi$

$\Rightarrow x\in\left[\frac{1}{4},\frac{5}{4}\right] \quad \Rightarrow \beta-\alpha=1$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

4 Solution of Triangle

Topicwise Questions

1. (*b*) A, B, C are in A. P. then angle $B = 60^o$,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\left\{\begin{array}{l}\text{since } A + B + C = 180^o \text{ and} \\ A + C = 2B \Rightarrow B = 60^o\end{array}\right\}$$

$$\Rightarrow \frac{1}{2} = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow a^2 + c^2 - b^2 = ac$$

$$\Rightarrow b^2 = a^2 + c^2 - ac$$

2. (*a*) $a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$$= k(\Sigma \sin A \sin(B-C) = k\{\Sigma \sin(B+C)\sin(B-C)\}$$

$$= k\left\{\Sigma \frac{1}{2}(\cos 2C - \cos 2B)\right\} = 0$$

3. (*a*) $(b^2 - c^2)\cot A = (b^2 - c^2)\frac{\cos A}{\sin A} = \frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2bc.ka}$

Hence L.H.S. $= \frac{1}{2k\ abc}$

$$[(b^4 - c^4) + (c^4 - a^4) + (a^4 - b^4) - \{a^2(b^2 - c^2)$$

$$+ b^2(c^2 - a^2) + c^2(a^2 - b^2)\}] = 0.$$

4. (*b*) Largest side is $\sqrt{p^2 + pq + q^2}$.

If largest angle is θ, then

$$\cos\theta = \frac{p^2 + q^2 - p^2 - pq - q^2}{2pq} = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

5. (*b*) $\sqrt{\frac{b+c}{4c}} = \sqrt{\frac{\sin 3C + \sin C}{4\sin C}}$

$$\Rightarrow \sqrt{\frac{2\sin 2C\cos C}{4\sin C}} = \cos C$$

$$\frac{b-c}{2c} = \frac{\sin 3C - \sin C}{2\sin C} = \frac{2\cos 2C \sin C}{2\sin C} = \cos 2C = \sin\frac{A}{2}$$

6. (*b*) $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \lambda$ (Let)

$$\therefore b + c = 11\lambda \quad \ldots.(i)$$

$$c + a = 12\lambda \quad \ldots.(ii)$$

$$\text{and } a + b = 13\lambda \quad \ldots.(iii)$$

From (i) + (ii) + (iii), $2(a + b + c) = 36\lambda$

$$\therefore a + b + c = 18\lambda \quad \ldots.(iv)$$

Now, (iv) – (i) gives, $a = 7\lambda$

(iv) – (ii) gives, $b = 6\lambda$

(iv) – (iii) gives, $c = 5\lambda$

7. (*d*) We have, $b = \sqrt{3}, c = 1, \angle A = 30^o$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow \frac{\sqrt{3}}{2} = \frac{(\sqrt{3})^2 + 1^2 - a^2}{2.\sqrt{3}.1}$$

$\therefore\ a = 1, b = \sqrt{3}, c = 1$

b is the largest side. Therefore, the largest angle B is given by $\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1+1-3}{2.1.1} = -\frac{1}{2}$

$\therefore\ B = 120°$

8. (*b*) $2ac\sin\frac{A - B + C}{2} = 2ac\sin\frac{\pi - 2B}{2} = 2ac\cos B$

$$2ac\frac{c^2 + a^2 - b^2}{2ca} = c^2 + a^2 - b^2$$

9. (*b*) $\cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2c}$

$$\Rightarrow b^2 + c^2 - a^2 - b^2 = 0 \Rightarrow c^2 = a^2$$

10. (*a*) $a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2 + 2a^2b^2 = 2a^2b^2$

$$\Rightarrow (a^2 + b^2 - c^2)^2 = (\sqrt{2}ab)^2 \Rightarrow a^2 + b^2 - c^2 = \pm\sqrt{2}ab$$

$$\Rightarrow \frac{a^2 + b^2 - c^2}{2ab} = \pm\frac{\sqrt{2}ab}{2ab} = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos C = \cos 45^o \text{ or } \cos 135^o \Rightarrow C = 45^o \text{ or } 135^o$$

11. (*a*) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$

$\Rightarrow \tan\left(\frac{90^\circ}{2}\right)=\frac{\sqrt{3}-1}{\sqrt{3}+1}\cot\frac{A}{2}$

$\Rightarrow \tan\left(\frac{A}{2}\right)=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{3+1-2\sqrt{3}}{2}=2-\sqrt{3}$

$\Rightarrow \frac{A}{2}=15^\circ \Rightarrow A=30^\circ$.

12. (*c*) $\sin^2 B-\sin^2 A=\sin^2 C-\sin^2 B$

$\therefore \sin(B+A)\sin(B-A)=\sin(C+B)\sin(C-B)$

or $\sin C(\sin B\cos A-\cos B\sin A)$

$=\sin A(\sin C\cos B-\cos C\sin B)$

Divide by $\sin A\sin B\sin C$

$\therefore \cot A-\cot B=\cot B-\cot C$. Hence the result.

13. (*a*) From the given relation

$\sin C=\frac{1-\cos A\cos B}{\sin A\sin B}\le 1$ .....(i)

$\Rightarrow 1\le \cos A\cos B+\sin A\sin B$

$\Rightarrow \cos(A-B)\ge 1$; $\because \cos\theta \ngtr 1$ .....(ii)

$\therefore A-B=0$ or $A=B$

Hence from (i), $\sin C=\frac{1-\cos^2 A}{\sin^2 A}=\frac{\sin^2 A}{\sin^2 A}=1$

$\therefore C=90^\circ \Rightarrow A+B=90^\circ$ or $A=B=45^\circ$ {by (ii)}

Hence, $a:b:c=\sin A:\sin B:\sin C=1:1:\sqrt{2}$.

14. (*d*) We have, $\frac{\tan\left(\frac{B}{2}\right)}{\cot\left(\frac{C-A}{2}\right)}=\frac{\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right)}{\cos\frac{B}{2}\cos\left(\frac{C-A}{2}\right)}$

$=\frac{\sin C-\sin A}{\sin C+\sin A}=\frac{kc-ka}{kc+ka}=\frac{c-a}{c+a}=\frac{a}{3a}=\frac{1}{3}$, {$\because c=2a$}

15. (*d*) $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$; $\frac{5}{\sin\left(\frac{\pi}{2}+B\right)}=\frac{4}{\sin B}$

$\frac{5}{\cos B}=\frac{4}{\sin B}$, $\therefore \tan B=\frac{4}{5}$

$\tan A=\tan\left(\frac{\pi}{2}+B\right)=-\cot B=\frac{-5}{4}$

$\tan C=\tan(\pi-(A+B))$

$=-\tan(A+B)$, $[A+B+C=\pi]$

$=-\frac{(\tan A+\tan B)}{1-\tan A.\tan B}=\frac{-\left(-\frac{5}{4}+\frac{4}{5}\right)}{1+1}=\frac{9}{40}$

$C=\tan^{-1}\left(\frac{\left(2.\frac{1}{9}\right)}{1-\left(\frac{1}{9}\right)^2}\right)$, $\therefore C=2\tan^{-1}\left(\frac{1}{9}\right)$

16. (*a*) We have, $2s=a+b+c, A^2=s(s-a)(s-b)(s-c)$

$\because$ A.M. $\ge$ G.M.

$\Rightarrow \frac{s-a+s-b+s-c}{3}\ge\sqrt[3]{(s-a)(s-b)(s-c)}$

$\Rightarrow \frac{3s-2s}{3}\ge\frac{(A^2)^{1/3}}{s^{1/3}}\Rightarrow\frac{s^3}{27}\ge\frac{A^2}{s}\Rightarrow A\le\frac{s^2}{3(\sqrt{3})}$

17. (*b*) $s-a=3\Rightarrow b+c-a=6$ ....(i)

$s-c=2\Rightarrow a+b-c=4$ ....(ii)

Adding these two equations, we get $b=5$

Since *B* is a right angle

$\therefore b^2=a^2+c^2\Rightarrow a^2+c^2=25$ ....(iii)

Solving, we get $a=3, c=4$

18. (*b*) We have $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow b^2-2bc\cos A+(c^2-a^2)=0$

It is given that b_1 and b_2 are roots of this equation.

Therefore $b_1+b_2=2c\cos A$ and $b_1b_2=c^2-a^2$

$\Rightarrow 3b_1=2c\cos A$, $2b_1^2=c^2-a^2$ ($\because b_2=2b_1$ (given)

$2\left(\frac{2c}{3}\cos A\right)^2=c^2-a^2\Rightarrow 8c^2(1-\sin^2 A)=9c^2-9a^2$

19. (*d*) $a^2\sin 2C+c^2\sin 2A$

$=a^2(2\sin C\cos C)+c^2(2\sin A\cos A)$

$=2a^2\left(\frac{2\Delta}{ab}\cos C\right)+2c^2\left(\frac{2\Delta}{bc}\cos A\right)$

($\because \Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A$, $\therefore \sin C=\frac{2\Delta}{ab}, \sin A=\frac{2\Delta}{bc}$)

$=4\Delta\left\{\frac{a\cos C+c\cos A}{b}\right\}=4\Delta\left(\frac{b}{b}\right)=4\Delta$

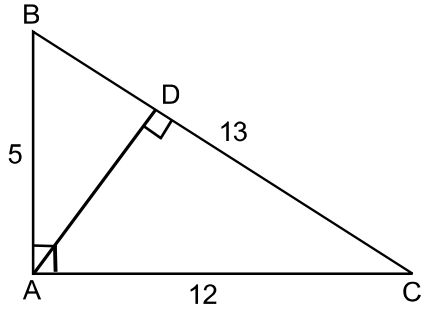
20. (*b*) A.M. $= \dfrac{a+b+c}{3}$

Altitudes are $\dfrac{2\Delta}{a}, \dfrac{2\Delta}{b}, \dfrac{2\Delta}{c}$

H.M. $= \dfrac{3}{\frac{a}{2\Delta}+\frac{b}{2\Delta}+\frac{c}{2\Delta}} = \dfrac{6\Delta}{(a+b+c)}$

(A.M.) $\times$ (H.M.) $= \dfrac{(a+b+c)}{3} \times \dfrac{3\times 2\Delta}{(a+b+c)} = 2\Delta$

21. (*b*) Given $c = 5, b = 12, a = 13$



$\cos A = \dfrac{12^2+5^2-13^2}{2(12)(5)} = 0$

i.e. $A = 90^o$

Altitude $AD = \dfrac{2\Delta}{a} = \dfrac{2\times 30}{13} = \dfrac{60}{13}$

$\{\Delta = \dfrac{1}{2}\times 5\times 12 = 30\}$

22. (*b*) $\cos\dfrac{\pi}{6} = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{\sqrt{3}}{2} \Rightarrow a = 1$

$2R = \dfrac{a}{\sin A} \Rightarrow R = 1$

23. (*c*) $\dfrac{1}{\sin^2\frac{A}{2}}, \dfrac{1}{\sin^2\frac{B}{2}}, \dfrac{1}{\sin^2\frac{C}{2}}$ are in A. P.

$\Rightarrow \dfrac{1}{\sin^2\frac{C}{2}} - \dfrac{1}{\sin^2\frac{B}{2}} = \dfrac{1}{\sin^2\frac{B}{2}} - \dfrac{1}{\sin^2\frac{A}{2}}$

$\Rightarrow \dfrac{ab}{(s-a)(s-b)} - \dfrac{ac}{(s-a)(s-c)} = \dfrac{ac}{(s-a)(s-c)} - \dfrac{bc}{(s-b)(s-c)}$

$\Rightarrow \left(\dfrac{a}{s-a}\right)\left(\dfrac{b(s-c)-c(s-b)}{(s-b)(s-c)}\right)$

$= \left(\dfrac{c}{s-c}\right)\left(\dfrac{a(s-b)-b(s-a)}{(s-a)(s-b)}\right)$

$\Rightarrow abs - abc - acs + abc = acs - abc - bcs + abc$

$\Rightarrow ab - ac = ac - bc \Rightarrow ab + bc = 2ac$

or $\dfrac{1}{c}+\dfrac{1}{a} = \dfrac{2}{b}$, *i.e.*, a, b, c are in H. P.

24. (*b*) $\dfrac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}} = \dfrac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} - \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}$

$= \dfrac{(s-b)\sqrt{s(s-c)} - (s-a)\sqrt{s(s-c)}}{(s-b)\sqrt{s(s-c)} + (s-a)\sqrt{s(s-c)}}$

$= \dfrac{\sqrt{s(s-c)}(s-b-s+a)}{\sqrt{s(s-c)}\ (s-b+s-a)} = \dfrac{a-b}{c}$

25. (*c*) $ab^2\cos A + ba^2\cos B + ac^2\cos A + ca^2\cos C$
$+bc^2\cos B + b^2c\cos C$
$= ab(b\cos A + a\cos B) + ac\ c(\cos A + a\cos C)$
$+bc(c\cos B + b\cos C)$
$= abc + abc + abc = 3abc$

26. (*c*) $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{a^2-(b^2-c^2)}{2ac}$

Now $AD = \dfrac{abc}{b^2-c^2} \therefore \cos B = \dfrac{a^2-\frac{abc}{AD}}{2ac}$

Also, $AD = b\sin 23^o \therefore \cos B = \dfrac{a-\frac{c}{\sin 23^o}}{2c}$

By sine formula, $\dfrac{a}{c} = \dfrac{\sin(B+23^o)}{\sin 23^o}$

$\therefore \cos B = \left(\dfrac{\sin(B+23^o)}{\sin 23^o} - \dfrac{1}{\sin 23^o}\right) \div 2$

$\Rightarrow \sin(23^o - B) = -1 = \sin(-90^o)$

$\Rightarrow 23^o - B = -90^o$ or $B = 113^o$

27. (*a*) $a = 3, b = 5, c = 4, s = \dfrac{a+b+c}{2} = \dfrac{12}{2} = 6$

$\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}} = \sqrt{\dfrac{2.3}{12}} = \sqrt{\dfrac{1}{2}}$

$\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}} = \sqrt{\dfrac{6.1}{12}} = \sqrt{\dfrac{1}{2}}$

$\therefore \sin\dfrac{B}{2} + \cos\dfrac{B}{2} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

28. (*c*) $\dfrac{b-c}{a} = \dfrac{\sin B - \sin C}{\sin A} = \dfrac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \dfrac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$

$\Rightarrow (b-c)\cos\dfrac{A}{2} = a\sin\dfrac{B-C}{2}$

29. (*b*) $\cot\frac{B}{2}.\cot\frac{C}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}.\frac{s(s-c)}{(s-a)(s-b)}}$

$=\frac{s}{s-a}$ {Since $3a=b+c$ or $a+b+c=2s=4a$}

$=2a/a=2$

30. (*c*) $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{1}{2R}\Rightarrow \sin A=\frac{a}{2R}$ and so on.

Therefore $2R^2\sin A\sin B\sin C=2R^2\frac{a}{2R}.\frac{b}{2R}.\frac{c}{2R}$

$=\frac{abc}{4R}=\Delta$

31. (*b*) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow r=4R\sin^3 30^\circ$, $\{\because A=B=C=60^\circ\}$

$\Rightarrow r=\frac{R}{2}$

32. (*a*) $a=5k, b=6k$ and $c=5k$

$s=\frac{a+b+c}{2}=\frac{5k+6k+5k}{2}=8k$

$r=\frac{\Delta}{s}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{s}}$

$r=\sqrt{\frac{8k(8k-5k)(8k-6k)(8k-5k)}{8k}}$

$r=\frac{3k}{2}\Rightarrow k=\frac{2r}{3}=\frac{2\times6}{3}=4$

33. (*b*) $R=\frac{abc}{4\Delta}$, where $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$a=13, b=12, c=5, s=\frac{30}{2}=15$

$\Delta=\sqrt{15(2)(3)10}=3\times2\times5=30$

$\therefore R=\frac{13\times12\times5}{4\times30}=\frac{13}{2}$

Learning Plus

1. (*c*) $\because$ $A:B:C=3:5:4$

$\therefore$ $A=45^\circ, B=75^\circ, C=60^\circ$

$\because$ from Sine - rule

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$\Rightarrow \frac{a}{\frac{1}{\sqrt{2}}}=\frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{c}{\frac{\sqrt{3}}{2}}=k$

($\because \sin 75^\circ=\sin(45^\circ+30^\circ)$)

$\therefore a=\frac{k}{\sqrt{2}}, b=\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$ and $c=\frac{k\sqrt{3}}{2}$

$\therefore a+b+c\sqrt{2}=\frac{k}{\sqrt{2}}+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k+\left(\frac{k\sqrt{3}}{2}\right)\sqrt{2}$

$=\frac{k}{2\sqrt{2}}\left[2+(\sqrt{3}+1)+2\sqrt{3}\right]=\frac{3k(\sqrt{3}+1)}{2\sqrt{2}}=3b$

2. (*c*) given $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ ..... (i)

$\because$ $a=k\sin A, b=k\sin B, c=k\sin C$

$\therefore$ (i) becomes

$\frac{\cot A}{k}=\frac{\cot B}{k}=\frac{\cot C}{k}$

$\therefore$ $A=B=C$

$\therefore$ ΔABC is an equilateral triangle

3. (*c*) $\because$ $\frac{bc\sin^2 A}{\cos A+\cos B\cos C}$

$=\frac{k^2\sin B\sin C\sin^2 A}{-\cos(B+C)+\cos B\cos C}$

$=\frac{k^2\sin B\sin C\sin^2 A}{\sin B\sin C}=k^2\sin^2 A=a^2.$

4. (*c*) $(a+b+c)(b+c-a)=k\,bc$

$\Rightarrow (b+c+a)(b+c-a)=kbc$

$\Rightarrow (b+c)^2-a^2=kbc$

$\Rightarrow b^2+c^2-a^2+2bc=kbc$

$\Rightarrow b^2+c^2-a^2=(k-2)bc$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc}=\frac{k-2}{2}$

$\because -1<\frac{k-2}{2}<1$

$\Rightarrow -2<k-2<2 \Rightarrow 0<k<4$

5. (*b*) $\because A = \frac{2\pi}{3}, b - c = 3\sqrt{3}$ and Area $= \frac{9\sqrt{3}}{2}$ cm^2

$\because \Delta = \frac{1}{2} bc \sin A \Rightarrow \frac{9\sqrt{3}}{2} = \frac{1}{2} bc \sin \frac{2\pi}{3}$

$\Rightarrow bc = 18$

$\because \cos \frac{2\pi}{3} = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{1}{2}$

$\Rightarrow \frac{(b-c)^2 + 2bc - a^2}{2bc} = -\frac{1}{2}$

$\Rightarrow a = 9$

6. (*c*) $AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$\Rightarrow 4AD^2 = 2b^2 + 2c^2 - a^2$

$\Rightarrow 4BE^2 = 2c^2 + 2a^2 - b^2$

$\Rightarrow 4CF^2 = 2a^2 + 2b^2 - c^2$

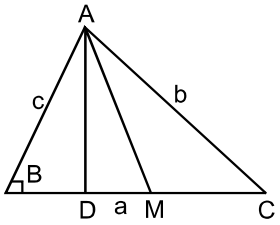
$\therefore 4(AD^2 + BE^2 + CF^2) = 3(a^2 + b^2 + c^2)$

$\Rightarrow \frac{AD^2 + DE^2 + CF^2}{BC^2 + CA^2 + AB^2} = \frac{3}{4} = 3:4$

7. (*b*) $BM = a/2$

$BD = c \cos B$

$DM = \frac{a}{2} - c \cos B$



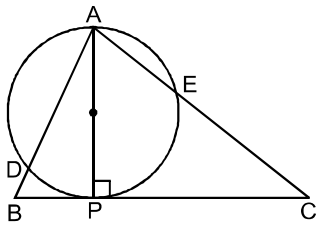
$= \frac{a}{2} - c\frac{(c^2 + a^2 - b^2)}{2ca}$

$= \frac{a^2 - c^2 - a^2 + b^2}{2a} = \frac{b^2 - c^2}{2a}$

8. (*d*) Sine Rule in ΔADE

$\frac{DE}{\sin A} = AP$

$\Rightarrow DE = \frac{2\Delta}{a} \sin A$



$= 2\Delta \times \frac{1}{R} = \frac{\Delta}{R} \quad \left\{\frac{\sin A}{a} = \frac{1}{2R}\right\}$

9. (*b*) Given In a ΔABC, $\sin A = \frac{2\Delta}{bc}$

$\Rightarrow \sin A = \frac{2[a^2 - b^2 - c^2 + 2bc]}{bc}$

$\Rightarrow \sin A = 2[2 - 2\cos A]$

$\Rightarrow \sin A = 4[1 - \cos A]$

$\Rightarrow 2 \sin \frac{A}{2} \cos \frac{A}{2} = 4 \cdot 2 \sin^2 \frac{A}{2}$

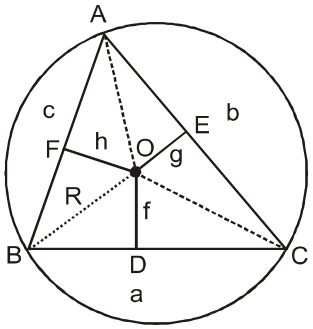
$\Rightarrow \tan \frac{A}{2} = \frac{1}{4}$

$\Rightarrow \tan A = \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} = \frac{2 \times \frac{1}{4}}{1 - \left(\frac{1}{4}\right)^2}$

$= \frac{\frac{1}{2}}{\frac{15}{16}} = \frac{1}{2} \times \frac{16}{15} \Rightarrow \tan A = \frac{8}{15}$

10. (*a*) In ΔBOD,

$\tan B = \frac{a}{\frac{2}{f}} = \frac{a}{2f}$



$\Sigma \tan A = \pi \tan A$

$\Rightarrow \frac{a}{2f} + \frac{b}{2g} + \frac{c}{2h} = \frac{abc}{8fgh}$

$\Rightarrow \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{abc}{4fgh}$

$\Rightarrow \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh} \quad \therefore \lambda = \frac{1}{4}$

11. (*b*) $\because s - a = 3$ ...(1)

and $s - c = 2$ ...(2)

by (1) – (2), we get

$c - a = 1$

(1) + (2), we get $2s - a - c = 5$

$\Rightarrow \quad b = 5$

$\because \Delta ABC$ is right angled at B

$\therefore a^2 + c^2 = 25$ ...(3)

$\therefore (c - a)^2 + 2ac = 25$

$ac = 12$ ...(4)

$\therefore\ a(1+a) = 12 \Rightarrow a^2 + a - 12 = 0$

$\Rightarrow (a+4)(a-3) = 0$

$\Rightarrow a = 3$ and $c = 4$.

12 (*a*) $\because\ b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c.$

$\Rightarrow b\frac{s(s-a)}{bc} + a\frac{s(s-b)}{ac} = \frac{3}{2}c.$

$\Rightarrow \frac{s}{c}[s-a+s-b] = \frac{3}{2}c \Rightarrow \frac{s}{c} \times c = \frac{3}{2}c$

$\Rightarrow \frac{a+b+c}{2} = \frac{3c}{2} \Rightarrow a+b = 2c$

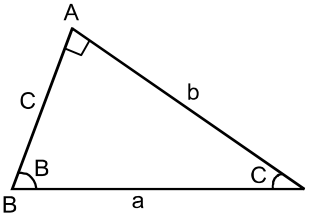
$\Rightarrow$ a, c, b are in A.P.

13. (*c*) $\because\ \cot\frac{B}{2}\cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$= \frac{s}{s-a} = \frac{2s}{2s-2a}$

$= \frac{a+b+c}{b+c-a} = \frac{4a}{2a} = 2\ (\because\ b+c = 3a)$

14. (*d*) $\Delta ABC, \angle A = \frac{\pi}{2}$



$\tan\frac{C}{2} = \frac{r}{s-c} = \frac{\Delta}{s(s-c)}\ \{\because \Delta = \frac{1}{2}bc\}$

$\tan\frac{C}{2} = \frac{\frac{1}{2}bc}{\frac{(a+b+c)}{2}\frac{(a+b-c)}{2}} = \frac{2bc}{2b^2+2ab}$

$\{\because a^2 = b^2 + c^2\}$

$= \frac{2bc}{2b(b+a)} = \frac{c}{a+b} \times \frac{a-b}{a-b}$

$= \frac{c(a-b)}{a^2-b^2} = \frac{c(a-b)}{c^2} = \frac{a-b}{c}$

15. (*b*) $\because\ \frac{b^2-c^2}{2aR} = \frac{4R^2(\sin^2 B - \sin^2 C)}{2.2R\sin A.R}$

$= \frac{\sin(B+C).\sin(B-C)}{\sin A} = \sin(B-C)$

16. (*a*) $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$

$= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2R(\sin A + \sin B + \sin C)}$

$= \frac{4\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{R}.$

17. (*b*) Given $a : b : c = 3 : 7 : 8$

We know $R = \frac{a}{2\sin A}$ & $r = \frac{\Delta}{s}$

$\Rightarrow \frac{R}{r} = \frac{as}{2\Delta\sin A} = \frac{a(a+b+c)}{4\times\frac{1}{2}bc\sin^2 A}$

$= \frac{a(a+b+c)}{2bc\sin^2 A}\ \{\Delta = \frac{1}{2}bc\sin A\}$

$\{\because a = 3k, b = 7k, c = 8k\}$

$\because\ \cos A = \frac{b^2+c^2-a^2}{2bc}$

$= \frac{k^2}{k^2}\ \frac{49+64-9}{2.7.8} = \frac{104}{2.7.8} = \frac{13}{14}$

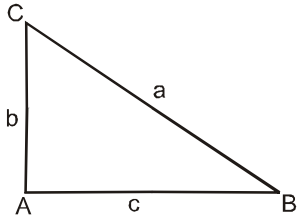
$\therefore \sin^2 A = 1 - \frac{13^2}{14^2} = \frac{27}{14^2}$

$\Rightarrow \frac{R}{r} = \frac{3(3+7+8)}{2\times7\times8\sin^2 A}\ \frac{k^2}{k^2} = \frac{3\times18\times14\times14}{2\times7\times8\times27}$

$\Rightarrow \frac{R}{r} = \frac{14}{4} = \frac{7}{2} \Rightarrow R : r = 7 : 2$

18. (*b*) $\because\ R = \frac{a}{2}$

$\because\ r = (s-a)\tan\frac{A}{2}$



$\Rightarrow r = (s-a) \Rightarrow r = s - 2R \Rightarrow R = \frac{s-r}{2}$

19. (*c*) Given $\Delta = 100$ sq. cm, $r_1 = 10$ cm, $r_2 = 50$ cm

$$r_1 = \frac{\Delta}{s-a} \Rightarrow s-a = \frac{100}{10} \Rightarrow s-a = 10 \ldots.(1)$$

$$r_2 = \frac{\Delta}{s-b} \Rightarrow s-b = \frac{100}{50} \Rightarrow s-b = 2 \ldots.(2)$$

$(1) - (2) \Rightarrow b-a = 10-2 = 8$

20. (*b*) $r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

$$\Rightarrow r \cdot r_1 \cdot r_2 \cdot r_3 = \frac{\Delta}{s} \cdot \frac{\Delta}{(s-a)} \cdot \frac{\Delta}{(s-b)} \cdot \frac{\Delta}{(s-c)}$$

$\Rightarrow r \cdot r_1 \cdot r_2 \cdot r_3 = \Delta^2 \quad \because \{\Delta^2 = s(s-a)(s-b)(s-c)\}$

21. (*a*) If $r_1 > r_2 > r_3 \Rightarrow \frac{\Delta}{s-a} > \frac{\Delta}{s-b} > \frac{\Delta}{s-c}$

$\Rightarrow (s-a) < (s-b) < (s-c)$

$\Rightarrow -a < -b < -c$

$\Rightarrow a > b > c$

22. (*a*) Given $\angle B = \frac{\pi}{2}$

$$r = (s-b)\tan\frac{B}{2} = (s-b)\tan\frac{\pi}{4}$$

$$= \frac{a+b+c}{2} - b = \frac{c+a-b}{2} = \frac{AB+BC-CA}{2}$$

23. (*a*) H is orthocentre of ΔABC

Radius of circumcircle of ABC,

BHC, CHA, AHB are same and equal to R

$\angle BHC = B + C$

In ΔBHC, by sin law

$$\Rightarrow \frac{a}{\sin(B+C)} = 2R \ \& \ \frac{a}{\sin A} = 2R$$

$\Rightarrow \sin(B+C) = \sin A \Rightarrow R_1 = R$

similarly $R_2 = R$ & $R_3 = R$

24. (*a*) $\frac{a-b}{b-c} = \frac{s-a}{s-c} \Rightarrow \frac{(s-b)-(s-a)}{(s-c)-(s-b)} = \frac{(s-a)}{(s-c)}$

$\Rightarrow (s-b)(s-c) - (s-a)(s-c)$

$= (s-a)(s-c) - (s-a)(s-b)$

$\Rightarrow (s-b)(s-c) + (s-a)(s-b) = 2(s-a)(s-c)$

divided by $(s-a)(s-b)(s-c)$

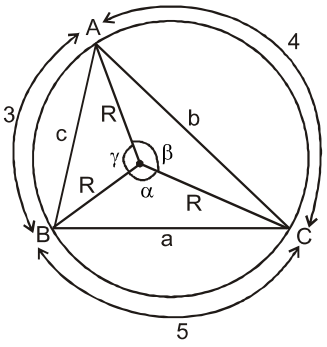
$$\Rightarrow \frac{1}{(s-a)} + \frac{1}{(s-c)} = \frac{2}{(s-b)}$$

$$\Rightarrow \frac{\Delta}{(s-a)} + \frac{\Delta}{(s-c)} = \frac{2\Delta}{(s-b)}$$

$\Rightarrow r_1 + r_3 = 2r_2 \Rightarrow r_1, r_2, r_3$ in A.P.

25. (*a*) $2\pi R = 3+4+5 \Rightarrow R = \frac{6}{\pi}$

$$\alpha = \frac{3\times 360}{3+4+5} = \frac{3\times 360}{12}$$



$\alpha = 90, \beta = 120, \gamma = 150$

$\Delta ABC = \Delta OBC + \Delta OCA + \Delta OAB$

$$\Delta ABC = \frac{1}{2}R.R\sin\alpha + \frac{1}{2}R.R\sin\beta + \frac{1}{2}R.R\sin\gamma$$

$$= \frac{1}{2}R^2[\sin 90^\circ + \sin 120^\circ + \sin 150^\circ]$$

$$= \frac{R^2}{2}\left[1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right] = \frac{1}{2}\left(\frac{6}{\pi}\right)^2\left[\frac{3+\sqrt{3}}{2}\right]$$

$$= \frac{36}{4\pi^2}(3+\sqrt{3}) = \frac{9\sqrt{3}}{\pi^2}(1+\sqrt{3})$$

26. (*b*) $r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}, s = \frac{a+b+c}{2}$

$$\frac{r}{r_1} = \frac{s-a}{s} = \frac{1}{2} \Rightarrow s = 2a$$

$\because \tan\frac{A}{2} = \frac{r}{s-a}$

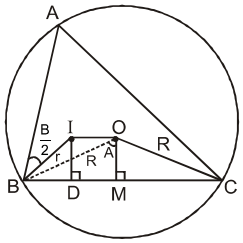
$$\tan\frac{A}{2}\left[\tan\frac{B}{2} + \tan\frac{C}{2}\right] = \frac{r}{s-a}\left[\frac{r}{s-b} + \frac{r}{s-c}\right]$$

$$= \frac{r^2(2s-b-c)}{(s-a)(s-b)(s-c)} = \frac{r^2(a+b+c-b-c)}{\left(\frac{\Delta^2}{s}\right)}$$

$$= \frac{\Delta^2}{s^2}\frac{s}{\Delta^2} \times a = \frac{a}{s} = \frac{a}{2a} = \frac{1}{2}$$

27. (*b*) In ΔOBM, $\cos A = \frac{r}{R}$

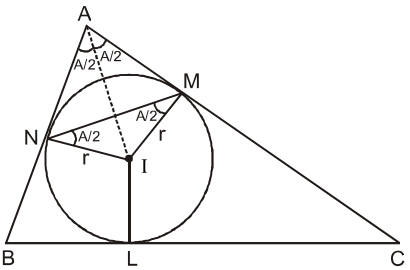
$\Rightarrow \cos A + \cos B + \cos C = 1 + 4\pi \sin \frac{A}{2}$



$\Rightarrow \frac{r}{R} + \cos B + \cos C = 1 + \frac{r}{R}$

$\Rightarrow \cos B + \cos C = 1$

28. (*c*) r is in radius



Circumradius is x

$x = \frac{r}{2\sin\frac{A}{2}}, y = \frac{r}{2\sin\frac{B}{2}}$ & $z = \frac{r}{2\sin\frac{C}{2}}$

$\Rightarrow xyz = \frac{r^3}{8\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)} = \frac{r^3}{8\frac{r}{4R}} = \frac{1}{2}r^2R$

Advanced Level Multiconcept Questions

1. (*a*, *b*)

(*a*) $\because \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$

$= \frac{a^2+b^2+c^2}{2abc}$

(*b*) $\because \frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c}$

$= \frac{a}{2R.a} + \frac{b}{2R.b} + \frac{c}{2R.c} = \frac{3}{2R}$

(*c*) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$

$\Rightarrow \cot A = \cot B = \cot C$

$\Rightarrow A = B = C$ true for equilateral triangle only

(*d*) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

$\Rightarrow \frac{2\sin A\cos A}{k^2\sin^2 A} = \frac{2\sin B\cos B}{k^2\sin^2 B} = \frac{2\sin C\cos C}{k^2\sin^2 C}$

$\Rightarrow \cot A = \cot B = \cot C$

$\Rightarrow A = B = C$ true for equilateral triangle only

2. (*a*, *b*, *c*)

(*a*) $\because \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ .....(i)

$\because \tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$

$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}}$ $\because a = 5$ and $b = 4$

$\therefore$ from equation (i), we get

$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right)\cot\frac{C}{2} \Rightarrow \frac{1}{3\sqrt{7}} = \frac{1}{9}\cot\frac{C}{2}$

$\Rightarrow \cot\frac{C}{2} = \frac{3}{\sqrt{7}}$

$\because \cos C = \frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$

$\because \cos C = \frac{b^2+a^2-c^2}{2ab}$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c = 6$

$(b), (c) \because$ Area $= \frac{1}{2}$ ab sin C $\because$ cosC $= \frac{1}{8}$

$\Rightarrow$ sinC $= \sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$

Area $= \frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$

Area $= \frac{15\sqrt{7}}{4}$ sq. unit. $\because$ From Sine rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \sin A = \frac{a \sin C}{c} = \frac{5 \times 3\sqrt{7}}{6 \times 8}$

$\therefore \sin A = \frac{5\sqrt{7}}{16}$

3. (a, c, d)

$\because \beta_a = \frac{2bc}{b+c} \cos \frac{A}{2}$

(a) correct

(b) incorrect

(c) $\frac{abc \operatorname{cosec} \frac{A}{2}}{2R(b+c)} = \frac{abc \operatorname{cosec} \frac{A}{2}}{\frac{a}{\sin A}.(b+c)}$

$= \frac{bc \, 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2}.(b+c)} = \frac{2bc}{(b+c)} \cos \frac{A}{2}$

$(d) \because \frac{2\Delta}{(b+c)} \operatorname{cosec} \frac{A}{2} = \frac{bc \sin A}{(b+c)} . \frac{1}{\sin \frac{A}{2}}$

$= \frac{2bc \sin \frac{A}{2} \cos \frac{A}{2}}{(b+c)} . \frac{1}{\sin \frac{A}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2}$

4. (b,d) $r_1 = 2r_2 = 3r_3$

$\Rightarrow \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c}$

$\Rightarrow \underset{\text{(i)}}{\frac{1}{s-a}} = \underset{\text{(ii)}}{\frac{2}{s-b}} = \underset{\text{(iii)}}{\frac{3}{s-c}}$

From (i) and (ii) we get $a - b = c/3$ ...(1)

From (i) and (iii), we get $2a - b = 2c$ ...(2)

From (ii) and (iii), we get $a - 5b = -5c$ ...(3)

let c = k, then from (1) and (2), we get

$a = \frac{5k}{3}$ and $b = \frac{4k}{3}$

$\therefore \frac{a}{b} = \frac{5}{4}$; $\frac{a}{c} = \frac{5}{3}$.

5. (a, b, c, d)

(a) $\frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{\Delta}{s-a} - \frac{\Delta}{s}$

$\Rightarrow \frac{1}{(s-b)(s-c)} = \frac{1}{s(s-a)}$

$\Rightarrow \tan^2 \frac{A}{2} = 1 \Rightarrow A = 90°$

(b) $4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = 8R^2$

$1 - (\cos^2 A - \sin^2 B) + \sin^2 C = 2$

$1 - \cos(A + B) \cos(A - B) + 1 - \cos^2 C = 2.$

$\cos C \cos(A - B) - \cos^2 C = 0$

$\cos C[\cos(A - B) - \cos C] = 0$

$\cos C[\cos(A - B) + \cos(A + B)] = 0$

$2 \cos A \cos B \cos C = 0$

$\Rightarrow A = 90°$ or $B = 90°$ or $C = 90°$

(c) $r_1 = s.$

$s \tan A/2 = s \Rightarrow \tan A/2 = 1 \Rightarrow A = 90°$

(d) $\frac{a}{\sin A} = \frac{a\Delta}{s(s-a)} \Rightarrow \frac{1}{\sin A} = \tan A/2$

$\Rightarrow \quad 2 \sin^2 A/2 = 1$

$\Rightarrow \quad 1 - \cos A = 1 \Rightarrow \cos A = 0 \Rightarrow A = 90°$

6. (A, B)

$\because \sin C = \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1$

$\Rightarrow \cos(A - B) \geq 1$

$\Rightarrow \cos(A - B) = 1$

$\Rightarrow A - B = 0 \qquad \Rightarrow A = B$

$\therefore \sin C = \frac{1 - \cos^2 A}{\sin^2 A} = 1$

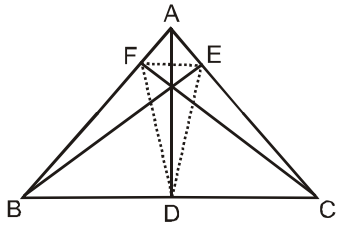
$\Rightarrow C = 90°$

7. (a, b, c, d)

$\because$ FE = a cos A = R sin 2A

DE = c cos C = R sin 2C

FD = b cos B = R sin 2B



$(a) = \dfrac{R\left(\sum \sin 2A\right)}{a+b+c}$

$= \dfrac{R\left(4\sin A \sin B \sin C\right)}{2R\left(4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\right)}$

$= \dfrac{8\left(\prod \sin\dfrac{A}{2}\right)\left(\prod \cos\dfrac{A}{2}\right)}{2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}} = \dfrac{r}{R}$

(b) $\because$ Area of ΔDEF

$= \dfrac{1}{2}$ FD $\times$ DE sin $(\pi - 2A)$

$= \dfrac{1}{2}$ b cos B.c cos C.sin 2A

$= \dfrac{1}{2}$ bc cos B.cos C.2 sin A.cos A

$= 2\left(\dfrac{1}{2}bc\sin A\right)$ cos A.cos B.cos C

$= 2\Delta$ cos A.cos B. cos C

(c) Area of $\Delta AEF = \dfrac{1}{2}$ AE $\times$AF sin A

$= \dfrac{1}{2}$ (c cos A) (b cos A) sin A

$= \left(\dfrac{1}{2}bc\sin A\right)$ $\cos^2 A = \Delta \cos^2 A$

(d) $R_{DEF} = \dfrac{FE \times DE \times FD}{4\Delta_{DEF}}$

$= \dfrac{abc\cos A\cos B\cos C}{4\times 2\Delta\cos A\cos B\cos C} = \dfrac{abc}{8\Delta} = \dfrac{4R\Delta}{8\Delta} = \dfrac{R}{2}$

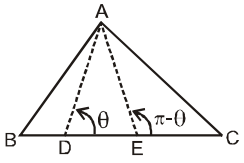
8. (b,d)

Product of distances of incenter from angular points

$= \dfrac{r^3}{\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}} = \dfrac{r^3}{r/4R} = 4Rr^2$

$= \dfrac{abc}{\Delta}r^2 = \dfrac{(abc)(r)}{\dfrac{\Delta}{r}} = \dfrac{(abc)(r)}{s}$.

9 (a, c, d)



if we apply m-n Rule in Δ ABE, we get

(1+1) cot θ = 1.cot B – 1.cot θ

2 cotθ = cot B – cotθ

3 cotθ = cot B

tan θ = 3 tan B .....(1)

Similarly, if we apply m-n Rule in Δ ACD, we get

(1+1) cot $(\pi - \theta)$ = 1.cotθ – 1.cotC.

cotC = 3 cotθ

$\Rightarrow$ tan θ = 3 tanC .....(2)

form (1) and (2) we can say that

tan B = tan C $\Rightarrow$ B = C

$\because$ A + B + C = π

$\therefore$ A = π – (B + C)

= π – 2B $\because$ B = C

$\therefore$ tan A = – tan2B

$= -\left(\dfrac{2\tan B}{1-\tan^2 B}\right) = -\dfrac{\dfrac{2\tan\theta}{3}}{1-\dfrac{\tan^2\theta}{9}}$

$\Rightarrow \tan A = \dfrac{6\tan\theta}{\tan^2\theta - 9}$

10. (a, c, d)

$\because$ $r_1 - r = \dfrac{\Delta}{s-a} - \dfrac{\Delta}{s} = \dfrac{\Delta a}{s(s-a)} = a\tan\dfrac{A}{2}$

$\therefore \Pi (r_1 - r) = abc \tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}$

$= abc\,\Pi\tan\dfrac{A}{2} = abc\,\dfrac{\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}}$

$= \dfrac{(abc)r}{4R.\dfrac{(\sin A + \sin B + \sin C)}{4}} = \dfrac{(abc)r}{R\left(\dfrac{a}{2R}+\dfrac{b}{2R}+\dfrac{c}{2R}\right)}$

$= \dfrac{2(abc)r}{2s} = \dfrac{4R\Delta r}{s} = 4Rr^2$

11. (c) $\dfrac{bx}{c} + \dfrac{cy}{a} + \dfrac{az}{b}$

$= b\sin B + c\sin C + a\sin A = \dfrac{b^2+c^2+a^2}{2R}$

$\therefore k = 2R$

12. (c) $\because$ cot A + cot B + cot C

$= \dfrac{R}{abc}(b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$

$= \dfrac{R}{abc}(b^2 + c^2 + a^2) = \dfrac{R}{abc}\left(\dfrac{4\Delta^2}{x^2}+\dfrac{4\Delta^2}{y^2}+\dfrac{4\Delta^2}{z^2}\right)$

$= \dfrac{4\Delta^2 R}{abc}\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}\right)$

$= \dfrac{4\Delta R}{abc}\cdot\Delta\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}\right) = \Delta\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}\right)$

$\therefore k = \Delta$

13. $(d)\ \sum\frac{c\sin B+b\sin C}{x}=\sum\frac{x+x}{x}=6$

14. $(d)\ \because\ PG=\frac{1}{3}AD$

$=\frac{1}{3}.\frac{2\Delta}{a}$

$=\frac{2}{3a}.\frac{1}{2}.ab\sin C$ or $=\frac{1}{3}b\sin C\ (\because\ \Delta=\frac{1}{2}ac\sin B)$

$\therefore\ PG=\frac{2}{3a}.\frac{1}{2}ac\sin B=\frac{1}{3}c\sin B$

15. $(b)\ \because$ Area of $\Delta GPL=\frac{1}{2}(PL)(PG)$

and Area of $\Delta ALD=\frac{1}{2}(DL)(AD)\ \because\ PL=\frac{1}{3}DL$

and $PG=\frac{1}{3}AD$

$\therefore\ \frac{\text{Area of }\Delta GPL}{\text{Area of }\Delta ALD}=\frac{\frac{1}{2}(PL)(PG)}{\frac{1}{2}(DL)(AD)}$

$=\frac{\frac{1}{3}(DL)\times\frac{1}{3}(AD)}{(DL)(AD)}=\frac{1}{9}$

16. $(b)\ \because$ Area of $\Delta PQR=$ Area of $\Delta PGQ+$ Area of $\Delta QGR+$ Area of ΔRGP ...(1)

$\because$ Area of $\Delta PGQ=\frac{1}{2}PG.GQ.\sin(\angle PGQ)$

$=\frac{1}{2}\times\frac{1}{3}AD\times\frac{1}{3}BE\sin(\pi-C)$

$=\frac{1}{18}\times\frac{2\Delta}{a}\times\frac{2\Delta}{b}\sin C$

$=\frac{2}{9ab}\times\frac{1}{2}bc\sin A\times\frac{1}{2}ac\sin B\times\sin C$

$=\frac{c^2}{18}\sin A.\sin B.\sin C$

Similarly Area of $\Delta QGR=\frac{a^2}{18}\sin A.\sin B.\sin C$ and

Area of $\Delta RGP=\frac{b^2}{18}\sin A.\sin B.\sin C$

$\therefore$ From equation (1), we get

Area of $\Delta PQR=\frac{1}{18}(a^2+b^2+c^2)\sin A.\sin B.\sin C$

17. $(a)\to(q),\ (b)\to(p),\ (c)\to(s),(d)\to(r)$

$(a)\ \because\ 2B=A+C\Rightarrow B=\frac{\pi}{3}$ and $A+C=\frac{2\pi}{3}$

$\because\ b^2=ac$

$\Rightarrow\sin^2B=\sin A.\sin C$

$\Rightarrow\sin A\sin C=\frac{3}{4}$

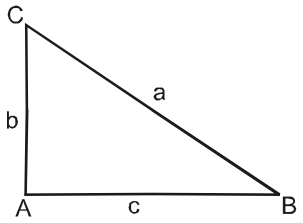
$\Rightarrow\cos(A-C)-\cos(A+C)=\frac{3}{2}\ \because\ A+C=\frac{2\pi}{3}$

$\Rightarrow\cos(A-C)=1\Rightarrow A=C=\frac{\pi}{3}=B\Rightarrow a=b=c$

$\therefore\ \frac{a^2(a+b+c)}{3abc}=1$

$(b)\ \because\ a^2=b^2+c^2$ and $2R=a$

$\therefore\ \frac{a^2+b^2+c^2}{R^2}=\frac{2a^2}{R^2}=8$



$(c)\ \because\ \Delta=\frac{1}{2}bc\sin A\Rightarrow\Delta=\frac{1}{2}.$

$9.\sin A=\frac{9}{2}\times\frac{a}{2R}\ \because\ a=2$

$\therefore\ 2R\Delta=9$

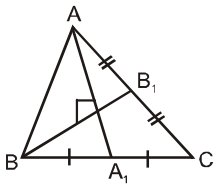
$(d)\ \because\ a=5,b=3$ and $c=7$

and because we know that

$b\cos C+c\cos B=a$

$\therefore\ 3\cos C+7\cos B=5$

18. $(a)\to(s),(b)\to(p),(c)\to(r),(d)\to(q)$



Match the column

$(a)\ \because\ AA_1$ and BB_1 are perpendicular

$\therefore\ a^2+b^2=5c^2$

$\therefore\ c^2=\frac{a^2+b^2}{5}=5\Rightarrow c=\sqrt{5}$

$\because \cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{16+9-5}{2\times4\times3} = \dfrac{5}{6}$

$\because \sin C = \dfrac{\sqrt{11}}{6}$

$\therefore \Delta = \dfrac{1}{2}\, ab \sin C = \sqrt{11}$

$\therefore \Delta^2 = 11$

(*b*) $\because$ line joining the circumcentre and orthocentre is parallel to side AC

$\Rightarrow R\cos B = 2R\cos A\cos C$

$\Rightarrow -\cos(A+C) = 2\cos A\cos C$

$\Rightarrow \sin A\sin C = 3\cos A\cos C$

$\Rightarrow \tan A\tan C = 3$

(*c*) $\because \tan^2\dfrac{C}{2} = \dfrac{(s-a)(s-b)}{s(s-c)}$ $\because a=5, b=4 \because 2s = 9+c$

$= \dfrac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \dfrac{c^2-1}{81-c^2}$

$\Rightarrow \dfrac{7}{9} = \dfrac{c^2-1}{81-c^2} \Rightarrow c^2 = 36 \Rightarrow c = 6$

(*d*) $\because 2a^2+4b^2+c^2 = 4ab+2ac.$

$\Rightarrow (a-2b)^2 + (a-c)^2 = 0$

$\Rightarrow a = 2b = c$

$\because \cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{7}{8}$

$\therefore 8\cos B = 7$

INTEGER TYPE

19. [0000]

Take, $a\sin(B-C) = a\sin B\cos C - a\cos B\sin C$

$= \dfrac{ab}{k}\left(\dfrac{b^2+a^2-c^2}{2ab}\right) - \dfrac{ac}{k}\left(\dfrac{c^2+a^2-b^2}{2ac}\right)$

$= \dfrac{b^2+a^2-c^2-c^2-a^2+b^2}{2k} = \dfrac{b^2-c^2}{k}$

Similarly, $b\sin(C-A) = \dfrac{c^2-a^2}{k}$

and $c\sin(A-B) = \dfrac{a^2-b^2}{k}$

Adding them, we get zero.

20. [7]

$\cos A = \dfrac{9+16-4}{2\cdot3\cdot4} = \dfrac{7}{8}$ $\qquad s = \dfrac{9}{2}$

$D = \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{3\sqrt{15}}{4}$

$HA = 2R\cos A = 2\cdot\left(\dfrac{abc}{4\Delta}\right)\cdot\cos A = 2\left(\dfrac{2\cdot3\cdot4}{4\cdot\dfrac{3\sqrt{5}}{4}}\right)\dfrac{7}{8}$

$= \dfrac{14}{\sqrt{15}} \Rightarrow \dfrac{\sqrt{15}\,HA}{2} = 7.$ Ans.

21. [0003]

We have $B+C = 180°-A$, $\quad B-C = 60°$

$\therefore B = 120-\dfrac{A}{2}$ and $C = 60°-\dfrac{A}{2}$

$\sin B = \sin(180°-B) = \sin\left(60+\dfrac{A}{2}\right)$

$\sin C = \sin\left(60°-\dfrac{A}{2}\right)$

Also area of the $\Delta = \dfrac{1}{2}ca\sin B$

$= \dfrac{1}{2}a\,2R\sin C\sin B = \dfrac{1}{2}a\left(\dfrac{a}{\sin A}\right)\sin B\sin C$

$= \dfrac{a^2\sin B\sin C}{2\sin A}$

$\therefore 12 = \dfrac{1}{2}\dfrac{(36)\sin(60°-\frac{A}{2})\sin(60°+\frac{A}{2})}{\sin A}$

$4\sin A = 3(\cos A - \cos 120°) = 3\cos A + \dfrac{3}{2}$

$\therefore 8\sin A - 6\cos A = 3.$

22. [0009]

$a:b:c = 4:5:6$

Let $a = 4k, b = 5k, c = 6k$

$S = \dfrac{15k}{2}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\left(\dfrac{15k}{2}\right)\left(\dfrac{15k}{2}-4k\right)\left(\dfrac{15k}{2}-5k\right)\times\left(\dfrac{15k}{2}-6k\right)}$

$= \sqrt{\left(\dfrac{15k}{2}\right)\left(\dfrac{7k}{2}\right)\left(\dfrac{5k}{2}\right)\left(\dfrac{3k}{2}\right)} = \dfrac{15\sqrt{7}}{4}\,k^2$

$r = \dfrac{\Delta}{s} = \dfrac{\frac{15\sqrt{7}}{4}k^2}{\left(\frac{15k}{2}\right)} = \dfrac{\sqrt{7}}{2}k.$

$$R = \frac{abc}{4\Delta} = \frac{(4k)(5k)(6k)}{4 \cdot \frac{15\sqrt{7}}{4}k^2} = \frac{8}{\sqrt{7}}k$$

$$\frac{R}{s} = \frac{\frac{8}{\sqrt{7}}}{\frac{\sqrt{7}}{2}} = \frac{16}{7}$$

$m = 16, n = 7, m - n = 9$. **Ans.**

23. [0004]

$$\cos B = \frac{a^2 + 16 - 8}{2 \times a \times 4}$$

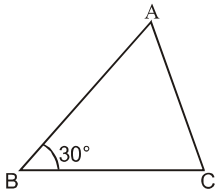
$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{a^2 + 8}{8a}$$

$$\Rightarrow a^2 - 4\sqrt{3}a + 8 = 0$$

$$\Rightarrow a_1 + a_2 = 4\sqrt{3}, a_2 a_2 = 8$$

$$\Rightarrow |a_1 - a_2| = 4$$

$$\Rightarrow |\Delta_1 - \Delta_2| = \frac{1}{2} \times \sin 30° \times 4 = 4$$



24. [0000]

$\because$ a, b, c are in A.P. as well as in G.P.

$b^2 = ac$

$a + c = 2b$

$(a + c)^2 = 4ac$

$(a - c)^2 = 0$

$a = c$

$\Rightarrow 2a = 2b, a = b$

$$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$$

$\therefore r_1 = r_2 = r_3 (\because a = b = c)$

$\therefore \frac{r_1}{r_2} = \frac{r_2}{r_3}$. Hence $\frac{r_1}{r_2} - \frac{r_2}{r_3} = 0$

25. [0002]

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 36 - 16}{60} = \frac{3}{4}$$

Similarly, $\cos B = \frac{9}{16}$ and $\cos C = \frac{1}{8}$

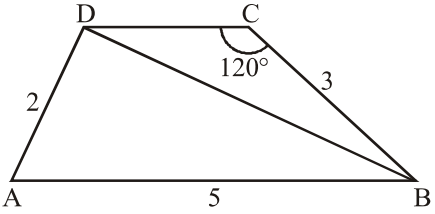
$$\therefore \cos A + \cos B + \cos C = \frac{23}{16} = \frac{K}{M} \text{ then } \left(\frac{K}{M}\right) = 2$$

(where $(.)$ denotes least integer)

26. [0002]

$$\cos 60° = \frac{2^2 + 5^2 - BD^2}{2(2)(5)}, BD^2 = 19$$

$$\text{Now } \cos 120° = \frac{x^2 + 3^2 - 19}{(2)(3)(x)}$$



$\Rightarrow x^2 + 3x - 10 = 0$

$\Rightarrow x = -5, 2$

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*)

$5, 5r, 5r^2$ sides of triangle

$5 + 5r > 5r^2$ .....(1)

$5 + 5r^2 > 5r$ .....(2)

$5r + 5r^2 > 5$ .....(3)

From $r^2 - r - 1 < 0$

$$\left[r - \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)\right] > 0 \quad \text{.....(4)}$$

From (2) $r^2 - r + 1 > \Rightarrow r \in R$ .....(5)

From (3) $r^2 + r - 1 > 0$

So, $\left(r + \frac{\sqrt{1-\sqrt{5}}}{2}\right)\left(r + \frac{1-\sqrt{5}}{2}\right) > 0$

$$r \in \left(-\infty, \frac{1+\sqrt{5}}{2}\right) \cup \left(-\frac{1-\sqrt{5}}{2}, \infty\right) \quad \text{.....(6)}$$

From (4), (5), (6) $r \in \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$

now check options

2. (*a*) $a=\sqrt{3}+1$

$b=\sqrt{3}-1$

$\frac{\sin A}{\sin B}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{3+1+2\sqrt{3}}{2}=2+\sqrt{3}$

$\frac{\sin A}{\sin(120-A)}=\sqrt{3}+2$

$\frac{\sin A}{\sin 12\cos A-\cos 12\sin A}=\sqrt{3}+2$

$\frac{1}{\frac{\sqrt{3}}{2}\cot A+\frac{1}{2}}=\sqrt{3}+2$

$\frac{\sqrt{3}\cot A+1}{2}=\frac{1}{\sqrt{3}+2}=\frac{\sqrt{3}-2}{-1}$

$\frac{\sqrt{3}\cot A+1}{2}=-\sqrt{3}+2$

$\sqrt{3}\cot A=4-2\sqrt{3}-1$

$\sqrt{3}\cot A=3-2\sqrt{3}$

$\cot A=\sqrt{3}-2$

$-\cot A=2-\sqrt{3}=\tan 15$

$\therefore A=105^{o}$

$\therefore B=15^{o}$

3. (*a*) $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}=\frac{a+b+c}{18}$

$\Rightarrow a=7k, b=6k, c=5k$

$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{1}{5}$

$\cos B=\frac{19}{25}, \cos C=\frac{5}{7}$

$\frac{1}{5\alpha}=\frac{18}{35\beta}=\frac{5}{7\lambda}$

$\Rightarrow \frac{7}{35\alpha}=\frac{19}{35\beta}=\frac{25}{35\gamma}$

$\alpha:\beta:\gamma=7:19:25$

4. (*b*) $x^2-c^2=y$

$(a+b)^2-c^2=ab$

$a^2+b^2-c^2=-ab$

$\frac{a^2+b^2-c^2}{2ab}=-\frac{1}{2}$

$\cos c=-\frac{1}{2}$

$c=\frac{2\pi}{3}$

$\sin C=\frac{\sqrt{3}}{2}$

$\frac{c}{\sin c}=2R \Rightarrow R=\frac{c}{2\sin c}=\frac{c}{\sqrt{3}}$

5. (*c*) $a<b<c$ are in A.P.

$\angle C=2\angle A$(Given)

$\Rightarrow \sin C=\sin 2A$

$\Rightarrow \sin C=2\sin A.\cos A$

$\Rightarrow \frac{\sin C}{\sin A}=2\cos A$

$\Rightarrow \frac{c}{a}=2\frac{b^2+c^2-a^2}{2bc}$

put $a=b-\lambda, c=b+\lambda, \lambda>0$

$\Rightarrow \lambda=\frac{b}{5}$

$\Rightarrow a=b-\frac{b}{5}=\frac{4}{5}b, c=b+\frac{b}{5}=\frac{6b}{5}$

$\Rightarrow a=b-\frac{b}{5}=\frac{4}{5}b, c=b+\frac{b}{5}=\frac{6b}{5}$

$\Rightarrow$ required ratio $=4:5:6$

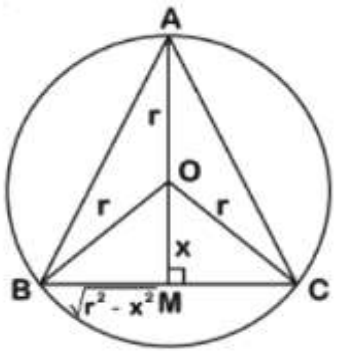
6. (*c*) $\angle B=\frac{\pi}{3}$, by sine Rule

$\sin A=\frac{1}{2}$

$\Rightarrow A=30^{o}, a=2, b=2\sqrt{3}, c=4$

$\Delta=\frac{1}{2}\times 2\sqrt{3}\times 2=2\sqrt{3}$ sq. cm

7. (*a*) Area of triangle *ABC*



$$A = \frac{1}{2} \times BC \times AM$$

$$= \frac{1}{2} \times 2\sqrt{r^2 - x^2} \times (r + x)$$

$$A = (r + x)\sqrt{r^2 - x^2}$$

$$\frac{dA}{dx} = \sqrt{r^2 - x^2} - \frac{x}{\sqrt{r^2 - x^2}} \times (r + x)$$

$$= \frac{r^2 - x^2 - rx - x^2}{\sqrt{r^2 - x^2}} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = \frac{-(x + r)(2x - r)}{\sqrt{r^2 - x^2}}$$

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{r}{2}$$

Sign change of $\frac{dA}{dx}$ at $x = \frac{r}{2}$

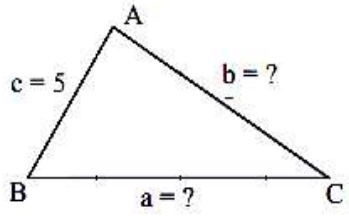
$\Rightarrow$ A has maximum at $x = \frac{r}{2}$

$$BC = 2\sqrt{r^2 - x^2} = \sqrt{3}r$$

$$AM = r + \frac{1}{2}r = \frac{3}{2}r$$

$$\Rightarrow AB = AC = \sqrt{3}r$$

8. (*c*)



$$As, \cos B = \frac{3}{5} \Rightarrow B = 53^\circ$$

$$As, R = 5 \Rightarrow \frac{c}{\sin C} = 2R$$

$$\Rightarrow \frac{5}{10} = \sin c \Rightarrow C = 30^\circ$$

Now, $\frac{b}{\sin \Delta} = 2R \Rightarrow b = 2(5)\left(\frac{4}{5}\right) = 8$

Now, by cosine formula

$$\cos B = \frac{a^2 + c^2 - b^2}{3ac} \Rightarrow \frac{3}{5} = \frac{a^3 + 25 - 64}{a(3)a}$$

$$\Rightarrow a^2 - 6a - 3 = 0$$

$$\therefore a = \frac{61\sqrt{12}}{2} = \frac{61\pi\sqrt{3}}{2}$$

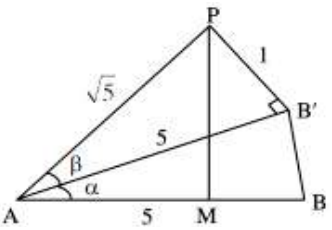
$\Rightarrow 3 + 4\sqrt{3}$ (Reject $a = 3 - 4\sqrt{3}$)

Now, $\Delta = \frac{abc}{4\pi} = \frac{(3 + 4\sqrt{3})(3)(3)}{4(5)} = 2(3 + 4\sqrt{3}))$

$\Rightarrow \Delta = (6 + 8\sqrt{3})$

Option (*c*) is correct.

9. (*a*)



From figure,

$$\sin \beta = \frac{1}{\sqrt{3}}$$

$$\therefore \tan \beta = \frac{1}{2}$$

$$\tan(\alpha + \beta) = 2$$

$$\frac{\tan \alpha + \tan \beta)}{1 - \tan \alpha \tan \beta} = 2$$

$$\frac{\tan \alpha + \frac{1}{\pi}}{1 - \tan \alpha\left(\frac{3}{x}\right)} = 2$$

$$\tan \alpha = \frac{3}{4}$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

10. (*b*) $\frac{\sin A}{\sin B} = \frac{\sin(A - C)}{\sin(C - B)}$

As A, B, C are angles of triangle.

$$A + B + C = \pi$$

$$A = \pi - (B + C)$$

From (1) and (2)

So, $\sin A = \sin(B + C)$ ...(1)

Similarly, $\sin B = \sin(A + C)$ ...(2)

$$\frac{\sin(B + C)}{\sin(A + C)} = \frac{\sin(A - C)}{\sin(C - B)}$$

$$\sin(C + B) \cdot \sin(C - B) = \sin(A - C)\sin(A + C)$$

$$\sin^2 C - \sin^2 B = \sin^2 A - \sin^2 C$$

$$\because \left\{\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y\right\}$$

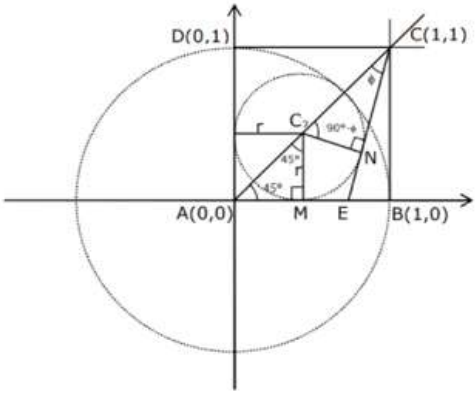
$$2\sin^2 C = \sin^2 A + \sin^2 B$$

By sine rule

$2c^2 = a^2 + b^2$

$\Rightarrow b^2$, c^2 and a^2 are in A.P.

11. (1)



(*i*) $\sqrt{2}r + r = 1$

$$r = \frac{1}{\sqrt{2}+1}$$

$$r = \sqrt{2}-1$$

(*ii*) $CC_2 = 2\sqrt{2} - 2 = 2(\sqrt{2}-1)$

From $\Delta CC_2N = \sin\phi = \dfrac{\sqrt{2}-1}{2(\sqrt{2}-1)}$

(*iii*) In ΔACE apply sine law

$$\frac{AE}{\sin\phi} = \frac{AC}{\sin 105^\circ}$$

$$AE = \frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3}+1} \cdot 2\sqrt{2}$$

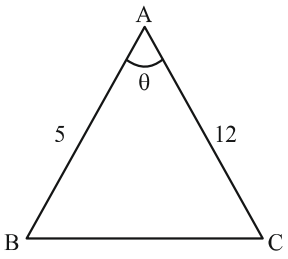
$$AE = \frac{2}{\sqrt{3}+1} = \sqrt{3}-1$$

$$\therefore EB = 1-(\sqrt{3}-1)$$

$$= 2-\sqrt{3}$$

$$\therefore \alpha = 2, \beta = -1 \Rightarrow \alpha+\beta = 1$$

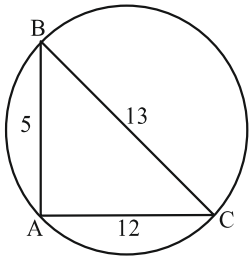
12. (15)



$$\text{Area} = \frac{1}{2}(5)(12)\sin\theta = 30$$

$$\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

Δ is right angle Δ



$$r = (s-a)\tan\frac{A}{2}$$

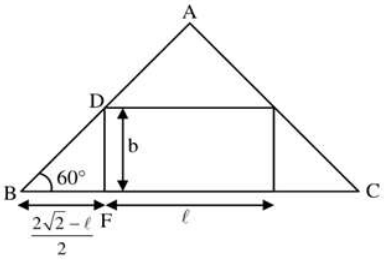
$$r = (s-a)\tan\frac{90}{2}$$

$$r = (s-a)$$

$$2R + r = s, (\text{ As } a = 2R)$$

$$2R + r = \frac{5+12+13}{2} = 15$$

13. (3)



In ΔDBF

$$\tan 60^\circ = \frac{2b}{2\sqrt{2}-l} \Rightarrow b = \frac{\sqrt{3}(2\sqrt{2}-l)}{2}$$

A = Area of rectangle $= l \times b$

$$A = l \times \frac{\sqrt{3}}{2}(2\sqrt{2}-l)$$

$$\frac{dA}{dl} = \frac{\sqrt{3}}{2}(2\sqrt{2}-l) - \frac{l\cdot\sqrt{3}}{2} = 0$$

$$l = \sqrt{2}$$

$$A = l \times b = \sqrt{2} \times \frac{\sqrt{3}}{2}(\sqrt{2}) = \sqrt{3}$$

$$\Rightarrow A^2 = 3$$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*c*) $a = 2 = QR$

$$b = \frac{7}{2} = PR$$

$$c = \frac{5}{2} = PQ \Rightarrow s = \frac{a+b+c}{2} = \frac{8}{4} = 4$$

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)}$$

$$= \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2}$$

$$= \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

2. (b, d)

3. (b) $a + b = x$

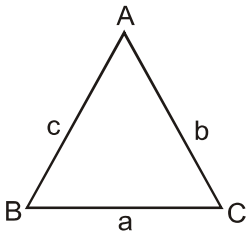
$ab = y$

$x^2 - c^2 = y$

$(a + b)^2 - c^2 = ab$

$a^2 + b^2 + ab = c^2$

$a^2 + b^2 - c^2 = -ab$



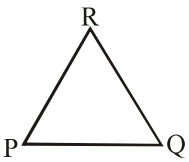
$$\frac{a^2+b^2-c^2}{2ab} = \frac{7}{2}$$

$$\cos C = \frac{-1}{2} \quad \Rightarrow C = \frac{2\pi}{3}$$

$$\frac{r}{R} = \frac{\Delta \times 4\Delta}{s \times abc} = \frac{4 \times \frac{1}{4}a^2b^2\sin^2 C}{(a+b+c)abc} = \frac{3ab}{4c(x+c)}$$

$$= \frac{3y}{4c(x+c)}$$

4. (a)



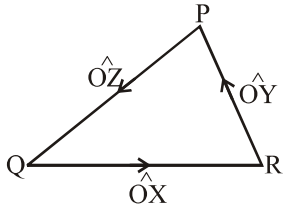
$\cos(P + Q) + \cos(Q + R) + \cos(R + P)$

$= -\cos R - \cos P - \cos Q$

In any Δ, $\max of \cos P + \cos Q + \cos R = \frac{3}{2}$

So minimum value of the given expression is $-\frac{3}{2}$

5. (a)



$\cos R = -\, O\hat{X} \cdot X\hat{Y}$

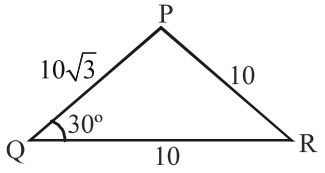
$\Rightarrow |\cos R| = |\, O\hat{X} \cdot X\hat{Y}\,|$

$|\, O\hat{X} \times O\hat{Y}\,| = |\sin R|$

$= |\sin(\pi - (P + Q))| = \sin(P + Q)| = \sin(P + Q)$

6. (b, c, d)

$$\cos 30^\circ = \frac{(10\sqrt{3})^2 + (10)^2 - (PR)^2}{2 \times 10\sqrt{3} \times 10}$$



$\Rightarrow PR = 10$

$\because QR = PR \Rightarrow \angle PQR = \angle QPR$

$\angle QPR = 30^\circ$

(b) area of $\Delta PQR = \frac{1}{2} \times 10\sqrt{3} \times 10 \times \sin 30^\circ$

$$= \frac{1}{2} \times 10 \times 10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3}$$

$\angle QRP = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$

$$(c)\ r = \frac{\Delta}{S} = \frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}}$$

$$= 5\sqrt{3}.(2-\sqrt{3}) = 10\sqrt{3} - 15$$

$$(d)\ R = \frac{a}{2\sin A} = \frac{10}{2\sin 30^\circ} = 10$$

$\therefore$ Area $= \pi R^2 = 100\pi$

7. (b,c,d)

$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$$

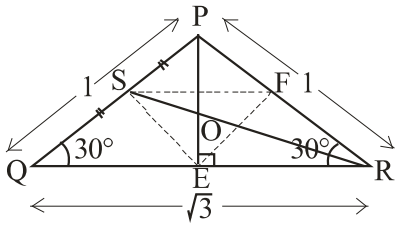
$$\Rightarrow P = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ and } Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since $p > q \Rightarrow P > Q$

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$ (not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$$



Now, area of $\Delta SEF = \frac{1}{4}$ area of ΔPQR

$\Rightarrow$ area of $\Delta SOE = \frac{1}{3}$ area of $\Delta SEF = \frac{1}{12}$ area of ΔPQR

$= \frac{1}{12}\cdot\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$

$RS = \frac{1}{2}\sqrt{2(3)+2(1)-1} = \frac{\sqrt{7}}{2}$

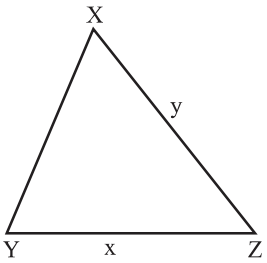
$OE = \frac{1}{3}PE = \frac{1}{3}\cdot\frac{1}{2}\sqrt{2(1)^2+2(1)^2-3} = \frac{1}{6}$

8. (*b*,*c*)

$\tan\frac{x}{2}+\tan\frac{z}{2} = \frac{2y}{x+y+z}$

$\frac{\Delta}{s(s-x)}+\frac{\Delta}{s(s-z)} = \frac{2y}{2s}$

$\frac{\Delta}{s}\left(\frac{2s-(x+z)}{(s-x)(s-z)}\right) = \frac{y}{s}$



$\Rightarrow \frac{\Delta y}{s(s-x)(s-z)} = \frac{y}{s}$

$\Rightarrow \Delta^2 = (s-x)^2(s-z)^2$

$\Rightarrow s(s-y) = (s-x)(s-z)$

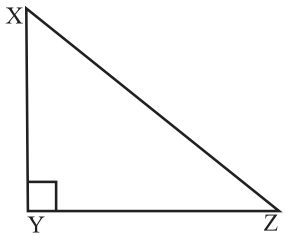
$\Rightarrow (x+y+z)(x+z-y) = (y+z-x)(x+y-z)$

$\Rightarrow (x+z)^2-y^2 = y^2-(z-x)^2$

$\Rightarrow (x+z)^2+(x-z)^2 = 2y^2$

$\Rightarrow x^2+z^2 = y^2 \Rightarrow \angle Y = \frac{\pi}{2}$

$\Rightarrow \angle Y = \angle X+\angle Z$



$\tan\frac{x}{2} = \frac{\Delta}{S(S-x)}$

$\tan\frac{x}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2-x^2}{4}}$

$\tan\frac{x}{2} = \frac{2xz}{y^2+z^2+2yz-x^2}$

$\tan\frac{x}{2} = \frac{2xz}{2z^2+2yz}$ (using $y^2=x^2+z^2$)

$\tan\frac{x}{2} = \frac{x}{y+2}$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

2 QUADRATIC EQUATION

QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

The expression, $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called as a quadratic expression in x. The quadratic expression when equated to zero, $ax^2 + bx + c = 0$, is called as a quadratic equation in x. Where the numbers a, b, c are called the coefficients of the equation.

1. The values of x which satisfy the quadratic equation is called roots (also called solutions or zeros) of the quadratic equation.
2. This equation has two roots which are given by

$$x = \frac{-b \pm \sqrt{D}}{2a},$$

where D (or Δ) = $b^2 - 4ac$ is called as discriminant of the quadratic equation.

3. If α and β denote the roots of $ax^2 + bx + c = 0$, then,

$$(i)\ \alpha + \beta = -\frac{b}{a} \quad (ii)\ \alpha\beta = \frac{c}{a} \quad (iii)\ |\alpha - \beta| = \frac{\sqrt{D}}{|a|}$$

NATURE OF ROOTS

- Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ & $a \neq 0$ then;
  - (*i*) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal)
  - (*ii*) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)
  - (*iii*) $D < 0 \Leftrightarrow$ roots are imaginary
  - (*iv*) If $p + iq$ is one root of a quadratic equation, then the other must be it's conjugate $p - iq$ & vice versa. $(p, q \in R$ & $i = \sqrt{-1})$
- Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in Q$ & $a \neq 0$ then ;
  - (*i*) If $D > 0$ & is a perfect square , then roots are rational & unequal.
  - (*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

TRAIN YOUR BRAIN

Example 1: Determine the values of m for which the equation $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$ will have

(*a*) equal roots

(*b*) product of the roots as 2

Sol. $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$

$$(5 + 4m)x^2 - 2(m + 2)x + (2 - m) = 0 \qquad \ldots(1)$$

(*a*) For equal roots, the discriminant of (1) should be equal to zero.

$$\therefore\ 4(m + 2)^2 - 4.(5 + 4m)(2 - m) = 0$$

$$\Rightarrow m^2 + 4m + 4 - (10 + 8m - 5m - 4m^2) = 0$$

$$\Rightarrow 5m^2 + m - 6 = 0 \qquad \Rightarrow (5m + 6)(m - 1) = 0$$

$$\Rightarrow m = -\frac{6}{5} \quad \text{or} \quad m = 1$$

(*b*) $\frac{2 - m}{5 + 4m} = 2$

$$\therefore\ 2 - m = 10 + 8m \Rightarrow m = -\frac{8}{9}$$

RELATION BETWEEN ROOTS AND CO-EFFICIENTS

(*i*) The solutions of quadratic equation, $ax^2 + bx + c = 0, (a \neq 0)$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*ii*) If α, β are the roots of quadratic equation,

$ax^2 + bx + c = 0$ ...(*i*)

then equation (*i*) can be written as

$a(x - \alpha)(x - \beta) = 0$

or $ax^2 - a(\alpha + \beta)x + a\alpha\beta = 0$ ...(*ii*)

$$\alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Hence we conclude that the quadratic equation whose roots are α & β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Now consider the following cases.

Case		Nature of roots
Case-I	$a > 0,\ b > 0, c > 0$ $\Rightarrow \alpha + \beta < 0, \alpha\beta > 0$	Both roots are negative.
Case-II	$a > 0,\ b > 0, c < 0$ $\Rightarrow \alpha + \beta < 0, \alpha\beta < 0$	Both roots are opposite in sign; Magnitude of negative root is more than the magnitude of positive root.
Case-III	$a > 0,\ b < 0, c > 0$ $\Rightarrow \alpha + \beta > 0,\ \alpha\beta > 0$	Both roots are positive.
Case-IV	$a > 0,\ b < 0, c < 0$ $\Rightarrow \alpha + \beta > 0, \alpha\beta < 0$	Roots are opposite in sign. Magnitude of positive root is more than magnitude of negative root.

Remember:

(*i*) Roots are rational $\Leftrightarrow$ D is a perfect square.

(*ii*) Roots are irrational $\Leftrightarrow$ D is positive but not a perfect square.

(*iii*) If $a + b + c = 0$, then 1 is a root of the equation $ax^2 + bx + c = 0$.

(*iv*) If a and c are of opposite sign, the roots must be of opposite sign.

(*v*) If the roots are equal in magnitude but opposite in sign, then $b = 0$, $ac < 0$.

(*vi*) If the roots are reciprocal of each other, then $c = a$.

(*vii*) The quadratic equation whose roots are reciprocals of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$ (*i.e.*, the coefficients are written in reverse order).

(*viii*) If $a = 1$, $b, c \in Z$ and the roots are rational numbers, then these roots must be integers.

(*ix*) If $ax^2 + bx + c = 0$ is satisfied by more than two values, it is an identity and $a = b = c = 0$ and vice-versa.

(*x*) If $P(x) = a_1x^2 + b_1x + c_1$ and $Q(x) = a_2x^2 + b_2x + c_2$, then $P(x).\ Q(x) = 0$ have atleast two real roots if $D_1 + D_2 > O$.

SYMMETRIC FUNCTION OF THE ROOTS

The following results may be useful.

(*i*) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(*ii*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$

(*iii*) $\alpha^4 + \beta^4 = (\alpha^3 + \beta^3)\,(\alpha + \beta) - \alpha\beta\,(\alpha^2 + \beta^2)$

(*iv*) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)\,(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$

(*v*) $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

(*vi*) $\alpha^2 - \beta^2 = (\alpha + \beta)\,(\alpha - \beta)$

(*vii*) $\alpha^3 - \beta^3 = (\alpha - \beta)\,[(\alpha + \beta)^2 - \alpha\beta]$

(*viii*) $\alpha^4 - \beta^4 = (\alpha + \beta)\,(\alpha - \beta)\,(\alpha^2 + \beta^2)$

QUADRATIC EXPRESSION

Let $y = ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right)$ ...(1)

1. Represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

   If $a > 0$, the parabola opens upward while if $a < 0$, the parabola opens downward. The parabola cuts the x-axis at points corresponding to roots of $ax^2 + bx + c = 0$. If this equation has

   (*i*) $D > 0$, the parabola cuts x-axis at two real and distinct points.

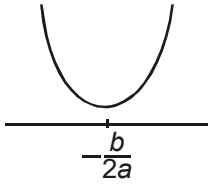
   (*ii*) $D = 0$, the parabola touches x-axis at $x = \frac{-b}{2a}$.

   (*iii*) $D < 0$, then;
   if $a > 0$, parabola lies above x-axis.
   if $a < 0$, parabola lies below x-axis.

GRAPHS OF QUADRATIC EXPRESSIONS

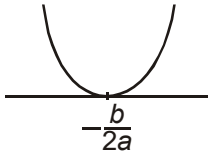
Let $f(x) = ax^2 + bx + c$ and $\alpha, \beta : \alpha < \beta$, be its roots

(*i*) $a > 0$ and $D < 0 \Leftrightarrow f(x) > 0 \quad \forall \quad x \in R$



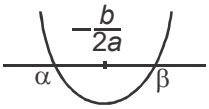
α and β are complex conjugates

(*ii*) $a > 0$ and $D = 0 \Leftrightarrow f(x) \geq 0 \quad \forall \quad x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

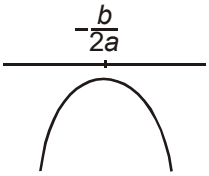
(*iii*) $a > 0$ and $D > 0$; then,



$f(x) > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$
$f(x) < 0 \ \forall \ x \in (\alpha, \beta)$

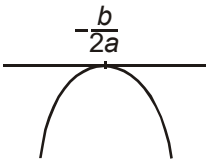
$$f(x)_{min} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}$$

(*iv*) $a < 0$ and $D < 0 \Leftrightarrow f(x) < 0 \quad \forall \quad x \in R$



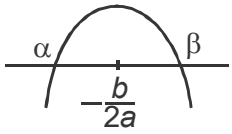
α and β are complex conjugates

(*v*) $a < 0$ and $D = 0 \Leftrightarrow f(x) \le 0 \quad \forall \quad x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

(*vi*) $a < 0$ and $D > 0$; then



$f(x) < 0 \quad \forall \quad x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0 \quad \forall \quad x \in (\alpha, \beta)$

$$f(x)_{max} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}$$

EQUATION V/S IDENTITY

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$ is:

A **quadratic equation** if $a \neq 0$ and An **identity** if $a = b = c = 0$ Infinite Roots

If a equation is satisfied by three distinct values of 'x', then it is an identity.

$(x + 1)^2 = x^2 + 2x + 1$ is an identity in x.

SOLUTION OF QUADRATIC INEQUALITIES

The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0 \ (a \neq 0)$ are:

(*i*) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $\alpha < \beta$.
$a > 0 \Rightarrow x \in (\infty, \alpha) \cup (\beta, \infty)$
$a < 0 \Rightarrow x \in (\alpha, \beta)$

(*ii*) If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.
$a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$
$a < 0 \Rightarrow x \in \phi$

(*iii*) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.
$a > 0 \Rightarrow x \in R$
$a < 0 \Rightarrow x \in \phi$

(*iv*) Inequalities of the form $\frac{P(x)}{Q(x)} \gtreqless 0$ can be solved using the method of intervals.

Example 2: Solve inequatility, $\frac{x^2 + x + 1}{|x + 1|} > 0$.

Sol. $\frac{x^2 + x + 1}{|x + 1|} > 0$ ...(*i*)

$\therefore \quad |x + 1| > 0, \ \forall \ x \in R - \{-1\}$
$\therefore \quad$ (i) becomes $x^2 + x + 1 > 0 \ x \neq -1$
$\because \quad x^2 + x + 1 > 0, \ \forall \ x \in R$ (as its $a > 0$ and $D < 0$)
$\therefore \quad$ Solution of (i) is $x \in R - \{-1\}$

Example 3: Solve $\left|\frac{x^2 - 3x - 1}{x^2 + x + 1}\right| < 3$.

Sol. $\frac{|x^2 - 3x - 1|}{x^2 + x + 1} < 3$.

$\because \quad (x^2 + x + 1) > 0, \forall \ x \in R$
$\therefore \quad |x^2 - 3x - 1| < 3(x^2 + x + 1)$
$\Rightarrow \quad (x^2 - 3x - 1)^2 - \{3(x^2 + x + 1)\}^2 < 0$
$\Rightarrow \quad (4x^2 + 2)(-2x^2 - 6x - 4) < 0$
$\Rightarrow \quad (2x^2 + 1)(x + 2)(x + 1) > 0$
$\Rightarrow \quad x \in (-\infty, -2) \cup (-1, \infty)$

COMMON ROOTS

One Root Common:

If $\alpha \neq 0$ is a common root of the equation

$$a_1x^2 + b_1x + c_1 = 0 \quad ...(i)$$

and

$$a_2x^2 + b_2x + c_2 = 0 \quad ...(ii)$$

then we have

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \text{ and}$$
$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

These give $\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1}$

$$= \frac{1}{a_1b_2 - a_2b_1} (a_1b_2 - a_2b_1 \neq 0).$$

Thus, the required condition for one common root is $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$ and the value of the common root is

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ or } \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}.$$

Both Roots Common

If the equations (i) and (ii) have both roots common, then these equations will be identical. Thus the required condition for both roots common is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \qquad \text{(If no root is equal to zero)}$$

Remember

(*i*) To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of *x* so obtained is the required common root.

(*ii*) If two quadratic equations with real and rational coefficients have a common imaginary or irrational root, then both roots will be common and the two equations will be identical. The required condition is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(*iii*) If α is a repeated root of the quadratic equation $f(x) = ax^2 + bx + c = 0$, then α is also a root of the equation $f'(x) = 0$.

Note:

If $f(x) = 0$ & $g(x) = 0$ are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv af(x) + bg(x) = 0$.

TRAIN YOUR BRAIN

Example 4: Find *m* and *n* in order that the equations $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$ may have both the roots common.

Sol. The equations are $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$. Since they have both the roots common,

$$\frac{m}{3} = \frac{5}{10} = \frac{2}{n} \qquad \because \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_1}$$

From the first-relation, $m = \frac{15}{10} = \frac{3}{2}$.

From the last-relation, $n = 4$.

RANGE OF QUADRATIC EXPRESSION

$$f(x) = ax^2 + bx + c$$

(*i*) **Range when $x \in R$:** If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \Rightarrow f(x) \in \left(-\infty, \frac{D}{4a}\right]$

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$ respectively

(*ii*) **Range in restricted domain :** Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$f(x) \in \left[\min\{f(x_1), f(x_2)\},\ \max\{f(x_1), f(x_2)\}\right]$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\},\ \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

TRAIN YOUR BRAIN

Example 5: Find the minimum value of $f(x) = x^2 - 5x + 6$.

Sol. Minimum value of $f(x) = -\frac{D}{4a} = -\left(\frac{25-24}{4}\right)$

$= -\frac{1}{4}$ $\left(\text{at } x = -\frac{b}{2a} = \frac{5}{2}\right)$. Hence range is $\left[-\frac{1}{4}, \infty\right)$.

Example 6: Find the range of $y = \frac{x+2}{2x^2+3x+6}$, if x is real.

Sol. $y = \frac{x+2}{2x^2+3x+6}$

$\Rightarrow 2yx^2 + 3yx + 6y = x + 2$

$\Rightarrow 2yx^2 + (3y-1)x + 6y - 2 = 0$ ...(*i*)

case-I:

if $y \neq 0$, then equation (*i*) is quadratic in x

$\because$ x is real

$\therefore D \geq 0$

$\Rightarrow (3y-1)^2 - 8y(6y-2) \geq 0$

$\Rightarrow (3y-1)(13y+1) \leq 0$

$y \in \left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$

case-II : if $y = 0$, then equation becomes

$x = -2$

which is possible as x is real

$\therefore$ Range $y \in \left[-\frac{1}{13}, \frac{1}{3}\right]$

THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation ; $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, ... a_n$ are all real & $a_0 \neq 0$ then,

$$\sum\alpha_1 = -\frac{a_1}{a_0},\ \sum\alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \sum\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0},\ .....,\ \alpha_1$$

$\alpha_2 \qquad \alpha_3 .....\alpha_n = (-1)^n \frac{a_n}{a_0}$

Note:

(*i*) If α is a root of the equation $f(x)=0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$ or $(x-\alpha)$ is a factor of $f(x)$ and conversely.

(*ii*) Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(*iii*) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs**.

(*iv*) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(*v*) If there be any two real numbers 'a' & 'b' such that $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(*vi*) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

TRAIN YOUR BRAIN

Example 7: If $x = 1$ and $x = 2$ are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and $a + b = 1$, then find the value of b.

Sol. $a + b + c = -1 \Rightarrow c = -2$ & $8 + 4a + 2b + c = 0$

$\Rightarrow 4a + 2b = -6 \Rightarrow 2a + b = -3$

$\Rightarrow a = -4,\ b = 5$

Example 8: A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by$(x-1)$; $(x+2)$ & $(x+1)$ respectively. Find the remainder, if the polynomial is divided by, $(x^2-1)(x+2)$.

Sol. $f(x) = Q_1(x-1)+2 = Q_2(x+2)+1 = Q_3(x+1)-1$

$\Rightarrow f(1) = 2\ ;\ f(-2) = 1\ ;\ f(-1) = -1$

Let $f(x) = Q_r(x^2-1)(x+2) + ax^2 + bx + c$

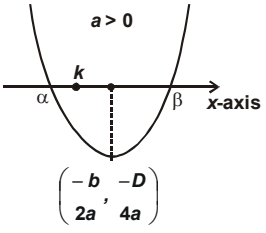
Hence $a + b + c = 2\ ;\ 4a - 2b + c = 1$

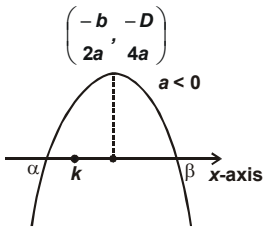
and $a - b + c = -1$

LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c,\ a, b, c \in R,\ a \neq 0$ and α, β be roots of $f(x) = 0$

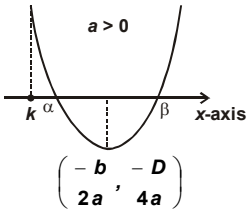
1. A real number k lies between the roots of $f(x) = 0$

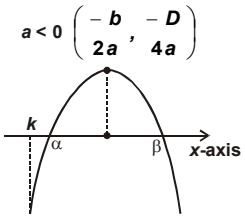




(*i*) $D > 0$, (*ii*) $af(k) < 0$, where $\alpha < \beta$

2. If both roots of quadratic equation $f(x) = 0$ are greater than k, then

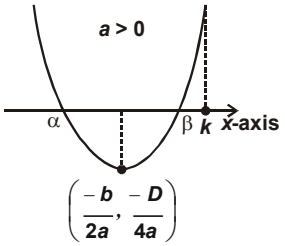


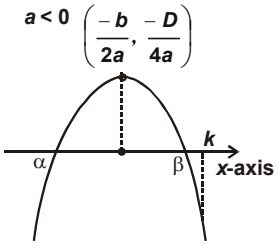


(*i*) $D \geq 0$, (*ii*) $af(k) > 0$

(*iii*) $k < \frac{-b}{2a}$, where $\alpha \leq \beta$

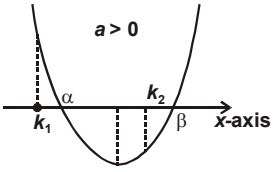
3. If both roots are less than real number k, then

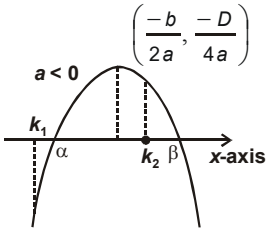




(*i*) $D \geq 0$, (*ii*) $af(k) > 0$ (*iii*) $k > \frac{-b}{2a}$, where $\alpha \leq \beta$

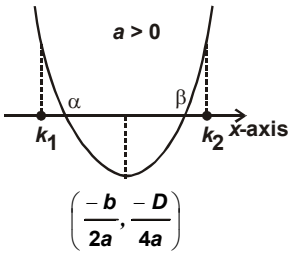
4. Exactly one root lies between real numbers k_1 and k_2, where $k_1 < k_2$.

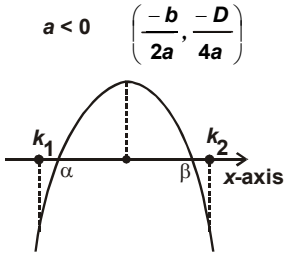




(i) $D > 0$, (ii) $f(k_1).f(k_2) < 0$, where $\alpha < \beta$

5. If both roots of $f(x) = 0$ are confined between real numbers k_1 and k_2, where $k_1 < k_2$.

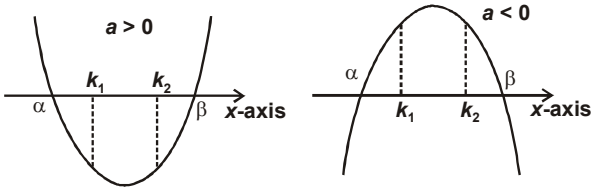




(*i*) $D \geq 0$, (*ii*) $af(k_1) > 0$

(*iii*) $af(k_2) > 0$, (*iv*) $k_1 < \frac{-b}{2a} < k_2$, where $\alpha \leq \beta$

6. If real numbers k_1 and k_2 lie between roots of $f(x) = 0$, where $k_1 < k_2$.



(*i*) $D > 0$, (*ii*) $af(k_1) < 0$

(*iii*) $af(k_2) < 0$, where $\alpha < \beta$

TRAIN YOUR BRAIN

Example 9: Let $x^2 - (m-3)x + m = 0$ ($m \in R$) be a quadratic equation, then find the values of 'm' for which

(*a*) both the roots are greater than 2.

(*b*) both roots are positive.

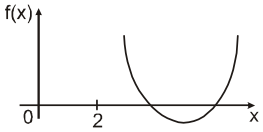
(*c*) one root is positive and other is negative.

(*d*) One root is greater than 2 and other smaller than 1

(*e*) Roots are equal in magnitude and opposite in sign.

(*f*) both roots lie in the interval (1, 2)

Sol. (*a*)



Condition - I: $D \geq 0 \Rightarrow (m-3)^2 - 4m \geq 0$

$\Rightarrow \quad m^2 - 10m + 9 \geq 0$

$\Rightarrow \quad (m-1)(m-9) \geq 0$

$\Rightarrow \quad m \in (-\infty, 1] \cup [9, \infty)$ ...(*i*)

Condition - II: $f(2) > 0$

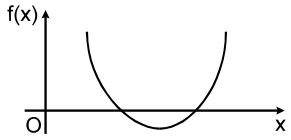
$\Rightarrow \quad 4 - (m-3)2 + m > 0$

$\Rightarrow \quad m < 10$ ...(*ii*)

Condition - III: $-\dfrac{b}{2a} > 2 \Rightarrow m > 7$ ...(*iii*)

Intersection of (*i*), (*ii*) and (*iii*) gives $m \in [9, 10)$

(*b*)



Condition - I $D \geq 0$

$\Rightarrow \quad m \in (-\infty, 1] \cup [9, \infty)$

Condition - II $f(0) > 0$

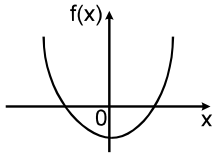
$\Rightarrow \quad m > 0$

Condition - III $-\dfrac{b}{2a} > 0 \Rightarrow \dfrac{m-3}{2} > 0$

$\Rightarrow \quad m > 3$

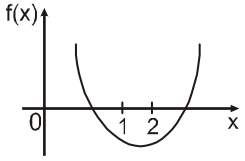
intersection gives $m \in [9, \infty)$ **Ans.**

(*c*)



Condition - I $f(0) < 0 \Rightarrow m < 0$ **Ans.**

(*d*)



Condition - I $f(1) < 0 \Rightarrow 4 < 0 \Rightarrow m \in \phi$

Condition - II $f(2) < 0 \Rightarrow m > 10$

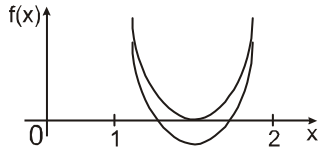
Intersection gives $m \in \phi$ **Ans.**

(*e*) sum of roots $= 0 \Rightarrow m = 3$

and $f(0) < 0 \Rightarrow m < 0$

$\therefore \quad m \in \phi$ **Ans.**

(*f*)



Condition - I $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition - II $f(1) > 0 \Rightarrow 1 - (m-3) + m > 0 \Rightarrow 4 > 0$ which is true $\forall\ m \in R$

Condition - III $f(2) > 0 \Rightarrow m < 10$

Condition - IV $1 < -\dfrac{b}{2a} < 2$

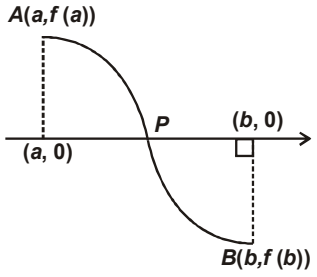
$\Rightarrow 1 < \dfrac{m-3}{2} < 2$

$\Rightarrow 5 < m < 7$

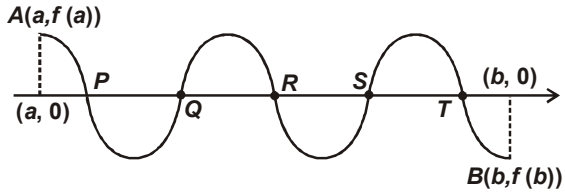
intersection gives $m \in \phi$ **Ans.**

NUMBER OF ROOTS OF A POLYNOMIAL EQUATION

1. If $f(x)$ is an increasing function in $[a, b]$, then $f(x) = 0$ will have atmost one root in $[a, b]$.
2. Let $f(x) = 0$ be a polynomial equation and a, b are two real numbers. Then $f(x) = 0$ will have at least one real root or odd number of real roots in (a, b) if $f(a)$ and $f(b)$ $(a < b)$ are of opposite signs.

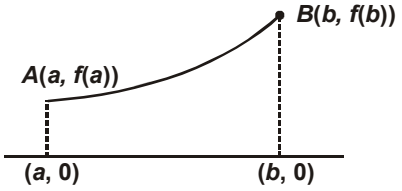


One real root

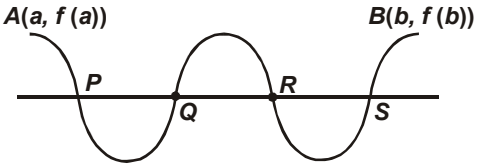


Odd number of real roots

But if $f(a)$ and $f(b)$ are of same signs, then either $f(x) = 0$ has no real root or even number of real roots in (a, b).



No real root



Even number of real roots

3. If the equation $f(x) = 0$ has two real roots a and b then $f'(x) = 0$ will have atleast one real root lying between a and b (using Rolle's Theorem).

EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

1. **Reciprocal equation** of the standard form can be reduced to an equation of half its dimension.
2. **Equations of the form**
   (a) $(x-a)(x-b)(x-c)(x-d) = A$ where $a < b < c < d$ can be solved by change of variable.
   $$y = x - \left(\frac{a+b+c+d}{4}\right)$$
   (b) $(x-a)(x-b)(x-c)(x-d) = Ax^2$ where $ab = cd$ can be solved by assumption $y = x + \frac{ab}{x}$.
3. **For the equation of the type $(x-a)^4 + (x-b)^4 = A$**
   Substitute $y = \frac{(x-a)+(x-b)}{2}$

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Example 10: Solve $2x^4 + x^3 - 11x^2 + x + 2 = 0$

Sol. Since $x = 0$ is not a solution hence, divide by x^2
We get,

$$2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$$

or $$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0$$

Let $x + \frac{1}{x} = y \Rightarrow 2(y^2 - 2) + y - 11 = 0$

or $2y^2 + y - 15 = 0 \Rightarrow y = -3, \frac{5}{2}$

Corresponding values of x are $\frac{1}{2}, 2, \frac{-3 \pm \sqrt{5}}{2}$

Example 11: $(x+4)(x+6)(x+8)(x+12) = 1680x^2$

Sol. Here $(-4 \times -12) = 48 = (-6) \times (-8)$

$(x+4)(x+12)(x+6)(x+8) = 1680x^2$

$\Rightarrow (x^2 + 16x + 48)(x^2 + 14x + 48) = 1680x^2$

$\Rightarrow \left(x + 16 + \frac{48}{x}\right)\left(x + 14 + \frac{48}{x}\right) = 1680$ $(\because x \neq 0)$

Let $y = x + \frac{48}{x}$ $\Rightarrow (y+16)(y+14) = 1680$

$\Rightarrow y^2 + 30y - 1680 + 224 = 0$

$\Rightarrow y^2 + 30y - 1456 = 0 \Rightarrow y^2 + 56y - 26y - 1456 = 0$

$\Rightarrow (y+56)(y-26) = 0$ $\Rightarrow y = 26, -56$

$\therefore x + \frac{48}{x} = 26$ and $x + \frac{48}{x} = -56$

$\Rightarrow x^2 - 26x + 48 = 0 \Rightarrow x^2 + 56x + 48 = 0$

$\Rightarrow x = 2, 24 \Rightarrow x = \frac{-56 \pm \sqrt{2944}}{2} = -28 \pm \sqrt{736}$

$= -28 \pm 4\sqrt{46}$

$\therefore x = \left\{2, 24, -28 - 4\sqrt{46}, -28 + 4\sqrt{46}\right\}$

IMPORTANT THEOREMS AND RESULTS

1. If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$ then α is a root of the equation $f(x) = 0$.
2. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first term is positive.
3. Every equation of an even degree whose last term is negative has at least two real roots, one positive and one negative, provided that the coefficient of the first term is positive.
4. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation are having positive sign.
5. If $x = \alpha$ is root repeated m times in $f(x) = 0$, ($f(x) = 0$ is an nth degree equation in x) then
   $$f(x) = (x - \alpha)^m g(x)$$
   when $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ times in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0$,...$(m - (m - 1))$ times in $f^{m-1}(x) = 0$.

6. The condition that a quadratic function $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factor if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$\text{or} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ is 0.}$$

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Example 12: If $0 \le x \le \pi$, then the solution of the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$ is given by x equal to

(*a*) $\frac{\pi}{6}, \frac{\pi}{3}$ (*b*) $\frac{\pi}{3}, \frac{\pi}{2}$

(*c*) $\frac{\pi}{6}, \frac{\pi}{2}$ (*d*) None of these

Sol. Let $16^{\sin^2 x} = y$, then $16^{\cos^2 x} = 16^{1-\sin^2 x} = \frac{16}{y}$

Hence, $y + \frac{16}{y} = 10$

$\Rightarrow y^2 - 10y + 16 = 0$ or $y = 2, 8$

Now $16^{\sin^2 x} = 2 \Rightarrow 4\sin^2 x = 1$

$\therefore \sin x = \pm\frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

Also $16^{\sin^2 x} = 8 \Rightarrow 4\sin^2 x = 3$

$\therefore \sin x = \pm\frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$

Hence **(a)** is the correct answer

Topicwise Questions

Quadratic Equation & Nature of Roots

1. If $a + b + c = 0$, and $a, b, c \in R$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are

(*a*) Equal (*b*) Imaginary

(*c*) Real (*d*) Both (*a*) and (*b*)

2. The roots of the given equation $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are
[Where $a \neq b$]

(*a*) Rational (*b*) Irrational

(*c*) Real (*d*) Imaginary

3. If $a, b, c \in Q$, then roots of the equation $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are

(*a*) Rational (*b*) Non-real

(*c*) Irrational (*d*) Equal

4. The value of k for which the quadratic equation, $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are

(*a*) $-11, -3$ (*b*) $5, 7$

(*c*) $5, -7$ (*d*) $-7, 25$

5. $x^2 + x + 1 + 2k(x^2 - x - 1) = 0$ is a perfect square for how many values of k

(*a*) 2 (*b*) 0

(*c*) 1 (*d*) 3

6. If $x^2 - 3x + 2$ be a factor of $x^4 - px^2 + q$, then $(pq) =$

(*a*) $(3, 4)$ (*b*) $(4, 5)$

(*c*) $(4, 3)$ (*d*) $(5, 4)$

7. The roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are

(*a*) $\frac{c-a}{b-c}, 1$ (*b*) $\frac{a-b}{b-c}, 1$

(*c*) $\frac{b-c}{a-b}, 1$ (*d*) $\frac{c-a}{a-b}, 1$

8. If a, b, c are integers and $b^2 = 4(ac + 5d^2)$, $d \in N$, then roots of the equation $ax^2 + bx + c = 0$ are

(*a*) Irrational (*b*) Rational & different

(*c*) Complex conjugate (*d*) Rational & equal

9. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has

(*a*) real roots (*b*) imaginary roots

(*c*) exactly one root (*d*) none of these

10. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots, is

(*a*) 4 (*b*) 6

(*c*) 8 (*d*) 3

11. The entire graph of the expression $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if

(*a*) $k < 7$ (*b*) $-5 < k < 7$

(*c*) $k > -5$ (*d*) None

12. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is:

(*a*) positive (*b*) negative

(*c*) zero (*d*) depends on the sign of b

13. If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is

(*a*) $\frac{3}{2}$ (*b*) $\frac{9}{4}$

(*c*) $-\frac{9}{4}$ (*d*) 1

14. If both roots of the quadratic equation $(2 - x)(x + 1) = p$ are distinct & positive, then p must lie in the interval

(*a*) $(2, \infty)$ (*b*) $(2, 9/4)$

(*c*) $(-\infty, -2)$ (*d*) $(-\infty, \infty)$

15. If $(1 - p)$ is root of quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are

(*a*) $0, 1$ (*b*) $-1, 1$

(*c*) $0, -1$ (*d*) $-1, 2$

Sum and Product of Roots

16. If one root of $5x^2 + 13x + k = 0$ is reciprocal of the other, then $k =$

(*a*) 0 (*b*) 5

(*c*) 1/6 (*d*) 6

17. If α and β are the roots of the equation $4x^2 + 3x + 7 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta} =$

(*a*) $-\frac{3}{7}$ (*b*) $\frac{3}{7}$

(*c*) $-\frac{3}{5}$ (*d*) $\frac{3}{5}$

18. If the sum of the roots of the equation $ax^2 + bx + c = 0$ be equal to the sum of their squares, then

(*a*) $a(a + b) = 2bc$ (*b*) $c(a + c) = 2ab$

(*c*) $b(a + b) = 2ac$ (*d*) $b(a + b) = ac$

19. If α, β be the roots of the equation $x^2 - 2x + 3 = 0$, then the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is

(*a*) $x^2 + 2x + 1 = 0$ (*b*) $9x^2 + 2x + 1 = 0$

(*c*) $9x^2 - 2x + 1 = 0$ (*d*) $9x^2 + 2x - 1 = 0$

20. If the product of roots of the equation, $mx^2 + 6x + (2m - 1) = 0$ is -1, then the value of m will be

(*a*) 1 (*b*) -1

(*c*) $\frac{1}{3}$ (*d*) $-\frac{1}{3}$

21. If α and β are the roots of the equation $x^2-4x+1=0$ the value of $\alpha^3+\beta^3$ is

(*a*) 76 (*b*) 52

(*c*) –52 (*d*) –76

22. If α, β, γ are the roots of the equation $x^3+x+1=0$, then the value of $\alpha^3\beta^3\gamma^3$

(*a*) 0 (*b*) – 3

(*c*) 3 (*d*) – 1

23. If a, b are the roots of quadratic equation $x^2+px+q=0$ and g, d are the roots of $x^2+px-r=0$, then $(a-g).(a-d)$ is equal to:

(*a*) $q+r$ (*b*) $q-r$

(*c*) $-(q+r)$ (*d*) $-(p+q+r)$

24. Two real numbers a & b are such that $a+b=3$ & $|a-b|=4$, then a & b are the roots of the quadratic equation:

(*a*) $4x^2-12x-7=0$ (*b*) $4x^2-12x+7=0$

(*c*) $4x^2-12x+25=0$ (*d*) None of these

25. Let conditions C_1 and C_2 be defined as follows: $C_1: b^2-4ac\geq 0$, C_2 : a, – b, c are of same sign. The roots of $ax^2+bx+c=0$ are real and positive, if

(*a*) both C_1 and C_2 are satisfied

(*b*) only C_2 is satisfied

(*c*) only C_1 is satisfied

(*d*) None of these

26. If α, β are roots of the equation $ax^2+bx+c=0$, then the value of $\alpha^3+\beta^3$ is

(*a*) $\frac{3abc+b^3}{a}$ (*b*) $\frac{a^3+b^3}{3abc}$

(*c*) $\frac{3abc-b^3}{a^3}$ (*d*) $\frac{-(3abc+b^3)}{a^3}$

Common Roots

27. If both the roots of $k(6x^2+3)+rx+2x^2-1=0$ and $6k(2x^2+1)+px+4x^2-2=0$ are common, then $2r-p$ is equal to

(*a*) –1 (*b*) 0

(*c*) 1 (*d*) 2

28. If the equation $x^2+px+q=0$ and $x^2+qx+p=0$, have a common root,(Where $p\neq q$) then $p+q+1=$

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) – 1

29. If a, b, p, q are non-zero real numbers, then the equations $2a^2x^2-2abx+b^2=0$ and $p^2x^2+2pqx+q^2=0$ have:

(*a*) no common root

(*b*) one common root if $2a^2+b^2=p^2+q^2$

(*c*) two common roots if $3pq=2ab$

(*d*) two common roots if $3qb=2ap$

Theory of Equation and Identity, Inequalities

30. If x is real and satisfies $x+2>\sqrt{x+4}$, then

(*a*) $x<-2$ (*b*) $x>0$

(*c*) $-3<x<0$ (*d*) $-3<x<4$

31. The set of all real numbers x for which $x^2-|x+2|+x>0$ is

(*a*) $(-\infty,-2)\cup(2,\infty)$ (*b*) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$

(*c*) $(-\infty,-1)\cup(1,\infty)$ (*d*) $(\sqrt{2},\infty)$

32. Number of values of 'p' for which the equation $(p^2-3p+2)x^2-(p^2-5p+4)x+p-p^2=0$ possess more than two roots, is:

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) None

33. The number of the integer solutions of $x^2+9<(x+3)^2<8x+25$ is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) None of these

34. The complete set of values of 'x' which satisfy the inequations : $5x+2<3x+8$ and $\frac{x+2}{x-1}<4$ is

(*a*) $(-\infty, 1)$ (*b*) $(2, 3)$

(*c*) $(-\infty, 3)$ (*d*) $(-\infty, 1)\cup(2, 3)$

35. The complete solution set of the inequality $\frac{x^4-3x^3+2x^2}{x^2-x-30}\geq 0$ is:

(*a*) $(-\infty,-5)\cup(1,2)\cup(6,\infty)\cup\{0\}$

(*b*) $(-\infty,-5)\cup[1,2]\cup(6,\infty)\cup\{0\}$

(*c*) $(-\infty,-5]\cup[1,2]\cup[6,\infty)\cup\{0\}$

(*d*) none of these

36. If the inequality $(m-2)x^2+8x+m+4>0$ is satisfied for all $x\in R$, then the least integral 'm' is:

(*a*) 4 (*b*) 5

(*c*) 6 (*d*) none

37. The complete set of real 'x' satisfying $||x-1|-1|\leq 1$ is:

(*a*) $[0,2]$ (*b*) $[-1,3]$

(*c*) $[-1,1]$ (*d*) $[1,3]$

38. If $\log_{1/3}\frac{3x-1}{x+2}$ is less than unity, then 'x' must lie in the interval:

(*a*) $(-\infty,-2)\cup(5/8,\infty)$ (*b*) $(-2, 5/8)$

(*c*) $(-\infty,-2)\cup(1/3, 5/8)$ (*d*) $(-2, 1/3)$

39. Solution set of the inequality $2-\log_2(x^2+3x)\geq 0$ is:

(*a*) $[-4, 1]$ (*b*) $[-4,-3)\cup(0, 1]$

(*c*) $(-\infty,-3)\cup(1,\infty)$ (*d*) $(-\infty,-4)\cup[1,\infty)$

40. The set of all the solutions of the inequality $\log_{1-x}(x-2)\geq 0$ is

(*a*) $(-\infty, 0)$ (*b*) $(2,\infty)$

(*c*) $(-\infty, 1)$ (*d*) ϕ

41. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
(*a*) $(2, \infty)$ (*b*) $(1, 2)$
(*c*) $(-2, -1)$ (*d*) None of these

42. If $\log_{0.5}\log_5(x^2-4) > \log_{0.5}1$, then 'x' lies in the interval
(*a*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3)$
(*b*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3\sqrt{5})$
(*c*) $(\sqrt{5}, 3\sqrt{5})$
(*d*) ϕ

43. The set of all solutions of the inequality $(1/2)^{x^2-2x} < 1/4$ contains the set
(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 1)$
(*c*) $(1, \infty)$ (*d*) $(3, \infty)$

44. If $\frac{6x^2-5x-3}{x^2-2x+6} \le 4$, then least and the highest values of $4x^2$ are
(*a*) 0 & 81 (*b*) 9 & 81
(*c*) 36 & 81 (*d*) None of these

45. If two roots of the equation $x^3 - px^2 + qx - r = 0$ are equal in magnitude but opposite in sign, then:
(*a*) $pr = q$ (*b*) $qr = p$
(*c*) $pq = r$ (*d*) None

46. If α, β & γ are the roots of the equation $x^3 - x - 1 = 0$ then, $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$ has the value equal to:
(*a*) zero (*b*) -1
(*c*) -7 (*d*) 1

47. For what value of a and b the equation $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ has four real positive roots?
(*a*) $(-6, -4)$ (*b*) $(-6, 5)$
(*c*) $(-6, 4)$ (*d*) $(6, -4)$

48. If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$ is
(*a*) $ab\,x^2 - (a+b)\,cx + (a+b)^2 = 0$
(*b*) $ac\,x^2 - (a+c)\,bx + (a+c)^2 = 0$
(*c*) $ac\,x^2\,(a+c)\,bx - (a+c)^2 = 0$
(*d*) None of these

49. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains
(*a*) $(-\infty, -3/2)$ (*b*) $(-3/2, 1/4)$
(*c*) $(-1/4, 1/2)$ (*d*) $(-1/2, 3)$

Max and Min Value, Factorization

50. If x is real, then the maximum and minimum values of the expression $\frac{x^2-3x+4}{x^2+3x+4}$ will be
(*a*) $2, 1$ (*b*) $5, \frac{1}{5}$
(*c*) $7, \frac{1}{7}$ (*d*) $2, 7$

51. The smallest value of $x^2 - 3x + 3$ in the interval $(-3, 3/2)$ is
(*a*) 3/4 (*b*) 5·
(*c*) –15 (*d*) –20

52. If $y = -2x^2 - 6x + 9$, (for $x \in R$) then
(*a*) maximum value of y is –11 and it occurs at $x = 2$
(*b*) minimum value of y is –11 and it occurs at $x = 2$
(*c*) maximum value of y is 13.5 and it occurs at $x = -1.5$
(*d*) minimum value of y is 13.5 and it occurs at $x = -1.5$

53. If 'x' is real and $k = \frac{x^2-x+1}{x^2+x+1}$, then:
(*a*) $\frac{1}{3} \le k \le 3$ (*b*) $k \ge 5$
(*c*) $k \le 0$ (*d*) None of these

54. Consider $y = \frac{2x}{1+x^2}$, where x is real, then the range of expression $y^2 + y - 2$ is
(*a*) $[-1, 1]$ (*b*) $[0, 1]$
(*c*) $[-9/4, 0]$ (*d*) $[-9/4, 1]$

55. The values of x and y besides y can satisfy the equation (x, y ∈ real numbers) $x^2 - xy + y^2 - 4x - 4y + 16 = 0$
(*a*) 2, 2 (*b*) 4, 4
(*c*) 3, 3 (*d*) None of these

56. If x is real, then $\frac{x^2-x+c}{x^2+x+2c}$ can take all real values if
(*a*) $c \in [0, 6]$ (*b*) $c \in [-6, 0]$
(*c*) $c \in (-\infty, -6) \cup (0, \infty)$ (*d*) $c \in (-6, 0)$

Location of Roots

57. The real values of 'a' for which the quadratic equation $2x^2 - (a^3 + 8a - 1)\,x + a^2 - 4a = 0$ possesses roots of opposite sign is given by:
(*a*) $a > 5$ (*b*) $0 < a < 4$
(*c*) $a > 0$ (*d*) $a > 7$

58. If a, b are the roots of the quadratic equation $x^2 - 2p(x-4) - 15 = 0$, then the set of values of 'p' for which one root is less than 1 & the other root is greater than 2 is:
(*a*) $(7/3, \infty)$ (*b*) $(-\infty, 7/3)$
(*c*) $x \in R$ (*d*) None of these

59. The values of k, for which the equation $x^2 + 2(k-1)\,x + k + 5 = 0$ possess atleast one positive root, are
(*a*) $[4, \infty)$ (*b*) $(-\infty, -1] \cup [4, \infty)$
(*c*) $[-1, 4]$ (*d*) $(-\infty, -1]$

Learning Plus

1. If $a > 0, b > 0, c > 0$ then both the roots of the equation $ax^2 + bx + c = 0$
(*a*) Are real and negative
(*b*) Have negative real parts
(*c*) Are rational numbers
(*d*) Both (*a*) and (*c*)

2. The value of k for which the equation $(k-2)x^2 + 8x + k + 4 = 0$ has both real, distinct and negative is -
(*a*) 0 (*b*) 2
(*c*) 3 (*d*) -4

3. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is
(*a*) $acx^2 + (a+c)bx + (a+c)^2 = 0$
(*b*) $abx^2 + (a+c)bx + (a+c)^2 = 0$
(*c*) $acx^2 + (a+b)cx + (a+c)^2 = 0$
(*d*) $acx^3 + (a+c)bx + (a+c)^3$

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{\alpha}{a\beta + b} + \frac{\beta}{a\alpha + b} =$
(*a*) $\frac{2}{a}$ (*b*) $\frac{2}{b}$
(*c*) $\frac{2}{c}$ (*d*) $-\frac{2}{a}$

5. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2 : 3, then m =
(*a*) $5\sqrt{10}$ (*b*) $3\sqrt{10}$
(*c*) $2\sqrt{10}$ (*d*) $10\sqrt{5}$

6. $x^2 - 11x + a$ and $x^2 - 14x + 2a$ will have a common factor, if $a =$
(*a*) 24 (*b*) 0, 24
(*c*) 3, 24 (*d*) 0, 3

7. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then $(\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1}$
(*a*) 2 (*b*) 3
(*c*) 4 (*d*) 5

8. If the roots of $x^2 + x + a = 0$ exceed a , then
(*a*) $2 < a < 3$ (*b*) $a > 3$
(*c*) $-3 < a < 3$ (*d*) $a < -2$

9. The value of *p* for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2, lies in
(*a*) (4/5, 2) (*b*) $(2, \infty)$
(*c*) $(-1, -4/5)$ (*d*) $(-\infty, -1)$

10. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has -
(*a*) No root (*b*) One root
(*c*) Two equal root (*d*) Infinitely many roots

11. The number of quardratic equations which are unchanged by squaring their roots
(*a*) 2 (*b*) 3
(*c*) 4 (*d*) None of these

12. For what value of the curve $y = x^2 + ax + 25$ touches the x-axis
(*a*) 0 (*b*) ± 5
(*c*) ± 10 (*d*) None of these

13. If $b^2 < 2ac$, then equation $ax^3 + bx^2 + cx + d = 0$ has
(*a*) exactly one real roots
(*b*) Has three real roots
(*c*) at least two roots
(*d*) None of these

14. Let α and β are the roots of the equation $x^2 + x + 1 = 0$ then
(*a*) $\alpha^2 + \beta^2 = 4$ (*b*) $(\alpha - \beta)^2 = 3$
(*c*) $\alpha^3 + \beta^3 = 2$ (*d*) $\alpha^4 + \beta^4 = 1$

15. If $\sec\alpha, \tan\alpha$ are roots of $ax^2 + bx + c = 0$, then
(*a*) $a^4 - b^4 + 4ab^2c = 0$ (*b*) $a^4 + b^4 - 4ab^2c = 0$
(*c*) $a^2 - b^2 = 4ac$ (*d*) $a^2 + b^2 = ac$

16. Let α, β be the roots of $ax^2 + bx + c = 0$, γ, δ be the roots of $px^2 + qx + r = 0$ and D_1 and D_2 be their respective discriminant. If $\alpha, \beta, \gamma, \delta$, are in A.P., then the ratio $D_1 : D_2$ is equal to
(*a*) $\frac{a^2}{b^2}$ (*b*) $\frac{a^2}{p^2}$
(*c*) $\frac{b^2}{q^2}$ (*d*) $\frac{c^2}{r^2}$

17. If one root of the equation $(l-m)x^2 + lx + 1 = 0$ is double the other and if *l* is real, then the greatest value of m is
(*a*) $\frac{9}{8}$ (*b*) $\frac{7}{8}$
(*c*) $\frac{8}{9}$ (*d*) $\frac{8}{9}$

18. If p, q, r are real numbers satisfying the condition $p + q + r = 0$, then the roots of the quadratic equation $3px^2 + 5qx + 7r = 0$ are

(a) Positive (b) Negative

(c) Real and distinct (d) Imaginary

19. If a, b, c are in G.P., then the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) $ab = cd$

20. If $a < b < c < d$ and $K > 0$, then the quadratic equation $(x - a)(x - c) + k(x - b)(x - d) = 0$ has

(a) All roots real and distinct

(b) All roots real but not necessarily distinct

(c) All root real and negative

(d) May be imaginary

21. The number of integers satisfying the inequality $\frac{x}{x+6} \le \frac{1}{x}$ is:

(a) 7 (b) 8

(c) 9 (d) 3

22. Sum of values of x and y satisfying the equation $3^x - 4^y = 77$; $3^{x/2} - 2^y = 7$ is:

(a) 2 (b) 3

(c) 4 (d) 5

23. If the value of $m^4 + \frac{1}{m^4} = 119$, then the value of $\left|m^3 - \frac{1}{m^3}\right| =$

(a) 11 (b) 18

(c) 24 (d) 36

24. The number of integral roots of the equation $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0$ is:

(a) 0 (b) 2

(c) 4 (d) 6

25. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots is:

(a) $-2(p^2 + q^2)$ (b) $-(p^2 + q^2)$

(c) $-\frac{(p^2 + q^2)}{2}$ (d) $-pq$

26. If $\alpha \neq \beta$ but $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, then the equation with roots $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ is:

(a) $3x^2 - 25x + 3 = 0$

(b) $x^2 + 5x - 3 = 0$

(c) $x^2 - 5x + 3 = 0$

(d) $3x^2 - 19x + 3 = 0$

27. Minimum possible number of positive root of the quadratic equation $x^2 - (1 + \lambda)x + \lambda - 2 = 0, \lambda \in R$:

(a) 2 (b) 0

(c) 1 (d) Can not be determined

28. If $\alpha, \beta, \gamma, \delta \in R$ satisty

$$\frac{(\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2}{\alpha + \beta + \gamma + \delta} = 4$$

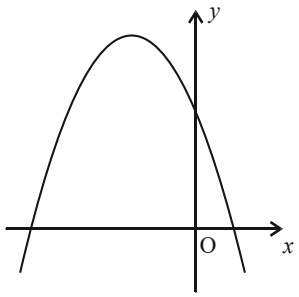
If biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has the roots $\left(\alpha + \frac{1}{\beta} - 1\right), \left(\beta + \frac{1}{\gamma} - 1\right), \left(\gamma + \frac{1}{\delta} - 1\right), \left(\delta + \frac{1}{\alpha} - 1\right)$.

Then the value of a_2/a_0 is:

(a) 4 (b) –4

(c) 6 (d) None the these

29. If graph of the quadratic $y = ax^2 + bx + c$ is given below:



(a) $a > 0, b > 0, c > 0$

(b) $a < 0, b > 0, c < 0$

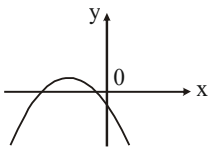
(c) $a < 0, b < 0, c > 0$

(d) $a < 0, b < 0, c < 0$

Advanced Level Multiconcept Questions

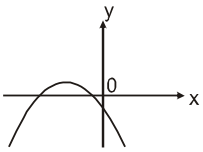
MCQ/COMPREHENSION/MATCHING/ NUMERICAL

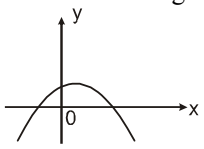
1. The graph of the quadratic polynomial $y = ax^2 + bx + c$ is as shown in the figure. Then:

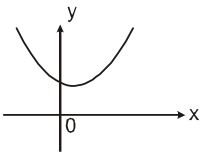


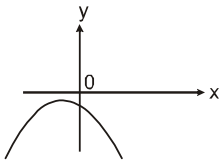
(*a*) $b^2 - 4ac > 0$ (*b*) $b < 0$
(*c*) $a > 0$ (*d*) $c < 0$

2. For which of the following graphs of the quadratic expression $y = ax^2 + bx + c$, the product a b c is negative?

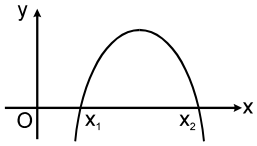
(*a*) 

(*b*) 

(*c*) 

(*d*) 

3. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then



(*a*) $a < 0$
(*b*) $b^2 < 4ac$
(*c*) $c > 0$
(*d*) a and b are of opposite sign

4. For the equation $|x|^2 + |x| - 6 = 0$, the correct statement (s) is (are):
(*a*) sum of roots is 0 (*b*) product of roots is – 4
(*c*) there are 4 roots (*d*) there are only 2 roots

5. If a, b are the roots of $ax^2 + bx + c = 0$, and a + h, b + h are the roots of $px^2 + qx + r = 0$, (where $h \neq 0$), then

(*a*) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ (*b*) $h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$

(*c*) $h = \frac{1}{2}\left(\frac{b}{a} + \frac{q}{p}\right)$ (*d*) $\frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$

6. If a, b are non-zero real numbers and α, β the roots of $x^2 + ax + b = 0$, then
(*a*) α^2, β^2 are the roots of $x^2 - (2b - a^2)x + a^2 = 0$
(*b*) $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $bx^2 + ax + 1 = 0$
(*c*) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $bx^2 + (2b - a^2)x + b = 0$
(*d*) $(\alpha - 1), (\beta - 1)$ are the roots of the equation $x^2 + x(a + 2) + 1 + a + b = 0$

7. $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d = 0$, then the real root of above equation is
$(a, b, c, d \in R)$
(*a*) $-d/a$ (*b*) d/a
(*c*) $(b - a)/a$ (*d*) $(a - b)/a$

8. If $\frac{1}{2} \leq \log_{0.1} x \leq 2$, then
(*a*) maximum value of x is $\frac{1}{\sqrt{10}}$
(*b*) x lies between $\frac{1}{100}$ and $\frac{1}{\sqrt{10}}$
(*c*) minimum value of x is $\frac{1}{\sqrt{10}}$
(*d*) minimum value of x is $\frac{1}{100}$

9. If the quadratic equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, then the equation containing their other roots is/are:
(*a*) $x^2 + a(b + c)x - a^2bc = 0$
(*b*) $x^2 - a(b + c)x + a^2bc = 0$
(*c*) $a(b + c)x^2 - (b + c)x + abc = 0$
(*d*) $a(b + c)x^2 + (b + c)x - abc = 0$

10. If the quadratic equations $ax^2 + bx + c = 0$ $(a, b, c \in R, a \neq 0)$ and $x^2 + 4x + 5 = 0$ have a common root, then a, b, c must satisfy the relations:
(*a*) $a > b > c$
(*b*) $a < b < c$
(*c*) $a = k; b = 4k; c = 5k$ $(k \in R, k \neq 0)$
(*d*) $b^2 - 4ac$ is negative.

11. If α, β are the real and distinct roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always
(*a*) two real roots
(*b*) two negative roots
(*c*) two positive roots
(*d*) one positive root and one negative root

12. If $(x^2+x+1)+(x^2+2x+3)+(x^2+3x+5)+.....+(x^2+20x+39)=4500$, then x is equal to
(*a*) 10 (*b*) -10
(*c*) 20.5 (*d*) -20.5

13. If roots of equation, $x^3 + bx^2 + cx - 1 = 0$ forms an increasing G.P., then
(*a*) $b + c = 0$
(*b*) $b \in (-\infty, -3)$
(*c*) one of the roots = 1
(*d*) one root is smaller than 1 & other > 1

14. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has
(*a*) exactly one real root in (2, 3)
(*b*) exactly one real root in (3, 4)
(*c*) at least one real root in (2, 3)
(*d*) None of these

Paragraph for question no. 15 & 16: A quadratic polynomial $f(x) = px^2 + qx + r$ has two distinct roots x_1 & x_2. If its vertex (of parabola) is V and x_1, x_3, x_2 are in A.P., then answer the following

Comprehension – 1 (No. 15 to 17)

Consider the equation $x^4 - \lambda x^2 + 9 = 0$. This can be solved by substituting $x^2 = t$ such equations are called as pseudo quadratic equations.

15. If the equation has four real and distinct roots, then λ lies in the interval
(*a*) $(-\infty, -6) \cup (6, \infty)$ (*b*) $(0, \infty)$
(*c*) $(6, \infty)$ (*d*) $(-\infty, -6)$

16. If the equation has no real root, then λ lies in the interval
(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 6)$
(*c*) $(6, \infty)$ (*d*) $(0, \infty)$

17. If the equation has only two real roots, then set of values of λ is
(*a*) $(-\infty, -6)$ (*b*) $(-6, 6)$
(*c*) $\{6\}$ (*d*) ϕ

18. Match the column.

Column – I	**Column – II**
(*a*) If α, $\alpha + 4$ are two roots of $x^2 - 8x + k = 0$, then possible value of k is	(P) 2
(*b*) Number of real roots of equation $x^2 - 5\lvert x\rvert + 6 = 0$ are 'n', then value of $\frac{n}{2}$ is	(Q) 3
(*c*) If $3 - i$ is a root of $x^2 + ax + b = 0$ ($a, b \in R$), then b is	(R) 12
(*d*) If both roots of $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then 'k' may be equal to	(S) 10

NUMERICAL BASED QUESTIONS

19. Find number of integer roots of equation $x(x+1)(x+2)(x+3) = 120$.

20. Find product of all real values of x satisfying $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$

21. The least prime integral value of '2a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$ is

22. If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$. Then find the value of $(a-c)(b-c)(a+d)(b+d)/(q^2-p^2)$.

23. α, β are roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 are the two values of λ for which the roots α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 4$, then the value of $\left(\frac{\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}}{14}\right)$ is

24. Let α, β be the roots of the equation $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 - ax + b - 2 = 0$. If $\alpha\beta\gamma\delta = 24$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{5}{6}$, then find the value of a.

25. The least value of expression $x^2 + 2xy + 2y^2 + 4y + 7$ is:

26. If $a > b > 0$ and $a^3 + b^3 + 27ab = 729$ then the quadratic equation $ax^2 + bx - 9 = 0$ has roots α, β ($\alpha < \beta$). Find the value of $4\beta - a\alpha$.

27. Let α and β be roots of $x^2 - 6(t^2 - 2t + 2)x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then find the minimum value of $\frac{a_{100} - 2a_{98}}{a_{99}}$ (where $t \in R$)

28. If roots of the equation $x^2 - 10ax - 11b = 0$ are c and d and those of $x^2 - 10cx - 11d = 0$ are a and b, then find the value of $\frac{a+b+c+d}{110}$. (where a, b, c, d are all distinct numbers)

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60}=1$ is: **[JEE Main-2016]**

(a) 3 (b) –4
(c) 6 (d) 5

2. If, for a positive integer n, the quadratic equation, $x(x+1)+(x+1)(x+2)+\ldots..+(x+\overline{n-1})(x+n)=10n$ has two consecutive integral solutions, then n is equal to: **[JEE Main-2017]**

(a) 11 (b) 12
(c) 9 (d) 10

3. If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2-x+1=0$, then $\alpha^{101}+\beta^{107}$ is equal to- **[JEE Main-2018]**

(a) 0 (b) 1
(c) 2 (d) –1

4. Let α and β be two roots of the equation $x^2+2x+2=0$, then $\alpha^{15}+\beta^{15}$ is equal to:- **[JEE Main-2019 (January)]**

(a) –256 (b) 512
(c) –512 (d) 256

5. If both the roots of the quadratic equation $x^2-mx+4=0$ are real and distinct and they lie in the interval [1, 5], then m lies in the interval **[JEE Main-2019 (January)]**

(a) (4, 5) (b) (3, 4)
(c) (5, 6) (d) (–5, –4)

6. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2-11x+\alpha=0$ are rational numbers is: **[JEE Main-2019 (January)]**

(a) 2 (b) 5
(c) 3 (d) 4

7. Consider the quadratic equation $(c-5)x^2-2cx+(c-4)=0, c\neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0, 2) and its other root lies in the interval (2, 3). Then the number of elements in S is: **[JEE Main-2019 (January)]**

(a) 18 (b) 12
(c) 10 (d) 11

8. If one real root of the quadratic equation $81x^2+kx+256=0$ is cube of the other root, then a value of k is: **[JEE Main-2019 (January)]**

(a) –81 (b) 100
(c) 144 (d) –300

9. The number of integral values of m for which the quadratic expression, $(1+2m)x^2-2(1+3m)x+4(1+m)$, $x\in R$, is always positive, is:- **[JEE Main-2019 (January)]**

(a) 3 (b) 8
(c) 7 (d) 6

10. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2+m(m-4)x+2=0$, then the least value of m for which $\lambda+\frac{1}{\lambda}=1$, is **[JEE Main-2019 (January)]**

(a) $2-\sqrt{3}$ (b) $4-3\sqrt{2}$
(c) $-2+\sqrt{2}$ (d) $4-2\sqrt{3}$

11. If α and β be the roots of the equation $x^2-2x+2=0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n=1$ is: **[JEE Main-2019 (April)]**

(a) 2 (b) 3
(c) 4 (d) 5

12. If three distinct numbers a,b,c are in G.P. and the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root, then which one of the following statements is correct? **[JEE Main-2019 (April)]**

(a) d, e, f are in A.P. (b) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in G.P.
(c) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P. (d) d, e, f are in G.P.

13. The number of integral values of m for which the equation $(1+m^2)x^2-2(1+3m)x+(1+8m)=0$ has no real root is: **[JEE Main-2019 (April)]**

(a) infinitely many (b) 2
(c) 3 (d) 1

14. Let $p, q \in R$. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2+px+q=0$, then: **[JEE Main-2019 (April)]**

(a) $q^2+4p+14=0$ (b) $p^2-4q-12=0$
(c) $q^2-4p-16=0$ (d) $p^2-4q+12=0$

15. If m is chosen in the quadratic equation $(m^2+1)x^2-3x+(m^2+1)^2=0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is:- **[JEE Main-2019 (April)]**

(a) $8\sqrt{3}$ (b) $4\sqrt{3}$
(c) $10\sqrt{5}$ (d) $8\sqrt{5}$

16. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12} + \beta^{12}}{\left(\alpha^{-12} + \beta^{-12}\right)(\alpha - \beta)^{24}}$ is equal to:

[*JEE Main-2019 (April)*]

(*a*) $\frac{2^6}{(\sin\theta + 8)^{12}}$ (*b*) $\frac{2^{12}}{(\sin\theta - 8)^6}$

(*c*) $\frac{2^{12}}{(\sin\theta - 4)^{12}}$ (*d*) $\frac{2^{12}}{(\sin\theta + 8)^{12}}$

17. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is: [*JEE Main-2019 (April)*]

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 1

18. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \alpha^r + \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \beta^r$ is equal to:

[*JEE Main-2019 (April)*]

(*a*) $\frac{21}{346}$ (*b*) $\frac{29}{358}$

(*c*) $\frac{1}{12}$ (*d*) $\frac{7}{116}$

19. If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to:

[*JEE Main-2019 (April)*]

(*a*) $\beta\gamma$ (*b*) 0

(*c*) $\alpha\gamma$ (*d*) $\alpha\beta$

20. Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If $p_k = (\alpha)^k + (\beta)^k$, $k \geq 1$, then which of the following statements is not true? [*JEE Main-2020 (January)*]

(*a*) $p_5 = p_2 \cdot p_3$ (*b*) $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

(*c*) $p_3 = p_5 - p_4$ (*d*) $p_5 = 11$

21. The least positive value of 'a' for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is _____.

[*JEE Main-2020 (January)*]

22. Let S be the set of all real roots of the equation,

[*JEE Main-2020 (January)*]

$3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$. Then S:

(*a*) contains exactly two elements

(*b*) is a singleton

(*c*) contains at least four elements

(*d*) is an empty set

23. The number of real roots of the equation,

[*JEE Main-2020 (January)*]

$e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$

(*a*) 3 (*b*) 1

(*c*) 4 (*d*) 2

24. Let a, $b \in R$, $a \neq 0$ be such that the equation, $ax^2 - 2bx + 5 = 0$ has a repeated root α, which is also a root of the equation, $x^2 - 2bx - 10 = 0$. If β is the other root of this equation, $\alpha^2 + \beta^2$ is equal to:

[*JEE Main-2020 (January)*]

(*a*) 25 (*b*) 26

(*c*) 24 (*d*) 28

25. Let f(x) be a quadratic polynomial such that $f(-1) + f(b) = 0$. If one of the roots of $f(x) = 0$ is 3, then its other root lies in: [*JEE Main-2020 (September)*]

(*a*) $(-1, 0)$ (*b*) $(-3, -1)$

(*c*) $(0, 1)$ (*d*) $(1, 3)$

26. Let α and β be the roots of the equation, $5x^2 + 6x - 2 = 0$. If $S_n = \alpha^n + \beta^n$, , n = 1, 2, 3, ..., then:

[*JEE Main-2020 (September)*]

(*a*) $5S_6 + 6S_5 = 2S_4$

(*b*) $6S_6 + 5S_5 + 2S_4 = 0$

(*c*) $6S_6 + 5S_5 = 2S_4$

(*d*) $5S_6 + 6S_5 + 2S_4 = 0$

27. The set of all real values of λ for which the quadratic equations, $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval (0,1) is:

[*JEE Main-2020 (September)*]

(*a*) $(-3, -1)$ (*b*) $(2, 4]$

(*c*) $(0, 2)$ (*d*) $(1, 3]$

28. If α and β are the roots of the equation $x^2 + px + 2 = 0$ and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $2x^2 + 2qx + 1 = 0$, then $\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$ is equal to:

[*JEE Main-2020 (September)*]

(*a*) $\frac{9}{4}(9 - q^2)$ (*b*) $\frac{9}{4}(9 + p^2)$

(*c*) $\frac{9}{4}(9 + q^2)$ (*d*) $\frac{9}{4}(9 - p^2)$

29. Let $\lambda \neq 0$ be in R. If α and β are the roots of the equation, $x^2 - x + 2\lambda = 0$ and α and γ are the roots of the equation, $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to:

[*JEE Main-2020 (September)*]

(*a*) 18 (*b*) 9

(*c*) 27 (*d*) 36

30. Let α and β be the roots of $x^2 - 3x + p = 0$ and γ and δ be the roots of $x - 6x + q = 0$. If $\alpha, \beta, \gamma, \delta$ form a geometric progression. Then ratio $(2q + p) : (2q - p)$ is:

[*JEE Main-2020 (September)*]

(*a*) 3 : 1 (*b*) 5 : 3

(*c*) 9 : 7 (*d*) 33 : 31

31. If α and β are the roots of the equation, $7x^2 - 3x - 2 = 0$, then the value of $\frac{\alpha}{1-\alpha^2} + \frac{\beta}{1-\beta^2}$ is equal to:

[JEE Main-2020 (September)]

(*a*) $\frac{1}{24}$ (*b*) $\frac{27}{32}$

(*c*) $\frac{3}{8}$ (*d*) $\frac{27}{16}$

32. The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$, is: **[JEE Main-2020 (September)]**

(*a*) $\frac{25}{9}$ (*b*) $\frac{25}{81}$

(*c*) $\frac{5}{9}$ (*d*) $\frac{5}{27}$

33. If α and β are the roots of the equation $2x(2x + 1) = 1$, then β is equal to: **[JEE Main-2020 (September)]**

(*a*) $2\alpha^2$ (*b*) $-2\alpha(\alpha + 1)$

(*c*) $2\alpha(\alpha - 1)$ (*d*) $2\alpha(\alpha + 1)$

34. If α and β be two roots of the equation $x^2 - 64x + 256 = 0$. Then the value of $\left(\frac{\alpha^3}{\beta^5}\right)^{1/8} + \left(\frac{\beta^3}{\alpha^5}\right)^{1/8}$ is:

[JEE Main-2020 (September)]

(*a*) 3 (*b*) 2

(*c*) 4 (*d*) 1

35. Let p and q be two positive number such that $p + q = 2$ and $p^4 + p^4 = 272$. Then p and q are roots of the equation: **[JEE Main-2021 (February)]**

(*a*) $x^2 - 2x + 2 = 0$ (*b*) $x^2 - 2x + 8 = 0$

(*c*) $x^2 - 2x + 136 = 0$ (*d*) $x^2 - 2x + 16 = 0$

36. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices (a, c), $(2, b)$ and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is:

[JEE Main-2021 (February)]

(*a*) $\frac{71}{256}$ (*b*) $-\frac{69}{256}$

(*c*) $\frac{69}{256}$ (*d*) $-\frac{71}{256}$

37. The number of the real roots of the equation $(x + 1)^2 + |x - 5| = \frac{27}{4}$ is **[JEE Main-2021 (February)]**

38. The coefficients a, b and c of the quadratic equation, $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is

[JEE Main-2021 (February)]

(*a*) $\frac{1}{54}$ (*b*) $\frac{1}{72}$

(*c*) $\frac{1}{36}$ (*d*) $\frac{5}{216}$

39. The integer 'k'. for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R is:

[JEE Main-2021 (February)]

(*a*) 3 (*b*) 2

(*c*) 4 (*d*) 0

40. If $\alpha, \beta \in R$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + az + \beta = 0$, then $(\alpha - \beta)$ is equal to:

[JEE Main-2021 (February)]

(*a*) 7 (*b*) –3

(*c*) 3 (*d*) –7

41. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = a^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is:

[JEE Main-2021 (February)]

(*a*) 4 (*b*) 1

(*c*) 2 (*d*) 3

42. The number of solutions of the equation $\log_4(x - 1) = \log_2(x - 3)$ is **[JEE Main-2021 (February)]**

43. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $P_n = (\alpha)^n + (\beta)^n$, $P_{n-1} = 11$ and $P_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of P_n^2 is:

[JEE Main-2021 (February)]

44. The number of elements in the set $\{x \in R: (|x| - 3)|x + 4| = 6\}$ is equal to

[JEE Main-2021 (March)]

(*a*) 3 (*b*) 2

(*c*) 4 (*d*) 1

45. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b + c)$ is equal to: **[JEE Main-2021 (March)]**

(*a*) 9 (*b*) 15

(*c*) 7 (*d*) 11

46. The value of $4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots\ldots\infty}}}}$

[JEE Main-2021 (March)]

(*a*) $2 + \frac{2}{5}\sqrt{30}$ (*b*) $2 + \frac{4}{\sqrt{5}}\sqrt{30}$

(*c*) $4 + \frac{4}{\sqrt{5}}\sqrt{30}$ (*d*) $5 + \frac{2}{5}\sqrt{30}$

47. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to— **[JEE Main-2021 (March)]**

JEE-ADVANCED
PREVIOUS YEARS

1. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has **[JEE Advanced-2014]**

(*a*) only purely imaginary roots
(*b*) all real roots
(*c*) two real and two purely imaginary roots
(*d*) neither real nor purely imaginary roots

2. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S? **[JEE Advanced-2015]**

(*a*) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$

(*b*) $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(*c*) $\left(0, \frac{1}{\sqrt{5}}\right)$

(*d*) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Comprehension–1 (3 and 4)

Let p, q_n be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$ where $\alpha \neq \beta$. For $n = 0, 1, 2, \ldots\ldots$let $a_n = p\alpha^n + q\beta^n$.

FACT: If a and b are rational numbers numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$. **[JEE Advanced-2017]**

3. $a_{12} =$

(*a*) $a_{11} + 2a_{10}$
(*b*) $2a_{11} + a_{10}$
(*c*) $a_{11} - a_{10}$
(*d*) $a_{11} + a_{10}$

4. If $a_4 = 28$, then $p + 2q =$

(*a*) 14 (*b*) 7
(*c*) 21 (*d*) 12

5. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; n \geq 1$$ **[JEE Advanced-2019]**

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct?

(*a*) $a_1 + a_2 + a_3 + \ldots\ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(*b*) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(*c*) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(*d*) $b_n = \gamma^n + \beta^n$ for all $n \geq 1$

6. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; n \geq 1, \; b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, \; n \geq 2.$$

Then which of the following options is/are correct? **[JEE Advanced-2019]**

(*a*) $a_1 + a_2 + a_3 + \ldots\ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(*b*) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(*c*) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(*d*) $b_n = \gamma^n + \beta^n$ for all $n \geq 1$

7. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c,d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is **[JEE(Advanced)-2020]**

(*a*) 0 (*b*) 8000
(*c*) 8080 (*d*) 16000

ANSWER KEY

Topicwise Questions

1. (*c*) **2.** (*d*) **3.** (*a*) **4.** (*c*) **5.** (*a*) **6.** (*d*) **7.** (*b*) **8.** (*a*) **9.** (*a*) **10.** (*c*) **11.** (*b*) **12.** (*a*)
13. (*c*) **14.** (*b*) **15.** (*c*) **16.** (*b*) **17.** (*a*) **18.** (*c*) **19.** (*b*) **20.** (*c*) **21.** (*b*) **22.**(*d*) **23.** (*c*) **24.** (*a*)
25. (*a*) **26.** (*c*) **27.** (*b*) **28.** (*a*) **29.** (*a*) **30.** (*b*) **31.** (*b*) **32.** (*b*) **33.** (*d*) **34.** (*d*) **35.** (*b*) **36.** (*b*)
37. (*b*) **38.** (*a*) **39.** (*b*) **40.** (*d*) **41.** (*a*) **42.** (*a*) **43.** (*d*) **44.** (*a*) **45.** (*c*) **46.** (*c*) **47.** (*d*) **48.** (*d*)
49. (*a*) **50.** (*c*) **51.** (*a*) **52.** (*c*) **53.** (*a*) **54.** (*c*) **55.** (*b*) **56.** (*d*) **57.** (*b*) **58.** (*b*) **59.** (*d*)

Learning Plus

1. (*b*) **2.** (*c*) **3.** (*a*) **4.** (*d*) **5.** (*a*) **6.** (*b*) **7.** (*c*) **8.** (*d*) **9.** (*d*) **10.** (*a*) **11.** (*c*) **12.** (*c*)
13. (*a*) **14.** (*c*) **15.** (*a*) **16.** (*b*) **17.** (*a*) **18.** (*c*) **19.** (*a*) **20.** (*a*) **21.** (*a*) **22.** (*d*) **23.** (*d*) **24.** (*a*)
25. (*c*) **26.** (*d*) **27.** (*c*) **28.** (*c*) **29.** (*c*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*,*b*,*d*) **2.** (*a*,*b*,*c*,*d*) **3.** (*a*,*d*) **4.** (*a*,*b*,*d*) **5.** (*b*,*d*) **6.** (*b*,*c*,*d*) **7.** (*a*,*d*) **8.** (*a*,*b*,*d*) **9.** (*b*,*d*) **10.** (*c*,*d*) **11.** (*a*,*d*)
12. (*a*,*d*) **13.** (*a*,*b*,*c*,*d*) **14.** (*a*,*b*) **15.** (*c*) **16.** (*b*) **17.** (*d*) **18.** (*a*) $\rightarrow$ (r),(*b*) $\rightarrow$ (p),(*c*) $\rightarrow$ (s), (*d*) $\rightarrow$ (p, q)
19. (2) **20.** (8) **21.** (11) **22.** (1) **23.** (73) **24.** (10) **25.** (3) **26.** (13) **27.** (6) **28.** (11)

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. (*a*) **2.** (*a*) **3.** (*b*) **4.** (*a*) **5.** (Bonus) **6.** (*c*) **7.** (*d*) **8.** (*d*) **9.** (*c*) **10.** (*b*) **11.** (*c*) **12.** (*c*)
13. (*a*) **14.** (*b*) **15.** (*d*) **16.** (*d*) **17.** (*d*) **18.** (*c*) **19.** (*a*) **20.** (*a*) **21.** 8 **22.** (*b*) **23.** (*b*) **24.** (*a*)
25. (*a*) **26.** (*a*) **27.** (*d*) **28.** (*d*) **29.** (*a*) **30.** (*c*) **31.** (*d*) **32.** (*b*) **33.** (*b*) **34.** (*b*) **35.** (*d*) **36.** (*d*)
37. (2) **38.** (*d*) **39.** (*a*) **40.** (*d*) **41.** (*c*) **42.** (1) **43.** (324) **44.** (*b*) **45.** (*c*) **46.** (*a*) **47.** (0)

JEE-ADVANCED
PREVIOUS YEARS

1. (*d*) **2.** (*a*, *d*) **3.** (*d*) **4.** (*d*) **5.** (*a*,*b*,*c*) **6.** (*a*,*b*,*c*) **7.** (*d*)

2 QUADRATIC EQUATION

Topicwise Questions

1. (*c*) We have $4ax^2 + 3bx + 2c = 0$ Let roots are α and β
Let $D = B^2 - 4AC = 9b^2 - 4(4a)(2c) = 9b^2 - 32ac$
Given that , $(a + b + c) = 0 \Rightarrow b = -(a + c)$
Putting this value, we get
$= 9(a + c)^2 - 32ac = 9(a - c)^2 + 4ac$
Hence roots are real.

2. (*d*) Given equation
$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$
Let $A = 2(a^2 + b^2)$, $B = 2(a + b)$ and $C = 1$
$B^2 - 4AC = 4(a^2 + b^2 + 2ab) - 4.2(a^2 + b^2)1$
$\Rightarrow B^2 - 4AC = -4(a - b)^2 < 0$
Thus given equation has imaginary roots.

3. (*a*) Here $(b + c - 2a) + (c + a - 2b) + (a + b - 2c) = 0$
Therefore the roots are rational.

4. (*c*) The quadratic is $(k + 11)x^2 - (k + 3)x + 1 = 0$
Accordingly, $(k + 3)^2 - 4(k + 11)(1) = 0 \Rightarrow k = -7, 5$

5. (*a*) Given equation $(1 + 2k)x^2 + (1 - 2k)x + (1 - 2k) = 0$
If equation is a perfect square then root are equal
i.e., $(1 - 2k)^2 - 4(1 + 2k)(1 - 2k) = 0$
i.e., $k = \frac{1}{2}, \frac{-3}{10}$. Hence total number of values = 2.

6. (*d*) $x^2 - 3x + 2$ be factor of $x^4 - px^2 + q = 0$
Hence $(x^2 - 3x + 2) = 0 \Rightarrow (x - 2)(x - 1) = 0$
$\Rightarrow x = 2, 1$ putting these values in given equation
so $4p - q - 16 = 0$ .....(i)
and $p - q - 1 = 0$ .....(ii)
Solving (i) and (ii), we get $(p, q) = (5, 4)$

7. (*b*) check by options
$x = 1$ is root
Let other root = a
$\therefore$ Product of the roots $= (1)(1) = \frac{a - b}{b - c}$
$\Rightarrow$ roots are $1, \frac{a - b}{b - c}$

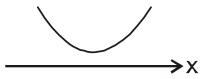
8. (*a*) $D = b^2 - 4ac = 20d^2$
$\sqrt{D} = 2\sqrt{5}d$ here $\sqrt{5}$ is irrational
So roots are irrational.

9. (*a*) $D = b^2 - 4ac = b^2 - 4a(-4a - 2b) = b^2 + 16a^2 + 8ab$
Since $ab > 0$
$\therefore D > 0$
So equation has real roots.

10. (*c*) For integral roots, D of equation should be perfect sq.
$\because D = 4(1 + n)$
By observation, for $n \in N$, D should be perfect sq. of even integer.
So $D = 4(1 + n) = 6^2, 8^2, 10^2, 12^2, 14^2, 16^2, 18^2, 20^2$
No. of values of n = 8.

11. (*b*) Here for $D < 0$, entire graph will be above x-axis
$(\because a > 0)$
$\Rightarrow (k - 1)^2 - 36 < 0$
$\Rightarrow (k - 7)(k + 5) < 0$
$\Rightarrow -5 < k < 7$

12. (*a*) Let $f(x) = ax^2 - bx + 1$
Given $D < 0$ & $f(0) = 1 > 0$
$\therefore$ possible graph is as shown



i.e. $f(x) > 0 \ \forall \ x \in R$
or $f(-1) > 0$
$f(-1) = a + b + 1 > 0$

13. (*c*) $x^2 + ax + b = 0$ 

$a + b = -a$
$\Rightarrow 2a + b = 0$
and $ab = b$
$ab - b = 0$
$b(a - 1) = 0$
$\Rightarrow$ Either $b = 0$ or $a = 1$
But $b \neq 0$(given)
$\therefore \quad a = 1$

$\therefore \quad b = -2$

$\therefore \quad f(x) = x^2 + x - 2$

Least value occurs at $x = -\frac{1}{2}$

Least value $= \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$

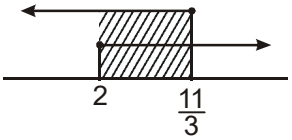
14. (*b*) $(2-x)(x+1) = p$

$(x-2)(x+1) + p = 0$

$\Rightarrow x^2 - x - 2 + p = 0$

$\frac{c}{a} > 0 \Rightarrow \; p - 2 > 0$

& $D > 0 \Rightarrow \; 1 - 4(p-2) > 0 \; \Rightarrow \; p < \frac{9}{4}$



$\frac{-b}{2a} > 0, \; \frac{-1}{2(2-p)} > 0, P \in (2, \infty)$

Taking intersection of all $p \in \left(2, \frac{9}{4}\right)$

15. (*c*) $x^2 + px + (1-p) = 0$

$(1-p)^2 + p(1-p) + (1-p) = 0$

$(1-p)[1-p+p+1] = 0 \Rightarrow p = 1$

Q.E. will be $\Rightarrow x^2 + x = 0 \; \Rightarrow \; x(x+1) = 0$

$\Rightarrow x = 0, -1$

Aliter

$\alpha + 1 - p = -p \Rightarrow \alpha = -1$

Satisfies

$1 - p + 1 - p = 0 \Rightarrow p = 1$

$\beta = 1 - p = 0 \Rightarrow \beta = 0$

16. (*b*) Let first root $= \alpha$ and second root $= \frac{1}{\alpha}$

Then $\alpha, \frac{1}{\alpha} = \frac{k}{5} \Rightarrow k = 5$

17. (*a*) Given equation $4x^2 + 3x + 7 = 0$, therefore

$\alpha + \beta = -\frac{3}{4}$ and $\alpha\beta = \frac{7}{4}$

Now $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-3/7}{7/4} = \frac{-3}{4} \times \frac{4}{7} = -\frac{3}{7}$

18. (*c*) Let α and β be two roots of $ax^2 + bx + c = 0$

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - 2\frac{c}{a}$

So under condition $\alpha + \beta = \alpha^2 + \beta^2 \; \alpha + \beta = a^2 + \beta^2$

$\Rightarrow \; -\frac{b}{a} = \frac{b^2 - 2ac}{a^2} \; \Rightarrow b(a+b) = 2ac$

19. (*b*) α, β be the roots of $x^2 - 2x + 3 = 0$, then $\alpha + \beta = 2$ and $\alpha\beta = 3$.

Now required equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ is

$x^2 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)x + \frac{1}{\alpha^2\beta^2} = 0$

$\Rightarrow x^2 - \left(-\frac{2}{9}\right)x + \frac{1}{9} = 0 \Rightarrow 9x^2 + 2x + 1 = 0$

20. (*c*) According to condition

$\frac{2m-1}{m} = -1 \; \Rightarrow 3m = 1 \Rightarrow m = \frac{1}{3}$

21. (*b*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (4)^3 - 3 \times 1(4) = 52$

22. (*d*) We know that the roots of the equation

$ax^3 + bx^2 + cx + d = 0$ follows $\alpha\beta\gamma = -d/a$

Comparing above equation with given equation

we get $d = 1, a = 1$

So, $\alpha\beta\gamma = -1$ or $\alpha^3\beta^3\gamma^3 = -1$.

23. (*c*) $a + b = -p$

$ab = q$

$g + d = -p$

$gd = -r$

$(a-g)(a-d) = a^2 - a(g+d) + gd$

$= a^2 + pa - r = a(a+p) - r = -ab - r$

$= -q - r = -(q + r)$

24. (*a*) $|a-b| = 4 \Rightarrow (a-b)^2 = 16$

$\Rightarrow (a+b)^2 - 4ab = 16$

$\Rightarrow 9 - 4ab = 16 \Rightarrow ab = -\frac{7}{4}$

$\Rightarrow$ equation is $x^2 - 3x - \frac{7}{4} = 0$

25. (*a*) $C_1 : b^2 - 4ac \geqslant 0,$

$ax^2 + bx + c = 0$ real roots C_1 satisfied

C_2 : a, –b, c are same sign

$\alpha + \beta > 0 \Rightarrow \frac{-b}{a} > 0$

$\alpha\beta > 0 \Rightarrow \frac{c}{a} > 0$

C_2 satisfied C_1 & C_2 are satisfied

26. (*c*) $ax^2 + bx + c = 0, \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

$\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

$\alpha^3 + \beta^3 = \left(\frac{-b}{a}\right)\left[\left(\frac{-b}{a}\right)^2 - 3\frac{c}{a}\right]$

$= \frac{-b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = \frac{-b}{a}\frac{(b^2 - 3ac)}{a^2} = \frac{3abc - b^3}{a^3}$

27. (*b*) Given equation can be written as

$(6k+2)x^2+rx+3k-1=0$ .....(i)

and $2(6k+2)x^2+px+2(3k-1)=0$ .....(ii)

Condition for common roots is

$$\frac{12k+4}{6k+2}=\frac{p}{r}=\frac{6k-2}{3k-1}=2 \text{ or } 2r-p=0$$

28. (a) Let α is the common root,

so $\alpha^2+p\alpha+q=0$ .....(i)

and $\alpha^2+q\alpha+p=0$ ....(ii)

from (i) – (ii),

$\Rightarrow (p-q)\alpha+(q-p)=0 \Rightarrow \alpha=1$

Put the value of α in (i), $p+q+1=0$.

29. (*a*) $D_1=4a^2b^2-8a^2b^2=-4a^2b^2<0$ img. root

$D_2=4p^2q^2-4p^2q^2=0$ equal, real roots

So no common roots.

30. (*b*) Given, $x+2>\sqrt{x+4} \Rightarrow (x+2)^2>(x+4)$

$\Rightarrow x+4x+4>x+4 \Rightarrow x^2+3x>0$

$\Rightarrow x(x+3)>0 \Rightarrow x<-3$ or $x>0 \Rightarrow x>0$

31. (*b*) **Case I:** When $x+2\geq 0$ i.e. $x\geq -2$

Then given inequality becomes

$x^2-(x+2)+x>0 \Rightarrow x^2-2>0 \Rightarrow |x|>\sqrt{2}$

$\Rightarrow x<-\sqrt{2}$ or $x>\sqrt{2}$

As $x\geq -2$, therefore, in this case the part of the solution set is $[-2,-\sqrt{2})\cup(\sqrt{2},\infty)$.

Case II: When $x+2\leq 0$ *i.e.* $x\leq -2$,

Then given inequality becomes $x^2+(x+2)+x>0$

$\Rightarrow x^2+2x+2>0 \Rightarrow (x+1)^2+1>0$, which is true for all real x

Hence, the part of the solution set in this case is $(-\infty,-2]$. Combining the two cases, the solution set is

$(-\infty,-2)\cup([-2,-\sqrt{2}]\cup(\sqrt{2},\infty)=(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$.

32. (*b*) For $(p^2-3p+2)x^2-(p^2-5p+4)x+p-p^2=0$ to be an identity

$p^2-3p+2=0 \Rightarrow p=1,2$ ...(1)

$p^2-5p+4=0 \Rightarrow p=1,4$ ...(2)

$p-p^2=0 \Rightarrow p=0,1$ ...(3)

For (1), (2) & (3) to hold simultaneously

$p=1$.

33. (*d*) $x^2+9<(x+3)^2<8x+25$

$x^2+9<x^2+6x+9 \Rightarrow x>0$

& $(x+3)^2<8x+25$

$x^2+6x+9-8x-25<0$

$x^2-2x-16<0$

$1-\sqrt{17}<x<1+\sqrt{17}$ & $x>0$

$\Rightarrow x\in(0,1+\sqrt{17})$

Integer $x=1,2,3,4,5$

No. of integer are = 5

34. (*d*) $5x+2<3x+8 \Rightarrow 2x<6 \Rightarrow x<3$ ...(i)

$$\frac{x+2}{x-1}<4 \Rightarrow \frac{x+2}{x-1}-4<0 \Rightarrow \frac{-3x+6}{x-1}<0$$

$$\Rightarrow \frac{x-2}{x-1}>0 \Rightarrow x\in(-\infty,1)\cup(2,\infty)$$ ...(ii)

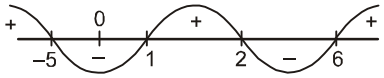
Taking intersection of (i) and (ii) $x\in(-\infty,1)\cup(2,3)$

35. (*b*) $\dfrac{x^2(x^2-3x+2)}{x^2-x-30}\geq 0$

$\Rightarrow \dfrac{x^2(x-1)(x-2)}{(x+5)(x-6)}\geq 0$

$x\neq -5,6$

$x\in(-\infty,-5)\cup[1,2]\cup(6,\infty)\cup\{0\}$



36. (*b* $\because$ $(m-2)x^2+8x+m+4>0 \ \forall x\in R$

$\Rightarrow m>2$ & $D<0$

$64-4(m-2)(m+4)<0$

$16-[m^2+2m-8]<0$

$\Rightarrow m^2+2m-24>0$

$\Rightarrow (m+6)(m-4)>0$

$m\in(-\infty,-6)\cup(4,\infty)$

But $m>2$

$\Rightarrow m\in(4,\infty)$

Then least integral m is m = 5.

37. (*b*) $-1\leq|x-1|-1\leq 1$

$\Rightarrow 0\leq|x-1|\leq 2$

$\Rightarrow 0\leq|x-1|$

$\Rightarrow x\in R$ ...(1)

and $|x-1|\leq 2$

$\Rightarrow -2\leq x-1\leq 2$

$\Rightarrow -1\leq x\leq 3$ ...(2)

$(1)\cap(2)$

$\Rightarrow x\in[-1,3]$.

38. (*a*) $\log_{1/3}\dfrac{3x-1}{x+2}<1$

$$\Rightarrow \frac{3x-1}{x+2}>0 \Rightarrow x\in(-\infty,-2)\cup\left(\frac{1}{3},\infty\right)$$ ....(i)

and $\dfrac{3x-1}{x+2}>\dfrac{1}{3}$

$$\Rightarrow \frac{8x-5}{x+2}>0$$

$$\Rightarrow x\in(-\infty,-2)\cup\left(\frac{5}{8},\infty\right)$$ .....(ii)

$$(i)\cap(ii) \Rightarrow x\in(-\infty,-2)\cup\left(\frac{5}{8},\infty\right)$$

39. (*b*) $2-\log_2(x^2+3x)\geq 0$

$\Rightarrow \log_2(x^2+3x)\leq 2$

$x^2+3x>0$

$\Rightarrow x\in(-\infty,-3)\cup(0,\infty)$ .....(i)

and $x^2+3x\leq 4$

$\Rightarrow (x-1)(x+4)\leq 0$

$\Rightarrow x\in[-4,1]$ .....(ii)

(i) $\cap$ (ii) $\Rightarrow x\in[-4,-3)\cup(0,1]$

40. (*d*) $\log_{1-x}(x-2)\geq 0$

$x>2$ .....(1)

(i) When $0<1-x<1 \Rightarrow 0<x<1$

So no common range comes out.

(ii) When $1-x>1\Rightarrow x<0$ but $x>2$

here, also no common range comes out. , hence no solution.

Finally, no solution

41. (*a*) $\log_{0.3}(x-1)<\log_{0.09}(x-1)$

$\log_{0.3}(x-1)<\dfrac{\log_{0.3}(x-1)}{2}$

$\Rightarrow \log_{0.3}(x-1)<0 \quad \Rightarrow \quad x-1>1 \quad \Rightarrow \quad x>2$

42. (*a*) $\log_{0.5}\log_5(x^2-4)>\log_{0.5}1$

$\log_{0.5}\log_5(x^2-4)>0$

$\Rightarrow x^2-4>0$

$\Rightarrow x\in(-\infty,-2)\cup(2,\infty)$ ....(i)

$\log_5(x^2-4)>0\Rightarrow x^2-5>0$

$\Rightarrow x\in(-\infty,-\sqrt{5})\cup(\sqrt{5},\infty)$ ....(ii)

$\log_5(x^2-4)<1$

$\Rightarrow x^2-9<0\Rightarrow x\in(-3,\ 3)$ ....(iii)

(i) $\cap$ (ii) $\cap$ (iii) $\Rightarrow x\in(-3,\sqrt{5})\cup(\sqrt{5},3)$

43. (*d*) $\left(\dfrac{1}{2}\right)^{x^2-2x}<\left(\dfrac{1}{2}\right)^2$ here base is less than zero so inequality change

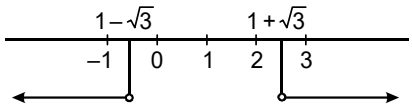
$\Rightarrow x^2-2x>2\Rightarrow x^2-2x-2>0$

$\alpha,\beta=\dfrac{2\pm\sqrt{4+8}}{2}=\dfrac{2\pm 2\sqrt{3}}{2}$

$a=1-\sqrt{3}$, $b=1+\sqrt{3}$

$(x-a)(x-b)>0$

$x\in\left(-\infty,\ 1-\sqrt{3}\right)\cup\left(1+\sqrt{3},\ \infty\right)$, x can be in $(3,\infty)$



44. (*a*) $\dfrac{6x^2-5x-3}{x^2-2x+6}\leq 4$

D^r is always >0

$6x^2-5x-3-4x^2+8x-24\leq 0$

$\Rightarrow 2x^2+3x-27\leq 0$

$\Rightarrow (2x+9)(x-3)\leq 0\Rightarrow x\in\left[\dfrac{-9}{2},3\right]$

least value of $4x^2=4.0^2=0$

Highest value of $4x^2$ is $=\max\left(4\cdot\left(-\dfrac{9}{2}\right)^2,\ 4\cdot 3^2\right)$

$=\max(81,36)=81$

45. (*c*) Let the roots be a, b, –b

then $\alpha+\beta-\beta=p$

$\Rightarrow \alpha=p$ ...(1)

and $\alpha\beta-\alpha\beta-\beta^2=q$

$\Rightarrow \beta^2=-q$ ...(2)

also $-\alpha\beta^2=r$

$\Rightarrow pq=r$ [using (1)].

46. (*c*) $x^3-x-1=0$ $\begin{matrix}\nearrow\alpha\\ \rightarrow\beta\\ \searrow\gamma\end{matrix}$

then $\alpha^3-\alpha-1=0$.....(1)

Let $\dfrac{1+\alpha}{1-\alpha}=y\ \Rightarrow\ \alpha=\dfrac{y-1}{y+1}$

from equation (1) $\left(\dfrac{y-1}{y+1}\right)^3-\left(\dfrac{y-1}{y+1}\right)-1=0$

$\Rightarrow\ y^3+7y^2-y+1=0$ $\begin{matrix}\nearrow\frac{1+\alpha}{1-\alpha}\\ \rightarrow\frac{1+\beta}{1-\beta}\\ \searrow\frac{1+\gamma}{1-\gamma}\end{matrix}$

then $\dfrac{1+\alpha}{1-\alpha}+\dfrac{1+\beta}{1-\beta}+\dfrac{1+\gamma}{1-\gamma}=-7$ Ans.

47. (*d*) $x^4-4x^3+ax^2+bx+1=0$

real & positive roots

$\alpha+\beta+r+\delta=4$ & $\alpha\beta r\delta=1$

$\Rightarrow \alpha=\beta=r=\delta=1$

$\Sigma\alpha\beta=a\Rightarrow a=6$

$\Sigma\alpha\beta r=-b\Rightarrow b=-4$

or $(x-1)^4=x^4-4x^3+6x^2-4x+1$

48. (*d*) $ax^2 + bx + c = 0 \begin{cases} \alpha \\ \beta \end{cases}$

sum of roots $= (2\alpha + 3\beta) + (3\alpha + 2\beta)$

$= 5(\alpha + \beta) = 5\left(-\frac{b}{a}\right)$

Product of roots $= 6\alpha^2 + 6\beta^2 + 13\alpha\beta = 6(\alpha + \beta)^2 + \alpha\beta$

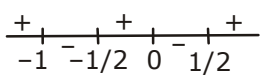
$= 6\left(\frac{-b}{a}\right)^2 + \frac{c}{a} = \frac{6b^2}{a^2} + \frac{c}{a}$

Q. E. $x^2 + \frac{5b}{a}x + \frac{6b^2}{a^2} + \frac{c}{a} = 0$

$a^2x^2 + 5abx + 6b^2 + ac = 0$

49. (*a*) $\Rightarrow \frac{(2x-1)}{x(2x^2+3x+1)} > 0$

$\Rightarrow \frac{(2x-1)}{x(x+1)(2x+1)} > 0$



consontains $\left(-\infty, \frac{-3}{2}\right)$

50. (*c*) Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow (y-1)x^2 + 3(y+1)x + 4(y-1) = 0$

For x is real $D \geq 0$

$\Rightarrow 9(y+1)^2 - 16(y-1)^2 \geq 0$ $9(y+1)^2 - 16(y-1)^2 \geq 0$

$\Rightarrow -7y^2 + 50y - 7 \geq 0 \Rightarrow 7y^2 - 50y + 7 \leq 0$

$\Rightarrow (y-7)(7y-1) \leq 0$

Now, the product of two factors is negative if one in –ve and one in +ve.

Case I : $(y-7) \geq 0$ and $(7y-1) \leq 0$

$\Rightarrow y \geq 7$ and $y \geq \frac{1}{7}$. But it is impossible

Case II : $(y-7) \leq 0$ and $(7y-1) \geq 0$

$\Rightarrow y \leq 7$ and $y \geq \frac{1}{7} \Rightarrow \frac{1}{7} \leq y \leq 7$

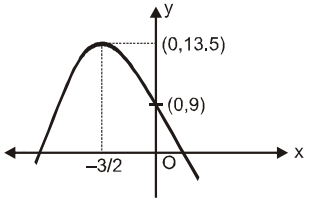
Hence maximum value is 7 and minimum value is $\frac{1}{7}$

51. (*a*) $x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$

Therefore, smallest value is $\frac{3}{4}$, which lie in $\left(-3, \frac{3}{2}\right)$

52. (*c*) $y = -2x^2 - 6x + 9$

$\therefore \frac{-b}{2a} = \frac{6}{2(-2)} = -\frac{3}{2} = -1.5$



& $D = 36 - 4(-2)(9) = 36 + 72 = 108$

$\therefore -\frac{D}{4a} = -\frac{108}{4(-2)} = +\frac{108}{8} = 13.5$

$\Rightarrow y \in (-\infty, 13.5]$

53. (*a*) $k = \frac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow (k-1)x^2 + (k+1)x + (k-1) = 0$

Q x is real

$\therefore D \geq 0$

$\Rightarrow (k+1)^2 - 4(k-1)^2 \geq 0$

$\Rightarrow (3k-1)(k-3) \leq 0$

$\Rightarrow k \in \left[\frac{1}{3}, 3\right]$

54. (*c*) $y = \frac{2x}{1+x^2}, x \in R$

$\Rightarrow yx^2 - 2x + y = 0$

$\Rightarrow D \geq 0 \Rightarrow 4 - 4y^2 \geq 0$

$\Rightarrow (y^2 - 1) \leq 0 \Rightarrow y \in [-1, 1]$

$\therefore$ Range of $f(y) = y^2 + y - 2$

Min value $= \frac{-D}{4a} = \frac{-9}{4}$ at $y = \frac{-b}{2a} = \frac{-1}{2}$

$y = \frac{-1}{2} \in [-1, 1]$

$f(-1) = 1 - 1 - 2 = -2$

$f(1) = 1 + 1 - 2 = 0$

max value is $= 0$

Range $\left[\frac{-9}{4}, 0\right]$

55. (*b*) $x^2 - xy + y^2 - 4x - 4y + 16 = 0, x, y \in R$

$x^2 - x(y+4) + (y^2 - 4y + 16) = 0$ …(1)

$x \in R \Rightarrow D \geq 0$

$(y+4)^2 - 4(y^2 - 4y + 16) \geq 0$

$\Rightarrow y^2 + 8y + 16 - 4y^2 + 16y - 64 \geq 0$

$\Rightarrow y^2 - 8y + 16 \le 0$

$\Rightarrow (y-4)^2 \le 0 \Rightarrow y = 4$

Put is given equation (i)

$x^2 - 8x + 16 = 0$

$\Rightarrow (x-4)^2 = 0 \Rightarrow x = 4$

56. (*d*) $(y-1)x^2 + (y+1)x + (2cy - c) = 0$

$D \ge 0 \therefore x \in R$

$\Rightarrow (y+1)^2 - 4(y-1)(2cy - c) \ge 0$

$y^2 + 2y + 1 - 8cy^2 + 12cy - 4c \ge 0$

$(1-8c)y^2 + (2+12c)y + (1-4c) \ge 0$

$\forall\, y \in R, D \le 0$

$(2+12c)^2 - 4(1-8c)(1-4c) \le 0$

$(1+6c)^2 - (1-8c)(1-4c) \le 0$

$4c^2 + 24c \le 0 \Rightarrow c \in [-6, 0]$

& N^r & D^r have no any common root

(i) both common factor (root) (not possible)

$\frac{1}{1} = \frac{-1}{+1} = \frac{c}{2c}$

(ii) If one common root is α

$(\alpha^2 - \alpha + c = 0) \times 2$

& $\alpha^2 + \alpha + 2c = 0$

$\alpha^2 - 3\alpha = 0$

$\alpha = 0 \Rightarrow c = 0$

or $\alpha = 3 \Rightarrow c = -6$

$\therefore \quad c \ne 0$ & $c \ne -6$

$\therefore \quad c \in (-6, 0)$

57. (*b*) $2x^2 - (a^3 + 8a - 1)x + (a^2 - 4a) = 0$

since the roots are of opposite sign, $f(0) < 0$

$\Rightarrow a^2 - 4a < 0$

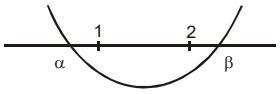
$\Rightarrow a(a-4) < 0$

$\Rightarrow a \in (0, 4)$

58. (*b*) $x^2 - 2px + (8p - 15) = 0$

$f(1) < 0$ and $f(2) < 0$

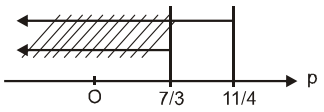
$\Rightarrow f(1) = 1 - 2p + 8p - 15 < 0$



$\Rightarrow p < 7/3$

and $f(2) = 4 - 4p + 8p - 15 < 0$

$\Rightarrow 4p - 11 < 0 \Rightarrow p < \frac{11}{4}$



Hence $p \in (-\infty, 7/3)$ Ans.

59. (*d*) $x^2 + 2(k-1)x + k + 5 = 0$

Case - I (i) $D \ldots 0$

$\Rightarrow 4(k-1)^2 - 4(k+5) \ldots 0$

$\Rightarrow k^2 - 3k - 4 \ldots 0 \Rightarrow (k+1)(k-4) \ldots 0$

$\Rightarrow k \in (-\infty, -1] \cup [4, \infty)$

& (ii) $f(0) > 0 \Rightarrow k + 5 > 0 \Rightarrow k \in (-5, \infty)$

& (iii) $\frac{-b}{2a} > 0 \Rightarrow \frac{-2(k-1)}{2} > 0$

$\Rightarrow k \in (-\infty, 1) \therefore k \in [-5, -1]$

Case - II $f(0) \le 0 \Rightarrow k + 5 \le 0$

$\Rightarrow k \in (-\infty, -5]$



Finally $k \in$ (Case - I) $\cup$ (Case - II)

$k \in (-\infty, -1]$

Learning Plus

1. (*b*) The roots of the equations are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) Let $b^2 - 4ac > 0, b > 0$

Now if $a > 0, c > 0$, $b^2 - 4ac < b^2$

$\Rightarrow$ the roots are negative.

(ii) Let $b^2 - 4ac < 0$, then the roots are given by

$$x = \frac{-b \pm i\sqrt{(4ac - b^2)}}{2a}, \ (i = \sqrt{-1})$$

Which are imaginary and have negative real part

$(\because b > 0)$

$\therefore$ In each case, the roots have negative real part.

2. (*c*) From options put $k = 3 \Rightarrow x^2 + 8x + 7 = 0$
$\Rightarrow (x+1)(x+7) = 0 \Rightarrow x = -1, -7$
means for $k = 3$ roots are negative.

3. (*a*) Here $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

If roots are $\alpha + \frac{1}{\beta}$, $\beta + \frac{1}{\alpha}$ then sum of roots are

$$= \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta} = \frac{b}{ac}(a + c)$$

and product $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

$$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta} = 2 + \frac{c}{a} + \frac{a}{c}$$

$$= \frac{2ac + c^2 + a^2}{ac} = \frac{(a+c)^2}{ac}$$

Hence required equation is given by

$$x^2 + \frac{b}{ac}(a+c)x + \frac{(a+c)^2}{ac} = 0$$

$\Rightarrow acx^2 + (a+c)bx + (a+c)^2 = 0$

Trick : Let $a = 1, b = -3, c = 2$, then $\alpha = 1, \beta = 2b = -3, c = 2$, then $\alpha = 1, \beta = 2$

$\therefore \alpha + \frac{1}{\beta} = \frac{3}{2}$ and $\beta + \frac{1}{\alpha} = 3$

Therefore, required equation must be
$(x-3)(2x-3) = 0$ i.e. $2x^2 - 9x + 9 = 0$
Here (1) gives this equation on putting
$a = 1, b = -3, c = 2$

4. (*d*) $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

and $\alpha^2 + \beta^2 = \frac{(b^2 - 2ac)}{a^2}$

Now $\frac{\alpha}{a\beta + b} + \frac{\beta}{a\alpha + b} = \frac{\alpha(a\alpha + b) + \beta(a\beta + b)}{(a\beta + b)(a\alpha + b)}$

$$= \frac{a(\alpha^2 + \beta^2) + b(\alpha + \beta)}{\alpha\beta a^2 + ab(\alpha + \beta) + b^2} = \frac{a\frac{(b^2 - 2ac)}{a^2} + b\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)a^2 + ab\left(-\frac{b}{a}\right) + b^2}$$

$$= \frac{b^2 - ac - b^2}{a^2c - ab^2 + ab^2} = \frac{-ac}{a^2c} = -\frac{2}{a}$$

5. (*a*) Let roots are α, β so, $\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2\beta}{3}$

$\therefore \alpha + \beta = \frac{m}{12}$

$\Rightarrow \frac{2\beta}{3} + \beta = \frac{m}{12} \Rightarrow \frac{5\beta}{3} = \frac{m}{12}$ .....(i)

and $\alpha\beta = \frac{5}{12} \Rightarrow \frac{2\beta}{3}.\beta = \frac{5}{12} \Rightarrow \beta^2 = \frac{5}{8}$

$\Rightarrow \beta = \sqrt{5/8}$

Put the value of β in (i), $\frac{5}{3}.\sqrt{\frac{5}{8}} = \frac{m}{12} \Rightarrow m = 5\sqrt{10}$.

6. (*b*) Expressions are $x^2 - 11x + a$ and $x^2 - 14x + 2a$ will have a common factor, then

$$\Rightarrow \frac{x^2}{-22a + 14a} = \frac{x}{a - 2a} = \frac{1}{-14 + 11}$$

$$\Rightarrow \frac{x^2}{-8a} = \frac{x}{-a} = \frac{1}{-3} \Rightarrow x^2 = \frac{8a}{3} \text{ and } x = \frac{a}{3}$$

$$\Rightarrow \left(\frac{a}{3}\right)^2 = \frac{8a}{3} \Rightarrow \frac{a^2}{9} = \frac{8a}{3} \Rightarrow a = 0, 24.$$

Trick : We can check by putting the values of *a* from the options.

7. (*c*) If α, β, γ are the roots of the equation.
$x^3 - px^2 + qx - r = 0$

$$\therefore (\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = \frac{p^2 + q}{pq - r}$$

Given, $p = 0, q = 4, r = -1$

$$\Rightarrow \frac{p^2 + q}{pq - r} = \frac{0 + 4}{0 + 1} = 4$$

8. (*d*) If the roots of the quadratic equation $ax^2 + bx + c = 0$ exceed a number k , then $ak^2 + bk + c > 0$ if $a > 0, b^2 - 4ac \geq 0$ and sum of the roots $> 2k$. Therefore, if the roots of $x^2 + x + a = 0$ exceed a number a , then
$a^2 + a + a > 0, 1 - 4a \geq 0$ and $-1 > 2a$

$\Rightarrow a(a+2) > 0, a \leq \frac{1}{4}$ and $a < -\frac{1}{2} \Rightarrow a > 0$ or

$a < -2, a < \frac{1}{4}$ and $a < -\frac{1}{2}$

Hence $a < -2$.

9. (*d*) Let
$f(x) = 4x^2 - 20px + (25p^2 + 15p - 66) = 0$ .....(i)
The roots of (i) are real if $b^2 - 4ac = 400p^2 - 16(25p^2 + 15p - 66) = 16(66 - 15p) \geq 0$
$\Rightarrow p \leq 22/5$ .....(ii)
Both roots of (i) are less than 2. Therefore $f(2) > 0$ and sum of roots < 4.

$\Rightarrow 4.2^2 - 20p.2 + (24p^2 + 15p - 66) > 0$ and $\frac{20p}{4} < 4$

$\Rightarrow p^2 - p - 2 > 0$ and $p < \frac{4}{5}$

$\Rightarrow (p+1)(p-2) > 0$ and $p < \frac{4}{5}$

$\Rightarrow p < -1$ or $p > 2$ and $p < \frac{4}{5} \Rightarrow p < -1$ .....(iii)

From (ii) and (iii), we get $p < -1$ i.e. $p \in (-\infty, -1)$.

10. (*a*) $x-\frac{2}{x-1}=1-\frac{2}{x-1}$

Since, $x-1$ is in denominator

$x-1\neq 0$

$x\neq 1$

$x-\frac{2}{x-1}=\ 1-\frac{2}{x-1}$

$\Rightarrow x=1$ But x can't be 1. So, No roots.

11. (*c*) If roots are α and β.

Then, $\alpha+\beta=\alpha^2+\beta^2$

$\alpha\beta=\alpha^2\beta^2$

$\Rightarrow \quad \alpha^2\beta^2-\alpha\beta=1$

$\alpha\beta(\alpha\beta-1)=0$

$\alpha\beta=0$ or $\alpha\beta=1$

$\Rightarrow \quad \alpha=0$ or $\beta=0$ or $\alpha\beta=1$

C1: if $\alpha=0$

$0+\beta=0+\beta^2\Rightarrow\beta^2-\beta=0$

$\beta(\beta-1)=0\Rightarrow\beta=0$ or 1.

Roots are 0,0 and 0,1

C2: Similarly, for $\beta=0\Rightarrow\alpha=0$ or 1

Roots will be 0,0 and 1,0

C3: $\alpha\beta=1$

$\alpha^2+\beta^2=\alpha+\beta$

$(\alpha+\beta)^2-2\alpha\beta=\alpha+\beta$

$(\alpha+\beta)^2-(\alpha+\beta)-2=0$

Let $\alpha+\beta=t$

$t^2-t-2=0$

$(t-2)(t+1)=0$

$t=2$ or $t=-1$

$\alpha+\beta=2$ or $\alpha+\beta=-1$

for $\alpha+\beta=2, \alpha\beta=1$

Roots are 1, 1

for $\alpha+\beta=-1, \alpha\beta=1$

Roots are w, w^2.

Hences total roots are possible.

0, 0

0, 1

1, 1

ω, ω^2

12. (*c*) for $y=x^2+ax+25$ to touch the x - axis, it should have equal & real roots, i.e D = 0

$a^2-4.1.25=0$

$a^2-100=0$

$(a-10)(a+10)=0$

$a=10$ or -10

13. (*a*) $ax^3+bx^2+cx+d=0$

Let roots be α, β, γ.

$\alpha+\beta+\gamma=-\frac{b}{a}$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$

$\alpha\beta\gamma=-\frac{d}{a}$

Now, $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$$=\left(\frac{-b}{a}\right)^2-\frac{2c}{a}=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$$

As $b^2<2ac$

$b^2-2ac<0 \quad \Rightarrow \ \alpha^2+\beta^2+\gamma^2<0$

$\downarrow$

Not possible if all α,β,γ are real.

And complex root occurs in pair, so two out of α,β,γ are complex & one is real.

14. (*c*) $x^2+x+1=0$

$$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=\text{w, w}^2$$

$\alpha=\omega, \beta=\omega^2$, ω is cube toot of unity.

(*a*) $\alpha^2+\beta^2=\omega^2+\omega^4=\omega^2+\omega=-1\ (1+\omega+\omega^2=0)$

(*b*) $(\alpha-\beta)^2=\alpha^2+\beta^2-2\alpha\beta$

$=-1-2.\omega^3$

$=-1-2=-3 \qquad (\omega^3=1)$

(*c*) $\alpha^3+\beta^3=\omega^3+(\omega^2)^3=1+1=2$

15. (*a*) $ax^2+bx+c=0$

$\sec\alpha$ and $\tan\alpha$ are roots.

$\sec\alpha+\tan\alpha=-\frac{b}{a}$; $\sec\alpha.\tan\alpha=\frac{c}{a}$

$(\sec\alpha-\tan\alpha)^2=(\sec\alpha+\tan\alpha)^2-4\sec\alpha\tan\alpha$

$$=\frac{b^2}{a^2}-\frac{4c}{a}=\frac{b^2-4ac}{a^2}$$

$$\sec\alpha-\tan\alpha=\sqrt{\frac{b^2-4ac}{a}}$$

Now, we know that,

$\sec^2\alpha-\tan^2\alpha=1 \ \Rightarrow (\sec\alpha-\tan\alpha)(\sec\alpha+\tan\alpha)=1$

$$\Rightarrow\left(\sqrt{\frac{b^2-4ac}{a}}\right)\left(\frac{-b}{a}\right)=1$$

$$\Rightarrow\frac{(b^2-4ac)b^2}{a^4}=1 \text{ (Squaring both side)}$$

$a^4-b^4+4ab^2c=0$

16. (*b*) $ax^2 + bx + c = 0$

$\alpha + \beta = \frac{-b}{a};\ \alpha\beta = \frac{c}{a};\ D1 = b^2 - 4ac$

$px^2 + qx + r = 0$

$\gamma + \delta = \frac{-q}{p};\ \gamma\delta = \frac{r}{p};\ D_2 = q^2 - 4pr$

$\alpha,\beta,\gamma,\delta$ are in AP.

$\alpha = \beta - \alpha = \delta - \gamma$

Common difference

$\Rightarrow \frac{\sqrt{D_1}}{a} = \frac{\sqrt{D_2}}{p}$ (Difference between roots).

$\sqrt{\frac{D_1}{D_2}} = \frac{a}{p}$

$\frac{D_1}{D_2} = \frac{a^2}{p^2}$

17. (*a*) $(l-m)x^2 + lx + 1 = 0$

Roots are α and 2α.

$\alpha + 2\alpha = \frac{-l}{l-m} \Rightarrow \alpha = \frac{-l}{3(l-3)}$ ...(i)

$\alpha.2\alpha = \frac{l}{l-m} \Rightarrow \alpha^2 = \frac{-l}{2(l-3)}$ ...(ii)

from (i) & (ii),

$\frac{l^2}{9(l-m)^2} = \frac{1}{2(l-3)}$

$\Rightarrow 2(l-m)\,l^2 = 9(l-m)^2$

$\Rightarrow 2l^2 - 9l + 9m = 0$

for real l, $D \geq 0$

$81 - 4.2.9m \geq 0 \quad \Rightarrow 9 - 8m \geq 0$

$m \leq \frac{9}{8}$

18. (*c*) $3px^2 + 5qx + 7r = 0$

$D = (5q)^2 - 4.3p.7r$

$= 25q^2 - 84pr$

$= 25\,(p+r)2 - 84pr$ $(p + q + r = 0)$

$= 25\,(p^2 + r^2 + 2pr) - 84pr$

$= 25p^2 + 25r^2 + 50pr - 84pr$

$= 25p^2 - 34pr + 25r^2$

$= (5p)^2 - 2.(5p)\,\frac{(17r)}{5} + \frac{(17r)^2}{5} - \frac{(17r)^2}{5} + 25r^2$

As $D > 0$

Roots are real & distinct.

19. (*a*) $ax^2 + 2bx + c = 0$

$D = (2b)^2 - 4ac = 4(b^2 - ac)$

As a,b,c are in G.P. So, $b^2 = ac$

$D = b^2 - ac = 0$

Hence, above eq^n has equal roots

$x = \frac{-2b}{2a} = \frac{-b}{a}$

This root will satisfy $dx^2 + 2ex + f = 0$

So, $d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$

$db^2 - 2aeb + a^2f = 0$

$dac - 2aeb + a^2f = 0$

$dac + a^2f = 2aeb \Rightarrow dc + af = 2eb$

Dividing by ac, both side

$\frac{d}{a} + \frac{f}{c} = \frac{2e}{b} \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

20. (*a*) $a < b < c < d$ and $k > 0$

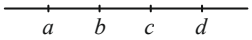
$f(x) = (x-a)\,(x-c) + k(x-b)\,(x-d)$

$f(a) = k(a-b)\,(a-b) \Rightarrow f(a) > 0$

$f(b) = (b-a)\,(b-c) \Rightarrow f(b) < 0$

$f(c) = (c-b)\,(c-d) \Rightarrow f(c) < 0$

$f(d) = (d-a)\,(d-c) \Rightarrow f(d) < 0$

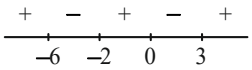


$f(x)$ has one root in (a,b) & another root in (c,d).

Hence, roots are real & distinct.

21. (*a*) $\frac{x}{x+6} - \frac{1}{x} \leq 0 \Rightarrow \frac{x^2 - x - 6}{x(x+6)} \leq 0$

$\frac{(x+2)(x-3)}{x(x+6)} \leq 0$



No. of integral values of $x = 7$

22. (*d*) Given, $3x - 4y = 77$

$3^{x/2} - 2^y = 7$

Let $3^{x/2} = p,\ 2^y = q$

$p - q = 7$

$p^2 - q^2 = 77 \Rightarrow (p-q)\,(p+q) = 77 \Rightarrow p + q = \frac{77}{7} = 11$

$\left.\begin{matrix} p - q = 7 \\ p + q = 11 \end{matrix}\right\} \Rightarrow p = 9,\ q = 2$

$3^{x/2} = 9 \Rightarrow \frac{x}{2} = 2 \Rightarrow x = 4$

$2^y = 2 \Rightarrow y = 1$

$x + y = 4 + 1 = 5$

23. (*d*) $\left(m^2+\frac{1}{m^2}\right)^2-2=119 \Rightarrow \left(m^2+\frac{1}{m^2}\right)^2=121$

$m^2+\frac{1}{m^2}=11 \Rightarrow \left(m-\frac{1}{m}\right)^2+2=11$

$\left(m-\frac{1}{m}\right)^2=9 \Rightarrow m-\frac{1}{m}=3$

$\left|m^3-\frac{1}{m^3}\right|=\left|\left(m-\frac{1}{m}\right)\left(m^2+\frac{1}{m^2}+1\right)\right|$

$=|3.12|=36$

24. (*a*) $x^8-24x^7-18x^5+39x^2+1155=0$

$x^2(x^6-24x^5-18x^3+39)=-1155$

$=-3\times5\times7\times11$

As $x \in Z$.

LHS contains x^2

But in RHS, there is no perfect square.

So, there is no integral value of x.

25. (*c*) $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$

$\frac{(x+q)+(x+p)}{(x+p)(x+q)}=\frac{1}{r}$

$(2x+p+q)r=x^2+(p+q)x+pq$

$x^2+(p+q-2r)x+pq-(p+q)r=0$

Roots are $\alpha_1-\alpha$.

$(\alpha)+(-d)=-(p+q-2r)$

$\Rightarrow 2r=p+q \Rightarrow r=\frac{p+q}{2}$

Product of roots $=\alpha.(-\alpha)=pq-(p+q)r$

$=pq-(p+q)\frac{(p+q)}{2}=-\frac{(p^2+q^2)}{2}$

26. (*d*) $\alpha^2=5\alpha-3;\ \beta^2=5\beta-3$

$\left.\begin{matrix}\alpha^2-5\alpha+3=0\\ \beta^2-5\beta+3=0\end{matrix}\right\}\alpha \& \beta$ are roots of $x^2-5x+3=0$

$\alpha+\beta=5$

$\alpha\beta=3$

Required Eq^n:

$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\left(\frac{\alpha}{\beta}.\frac{\beta}{\alpha}\right)=0$

$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0$

$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{25-2.3}{3}=\frac{19}{3}$

$x^2-\frac{19}{3}x+1=0$

$3x^2-19x+1=0$

27. (*c*) $x^2-(1+\gamma)x+\gamma-2=0$

α, β are roots.

$\alpha+\beta=(1+\gamma)$

$\alpha\beta=\gamma-2$

$\alpha+\beta-\alpha\beta=3$

$\alpha\beta-\gamma-\beta+3=0$

$\alpha\beta-\alpha-\beta+1+2=0$

$\alpha(\beta-1)-(\beta-1)=-2$

$(\alpha-1)(\beta-1)=-2$

Since, product is (*–ve*)

At least one root is *+ve*.

28. (*c*) $\frac{(\alpha+1)^2+(\beta+1)^2+(\gamma+1)^2(\delta+1)^2}{\alpha+\beta+\gamma+\delta}=4$

$(\alpha+1)^2+(\beta+1)^2+(\gamma+1)^2+(\delta+1)^2$

$=4\alpha+4\beta+4\gamma+4\delta$

$(\alpha+1)^2-4\alpha+(\beta+1)^2-4\beta+(\gamma+1)^2$

$-4\gamma+(\delta+1)^2-4\delta=0$

$(\alpha-1)^2+(\beta-1)^2+(\gamma-1)^2+(\delta-1)^2=0$

$\alpha=\beta=\gamma=\delta=1$

$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$

Roots are 1, 1, 1, 1

$\frac{a_2}{a_0}=S_2=$ Sum of product of roots taken two at a time.

$\frac{a_2}{a_0}=6$

29. (*c*) $ax^2+bx+c=0$

(i) Since downward parabola, $a<0$

(ii) As graph cuts *+ve* y-axis, $c>0$

(iii) Vertex lies in 2nd Quardrant,

$\frac{-b}{2a}<0 \Rightarrow \frac{b}{2a}>0$

$\Rightarrow b$ & a must have same sign.

$b<0$

Hence, $a<0, b<0, c>0$

Advanced Level Multiconcept Questions

1. (a, b, d)

$y = ax^2 + bx + c$

Clearly $a < 0$

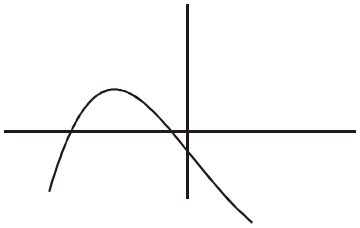
and $\frac{-b}{2a} < 0$

$\Rightarrow b < 0$

also $f(0) < 0 \Rightarrow c < 0$

and $D > 0$

$\therefore$ (A), (B) and (D).



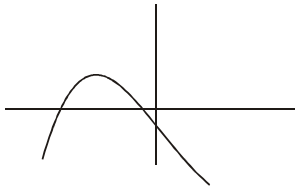
2. (a,b,c,d)

(a) $a < 0$,

$-\frac{-b}{2a} < 0 \Rightarrow b < 0$

& $f(0) < 0 \Rightarrow c < 0$

$\therefore$ $abc < 0$

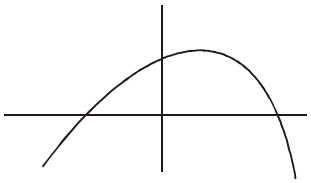


(b) $a < 0$,

$\frac{-b}{2a} > 0 \Rightarrow b > 0$

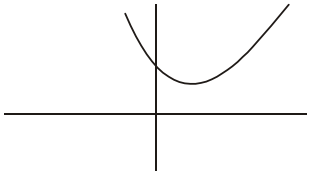
$f(0) > 0 \Rightarrow c > 0$

$\Rightarrow abc < 0$



(c) $a > 0$

$\frac{-b}{2a} > 0 \Rightarrow b < 0$



$f(0) > 0$ $c > 0$

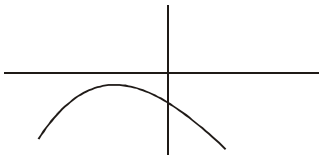
$\Rightarrow abc < 0$

(d) $a < 0$

$\frac{-b}{2a} < 0 \Rightarrow b < 0$

$f(0) < 0 \Rightarrow c < 0$

$\therefore$ $(a), (b), (c), (d)$



3. (a,d)

Clearly $a < 0$

$\frac{-b}{2a} > 0 \Rightarrow b > 0$

$\therefore$ $(a), (d)$

4. (a, b, d)

$|x|^2 + |x| - 6 = 0 \Rightarrow |x| = -3, 2 \Rightarrow |x| = 2$

$\Rightarrow x = \pm 2$

5. (b, d)

$ax^2 + bx + c = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$a + b = -b/a$, $ab = c/a$

$px^2 + qx + r = 0 \begin{matrix} \nearrow \alpha + h \\ \searrow \beta + h \end{matrix}$

$(a + b) + 2h = \frac{-q}{p}$

$h = \frac{\frac{-q}{p} + \frac{b}{a}}{2} = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$ Ans.

$|\alpha - \beta| = |(\alpha + h) - (\beta + h)|$

$= \sqrt{[(\alpha + h) + (\beta + h)]^2 - 4(\alpha + h)(\beta + h)}$

$= \frac{b^2}{a^2} - \frac{4c}{a} = \frac{q^2}{p^2} - \frac{4r}{p} \Rightarrow \frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$ Ans.

6. (b,c,d)

(a) $S = a^2 + b^2 = a^2 - 2b$

$P = a^2 b^2 = b^2$

$\therefore$ equation is $x^2 - (a^2 - 2b)x + b^2 = 0$

(b) $S = \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{a}{b}$, $P = \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{b}$

$\therefore x^2 + \frac{a}{b}x + \frac{1}{b} = 0$

$\Rightarrow bx^2 + ax + 1 = 0$

(c) $S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{a^2 - 2b}{b}$

$P = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

$x^2 - \frac{a^2 - 2b}{b}x + 1 = 0 \Rightarrow bx^2 - (a^2 - 2b)x + b = 0$

(*d*) $S = a + b - 2 = -a - 2$

$P = (a-1)(b-1)$

$= ab - (a+b) + 1$

$= b + a + 1$

$\therefore$ equation is

$x^2 + (a+2)x + (a+b+1) = 0.$

7. (*a*,*d*)

$ax^3 + bx^2 + cx + d = 0 \begin{cases} \alpha \\ \beta \\ \gamma \end{cases}$

Let $ax^3 + bx^2 + cx + d \equiv (x^2 + x + 1)(Ax + B)$

Roots of $x^2 + x + 1 = 0$ are imaginary, Let these are α, β

So the third root 'γ' will be real.

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$-1 + \gamma = \frac{-b}{a} \Rightarrow \gamma = \frac{a-b}{a}$$

Also $\alpha\beta\gamma = \frac{-d}{a}$

But $\alpha\beta = 1$

$$\therefore \gamma = \frac{-d}{a}$$

$\therefore$ Ans are (*a*) & (*d*).

8. (*a*, *b*, *d*)

$$\frac{1}{2} \le \log_{1/10} x \le 2$$

$$\Rightarrow \frac{1}{100} \le x \le \frac{1}{\sqrt{10}}$$

9. (*b*, *d*)

$x^2 + abx + c = 0 \begin{cases} \alpha \\ \beta \end{cases}$ ...(1)

$\alpha + \beta = -ab, \alpha\beta = c$

$x^2 + acx + b = 0 \begin{cases} \alpha \\ \delta \end{cases}$ ...(2)

$\alpha + \delta = -ac, \alpha\delta = b$

$\alpha^2 + ab\alpha + c = 0$

$\alpha^2 + ac\alpha + b = 0$

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{c-b} = \frac{1}{ac - ab}$$

$$\Rightarrow \alpha^2 = \frac{a(b^2 - c^2)}{a(c-b)} = -(b+c)$$

$$\& \ \alpha = \frac{c-b}{a(c-b)} = \frac{1}{a} \ \therefore \text{common root, } \alpha = \frac{1}{a}$$

$$\therefore -(b+c) = \frac{1}{a^2} \Rightarrow a^2(b+c) = -1$$

Product of the roots of equation (1) & (2) gives

$$\beta \times \frac{1}{a} = c \Rightarrow \beta = ac$$

$$\& \ \delta \times \frac{1}{a} = b \Rightarrow \delta = ab.$$

$\therefore$ equation having roots β, δ is

$x^2 - a(b+c)x + a^2bc = 0$

$a(b+c)x^2 - a^2(b+c)^2 x + a.(b+c)a^2bc = 0$

$a(b+c)x^2 + (b+c)x - abc = 0.$

10. (*c*, *d*)

$\because$ D of $x^2 + 4x + 5 = 0$ is less than zero

$\Rightarrow$ both the roots are imaginary

$\Rightarrow$ both the roots of quadratic are same

$$\Rightarrow b^2 - 4ac < 0 \ \& \ \frac{a}{1} = \frac{b}{4} = \frac{c}{5} = k$$

$\Rightarrow a = k, b = 4k, c = 5k.$

11. (*a*, *d*)

$x^2 + px + q = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$\alpha + \beta = -p, \alpha\beta = q$ and $p^2 - 4q > 0$

$x^2 - rx + s = 0 \begin{cases} \alpha^4 \\ \beta^4 \end{cases}$ .....(1)

Now $\alpha^4 + \beta^4 = r$

$\Rightarrow \alpha^4 + \beta^4 = r, (\alpha\beta)^4 = s = q^4$

$\therefore (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = r$

$\Rightarrow [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2 = r$

$\Rightarrow (p^2 - 2q)^2 - 2q^2 = r$

$\Rightarrow (p^2 - 2q)^2 = 2q^2 + r > 0$ .....(2)

Now, for $x^2 - 4qx + 2q^2 - r = 0$

$D = 16q^2 - 4(2q^2 - r)$ by equation (2)

$= 8q^2 + 4r = 4(2q^2 + r) > 0$

$\Rightarrow$ D > 0 two real and distinct roots

Product of roots $= 2q^2 - r$

$= 2q^2 - [(p^2 - 2q)^2 - 2q^2]$

$= 4q^2 - (p^2 - 2q)^2$

$= -p^2(p^2 - 4q) < 0$ from (1)

So product of roots is – ve

hence roots are opposite in sign

12. (*a*, *d*)

$20x^2 + 210x + 400 = 4500 \Rightarrow 2x^2 + 21x - 410 = 0$

$\Rightarrow (2x + 41)(x - 10) = 0$

$$\Rightarrow x = \frac{-41}{2}, \ x = 10 \ \Rightarrow x = -20.5, \ x = 10$$

13. (*a*,*b*,*c*,*d*)

$x^3+bx^2+cx-1=0 \begin{cases} \alpha=\frac{a}{r} \\ \beta=a \\ r=ar \end{cases}$

$\frac{a}{r}+a+ar=-b \Rightarrow a\left(\frac{1}{r}+1+r\right)=-b$

& $\frac{a}{r}\times a\times ar=1$

$a^3=1\Rightarrow a=1$

& $\frac{a}{r}a+a\cdot ar+\frac{a}{r}\cdot ar=c$

$a^2\left(\frac{1}{r}+r+1\right)=c$

$\frac{1}{r}+r+1=-b$ & $\frac{1}{r}+r+1=c\Rightarrow b+c=0$

we know $\frac{1}{r}+r>2\Rightarrow \left(\frac{1}{r}+r+1\right)>3$

$-b>3\Rightarrow b<-3\Rightarrow b\in(-\infty,-3)$

& other two roots are $\frac{1}{r}$ & r

if $\frac{1}{r}>1\Rightarrow r<1$ if $r>1\Rightarrow r<1$

14. (*a*,*b*)

$f(x)=\frac{3}{(x-2)}+\frac{4}{(x-3)}+\frac{5}{(x-4)}=0$

$6x^2-14x-21x+49=0$

$(3x-7)(2x-7)=0$

$x=\frac{7}{2}, x=\frac{7}{2}$

$2<\frac{7}{2}<3<3<\frac{7}{2}<4$

2nd Method

$g(x)=3(x-3)(x-4)+4(x-2)(x-4)+5(x-2)(x-3)=0$

$g(2)>0; g(3)<0, g(4)>0$

one root lie b/w (2, 3) & other root lie b/w (3, 4)

15. (*c*)

16. (*b*)

17. (*d*)

Sol. (15 to 16)

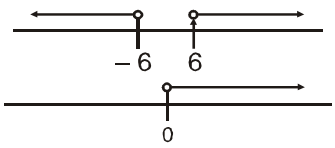
$x^4-\lambda x^2+9=0\Rightarrow \quad x^2=t\geq 0\Rightarrow f(t)=t^2-\lambda t+9=0$

15. given equation has four real & distinct roots



$D>0$

$\Rightarrow \lambda^2-36>0$



$\frac{-b}{2a}>0$

$\Rightarrow \frac{\lambda}{2}>0$

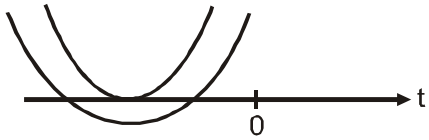
$\Rightarrow \lambda>0$

$f(0)>0$

$\Rightarrow 9>0$

$\therefore \quad \lambda\in(6,\infty)$

16. Equation has no real roots.



case-I $D\geq 0\Rightarrow \lambda^2-36\geq 0$

$\frac{-b}{2a}<0 \quad \Rightarrow \lambda<0$



$f(0)>0 \quad \Rightarrow \quad 9>0.$

$\therefore \quad \lambda\in(-\infty,-6]$

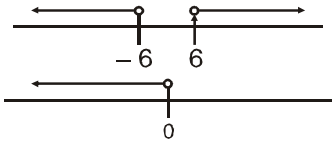
case-II $D<0$

$\Rightarrow \quad \lambda^2-36<0$

$\Rightarrow \quad \lambda\in(-6,6)$

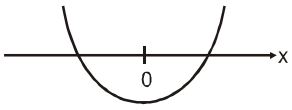
union of both cases gives

$\lambda\in(-\infty,6)$



17. Equation has only two real roots

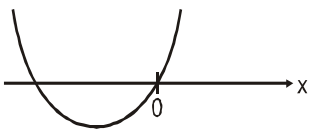
case-I $f(0)<0 \quad 9<0$



which is false

case-II $f(0)=0$

and $\frac{-b}{2a}<0$



$\therefore$ No solution

$\therefore$ Final answer is ϕ

18. $(a) \to (r), (b) \to (p), (c) \to (s), (d) \to (p, q)$

(a) $x^2 - 8x + k = 0$ $\begin{matrix}\nearrow \alpha \\ \searrow \alpha + 4 = \beta\end{matrix}$

$\because (\beta - \alpha)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow 16 = 64 - 4k \Rightarrow 4k = 48 \Rightarrow k = 12$

$(b) \because (|x| - 2)(|x| - 3) = 0$

$\Rightarrow x = \pm 2 ; \ x = \pm 3$

$\therefore n = 4 \therefore \dfrac{n}{2} = 2$

$(c) \because b = (3 - i)(3 + i)$

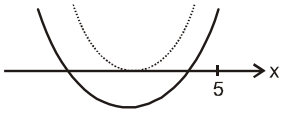
$b = 10$

(d) $x^2 - 2kx + (k^2 + k - 5) = 0$

(i) $D \geq 0$

$\Rightarrow 4k^2 - 4(k^2 + k - 5) \geq 0$

$\Rightarrow k - 5 \leq 0$

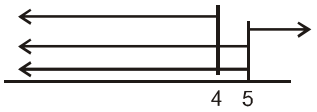


(ii) $f(5) > 0$

$\Rightarrow \quad 25 - 10k + k^2 + k - 5 > 0$

$\Rightarrow \quad k^2 - 9k + 20 > 0 \Rightarrow (k - 5)(k - 4) > 0$

$(iii) - \dfrac{b}{2a} < 5 \Rightarrow k < 5$



$\Rightarrow \quad k \in (-\infty, 4)$

So k may be 2, 3.

NUMERICAL VALUE BASED

19. [2] $\left(x^2 + 3x + 2\right)\left(x^2 + 3x\right) = 120$

Let $x^2 + 3x = y$

$\Rightarrow y^2 + 2y - 120 = 0$

$\Rightarrow (y + 12)(y - 10) = 0$

$\Rightarrow y = -12 \Rightarrow x^2 + 3x + 12 = 0$

$\Rightarrow x \in \phi$

$y = 10 \quad \Rightarrow x^2 + 3x - 10 = 0$

$\Rightarrow (x + 5)(x - 2) = 0 \quad \Rightarrow x = \{-5, 2\}$

$x = 2, -5$ are only two integer roots.

20. [8] $\left(5 + 2\sqrt{6}\right)^{x^2 - 3} + \dfrac{1}{\left(5 + 2\sqrt{6}\right)^{x^2 - 3}} = 10$

$\Rightarrow \quad t + \dfrac{1}{t} = 10$

$\Rightarrow \quad t^2 - 10t + 1 = 0 \quad t = \dfrac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6}$

$\Rightarrow \quad (5 + 2\sqrt{6})^{x^2 - 3} = (5 + 2\sqrt{6}) \quad \text{or} \quad \dfrac{1}{5 + 2\sqrt{6}}$

$\Rightarrow \quad x^2 - 3 = 1 \quad \text{or} \quad x^2 - 3 = -1$

$\Rightarrow \quad x = 2 \text{ or } -2 \quad \text{or} \quad -\sqrt{2} \text{ or } \sqrt{2}$

Product 8

21. [11] $2x^2 + 6x + a = 0$

Its roots are α, β

$\Rightarrow \quad \alpha + \beta = -3 \qquad \& \ \alpha\beta = \dfrac{a}{2} \Rightarrow \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} < 2$

$\Rightarrow \quad \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} < 2$

$\Rightarrow \quad \dfrac{9 - a}{a} < 1$

$\Rightarrow \quad \dfrac{2a - 9}{a} > 0$

$\Rightarrow \quad a \in (-\infty, 0) \cup \left(\dfrac{9}{2}, \infty\right)$

$\Rightarrow \quad 2a = 11$ is least prime.

22. [1]

$x^2 + px + 1 = 0$ $\begin{matrix}\nearrow a \\ \searrow b\end{matrix}$ $a + b = -p, \ ab = 1 \ ; \ x^2 + qx +$

$1 = 0$ $\begin{matrix}\nearrow c \\ \searrow d\end{matrix}$ $c + d = -q, \ cd = 1$

$a + b = -p, \ ab = 1 \Rightarrow c + d = -q, \ cd = 1$

RHS $= (a - c)(b - c)(a + d)(b + d)$

$= (ab - ac - bc + c^2)(ab + ad + bd + d^2)$

$= (1 - ac - bc + c^2)(1 + ad + bd + d^2)$

$= 1 + ad + bd + d^2 - ac - a^2cd - abcd - acd^2 - bc$

$\quad - abcd - b^2cd - bcd^2 + c^2 + adc^2 + bdc^2 + c^2d^2$

$= 1 + ad + bd + d^2 - ac - a^2 - 1 - ad - bc - 1 - b^2$

$\quad - bd + c^2 + ac + bc + 1$

$[\because ab = cd = 1]$

$= c^2 + d^2 - a^2 - b^2 = (c + d)^2 - 2cd - (a + b)^2 + 2ab$

$= q^2 - 2 - p^2 + 2 = q^2 - p^2 = \text{LHS}.$ **Proved.**

2nd Method :

RHS $= (ab - c(a + b) + c^2)(ab + d(ab + d(a + b) + d^2)$

$= (c^2 + pc + 1)(1 - pd + d^2) \qquad ...(1)$

Since c & d are the roots of the equation $x^2 + qx + 1 = 0$

$\therefore \ c^2 + qc + 1 = 0 \Rightarrow c^2 + 1 = -qc \ \& \ d^2 + qd + 1 = 0$

$\Rightarrow d^2 + 1 = -qd.$

$\therefore$ (i) Becomes $= (pc - qc)(-pd - qd) = c(p - q)(-d)(p + q) = -cd(p^2 - q^2)$

$= cd(q^2 - p^2) = q^2 - p^2 = \text{LHS}.$ **Proved.**

23. [73] $\because \alpha, \beta$ are roots of $\lambda x^2 - (\lambda - 1)x + 5 = 0$

$\therefore \alpha + \beta = \dfrac{\lambda - 1}{\lambda}$ and $\alpha\beta = \dfrac{5}{\lambda}$

$\because \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = 4 \qquad \Rightarrow \qquad \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = 4$

$\Rightarrow (\alpha+\beta)^2 = 6\alpha\beta \Rightarrow \frac{(\lambda-1)^2}{\lambda^2} = \frac{30}{\lambda}$

$\Rightarrow \lambda^2 - 32\lambda + 1 = 0$ ...(1)

$\because \lambda_1, \lambda_2$ are roots of (1)

$\therefore \lambda_1 + \lambda_2 = 32$ and $\lambda_1\lambda_2 = 1$

$\therefore \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = \frac{(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2} = \frac{(32)^2-2}{1}$

$= 1022 \Rightarrow \left(\frac{\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}}{14}\right) = 73$

24. [10] $\alpha.\beta = b$; $\gamma\delta = b-2$

$\Rightarrow \alpha\beta\gamma\delta = b(b-2) = 24$

$\therefore bx^2 + ax + 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}$

$\Rightarrow \frac{1}{\alpha}+\frac{1}{\beta} = \frac{-a}{b}$

$(b-2)x^2 - ax + 1 = 0$ has root $\frac{1}{\gamma}, \frac{1}{\delta} \Rightarrow \frac{1}{\gamma}+\frac{1}{\delta} = \frac{a}{b-2}$

$\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{-a}{b}+\frac{a}{b-2} = \frac{5}{6}$; $\frac{+2a}{b(b-2)} = \frac{5}{6}$;

$\frac{+2a}{24} = \frac{5}{6}$; $a = 10$.

25. [3] $x^2 + px + 1 = 0$

Roots = a,b

$a + b = -p$; $ab = 1$ ...(i)

$x^2 + qx + 1 = 0 \Rightarrow c + d = -q$; $cd = 1$ ...(ii)

Roots = c,d

Also, $c^2 + qc + 1 = 0$; $d^2 + qd + 1 = 0$ ...(iii)

$(a-c)(b-c)(a+b)(b+d) = (ab - (a+b)c + c^2)(ab + (a+b)d + d^2)$

$= (1 + p.c + c^2)(1 - pd + d^2)$ from ...(i)

$= (c^2 + 1 + pc)(d^2 + 1 - pd)$

$= (-qc + pc)(-qd - pd)$ from ...(iii)

$= -c(p-q)(p+q)d$

$= -cd(p^2 - q^2)$

$= -1(p^2 - q^2) = q^2 - p^2$

Required value = $\frac{(a-c)(b-c)(a+d)(b+d)}{q^2-p^2}$

$= \frac{q^2-p^2}{q^2-p^2} = 1$

26. [13] $a^3 + b^3 + (-9)^3 = 3 \cdot a \cdot b(-9)$

$\Rightarrow a + b - 9 = 0$ or

$a = b = -9$. Which is rejected.

As $a > b > -9$

$\Rightarrow a + b - 9 = 0 \Rightarrow x = 1$ is a root

other root = $\frac{-9}{a}$. $\therefore \alpha = \frac{-9}{a}$, $\beta = 1$

$\Rightarrow 4\beta - a\alpha = 4 - a\left(\frac{-9}{a}\right) = 4 + 9 = 13.$

27. [6] Let $t^2 - 2t + 2 = k$

$\Rightarrow \alpha^2 - 6k\alpha - 2 = 0$

$\Rightarrow \alpha^2 - 2 = 6k\alpha$

$a_{100} - 2a_{98} = \alpha^{100} - 2.\alpha^{98} - \beta^{100} + 2.\beta^{98}$

$= \alpha^{98}(\alpha^2 - 2) - \beta^{98}(\beta^2 - 2) = 6k(\alpha^{99} - \beta^{99})$

$a_{100} - 2a_{98} = 6k.a_{99}$

$\frac{a_{100} - 2a_{98}}{a_{99}} = 6k = 6(t^2 - 2t + 2) = 6[(t-1)^2 + 1]$

$\therefore$ min. value of $\frac{a_{100} - 2a_{98}}{a_{99}}$ is 6.

28. [11] Given that, roots of equation $x^2 - 10ax - 11b = 0$ are c, d

So $c + d = 10a$ and $cd = -11b$ and a, b are the roots of equation $x^2 - 10cx - 11d = 0$

$\therefore a + b = 10c$, $ab = -11d$

So $a + b + c + d = 10(a + c)$ and $(c + d) - (a + b) = 10(a - c)$

$(c - a) - (b - d) + 10(c - a) = 0$

$\Rightarrow b + d = 9(a + c)$ ....(i)

$abcd = 121\ bd$

$\Rightarrow ac = 121$ ....(ii)

$b - d = 11(c - a)$ ....(iii)

c & a satisfies the equation $x^2 - 10ax - 11b = 0$ and $x^2 - 10cx - 11d = 0$ respectively

$\therefore c^2 - 10ac - 11b = 0$

$a^2 - 10ca - 11d = 0$

$(c^2 - a^2) - 11(b - d) = 0$

$(c - a)(c + a) = 11(b - d) = 11.11(c - a)$ (by equation (iii))

$c + a = 121$

$\Rightarrow a + b + c + d = 10(c + a)$

$\Rightarrow 10.121 \Rightarrow \frac{a + b + c + d}{110} = 11.$

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*a*) $(x^2-5x+5)^{x^2+4x-60}=1$

$x^2-5x+5=1\Rightarrow x=1,4$

or, $x^2+4x-60=0\ x=-10,6$

or , $x^2-5x+5=-1\Rightarrow x=2,3$

But for $x=3,\ x^2+4x-60$ is odd

So Required values of x are $1,4,-10,6,2$ Sum = 3

2. (*a*) We have $\sum_{r=1}^{n}(x+r-1)(x+r)=10n$

$\Rightarrow \sum_{r=1}^{n}\left(x^2+(2r-1)x+\left(r^2-r\right)\right)=10n$

$\therefore$ On solving, we get

$x^2+nx+\left(\frac{n^2-31}{3}\right)=0 \qquad \therefore (2\alpha+1)=-n$

$\alpha \quad \alpha+1$

$\Rightarrow \alpha=\frac{-(n+1)}{2}$ .....(1)

and $\alpha(\alpha+1)=\frac{n^2-31}{3}$ .....(2)

$\Rightarrow \quad n^2=121$ (using (1) in (2))

or $\quad n=11$

3. (*b*) α, β are roots of $x^2-x+1=0$

$\therefore \alpha=-\omega$ and $\beta=-\omega^2$

where ω is non-real cube root of unity

so, $a^{101}+\alpha^{107}$

$\Rightarrow (-\omega)^{101}+(-\omega^2)^{107}$

$\Rightarrow -[\omega^2+\omega]$

$\Rightarrow -[-1]=1$

(As $1+\omega+\omega^2=0$ & $\omega^3=1$)

4. (*a*) $x^2+2x+2=0\Rightarrow (x+1)^2=-1$

$x=-1\pm i=\sqrt{2}e^{i\left(\pm\frac{3\pi}{4}\right)}$

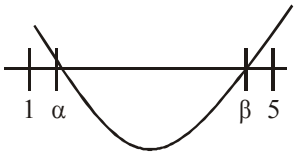
$\therefore \alpha^{15}, \beta^{15}=\left(\sqrt{2}\right)^{15}\times 2\cos\left(15.\frac{3\pi}{4}\right)$

$=2^8\sqrt{2}\times\left(-\frac{1}{\sqrt{2}}\right)=-256$

5. (Bonus)

$x^2-mx+4=0$

$\alpha, \beta \in [1,5]$



(*a*) $D>0\Rightarrow m^2-16>0$

$\Rightarrow m\in(-\infty,-4)\cup(4,\infty)$

(*b*) $f(1)\geq 0\Rightarrow 5-m\geq 0\Rightarrow m\in(-\infty,5)$

(*c*) $f(5)\geq 0\Rightarrow 29-5m\geq 0\Rightarrow m\in\left(-\infty,\frac{29}{5}\right]$

(*d*) $1<\frac{-b}{2a}<5\Rightarrow 1<\frac{m}{2}<5\Rightarrow m\in(2,10)$

$\Rightarrow m\in(4,5)$

No option correct : Bonus

* If we consider $\alpha, \beta\in(1,5)$ then option (1) is correct.

6. (*c*) D must be perfect square

$\Rightarrow 121-24\alpha=\lambda^2$

$\Rightarrow$ maximum value of α is 5

$\alpha=1\Rightarrow\lambda\notin 1$

$\alpha=2\Rightarrow\lambda\notin 1$

$\alpha=3\Rightarrow\lambda\in 1$

$\Rightarrow$ 3 integral values

$\alpha=4\Rightarrow\lambda\in 1$

$\alpha=5\Rightarrow\lambda\in 1$

7. (*d*) Case - 1

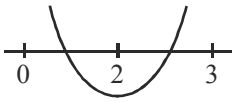
$c-5>0$ ....(i)

$f(0)>0$

$c-4>0$ ...(ii)

$f(2)<0$

$4(c-5)-4c+c-4<0$



$c<24$ ...(iii)

$f(2)>0$

$9(c-5)-6c+c-4>0$

$4c-49>0\Rightarrow c>\frac{49}{4}$ ...(iv)

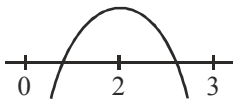
Here (i) $\cap$ (ii) $\cap$ (iii) $\cap$ (iv)

$c \in \left(\frac{49}{4}, 24\right)$

Case - II

$c - 5 < 0$ ...(i)

$f(0) < 0$



$c < 4$ ...(ii)

$f(2) > 0 \Rightarrow c > 24$ ...(iii)

$f(3) < 0 \Rightarrow c > 49$ ...(iv)

$4 \Rightarrow c \in \phi$

$c \in \left(\frac{49}{4}, 24\right)$.

8. (*d*) $\alpha + \alpha^3 = -\frac{K}{81}$ ....(1)

$\alpha^4 = \frac{256}{81}$

$\alpha = \pm \frac{4}{3}$ ....(2)

From (1) and (2)

$\frac{4}{3} + \frac{64}{27} = \frac{-K}{81}$

$K = -300$

9. (*c*) Expression is always positive if $2m + 1 > 0 \Rightarrow m > -\frac{1}{2}$

and $D < 0 \Rightarrow m^2 - 6m - 3 < 0$

$3\sqrt{12} < m < 3 + \sqrt{12}$

$\therefore$ common interval is $3 - \sqrt{12} < m < 3 + \sqrt{12}$

$\therefore$ Integral value of m {0,1,2,3,4,5,6}.

10. (*b*) Let roots are α and β now

$\lambda + \frac{1}{\lambda} = 1 \Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 1 \Rightarrow \alpha^2 + \beta^2 = \alpha\beta$

$(\alpha + \beta)^2 = 2\alpha\beta$

$\left(\frac{-m(m-4)}{3m^2}\right)^2 = 3 \cdot \frac{2}{3m^2}$

$m^2 - 8m - 2 = 0$

$m = 4 \pm 3\sqrt{2}$

So least value of $m = 4 - 3\sqrt{2}$

11. (*c*) $(x-1)^2 + 1 = 0 \Rightarrow x = 1 + i, 1 - i$

$\therefore \quad \left(\frac{\alpha}{\beta}\right)^n = 1 \Rightarrow (\pm 1)^n = 1$

$\therefore$ n (least natural number) = 4

12. (*c*) a, b, c, in G.P.

say a, ar, ar^2

satisfies $ax^2 + 2bx + c = 0 \Rightarrow x = -r$

$x = -r$ is the common root, satisfies second equation

$d(-r)^2 + 2e(-r) + f = 0$

$\Rightarrow d.\frac{c}{a} - \frac{2ce}{b} + f = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$

13. (*a*) $D < 0$

$4(1 + 3m)^2 - 4(Hm^2)(1 + 8m) < 0$

$\Rightarrow m(2m - 1)^2 > 0 \Rightarrow m > 0$

14. (*b*) In given question p, q $\in$ R. If we take other root as any real number α,

then quadratic equation will be

$x^2 - (\alpha + 2 - \sqrt{3})x + \alpha.(2 - \sqrt{3}) = 0$

Now, we can have none or any of the options can be correct depending upon 'α' Instead of p, q $\in$ R it should be p, q $\in$ Q then other root will be $2 + \sqrt{3}$

$\Rightarrow p = -(2 + \sqrt{3} - 2 - \sqrt{3}) = -4$

and $q = (2 + \sqrt{3})(2 - \sqrt{3}) = 1$

$\Rightarrow p^2 - 4q - 12 = (-4)^2 - 4 - 12$

$= 16 - 16 = 0$

Option (*b*) is correct

15. (*d*) $SOR = \frac{3}{m^2 + 1} \Rightarrow (S.O.R)_{max} = 3$

when $m = 0$

$x^2 - 3x + 1 = 0$ α, β

$\alpha + \beta = 3$

$\alpha\beta = 1$

$|\alpha^3 - \beta^2| = ||\alpha - \beta|(\alpha^2 + \beta^2 + \alpha\beta)|$

$= \left|\sqrt{(\alpha - \beta)^2 - \alpha\beta}\,((\alpha + \beta)^2 - \alpha\beta\right|$

$= \left|\sqrt{9 - 4}\,(9 - 1)\right|$

$= \sqrt{5} \times 8$

16. (*d*) $\frac{\alpha^{12} + \beta^{12}}{\left(\frac{1}{\alpha^{12}} + \frac{1}{\beta^{12}}\right)(\alpha - \beta)^{24}} = \frac{(\alpha\beta)^{12}}{(\alpha - \beta)^{24}}$

$= \frac{(\alpha\beta)^{12}}{\left[(\alpha + \beta)^2 - 4\alpha\beta\right]^{12}} = \left[\frac{\alpha\beta}{(\alpha + \beta)^2 - 4\alpha\beta}\right]^{12}$

$= \left(\frac{-2\sin\theta}{\sin^2\theta + 8\sin\theta}\right)^{12} = \frac{2^{12}}{(\sin\theta + 8)^{12}}$

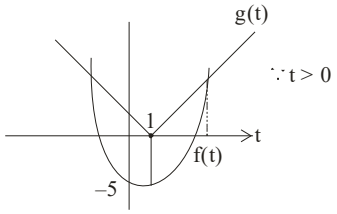
17. (*d*) Let $2^x = t$

$5 + |t-1| = t^2 - 2t$

$\Rightarrow |t-1| = (t^2 - 2t - 5)$

g(t) f(t)

From the graph



So, number of real root is 1.

18. (*c*) $375x^2 - 25x - 2 = 0$

$\alpha + \beta = \frac{25}{375}, \alpha\beta = \frac{-2}{375}$

$\Rightarrow (\alpha + \alpha^2 + \ldots..$ upto infinite terms) + $(\beta + \beta^2 + \ldots..$ upto infinite terms) $= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} = \frac{1}{12}$

19. (*a*) $\alpha x^2 + 2\beta x + \gamma = 0$

Let $\beta = \alpha t, \gamma = \alpha t^2$

$\therefore \alpha x^2 + 2\alpha t x + \alpha t^2 = 0$

$\Rightarrow x^2 + 2tx + t^2 = 0$

$\Rightarrow (x + t)^2 = 0$

$\Rightarrow x = -t$

it must be root of equation $x^2 + x - 1 = 0$

$\therefore t^2 - t - 1 = 0$ .....(1)

Now

$\alpha(\beta + \gamma) = \alpha^2 (t + t^2)$

Option 1 $\beta\gamma = \alpha t \,.\, \alpha t^2 = \alpha^2 t^3 = a^2 (t^2 + t)$

from equation 1

20. (*a*) $\alpha^5 = 5\alpha + 3$

$\beta^5 = 5\beta + 3$

$p_5 = 5(\alpha + \beta) + 6$

$= 5(1) + 6$

$P_5 = 11$ and $p_5 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1$

$P_2 = 3$ and $p_3 = \alpha^3 + \beta^3 = 2\alpha + 1 + 2\beta + 1 = 2(1) + 2 = 4$

$P_2 \times P_3 = 12$ and $P_5 = 11 \Rightarrow P_5 \neq P_2 \times P_3$

21. 8 $D \geq 0$

$(a - 10)^2 - 4(2)\left(\frac{33}{2} - 2a\right) \geq 0$

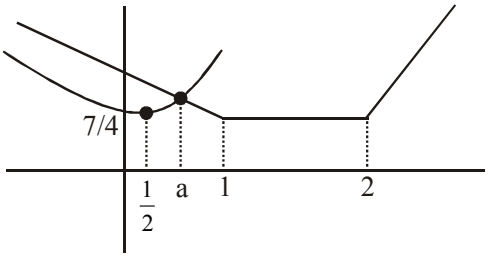
$(a - 10)^2 - 4(33 - 4a) \geq 0$

$a^2 - 4a - 32 \geq 0 \Rightarrow a \in (-\infty, -4] \cup [8, \infty)$.

22. (*b*) Let $3^x = t$

$t(t-1) + 2 = |t-1| + |t-2|$

$t^2 - t + 2 = |t-1| + |t-2|$



are positive solution

$t = a$

$3^x = a$

$x = \log_3$ is singleton set.

23. (*b*) Let $e^x = t \in (0, \infty)$

Given equation

$t^4 + t^3 - 4t^2 + t + 1 = 0$

$t^2 + t - 4 + \frac{1}{t} + \frac{1}{t^2} = 0$

$\left(t^2 + \frac{1}{t^2}\right) + \left(t + \frac{1}{t}\right) - 4 = 0$

Let $t + \frac{1}{t} = \alpha$

$(\alpha^2 - 2) + \alpha - 4 = 0$

$\alpha^2 + \alpha - 6 = 0$

$\alpha^2 + \alpha - 6 = 0$

$\alpha = -3, 2 \quad \Rightarrow \alpha = 2 \quad \Rightarrow e^x + e^{-x} = 2$

$x = 0$ only solution

24. (*a*) $2\alpha = \frac{2b}{a} \Rightarrow \alpha = \frac{b}{a}$ and $\alpha^2 = \frac{5}{a} \Rightarrow \frac{b^2}{a^2} = \frac{5}{a}$

$\Rightarrow b^2 = 5a$ .....(i) $(a \neq 0)$

$\alpha + \beta = 2b$ .....(ii)

$\alpha\beta = -10$ .....(iii)

$\alpha = \frac{b}{a}$ is also root of $x^2 - 2bx - 10 = 0$

$\Rightarrow b^2 - 2ab^2 - 10a^2 = 0$

by (i) $\Rightarrow 5a - 10a^2 - 10a^2 = 0$

$\Rightarrow 20a^2 = 5a$

$\Rightarrow a = \frac{1}{4}$ and $b^2 = \frac{5}{4}$

$\alpha^2 = 20$ and $\beta^2 = 5$

Now $\alpha^2 + \beta^2$

$= 5 + 20$

$= 25$

25. (*a*) Let $f(x) = ax^2 + bx + c$

Let roots are 3 and α

and $f(-1) + f(2) = 0$

$4a + 2b + c + a - b + c = 0$

$5a + b + 2c = 0$ ...(i)

$\because f(3) = 0 \Rightarrow 9a + 3b + c = 0$ ...(ii)

From equation (i) and (ii)

$$\frac{a}{1-6} = \frac{b}{18-5} = \frac{c}{15-9} \Rightarrow \frac{a}{-5} = \frac{b}{13} = \frac{c}{6}$$

$\therefore f(x) = k(-5x^2 + 13x + 6)$

$= -k(5x + 2)(x - 3)$

= Root are 3 and $-\frac{2}{5}$

$\therefore -\frac{2}{5}$ lies in interval $(-1, 0)$

26. (*a*) $\because \alpha$ is a root of given equation, then

$5\alpha^2 + 6\alpha = 2$

$\Rightarrow 5\alpha^6 + 6\alpha^5 = 2\alpha^4$ ...(1)

Similarly $5\beta^6 + 6\beta^5 = 2\beta^4$ ...(2)

Adding (1) and (2), we get

$5S_6 + 6S_5 = 2S_4$

27. (*d*) $\because$ Equation is : $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$

$\because$ One root in interval (0,1)

$\therefore f(0).\ f(1) < 0$

$2.(\lambda^2 + 1 - 4\lambda + 2) < 0$

$(\lambda - 3)(\lambda - 1) < 0$

$\therefore \lambda \in (1, 3)$

If $\lambda = 3$, then roots are 1 and $\frac{1}{5}$

$\therefore \lambda \in (1, 3]$

28. (*d*) $\alpha.\beta = 2$ and $\alpha + \beta = -p$ also $\frac{1}{\alpha} + \frac{1}{\beta} = -q$

$\Rightarrow p = 2q$

$$\text{Now } \left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$$

$$= \left[\alpha\beta + \frac{1}{\alpha\beta} - \frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right]\left[\alpha\beta + \frac{1}{\alpha\beta} + 1 + 1\right]$$

$$= \frac{9}{2}\left[\frac{5}{2} - \frac{\alpha^2 + \beta^2}{2}\right] = \frac{9}{4}\left[5 - (p^2 - 4)\right]$$

$$= \frac{9}{4}(9 - p^2)$$

29. (*a*) Roots of $x^2 - x + 2\lambda = 0$ are α and β

and roots of $3x^2 - 10x + 27\lambda = 0$ and α and γ

Here,

$3\alpha^2 - 10\alpha + 27\lambda = 0$ ....(i)

$3\alpha^2 - 3\alpha + 6\lambda = 0$ ....(ii)

$\therefore \alpha = 3\lambda$

Now,

$3\lambda + \beta = 1$ and $3\lambda \cdot \beta = 2\lambda$

and, $3\lambda + \gamma = \frac{10}{3}$ and $3\lambda.\gamma = 9\lambda$

$\therefore \gamma = 3, \alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}, \lambda = \frac{1}{9}$

$\frac{\beta\gamma}{\lambda} = 18$

30. (*c*) $\because \alpha, \beta, \gamma, \delta$ are in G.P, so $\alpha\delta = \beta\gamma$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \left|\frac{\alpha - \beta}{\alpha + \beta}\right| = \left|\frac{\gamma - \delta}{\gamma + \delta}\right|$$

$$\Rightarrow \sqrt{\frac{9 - 4p}{3}} = \sqrt{\frac{36 - 4p}{6}}$$

$\Rightarrow 36 - 16p = 36 - 4q$

$\Rightarrow q = 4p$

So, $\frac{2q + p}{2q - p} = \frac{9p}{7p} = \frac{9}{7}$

31. (*d*) $7x^2 - 3x - 2 = 0 \Rightarrow \alpha + \beta = \frac{3}{7}$, $\alpha\beta = \frac{-2}{7}$

$$\text{Now } \frac{\alpha}{1 - \alpha^2} + \frac{\beta}{1 - \beta^2}$$

$$= \frac{\alpha - \alpha\beta(\alpha + \beta) + \beta}{1 - (\alpha^2 + \beta^2) + (\alpha\beta)^2}$$

$$= \frac{(\alpha + \beta) - \alpha\beta(\alpha + \beta)}{1 - (\alpha^2 + \beta^2) + 2\alpha\beta + (\alpha\beta)^2}$$

$$= \frac{\frac{3}{7} + \frac{2}{7} \times \frac{3}{7}}{1 - \frac{9}{49} + 2 \times \frac{-2}{7} + \frac{4}{49}} = \frac{21 + 6}{49 - 9 - 28 + 4} = \frac{27}{16}$$

32. (*b*) Let $|x| = t$ we have

$9t^2 - 18t + 5 = 0$

$9t^2 - 15t - 3t + 5 = 0$

$(3t - 1)(3t - 5) = 0$

$\Rightarrow t = \frac{1}{3}$ or $\frac{5}{3} \Rightarrow |x| = \frac{1}{3}$ or $\frac{5}{3}$

Roots are $\pm\frac{1}{3}$ and $\pm\frac{5}{3}$

Product = $\frac{25}{81}$

33. (*b*) $\alpha+\beta=-\frac{1}{2}\Rightarrow -1=2\alpha+2\beta$

and $4\alpha^2+2\alpha-1=0$

$\Rightarrow 4\alpha^2+2\alpha+2\alpha+2\beta=0$

$\Rightarrow \beta=-2\alpha(\alpha+1)$

34. (*b*) $\frac{\alpha^{3/8}}{\beta^{5/8}}+\frac{\beta^{3/8}}{\alpha^{5/8}}=\frac{\alpha+\beta}{(\alpha\beta)^{5/8}}$

For $x^2-64x+256=0$

$\alpha+\beta=64$

$\alpha\beta=256$

$\therefore \frac{\alpha+\beta}{(\alpha\beta)^{5/8}}=\frac{64}{\left(2^8\right)^{5/8}}=\frac{64}{32}=2$

35. (*d*) Given, $p+q=2, p^4+q^4=272$

$\Rightarrow (p^2+q^2)-2p^2q^2=272$

$\Rightarrow ((p+q)^2-2pq)^2=2p^2q^2=272$

$\Rightarrow (4+2pq)^2-2(pq)^2=272$

$16+4(pq)^2-16pq-2(pq)^2=272$

$2(pq)^2-16(pq)-256=0$

$(pq)^2-8(pq)-128=0$

$pq=\frac{8\pm\sqrt{64+4.128}}{2}=\frac{8\pm 24}{2}=16$

As, p,q are $+ve_1$ So, $pq\neq -8$

$pq=16$

Quadrant equation is

$x^2-(p+q)x+pq=0$

$x^2-2x+16=0$

36. (*d*) $a,b,c\rightarrow \text{AP}\Rightarrow 2b=a+c$ ...(i)

Centroid is $\left(\frac{10}{3},\frac{7}{3}\right)$

$\frac{a+2+a}{3}=\frac{10}{3}$ & $\frac{c+b+b}{3}=\frac{7}{3}$

$2a+2=10$ & $c+2b=7$

$2a=8$

a = 4; ...(ii)

from (i) & (ii)

$2b=c+4$

$c+2b=7$

$c+c+4=7$

$2c=3$

$c=\frac{3}{2}$

$2b=c+4=\frac{11}{2}$

$b=\frac{11}{4}$

$ax^2+bx+4=0\Rightarrow 4x^2+\frac{11x}{4}+1=0$

$\alpha+\beta=\frac{-11}{4.4}=\frac{-11}{16}; \alpha\beta=\frac{1}{4}$x

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\alpha^2+\beta^2-\alpha\beta=(\alpha+\beta)^2-3\alpha\beta=\frac{121}{256}-\frac{3}{4}=\frac{-71}{256}$

37. (2) $(x+1)^2+|x-5|=\frac{27}{4}$

Case 1: if $x\geq 5$

$(x+1)^2+x-5=\frac{27}{4}$

$x^2+3x-4=\frac{27}{4}\Rightarrow x^2\,3x-\frac{43}{4}=0$

$4x^2+12x-43=0$

$x=\frac{-12\pm\sqrt{144-4.4.(-43)}}{2.4}=\frac{-12\pm\sqrt{832}}{8}$

$=\frac{-12\pm 28.8}{8}$

$=\frac{-3\pm 7.2}{2}=\frac{-10.2}{2},\frac{+4.2}{2}$

But $x\geq 5$, So, no real root.

Case 2: if $x<5$

$(x+1)^2+(-(x-5))=\frac{27}{4}\Rightarrow x^2+1+2x-x+5=\frac{27}{4}$

$x^2+x+6-\frac{27}{4}=0\Rightarrow 4x^2+4x-3=0$

$x=\frac{-4\pm\sqrt{16+4.4.3}}{2.4}=\frac{-4\pm 8}{8}=-\frac{12}{8},\frac{4}{8}$

Two real roots.

38. (*d*) $ax^2+bx+c=0$

$a,b,c\in\{1,2,3,4,5,6,\}$

$n(s)=6\times 6\times 6=216$

for equal roots, $D=0\Rightarrow b^2-4ac=0$

$\Rightarrow ac=\frac{b^2}{4}\Rightarrow b^2$ must be divisible by 4.

(*a*) $b^2=4\Rightarrow b=2$ & $ac=\frac{4}{4}=1\Rightarrow a=1, c=1$

(*b*) $b^2=16\Rightarrow b=4$ & $ac=\frac{16}{4}=4\Rightarrow a=4, c=1$

$a=2, c=2$

or

$a=1, c=4$

(*d*) $b^2=36\Rightarrow b=6$ & $oc=\frac{36}{4}=9$

$\Rightarrow a=3, c=3$

Total favourable case = 5

Req. Probability = $\frac{5}{216}$

39. (*a*) for $x^2 - 2(3k-1)x + 8k^2 - 7 > 0$

$D < 0$

$(-2(3k-1))^2 - 4.1.(8k^2-7) < 0$

$4(9k^2 + 1 - 6k) - 4(8k^2 - 7) < 0$

$k^2 - 6k + 8 < 0$

$(k-2)(k-4) < 0$

$k \in (2, 4)$

Integer value of $k = 3$

40. (*d*) $z^2 + \alpha z + \beta = 0$

$z = 1 - 2i$ is root

So, $(1-2i)^2 + \alpha(1-2i) + \beta = 0$

$1 + 4i^2 - 4i + \alpha - 2\alpha i + \beta = 0$

$(\alpha + \beta - 3) + i(4 + 2\alpha) = 0$

$\alpha + \beta - 3 = 0$ & $4 + 2\alpha = 0$

$2\alpha = -4$

$\alpha = -2$

$\alpha + \beta - 3 = 0$

$-2 + \beta - 3 = 0$

$\beta = 5$

$\alpha - \beta = -2 - 5 = -7$

41. (*c*) $x^2 - 6x - 2 = 0$

α, β roots

$\alpha + \beta = 6; \alpha\beta = -2$ ...(i)

$\left.\begin{aligned}\alpha^2 - 6\alpha - 2 = 0\\ \beta^2 - 6\beta - 2 = 0\end{aligned}\right\}$ ...(ii)

$$\frac{a_{10} - 2a_8}{3a_9} = \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{3(\alpha^9 - \beta^9)} = \frac{\alpha^8 \cdot 6\alpha - \beta^8 \cdot 6\beta}{3(\alpha^9 - \beta^9)} \text{ (from-(ii))}$$

$$= \frac{6(\alpha^9 - \beta^9)}{3(\alpha^9 - \beta^9)} = 2$$

42. (1) $\log_4(x-1) = \log_2(x-3)$

$x > 1$ & $x > 3 \Rightarrow x > 3$ (according to definition of log)

$\Rightarrow \log_2(x-1) = \log_2(x-3)$

$\frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\log_2(x-1) = \log_2(x-3)^2$

$(x-1) = (x-3)^2$

$x^2 + 9 - 6x = x - 1$

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0$

$x = 2$ or 5

But $x > B$

So, $x = 5$

Only one solution.

43. (324) $\alpha + \beta = 1; \alpha\beta = -1$

Quad. Eq^n: $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - x - 1 = 0$

α, β roots

$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^2 = \alpha + 1$

Multiplying both side by, α^{n-1}

$\alpha^{n-1}, \alpha^2 = \alpha^{n-1}(\alpha + 1)$

$\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ ...(i)

Similarly, $\beta^{n+1} = \beta^n + \beta^{n-1}$ ...(ii)

Adding (i) & (ii)

$\alpha^{n+1} + \beta^{n+1} = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})$

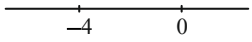
$P_{n+1} = P_n + P_{n-1}$

$29 = P_n + 11$

$P_n = 18$

$P_n^2 = 18^2 = 324$

44. (*b*) $(|x| - 3)(|x+4|) = 6$



Case I:	**Case II:**	**Case III**
$x \geq 0$	$-4 \leq x < 0$	$x < -4$
$(x-3)(x+4) = 6$	$(-(x)-3)(x+4) = 6$	$(-x-3)(-x-3)(-(x+4)) = 6$
$x^2 + x - 18 = 0$	$(x+3)(x+4) = -6$	$(x+3)(x+4) = 6$

$x = \frac{-1 \pm \sqrt{1 + 4.18}}{2}$ $x^2 + 7x + 18 = 0$ $x^2 + 7x + 6 = 0$

$x = \frac{-1 \pm \sqrt{73}}{2}$ $x = \frac{-7 \pm \sqrt{49 - 72}}{2}$ $(x+1)(x+6) = 0$

But $x \geq 0$ No real root $x = -1$ or -6

$x = \frac{-1 \pm \sqrt{73}}{2}$ But $x < -4$

Only 1 solution $x = -6$

Only 1 solution

In total, there are 2 real solution.

45. (*c*) $\int_0^1 P(x)dx = 1 \Rightarrow \int_0^1 (x^2 + bx + c)dx = 1$

$$\left[\frac{x^3}{3} + \frac{bx^2}{2} + c\right]_0^1 = 1 \Rightarrow \frac{1}{3} + \frac{b}{2} + c = 1$$

$3b + 6c = 4$ ...(i)

Also, $P(2) = 5$

$(2)^2 + b(2) + c = 5$

$12b + 6c = 6$ ...(ii)

from (i) & (ii)

$b = \frac{2}{9}$ & $c = \frac{5}{9}$

$a(b+c) = \frac{9(2+5)}{9} = 7$

46. (*a*) Let $x = 4+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{4+\ldots\ldots\infty}}}}$

Then,

$$x = 4+\frac{1}{5+\frac{1}{x}} \Rightarrow x-4 = \frac{x}{5x+1}$$

$$(x-4)(5x+1) = x$$

$$5x^2 - 20x - 4 = 0$$

$$x = \frac{20 \pm \sqrt{400+4.5.4}}{2.5} = \frac{20 \pm \sqrt{480}}{10}$$

But x is +*ve*.

So, $x = \frac{20+\sqrt{480}}{10} = \frac{2+4\sqrt{30}}{10}$

$$2+\frac{2\sqrt{30}}{5}$$

47. (0) $P(x) = f(x^3) + xg(x^3)$

$P(1) = f(1) + g(1)$ ...(i)

$x^2 + x + 1 = 0$

$x = \omega, \omega^2$ (cube root of unity)

$P(x)$ is divisible by $x^2 + x + 1$

$P(\omega) = 0; P(\omega^2) = 0$

$P(\omega) = 0 \Rightarrow f(\omega^3) + \omega g(\omega^3) = 0$

$f(1) + \omega g(1) = 0$ ...(ii)

$P(\omega^2) = 0 \Rightarrow f(\omega^6) + \omega^2(g(\omega^6)) = 0$

$f(1) + \omega^2(g(1)) = 0$ ...(iii)

(ii) – (iii)

$(\omega - \omega^2) g(1) = 0$

As $(\omega - \omega^2) \neq 0$, So, only possibility is $g(1) = 0$

Putting value of $g(1)$ in –(*ii*),

$f(1) + 0 = 0 \Rightarrow f(1) = 0$

$P(1) = f(1) + g(1) = 0 + 0 = 0$

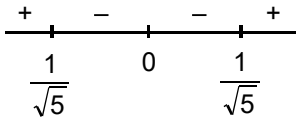
JEE-ADVANCED PREVIOUS YEARS

1. (*d*) p(x) will be of the form $ax^2 + c$. Since it has purely imaginary roots only.
Since p(x) is zero at imaginary values while $ax^2 + c$ takes real value only at real 'x', no root is real.
Also $p(p(x)) = 0 \Rightarrow p(x)$ is purely imaginary
$\Rightarrow$ $ax^2 + c$ = purely imaginary
Hence x can not be purely imaginary since x^2 will be negative in that case and $ax^2 + c$ will be real.
Thus. (*d*) is correct.

2. (*a*, *d*)

$(x_1 + x_2)^2 - 4x_1x_2 < 1$

$$\frac{1}{\alpha^2} - 4 < 1 \quad \Rightarrow \quad 5 - \frac{1}{\alpha^2} > 0 \Rightarrow \frac{5\alpha^2 - 1}{\alpha^2} > 0$$



$$\alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right) \quad ...(1)$$

$D > 0$

$1 - 4\alpha^2 > 0$

$$\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad ...(2)$$

(a) & (2)

$$\alpha \in \left(-\frac{1}{2}, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{2}}\right)$$

3. (*d*) As α and β are roots of equation $x^2 - x - 1 = 0$, we get:

$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^2 = \alpha + 1$

$\beta^2 - \beta - 1 = 0 \Rightarrow \beta^2 = \beta + 1$

$\therefore a_{11} + a_{10} = p\alpha^{11} + q\beta^{11} + p\alpha^{10} + q\beta^{10}$

$= p\alpha^{10}(\alpha + 1) + q\beta^{10}(\beta + 1)$

$= p\alpha^{10} \times \alpha^2 + q\beta^{10} \times \beta^2$

$= p\alpha^{12} + q\beta^{12} = \alpha_{12}$

4. (*d*) $a_{n+2} = a_{n+1} + a_n$

$a_4 = a_3 + a_2 = 3a_1 + 2a_0 = 3p\alpha + 3q\beta + 2(p + q)$

As $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$, we get

$$a_4 = 3p\left(\frac{1+\sqrt{5}}{2}\right) + 3q\left(\frac{1-\sqrt{5}}{2}\right) + 2p + 2q = 28$$

$$\Rightarrow \left(\frac{3p}{2} + \frac{3q}{2} + 2p + 2q - 28\right) = 0 \quad ...(i)$$

and $\Rightarrow \frac{3p}{2} - \frac{3q}{2} = 0$. ....(ii)

$\Rightarrow$ $p = q$ (from (ii))

$\Rightarrow$ $7p = 28$ (from (i) and (ii))

$\Rightarrow$ $p = 4$

$\Rightarrow$ $q = 4$

$\Rightarrow$ $p + 2q = 12$

5. (*a*,*b*,*c*)

α, β are roots of $x^2 - x - 1$

$$a_{r+2} - a_r = \frac{(\alpha^{r+2} - \beta^{r+2}) - (\alpha^r - \beta^r)}{\alpha - \beta}$$

$$= \frac{(\alpha^{r+2} - \alpha^r) - (\beta^{r+2} - \beta^r)}{\alpha - \beta}$$

$$= \frac{\alpha^r(\alpha^2 - 1) - \beta^r(\beta^2 - 1)}{\alpha - \beta} = \frac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta}$$

$$= \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}$$

$$\Rightarrow a_{r+2} - a_{r+1} = a_r$$

$$\Rightarrow \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta}$$

$$= a_{n+2} - (\alpha + \beta) = a_{n+2} - 1$$

Now $\sum_{r=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{r=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{r=1}^{\infty}\left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$

$$\frac{\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}}}{\alpha - \beta} = \frac{\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta}}{(\alpha - \beta)} = \frac{10}{(10 - \alpha)(10 - \beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10} +}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{(\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n$$

6. (*a*, *b*, *c*)

α, β are roots of $x^2 - x - 1$

$$a_{r+2} - a_r = \frac{(\alpha^{r+2} - \beta^{r+2}) - (\alpha^r - \beta^r)}{\alpha - \beta}$$

$$= \frac{(\alpha^{r+2} - \alpha^r) - (\beta^{r+2} - \beta^r)}{\alpha - \beta}$$

$$= \frac{\alpha^r(\alpha^2 - 1) - \beta^r(\beta^2 - 1)}{\alpha - \beta} = \frac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta} = \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} =$$

$$\Rightarrow a_{r+2} - a_{r+1} = a_r$$

$$\Rightarrow \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta}$$

$$= a_{n+2} - (\alpha + \beta) = a_{n+2} - 1$$

Now $\sum_{r=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{r=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{r=1}^{\infty}\left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$

$$\frac{\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}}}{\alpha - \beta} = \frac{\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta}}{(\alpha - \beta)} = \frac{10}{(10 - \alpha)(10 - \beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10} +}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{(\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n$$

7. (*d*) $x^2 + 20x - 2020 = 0$ has two roots $a, b \in R$

$x^2 - 20x + 2020 = 0$ has two roots $c, d \in$ complex

$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$

$= a^2(c + d) + b^2(c + d) - c^2(a + b) - d^2(a + b)$

$= (c + d)(a^2 + b^2) - (a + b)(c^2 + d^2)$

$= (c + d)((a + b)^2 - 2ab) - (a + b)((c + d)^2 - 2cd)$

$= 20[(20)^2 + 4040] + 20[(20)^2 - 4040]$

$= 20[(20)^2 + 4040 + (20)^2 - 4040]$

$= 20 \times 800 = 16000$

6 SEQUENCE AND SERIES

Types of Sequence: On the basis of the number of terms there are two types of sequence.

(*i*) Finite sequence: A sequence is said to be finite if it has finite number of terms.

(*ii*) Infinite sequence: A sequence is said to be infinite if it has infinite number of terms.

Series: By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots\ldots, a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n$ is a series.

Example:

(*i*) $1+2+3+4+\ldots\ldots\ldots+n$

(*ii*) $2+4+8+16+\ldots\ldots\ldots$

Progression: It is not necessary that the terms of a sequence always described by some explicit formula of the nth term. Those sequences whose terms follow certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

If a is the first term & d the common difference, then AP can be written as $a, a+d, a+2d, \ldots\ldots a+(n-1)d, \ldots\ldots$.

n^{th} term of this AP, $t_n = a+(n-1)d$, where $d = a_n - a_{n-1}$.

The sum of the first n terms of the AP is given by ;

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l].$$

where l is the last term.

Remarks

(*i*) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(*ii*) Three numbers in AP can be taken as $a-d, a, a+d$; four numbers in AP can be taken as $a-3d, a-d, a+d, a+3d$; five numbers in AP are $a-2d, a-d, a, a+d, a+2d$ & six terms in AP are $a-5d, a-3d, a-d, a+d, a+3d, a+5d$ etc.

(*iii*) The common difference can be zero, positive or negative.

(*iv*) The sum of the terms in an AP which are equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(*v*) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

$a_n = 1/2(a_{n-k}+a_{n+k})$, $k<n$.

For $k=1$, $a_n = (1/2)(a_{n-1}+a_{n+1})$;

For $k=2$, $a_n = (1/2)(a_{n-2}+a_{n+2})$ and so on.

(*vi*) $t_r = S_r - S_{r-1}$

(*vii*) If a, b, c are in AP $\Rightarrow 2b = a+c$.

TRAIN YOUR BRAIN

Example 1: Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Sol. All these numbers are 101, 108, 115,..... 997, to find n.

$997 = 101 + (n-1)7$

$\Rightarrow n = 129$ so $S = \frac{129}{2}[101+997]$

$= 70821.$

Example 2: The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively, then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

Example 3: If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., prove that a^2, b^2, c^2 are also in A.P.

Sol. $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b} \Rightarrow \quad b^2-a^2 = c^2-b^2$$

$\Rightarrow$ a^2, b^2, c^2 are in A.P.

GEOMETRIC PROGRESSION (GP)

In a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the G.P. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with a as the first term & r as common ratio.

(*i*) n^{th} term of a G.P. $= ar^{n-1}$

(*ii*) Sum of the Ist n terms i.e. $S_n = \frac{a(r^n-1)}{r-1}$, if $r \neq 1$.

(*iii*) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$

$r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \frac{a}{1-r}(|r|<1)$.

(*iv*) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(*v*) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(*vi*) If a, b, c are in GP $\Rightarrow b^2 = ac$.

TRAIN YOUR BRAIN

Example 4: Find three numbers in G.P. having sum 19 and product 216.

Sol. Let the three numbers be $\frac{a}{r}$, a, ar

so $\quad a\left[\frac{1}{r}+1+r\right] = 19 \qquad ...(i)$

and $\quad a^3 = 216 \quad \Rightarrow \quad a = 6$.

so from (*i*) $6r^2 - 13r + 6 = 0. \quad \Rightarrow \quad r = \frac{3}{2}, \frac{2}{3}$

Hence the three numbers are 4, 6, 9.

Example 5: Evaluate $7 + 77 + 777 +$ ..... upto n terms.

Sol. Let $S = 7 + 77 + 777 +$ .....upto n terms.

$$= \frac{7}{9}[9 + 99 + 999 + ...]$$

$$= \frac{7}{9}[(10-1)+(10^2-1)+(10^3-1) + ... + \text{upto n terms}]$$

$$= \frac{7}{9}[10 + 10^2 + 10^3 + ... + 10^n - n]$$

$$= \frac{7}{9}\left(\frac{10(10^n-1)}{9} - n\right) = \frac{7}{81}[10^{n+1} - 9n - 10]$$

ARITHMETICO-GEOMETRIC SERIES

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the Arithmetico-Geometric Series . e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in AP & 1, x, x^2, x^3 ..... are in GP.

Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a+(a+d)r+(a+2d)r^2 + ...+[a+(n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$, $r \neq 1$.

Sum to Infinity: If $|r| < 1$ & $n \to \infty$ then $\underset{n\to\infty}{\text{Limit}}\ r^n = 0$.

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

TRAIN YOUR BRAIN

Example 6: Find the sum to n terms of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+$..... Also find the sum if it exist if $n \to \infty$.

Sol. $S = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....+\frac{n}{2^n}$ ...(1)

$$\frac{S}{2} = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+.....+\frac{n-1}{2^n}+\frac{n}{2^{n+1}} \quad ...(2)$$

By (1) − (2)

$$\frac{S}{2} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+.....+\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$= \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$S_n = 2\left[\frac{2^{n+1}-2-n}{2^{n+1}}\right] = \frac{2^{n+1}-2-n}{2^n}.$$

If $n \to \infty$ then $S_\infty = \underset{n\to\infty}{\text{Lim}}\left[2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}\right] = 2$

HARMONIC PROGRESSION (HP)

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP . For HP whose first term is a & second term is b, the nth term is $t_n = \frac{ab}{b+(n-1)(a-b)}$. If a, b, c are in HP

$$\Rightarrow \quad b = \frac{2ac}{a+c} \quad \text{or} \quad \frac{a}{c} = \frac{a-b}{b-c}.$$

Example 7: If p^{th}, q^{th}, r^{th} terms of an H.P. be a, b, c respectively, prove that $(q-r)bc + (r-p)ac + (p-q)ab = 0$

Sol. Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $$\frac{1}{a} = x + (p-1)d \quad ...(i)$$

$$\frac{1}{b} = x + (q-1)d \quad ...(ii)$$

$$\frac{1}{c} = x + (r-1)d \quad ...(iii)$$

$(i) - (ii) \Rightarrow ab(p-q)d = b - a$ ...(iv)

$(ii) - (iii) \Rightarrow bc(q-r)d = c - b$ ...(v)

$(iii) - (i) \Rightarrow ac(r-p)d = a - c$ ...(vi)

$(iv) + (v) + (vi)$ gives

$$bc(q-r) + ac(r-p) + ab(p-q) = 0.$$

ARITHMETIC, GEOMETRIC & HARMONIC MEANS (AM, GM & HM)

Arithmetic Mean: If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

So, A.M. of two numbers a and b is $\frac{a+b}{2}$

AM for any n positive number $a_1, a_2, \ldots, a_n$ is ;

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

n - Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b.

$$A_1 = a + d = a + \frac{b-a}{n+1},$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$

$$, \ldots\ldots ,$$

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

Note: Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

Geometric Means: If a, b, c are in GP, b is the GM between a & c.

$$b^2 = ac, \text{ therefore } b = \sqrt{ac}\ ;\ a > 0,\ c > 0.$$

Geometric means of n positive numbers $a_1, a_2, \ldots a_n$ is $(a_1.a_2 \ldots a_n)^{1/n}$

n-Geometric Means Between a, b: If a, b are two given numbers & $a, G_1, G_2, \ldots, G_n, b$ are in GP. Then $G_1, G_2, G_3, \ldots,$ G_n are n GMs between a & b.

$G_1 = ar = a(b/a)^{1/n+1}$, $G_2 = ar^2 = a(b/a)^{2/n+1}$, ... , $G_n = ar^n = a(b/a)^{n/n+1}$

Note: The product of n GMs between a & b is equal to the nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

Harmonic Mean: If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a+c]$.

If a, b are two given number such that

$a, H_1, H_2, H_3 \ldots H_n, b$ are in H.P.

then $H_1, H_2, H_3 \ldots H_n$, are n harmonic means between a and b.

$\because a, H_1, H_2, H_3 \ldots H_n, b \rightarrow$ H.P.

$$\therefore \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} \ldots \frac{1}{H_n}, \frac{1}{b} \rightarrow \text{A.P.}$$

$$\text{Common difference (d)} = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$$

$$\text{So, } \frac{1}{H_1} = \frac{1}{a} + d,\ \frac{1}{H_2} = \frac{1}{a} + 2d \ldots \frac{1}{H_n} = \frac{1}{a} + nd$$

INEQUALITIES

(i) A.M. ≥ G.M. ≥ H.M.

Let $a_1, a_2, \ldots\ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean

(A), geometric mean (G) and harmonic mean (H) as

$$A = \frac{a_1 + a_2 + \ldots\ldots\ldots + a_n}{n},$$

$$G = (a_1 a_2 \ldots\ldots a_n)^{1/n} \text{ and } H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if

$$a_1 = a_2 = \ldots\ldots\ldots\ldots = a_n$$

(ii) Cauchy's Schwartz Inequality:

If $a_1, a_2, \ldots\ldots a_n$ and $b_1, b_2, \ldots\ldots\ldots, b_n$ are 2n real numbers, then $(a_1b_1 + a_2b_2 + \ldots\ldots\ldots\ldots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \ldots\ldots\ldots\ldots + a_n^2)(b_1^2 + b_2^2 + \ldots\ldots\ldots + b_n^2)$ with the equality holding if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots\ldots\ldots = \frac{a_n}{b_n}.$$

TRAIN YOUR BRAIN

Example 8: If a, b, c > 0, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$

Sol. Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\geq\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

ARITHMETIC MEAN OF M^TH POWER

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and let m be a real number, then

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}\geq\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m,$$

if $m \in R-[0,1]$

However if $m \in (0,1)$, then

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}\leq\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$$

Obviously if $m \in \{0,1\}$, then

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}=\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$$

Example 9: Prove that $a^4+b^4+c^4 \geq abc(a+b+c)$, $[a,b,c>0]$

Sol. Using mth power inequality, we get

$$\frac{a^4+b^4+c^4}{3}\geq\left(\frac{a+b+c}{3}\right)^4$$

$$=\left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3$$

$$\geq\left(\frac{a+b+c}{3}\right)[(abc)^{1/3}]^3$$

($\because$ A.M ≥ G.M)

or $$\frac{a^4+b^4+c^4}{3}\geq\left(\frac{a+b+c}{3}\right)abc$$

$\therefore\ a^4+b^4+c^4 \geq abc(a+b+c)$.

SIGMA NOTATIONS

Sigma (Σ) notation

Σ indicates sum i.e., $\sum_{i=1}^{n} i = \sum n = 1+2+3+\ldots\ldots+n$

Therorems

(a) $\sum_{r=1}^{n}(a_r \pm b_r)=\sum_{r=1}^{n}a_r \pm \sum_{r=1}^{n}b_r$

(b) $\sum_{r=1}^{n}ka_r=\sum_{r=1}^{n}a_r$

(c) $\sum_{r=1}^{n}k = nk$: where k is a constant.

Important Results

(*i*) Sum of first *n* natural numbers

$$\Sigma n = 1+2+\ldots.+n$$

$$=\frac{n(n+1)}{2}$$

(*ii*) Sum of squares of first *n* natural numbers

$$\Sigma n^2 = 1^2+2^2+\ldots..+n^2$$

$$=\frac{n(n+1)(2n+1)}{6}$$

(*iii*) Sum of cubes of first *n* natural numbers

$$\Sigma n^3 = 1^3+2^3+3^3+\ldots..+n^3$$

$$=\left(\frac{n(n+1)}{2}\right)^2=(\Sigma n)^2$$

(*iv*) Sum of *n* terms of a sequence $T_n = an^3+bn^2+cn+d$

$$\boxed{S_n = a\Sigma n^3+b\Sigma n^2+c\Sigma n+dn}$$

Example 10: Find the sum of the series $3.5+6.8+9.11+\ldots..$ + upto *n* terms

Sol. ***n^th term of 3, 6, 9, ... is 3n***

n^{th} term of 5, 8, 11, ... is $(3n+2)$

$\therefore\ T_n = 3n(3n+2) = 9n^2+6n$

$\therefore\ S_n = 9\Sigma n^2+6\Sigma n$

$$=\frac{9n(n+1)(2n+1)}{6}+\frac{6n(n+1)}{2}$$

$$=\frac{3}{2}n(n+1)[2n+1+2]=\frac{3n(n+1)(2n+3)}{2}$$

METHOD OF DIFFERENCES

Let $u_1, u_2, u_3, \ldots\ldots$ be a sequence, such that $u_2-u_1, u_3-u_2, \ldots\ldots\ldots$ is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ...(i)

$S = u_1 + u_2 + ...+ u_{n-1} + u_n$ ...(ii)

(i) – (ii)

$\Rightarrow u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum_{r=1}^{k} u_r$

Remark: It is not always necessary that the series of first order of differences i.e. $u_2 - u_1$, $u_3 - u_2$ ... $u_n - u_{n-1}$ is always either in A.P. or in G.P. in such case let $u_1 = T_1$, $u_2 - u_1 = T_2$, $u_3 - u_2 = T_3$ ....., $u_n - u_{n-1} = T_n$.

So $u_n = T_1 + T_2 + ... + T_n$ ....(i)

$u_n = T_1 + T_2 + ...+ T_{n-1} + T_n$ ....(ii)

(i) – (ii)

$\Rightarrow T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n + T_{n-1})$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P., then $u_n = u_1 + \Sigma T_r$

Otherwise in the similar way we find series of higher order of differences and the nth term of the series.

If possible express r^{th} term as difference of two terms as $t_r = f(r) - f(r \pm 1)$. This can be explained with help of examples given below.

TRAIN YOUR BRAIN

Example 11: Find the sum to n-terms $3 + 7 + 13 + 21 + ...$

Sol. Let $S = 3 + 7 + 13 + 21 + ... + T_n$ ...(i)

$S = 3 + 7 + 13 + ... + T_{n-1} + T_n$ ...(ii)

(i) – (ii) $\Rightarrow T_n = 3 + 4 + 6 + 8 + ... + (T_n - T_{n-1})$

$$= 3 + \frac{n-1}{2}[8 + (n-2)2] = 3 + (n-1)(n+2) = n^2 + n + 1$$

Hence $S = \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$$

Example 12:

(a) Sum the following series to infinity

$$\frac{1}{1\cdot4\cdot7} + \frac{1}{4\cdot7\cdot10} + \frac{1}{7\cdot10\cdot13} + ...$$

(b) Sum the following series upto n-terms

$1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 + ...$

Sol. *(a)* $T_n = \frac{1}{[1+(n-1)3][1+3n][4+3n]}$

$$= \frac{1}{(3n-2)(2n+1)(3n+4)}$$

$$= \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$T_1 = \frac{1}{6}\left[\frac{1}{1\cdot4} - \frac{1}{4\cdot7}\right]$$

$$T_2 = \frac{1}{6}\left[\frac{1}{4\cdot7} - \frac{1}{7\cdot10}\right]$$

$$\vdots \quad \vdots \quad \vdots$$

$$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$\therefore S_n = \sum T_n = \frac{1}{6}\left[\frac{1}{1\cdot4} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \left[\frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}\right] \text{ as } n \to \infty$$

$$\therefore S_\infty = \frac{1}{24}$$

(b) $1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 + ...$

$$T_n = \frac{1}{5}n(n+1)(n+2)(n+3)[(n+4) - (n-1)]$$

$$\therefore \quad T_1 = \frac{1}{5}[1\cdot2\cdot3\cdot4\cdot5 - 0]$$

$$T_2 = \frac{1}{5}[2\cdot3\cdot4\cdot5\cdot6 - 1\cdot2\cdot3\cdot4\cdot5]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)]$$

$$\therefore \quad S_n = \sum T_n$$

$$= \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]$$

$$= \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$$

Topicwise Questions

Arithmetic Progression and its Properties

1. The sum of 24 terms of the following series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+.....$ is

(a) 300 (b) $300\sqrt{2}$

(c) $200\sqrt{2}$ (d) $100\sqrt{2}$

2. If the pth, q^{th} and r^{th} term of an arithmetic sequence are a, b and c respectively, then the value of $[a(q-r)+b(r-p)+c(p-q)]=$

(a) 1 (b) –1

(c) 0 (d) 1/2

3. Let T_r be the rth term of an A.P. for $r=1, 2, 3,....$. If for some positive integers m, n we have $T_m=\frac{1}{n}$ and $T_n=\frac{1}{m}$, then T_{mn} equals

(a) $\frac{1}{mn}$ (b) $\frac{1}{m}+\frac{1}{n}$

(c) 1 (d) 0

4. If $1, \log_9(3^{1-x}+2), \log_3(4.3^x-1)$ are in A.P. then x equals

(a) $\log_3 4$ (b) $1-\log_3 4$

(c) $1-\log_4 3$ (d) $\log_4 3$

5. If the ratio of the sum of n terms of two A.P.'s be $(7n+1):(4n+27)$, then the ratio of their 11th terms will be

(a) 2 : 3 (b) 3 : 4

(c) 4 : 3 (d) 5 : 6

6. The ratio of the sums of first n even numbers and odd numbers will be

(a) 1 : n (b) (n + 1) : 1

(c) (n + 1) : n (d) (n – 1) : 1

7. If S_n denotes the sum of n terms of an arithmetic progression, then the value of $(S_{2n}-S_n)$ is equal to

(a) $2S_n$ (b) S_{3n}

(c) $\frac{1}{3}S_{3n}$ (d) $\frac{1}{2}S_n$

8. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is

(a) 2489 (b) 4735

(c) 2317 (d) 2632

9. Let S_n denotes the sum of n terms of an A.P. If $S_{2n}=3S_n$, then ratio $\frac{S_{3n}}{S_n}=$

(a) 4 (b) 6

(c) 8 (d) 10

10. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(a) 2550 (b) 1050

(c) 3050 (d) None of these

Arithmetic Mean

11. A number is the reciprocal of the other. If the arithmetic mean of the two numbers be $\frac{13}{12}$, then the numbers are

(a) $\frac{1}{4}, \frac{4}{1}$ (b) $\frac{3}{4}, \frac{4}{3}$

(c) $\frac{2}{5}, \frac{5}{2}$ (d) $\frac{3}{2}, \frac{2}{3}$

Geometric Progression and its Properties

12. If $x, 2x+2, 3x+3$, are in G.P., then the fourth term is

(a) 27 (b) –27

(c) 13.5 (d) –13.5

13. The 6th term of a G.P. is 32 and its 8th term is 128, then the common ratio of the G.P. is

(a) –1 (b) 2

(c) 4 (d) –4

14. The sum of 100 terms of the series $.9+.09+.009.....$ will be

(a) $1-\left(\frac{1}{10}\right)^{100}$ (b) $1+\left(\frac{1}{10}\right)^{106}$

(c) $1-\left(\frac{1}{10}\right)^{106}$ (d) $1+\left(\frac{1}{10}\right)^{100}$

15. The sum of the series $6+66+666+.....$ upto n terms is

(a) $(10^{n-1}-9n+10)/81$ (b) $2(10^{n+1}-9n-10)/27$

(c) $2(10^n-9n-10)/27$ (d) $(10^{n+1}-9n-10)/27$

16. Consider an infinite G.P. with first term a and common ratio r, its sum is 4 and the second term is 3/4, then

(a) $a=\frac{7}{4}, r=\frac{3}{7}$ (b) $a=\frac{3}{2}, r=\frac{1}{2}$

(c) $a=2, r=\frac{3}{8}$ (d) $a=3, r=\frac{1}{4}$

17. If a, b, c are in G.P., then

(a) a^2, b^2, c^2 are in G.P.

(b) $a^2(b+c), c^2(a+b), b^2(a+c)$ are in G.P.

(c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P.

(d) a^2, b^2, c^2 are in A.P.

18. The third term of a G.P. is 4. The product of the first five terms is

(a) 4^3 (b) 4^5

(c) 4^4 (d) None of these

19. If sum of the infinite G.P., $p, 1, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots\ldots$ is $\frac{9}{2}$, then value of p is

(a) $3, \frac{3}{2}$ (b) $\frac{2}{3}, 2$

(c) $\frac{3}{2}, 1$ (d) $\frac{1}{3}, 2$

Geometric Mean

20. If G be the geometric mean of x and y, then $\frac{1}{G^2-x^2}+\frac{1}{G^2-y^2}=$

(a) G^2 (b) $\frac{1}{G^2}$

(c) $\frac{2}{G^2}$ (d) $3G^2$

A.G.P.

21. $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\ldots\ldots\infty$ is equal to

(a) 3 (b) 6

(c) 9 (d) 12

22. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots\ldots$ is

(a) 2^n-n-1 (b) $1-2^{-n}$

(c) $n+2^{-n}-1$ (d) 2^n-1

Harmonic Progression and Harmonic Mean

23. Which number should be added to the numbers 13, 15, 19 so that the resulting numbers be the consecutive terms of a H.P.

(a) 7 (b) 6

(c) –6 (d) –7

24. If 5th term of a H.P. is $\frac{1}{45}$ and 11th term is $\frac{1}{69}$, then its 16th term will be

(a) 1/89 (b) 1/85

(c) 1/80 (d) 1/79

25. If the 7th term of a H.P. is $\frac{1}{10}$ and the 12th term is $\frac{1}{25}$, then the 20th term is

(a) $\frac{1}{37}$ (b) $\frac{1}{41}$

(c) $\frac{1}{45}$ (d) $\frac{1}{49}$

26. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is

(a) 2 (b) $\frac{pq}{p+q}$

(c) $\frac{p+q}{pq}$ (d) $\frac{2pq}{p+q}$

Relation between A.M., G.M., H.M.

27. If the arithmetic mean of two numbers be A and geometric mean be G, then the numbers will be

(a) $A \pm (A^2-G^2)$

(b) $\sqrt{A} \pm \sqrt{A^2-G^2}$

(c) $A \pm \sqrt{(A+G)(A-G)}$

(d) $\frac{A \pm \sqrt{(A+G)(A-G)}}{2}$

28. If the arithmetic, geometric and harmonic means between two positive real numbers be A, G and H, then

(a) $A^2 = GH$ (b) $H^2 = AG$

(c) $G = AH$ (d) $G^2 = AH$

29. If a^2, b^2, c^2 be in A.P., then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in

(a) A.P. (b) G.P.

(c) H.P. (d) A.G.P.

Miscelleneous Series

30. The sum of the series $1+(1+2)+(1+2+3)+\ldots\ldots$ upto n terms, will be

(a) n^2-2n+6 (b) $\frac{n(n+1)(2n-1)}{6}$

(c) n^2+2n+6 (d) $\frac{n(n+1)(n+2)}{6}$

31. $\frac{1^3+2^3+3^3+4^3+.....12^3}{1^2+2^2+3^2+4^2+.....12^2} =$

(a) $\frac{234}{25}$ (b) $\frac{243}{35}$

(c) $\frac{263}{27}$ (d) $\frac{236}{35}$

32. If the nth term of a series be $3+n(n-1)$, then the sum of n terms of the series is

(a) $\frac{n^2+n}{3}$ (b) $\frac{n^3+8n}{3}$

(c) $\frac{n^2+8n}{5}$ (d) $\frac{n^2-8n}{3}$

33. If the sum of $1+\frac{1+2}{2}+\frac{1+2+3}{3}+.....$ to n terms is S, then S is equal to

(a) $\frac{n(n+3)}{4}$

(b) $\frac{n(n+2)}{4}$

(c) $\frac{n(n+1)(n+2)}{6}$

(d) n^2

Learning Plus

1. The first term of an A.P. of consecutive integer is p^2+1. The sum of $(2p+1)$ terms of this series can be expressed as
(a) $(p+1)^2$ (b) $(2p+1)(p+1)^2$
(c) $(p+1)^3$ (d) $p^3+(p+1)^3$

2. If $a_1, a_2, a_3,.....$ are in A.P. such that $a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=225$, then $a_1+a_2+a_3+.....+a_{23}+a_{24}$ is equal to
(a) 909 (b) 75
(c) 750 (d) 900

3. The interior angles of a polygon are in A.P. If the smallest angle is 120° & the common difference is 5°, then the number of sides in the polygon is:
(a) 7 (b) 9
(c) 16 (d) None of these

4. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is
(a) $\frac{2355}{1001}$ (b) $\frac{2379}{997}$
(c) $\frac{2355}{999}$ (d) None of these

5. Consider an A.P. with first term 'a' and the common difference 'd'. Let S_k denote the sum of its first K terms. If $\frac{S_{kx}}{S_x}$ is independent of x, then
(a) $a=d/2$ (b) $a=d$
(c) $a=2d$ (d) None of these

6. If $x \in R$, the numbers $5^{1+x}+5^{1-x}$, $a/2$, 25^x+25^{-x} form an A.P. then 'a' must lie in the interval:
(a) [1, 5] (b) [2, 5]
(c) [5, 12] (d) $[12,\infty)$

7. The sum of the series $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+$
(a) $\frac{1}{2}n(n+1)$ (b) $\frac{1}{12}n(n+1)(2n+1)$
(c) $\frac{1}{n(n+1)}$ (d) $\frac{1}{4}n(n+1)$

8. Sum of the series $S=1^2-2^2+3^2-4^2+.....-2002^2+2003^2$ is
(a) 2007006 (b) 1005004
(c) 2000506 (d) None of these

9. If a and b are p^{th} and q^{th} terms of an AP, then the sum of its $(p+q)$ terms is
(a) $\frac{p+q}{2}\left[a-b+\frac{a+b}{p-q}\right]$
(b) $\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$
(c) $\frac{p-q}{2}\left[a+b+\frac{a+b}{p+q}\right]$
(d) None of these

10. There are n A.M's between 3 and 54, such that the 8th mean: $(n-2)^{th}$ mean:: 3: 5. The value of n is.
(a) 12 (b) 16
(c) 18 (d) 20

11. The sum to 10 terms of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+\sqrt{54}+...$ is
(a) $121(\sqrt{6}+\sqrt{2})$ (b) $\frac{121}{2}(\sqrt{3}+1)$
(c) $243(\sqrt{3}+1)$ (d) $243(\sqrt{3}-1)$

12. If S is the sum of infinity of a G.P. whose first term is 'a', then the sum of the first n terms is

(a) $S\left(1-\frac{a}{S}\right)^n$ (b) $S\left[1-\left(1-\frac{a}{S}\right)^n\right]$

(c) $a\left[1-\left(1-\frac{a}{S}\right)^n\right]$ (d) None of these

13. For a sequence $\{a_n\}$, $a_1=2$ and $\frac{a_{n+1}}{a_n}=\frac{1}{3}$. Then $\sum_{r=1}^{20} a_r$ is

(a) $\frac{20}{2}[4+19\times 3]$ (b) $3\left(1-\frac{1}{3^{20}}\right)$

(c) $2(1-3^{20})$ (d) None of these

14. α, β be the roots of the equation $x^2-3x+a=0$ and γ, δ the roots of $x^2-12x+b=0$ and numbers $\alpha, \beta, \gamma, \delta$ (in this order) form an increasing G.P., then

(a) $a=3, b=12$ (b) $a=12, b=3$

(c) $a=2, b=32$ (d) $a=4, b=16$

15. In a G.P. of positive terms, any term is equal to the sum of the next two terms. The common ratio of the G.P. is

(a) $2\cos 18^\circ$ (b) $\sin 18^\circ$

(c) $\cos 18^\circ$ (d) $2\sin 18^\circ$

16. If p is positive, then the sum to infinity of the series,

$\frac{1}{1+p}-\frac{1-p}{(1+p)^2}+\frac{(1-p)^2}{(1+p)^3}-\ldots\ldots$ is

(a) 1/2 (b) 3/4

(c) 1 (d) None of these

17. If $x>0$, and $\log_2 x+\log_2(\sqrt{x})+\log_2(\sqrt[4]{x})+\log_2(\sqrt[8]{x})+\log_2(\sqrt[16]{x})+\ldots.=4$, then x equals

(a) 2 (b) 3

(c) 4 (d) 5

18. If $3+\frac{1}{4}(3+d)+\frac{1}{4^2}(3+2d)+\ldots.+\text{upto}\,\infty=8$, then the value of d is

(a) 9 (b) 5

(c) 1 (d) None of these

19. If the sum of the roots of the quadratic equation, $ax^2+bx+c=0$ is equal to sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

20. If $x=\sum_{n=0}^{\infty} a^n$, $y=\sum_{n=0}^{\infty} b^n$, $z=\sum_{n=0}^{\infty} c^n$ where a, b, c are in AP and $|a|<1, |b|<1, |c|<1$, then x, y, z are in

(a) HP

(b) Arithmetic-Geometric Progression

(c) AP

(d) GP

21. If a, b, c are in A.P. p, q, r are in H.P. and ap, bq, cr are in G.P., then $\frac{p}{r}+\frac{r}{p}$ is equal to

(a) $\frac{a}{c}+\frac{c}{a}$ (b) $\frac{a}{c}-\frac{c}{a}$

(c) $\frac{b}{q}+\frac{q}{b}$ (d) $\frac{b}{q}-\frac{a}{p}$

22. If $a^x=b^y=c^z=d^t$ and a, b, c, d are in G.P., then x, y, z, t are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

23. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b, & c is given by:

(a) $x^3-3Ax^2+3G^3x-G^3=0$

(b) $x^3-3Ax^2+3(G^3/H)x-G^3=0$

(c) $x^3+3Ax^2+3(G^3/H)x-G^3=0$

(d) $x^3-3Ax^2-3(G^3/H)x+G^3=0$

24. The sum $\sum_{r=2}^{\infty}\frac{1}{r^2-1}$ is equal to

(a) 1 (b) 3/4

(c) 4/3 (d) None of these

25. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots$ upto $\infty=\frac{\pi^2}{6}$, then

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots.=$

(a) $\pi^2/12$ (b) $\pi^2/24$

(c) $\pi^2/8$ (d) None of these

26. The sum of the first n-terms of the series $1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+\ldots\ldots\ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is

(a) $\frac{n(n+1)^2}{4}$ (b) $\frac{n^2(n+2)}{4}$

(c) $\frac{n^2(n+1)}{2}$ (d) $\frac{n(n+2)^2}{4}$

27. $1^2+2^2+\ldots\ldots+n^2=1015$, then value of n is

(a) 15 (b) 14

(c) 13 (d) None of these

Advanced Level Multiconcept Questions

1. If $a_1, a_2, \ldots\ldots, a_n$ are distinct terms of an A.P., then
   (*a*) $a_1 + 2a_2 + a_3 = 0$
   (*b*) $a_1 - 2a_2 + a_3 = 0$
   (*c*) $a_1 + 3a_2 - 3a_3 - a_4 = 0$
   (*d*) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

2. If $x, |x+1|, |x-1|$ are three terms of an A.P., then its sum upto 20 terms is –
   (*a*) 180 (*b*) 350
   (*c*) 90 (*d*) 720

3. The sides of a right triangle form a G.P. The tangent of the smallest angle is
   (*a*) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (*b*) $\sqrt{\frac{\sqrt{5}-1}{2}}$
   (*c*) $\sqrt{\frac{2}{\sqrt{5}+1}}$ (*d*) $\sqrt{\frac{2}{\sqrt{5}-1}}$

4. If b_1, b_2, b_3 $(b_i > 0)$ are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by
   (*a*) $r > 3$ (*b*) $0 < r < 1$
   (*c*) $r = 3.5$ (*d*) $r = 5.2$

5 If positive numbers a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then
   (*a*) $a = b = c$ (*b*) $2b = a + c$
   (*c*) $b^2 = \sqrt{\frac{ac}{8}}$ (*d*) None of these

6. If the arithmetic mean of two positive numbers a & b $(a > b)$ is twice their geometric mean, then a: b is:
   (*a*) $2+\sqrt{3} : 2-\sqrt{3}$ (*b*) $7+4\sqrt{3} : 1$
   (*c*) $1 : 7-4\sqrt{3}$ (*d*) $2 : \sqrt{3}$

7. Let a, x, b be in A.P; a, y, b be in G.P. and a, z, b be in H.P. If $x = y + 2$ and $a = 5z$ then
   (*a*) $y^2 = xz$ (*b*) $x > y > z$
   (*c*) $a = 9, b = 1$ (*d*) $a = 1/4, b = 9/4$

8. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then
   (*a*) $a + c = b + d$
   (*b*) $e = 0$
   (*c*) $a, b - 2/3, c - 1$ are in A.P.
   (*d*) c/a is an integer

9. The value of $\sum_{r=1}^{n} \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}$ is
   (*a*) $\frac{n}{\sqrt{a}+\sqrt{a+nx}}$ (*b*) $\frac{n}{\sqrt{a}-\sqrt{a+nx}}$
   (*c*) $\frac{\sqrt{a+nx}-\sqrt{a}}{x}$ (*d*) $\frac{\sqrt{a}+\sqrt{a+nx}}{x}$

Comprehension – 01 (No. 10 to 12)

We know that $1 + 2 + 3 + \ldots\ldots = \frac{n(n+1)}{2} = f(n)$,

$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = g(n)$,

$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = h(n)$

10. $g(n) - g(n-1)$ must be equal to
    (*a*) n^2 (*b*) $(n-1)^2$
    (*c*) $n - 1$ (*d*) n^3

11. Greatest even natural number which divides $g(n) - f(n)$, for every $n \geq 2$, is
    (*a*) 2 (*b*) 4
    (*c*) 6 (*d*) None of these

12. $f(n) + 3\,g(n) + h(n)$ is divisible by $1 + 2 + 3 + \ldots\ldots + n$
    (*a*) only if $n = 1$ (*b*) only if n is odd
    (*c*) only if n is even (*d*) for all $n \in N$

Comprehension – 02 (No. 13 to 15)

The sum of squares of three distinct real numbers which form an increasing GP is S^2 (common ratio is r). If sum of numbers is αS, then

13. If $r = 3$ then α^2 cannot lie in
    (*a*) $\left(\frac{1}{3}, 1\right)$ (*b*) $(1, 2)$
    (*c*) $\left(\frac{1}{3}, 3\right)$ (*d*) $(1, 3)$

14. If $\alpha^2 = 2$, then the value of common ratio r is greater than
    (*a*) 9 (*b*) 4
    (*c*) 2 (*d*) 3

15. If r = 2, then the value of α^2 is $\frac{a}{b}$ (where a and b are coprime) then a + b is
(a) 3 (b) 5
(c) 8 (d) 10

16. **Match the column**

Column – I	Column – II
(a) If $\log_5 2$, $\log_5(2^x - 5)$ and $\log_5(2^x - 7/2)$ are in A.P., then value of 2x is equal to	(p) 6
(b) Sum of infinite series $4 + \frac{8}{3} + \frac{12}{3^2} + \frac{16}{3^3} + \text{---}$ is	(r) 3
(c) The length, breadth, height of a rectangular box are in G.P. The volume is 27, the total surface area is 78. Then the length is	(s) 1

17. **Match the column**

Column – I	Column – II
(a) If $\log_x y$, $\log_z x$, $\log_y z$ are in G.P., xyz = 64 and x^3, y^3, z^3 are in A.P., then $\frac{3x}{y}$ is equal to	(p) 2
(b) The value of $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}}$ --- ∞ is equal to	(q) 1
(c) If x, y, z are in A.P., then $(x + 2y - z)(2y + z - x)(z + x - y) = kxyz$, where $k \in N$, then k is equal to	(r) 3
(d) There are m A.M. between 1 and 31. If the ratio of the 7th and $(m-1)^{th}$ means is 5 : 9, then $\frac{m}{7}$ is equal to	(s) 4

NUMERICAL BASED QUESTIONS

18. The value of $2^{1/4} \times 4^{1/8} \times 8^{1/6} \ldots\ldots \infty$ is

19. An infinite G.P. is selected from $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to converge to 1/7. If $1/2^a$ is the first term of such a G.P., find a.

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

20. First term of an A.P. of non-constant terms is 3 and its second, tenth and thirty-fourth terms form a G.P., find the common difference.

21. If x > 0 and $\log_2 x + \log_2\left(\sqrt{x}\right) + \log_2\left(\sqrt[4]{x}\right) + \log_2\left(\sqrt[8]{x}\right) + \log_2\left(\sqrt[16]{x}\right) + \ldots\ldots = 4$ then x equals

22. If x, y, z be respectively the pth, qth and rth terms of G.P., then

$(q - r)\log x + (r - p)\log y + (p - q)\log z =$

23. Consider an infinitely decreasing geometric progression $b_1, b_2, b_3 \ldots, b_n, \ldots$ whose sum is 3 and sum of its cubes is $\frac{108}{13}$, find the value of b_2.

24. $\frac{1}{2.4} + \frac{1.3}{2.4.6} + \frac{1.3.5}{2.4.6.8} + \frac{1.3.5.7}{2.4.6.8.10} +$ .....∞ is equal to

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is: [*JEE Main-2016*]
(a) 8/5 (b) 4/3
(c) 1 (d) 7/4

2. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$, is $\frac{16}{5}m$, then m is equal to: [*JEE Main-2016*]
(a) 102 (b) 101
(c) 100 (d) 99

3. For any three positive real numbers a, b and c, then [*JEE Main-2017*]
(a) a, b and c are in G.P.
(b) b, c and a are in G.P.
(c) b, c and a are in A.P.
(d) a, b and c are in A.P.

4. Let a, b, $c \in R$. If $f(x) = ax^2 + bx + c$ is such that a + b + c = 3 and $f(x + y) = f(x) + f(y) + xy$, $\forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to : [*JEE Main-2017*]
(a) 255 (b) 330
(c) 165 (d) 190

5. Let $a1, a_2, a_3, \ldots, a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots + a_{17}^2 = 140m$, then m is equal to [JEE Main-2018]

(a) 68 (b) 34

(c) 33 (d) 66

6. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series [JEE Main-2018]

$1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$. If $B - 2A = 100\lambda$, then λ is equal to :

(a) 248 (b) 464

(c) 496 (d) 232

7. If a, b and c be three distinct numbers in G.P. and a+b+c = xb then x can not be [JEE Main-2019 (January)]

(a) –2 (b) –3

(c) 4 (d) 2

8. Let $a_1, a_2, \ldots, a_{30}$ be an A.P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{2i-1}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to [JEE Main-2019 (January)]

(a) 52 (b) 57

(c) 47 (d) 42

9. The sum of the following series $1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + \ldots + 5^2)}{11} + \ldots$ upto 15 terms, is : [JEE Main-2019 (January)]

(a) 7820 (b) 7830

(c) 7520 (d) 7510

10. Let a, b and c be the 7th, 11th and 13th terms respectively of a non-constant A.P. If these are also the three consecutive terms of a G.P. then $\frac{a}{c}$ is equal to: [JEE Main-2019 (January)]

(a) 1/2 (b) 4

(c) 2 (d) 7/13

11. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is: [JEE Main-2019 (January)]

(a) 1256 (b) 1465

(c) 1365 (d) 1356

12. If 19th terms of non – zero A. P is zero, then its (49th term) : (29th term) is : [JEE Main-2019 (January)]

(a) 4 : 1 (b) 1 : 3

(c) 3 : 1 (d) 2 : 1

13. The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{27}{19}$. Then the common ratio of this series is:- [JEE Main-2019 (January)]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{2}{9}$ (d) $\frac{4}{9}$

14. Let $a_1, a_2, \ldots, a_{10}$ be a G. P. If $\frac{a_3}{a_1} = 25$, then $\frac{a_9}{a_5}$ equals :- [JEE Main-2019 (January)]

(a) 5^4 (b) $4(5^2)$

(c) 5^3 (d) $2(5^2)$

15. The product of three consecutive terms of a G.P. is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an A.P. Then the sum of the original three terms of the given G.P. is: [JEE Main-2019 (January)]

(a) 36 (b) 32

(c) 24 (d) 28

16. Let $S_k = \frac{1 + 2 + 3 + \ldots + k}{k}$. If $S_1^2 + S_2^2 + \ldots + S_{10}^2 = \frac{5}{12}A$, then A equal to: [JEE Main-2019 (January)]

(a) 283 (b) 301

(c) 303 (d) 156

17. If the sum of the first 15 terms of the series $\left(\frac{3}{4}\right)^3 + \left(1\frac{1}{2}\right)^3 + \left(2\frac{1}{4}\right)^3 + 3^3 + \left(3\frac{3}{4}\right)^3 + \ldots$ is equal to 225 k, then k is equal to : [JEE Main-2019 (January)]

(a) 108 (b) 27

(c) 54 (d) 9

18. The sum of all natural numbers 'h' such that $100 < n < 200$ and H.C.F. $(91, n) > 1$ is : [JEE Main-2019 (April)]

(a) 3221 (b) 3121

(c) 3203 (d) 3303

19. The sum $\sum_{k=1}^{20} k\frac{1}{2^k}$ is equal to - [JEE Main-2019 (April)]

(a) $2 - \frac{3}{2^{17}}$ (b) $2 - \frac{11}{2^{19}}$

(c) $1 - \frac{11}{2^{20}}$ (d) $2 - \frac{21}{2^{20}}$

20. Let the sum of the first n terms of a non-constant A.P., a_1, a_2, a_3, ...be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this A.P., then the ordered pair (d, a_{50}) is equal to [JEE Main-2019 (April)]

(a) $(A, 50 + 46A)$ (b) $(A, 50 + 45A)$

(c) $(50, 50 + 46A)$ (d) $(50, 50 + 45A)$

21. If the sum and product of the first three term in an A.P. are 33 and 1155, rtespectively, then a value of its 11th term is:- [JEE Main-2019 (April)]

(a) -25 (b) 25

(c) -36 (d) -35

22. The sum of the series $1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + ...$ upto 11th term is : [JEE Main-2019 (April)]

(a) 915 (b) 946

(c) 945 (d) 916

23. The sum $1 + \frac{1^3 + 2^3}{1+2} + \frac{1^3 + 2^3 + 3^3}{1+2+3} + ...$

$+ \frac{1^3 + 2^3 + 3^3 + ... + 15^3}{1 + 2 + 3 + ... + 15} - \frac{1}{2}(1 + 2 + 3 + ... + 15)]$

[JEE Main-2019 (April)]

(a) 1240 (b) 1860

(c) 660 (d) 620

24. Let a, b and c be in G.P. with common ratio r, where $a \neq$ and $0 < r \leq \frac{1}{2}$. If 3a, 7b and 15c are the first three terms of an A.P. then the 4th term of this A.P. is:

[JEE Main-2019 (April)]

(a) $\frac{7}{3}$ (b) a

(c) $\frac{2}{3}$a (d) 5a

25. The sum $\frac{3 \times 1^3}{1^2} + \frac{5 \times (1^3 + 2^3)}{1^2 + 2^2} + \frac{7 \times (1^3 + 2^3 + 3^3)}{1^2 + 2^2 + 3^2} +$ .....up to 10 terms [JEE Main-2019 (April)]

(a) 660 (b) 620

(c) 680 (d) 600

26. If $a_1, a_2, a_3, ...a_n$ are in A.P. and $a_1 + a_4 + a_7 + ...+ a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to:

[JEE Main-2019 (April)]

(a) 38 (b) 98

(c) 76 (d) 64

27. Let S_n denote the sum of the first n terms of an A.P. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to:

[JEE Main-2019 (April)]

(a) -320 (b) -260

(c) -380 (d) -410

28. For $x \in R$, let [x] denote the greatest integer $\leq x$, then the sum of the series

$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + ... + \left[-\frac{1}{3} - \frac{99}{100}\right]$ is

[JEE Main-2019 (April)]

(a) -153 (b) -133

(c) -131 (d) -135

29. If $a_1, a_2, a_3, ...$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is

[JEE Main-2019 (April)]

(a) 200 (b) 280

(c) 120 (d) 150

30. The greatest positive integer k, for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + ... + 49^2 + 49 + 1$, is

[JEE Main-2020 (January)]

(a) 32 (b) 63

(c) 65 (d) 60

31. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $\frac{-1}{2}$, then the greatest number amongst them is [JEE Main-2020 (January)]

(a) $\frac{21}{2}$ (b) 16

(c) 27 (d) 7

32. If the sum of the first 40 terms of the series, $3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + ...$is $(102)^m$, then m is equal to: [JEE Main-2020 (January)]

(a) 25 (b) 20

(c) 10 (d) 5

33. Let $a_1, a_2, a_3, ...$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to:

[JEE Main-2020 (January)]

(a) 171 (b) -513

(c) $\frac{511}{3}$ (d) -171

34. Let $f : R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, f(x) and $(3^x + 3^{-x})$ are in A.P., then the minimum value of f(x) is

[JEE Main-2020 (January)]

(a) 0 (b) 2

(c) 4 (d) 3

35. The sum $\sum_{k=1}^{20}(1+2+3+.....+k)$ is

[JEE Main-2020 (January)]

36. If the 10th term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 term is

[JEE Main-2020 (January)]

(a) 100 (b) $50\frac{1}{4}$

(c) $100\frac{1}{2}$ (d) 50

37. The sum, $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to

[JEE Main-2020 (January)]

38. The product $2^{\frac{1}{4}}, 4^{\frac{1}{16}}, 8^{\frac{1}{48}}, 16^{\frac{1}{128}}$ ..... to ∞ is equal to

[JEE Main-2020 (January)]

(a) $2^{\frac{1}{4}}$ (b) $2^{\frac{1}{2}}$

(c) 1 (d) 2

39. Let a_n be the nth term of G.P. of positive terms. If $\sum_{n=1}^{100}a_{2n+1}=200$ and $\sum_{n=1}^{100}a_{2n}=100$, then $\sum_{n=1}^{200}a_n$ is equal to:

[JEE Main-2020 (January)]

(a) 175 (b) 150

(c) 300 (d) 225

40. The number of terms common the two A.P. s 3, 7, 11....., 407 and 2, 9, 16, ..., 709 is_____.

[JEE Main-2020 (January)]

41. Let S be the sum of the first 9 terms of the series: $\{x+ka\}+\{x^2+(k+2)a\}+\{x^3+(k+4)a\}+\{x^4+(k+6)a\}+...$ where $a \neq 0$ and $x \neq 1$.

If $S=\frac{x^{10}-x+45a(x-1)}{x-1}$, then k is equal to

[JEE Main-2020 (September)]

(a) -3 (b) 1

(c) -5 (d) 3

42. If the sum of first 11 terms of an A.P., $a_1, a_2, a_3...$ is 0 ($a_1 \neq 0$), then the sum of the A.P., , a_1, a_3, a_5,..., a_{23} is ka_1, where k is equal to:

[JEE Main-2020 (September)]

(a) $-\frac{121}{10}$ (b) $-\frac{72}{5}$

(c) $\frac{72}{5}$ (d) $\frac{121}{10}$

43. The sum of the first three terms of a G.P. is S and their product is 27. Then all such S lie in:

[JEE Main-2020 (September)]

(a) $(-\infty,-9]\cup[3,\infty)$ (b) $[-3,\infty)$

(c) $(-\infty,-3]\cup[9,\infty)$ (d) $(-\infty,9]$

44. If $|x|<1$, $|y|<1$ and $x \neq y$, then the sum to infinity of the following series $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+.....$ is:

[JEE Main-2020 (September)]

(a) $\frac{x+y+xy}{(1+x)(1+y)}$ (b) $\frac{x+y-xy}{(1-x)(1-y)}$

(c) $\frac{x+y-xy}{(1+x)(1+y)}$ (d) $\frac{x+y+xy}{(1-x)(1-y)}$

45. If m arithmetic means (A. Ms) and three geometric means (G.Ms) are inserted between 3 and 243 such that 4th A.M. is equal to 2nd G.M., then m is equal to _____.

[JEE Main-2020 (September)]

46. If the sum of the series

$20+19\frac{3}{5}+19\frac{1}{5}+18\frac{4}{5}+...$ upto n^{th} term is 488 and the nth term is negative, then

[JEE Main-2020 (September)]

(a) $n=41$ (b) n^{th} term is $-4\frac{2}{5}$

(c) $n=60$ (d) n^{th} term is -4

47. The value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+....\infty\right)}$ equal to _____.

[JEE Main-2020 (September)]

48. If the first term of an A.P. is 3 and the sum of its first 25 terms is equal to the sum of its next 15 terms, then the common difference of this A.P. is:

[JEE Main-2020 (September)]

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$

(c) $\frac{1}{7}$ (d) $\frac{1}{5}$

49. Let $a_1, a_2, ...a_n$ be a given A.P. whose common difference is an integer and $S_n=a_1+a_2+.....+a_n$. If $a_1=1$, $a_n=300$ and $15 \le n \le 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to

[JEE Main-2020 (September)]

(a) (2490, 249) (b) (2480, 249)

(c) (2490, 248) (d) (2480, 248)

50. If $1+(1-2^2\cdot 1)+(1-4^2\cdot 3)+(1-6^2\cdot 5)+.....+(1-20^2\cdot 19)=\alpha-220\beta$, then an ordered pair (α, β) is equal to:

[JEE Main-2020 (September)]

(a) (10, 103) (b) (10, 97)

(c) (11, 97) (d) (11, 103)

51. If the sum of the first 20 terms of the series $\log_{(7^{1/2})} x + \log_{(7^{1/3})} x + \log_{(7^{1/4})} x + \ldots$ is 460, then x is equal to [JEE Main-2020 (September)]

(a) 7^2 (b) e^2

(c) $7^{1/2}$ (d) $7^{46/21}$

52. If the sum of the second, third and fourth terms of a positive term G.P. is 3 and the sum of its sixth, seventh and eighth terms is 243, then the sum of the first 50 terms of this G.P. is: [JEE Main-2020 (September)]

(a) $\frac{2}{13}(3^{50}-1)$ (b) $\frac{1}{13}(3^{50}-1)$

(c) $\frac{1}{26}(3^{49}-1)$ (d) $\frac{1}{26}(3^{50}-1)$

53. If $2^{10}+2^9\cdot 3^1+2^8\cdot 3^2+\ldots.+2\cdot 3^9+3^{10}=S-2^{11}$, then S is equal to : [JEE Main-2020 (September)]

(a) $2\cdot 3^{11}$ (b) $3^{11}-2^{12}$

(c) $\frac{3^{11}}{2}+2^{10}$ (d) 3^{11}

54. If $3^{2\sin 2\alpha-1}$, 14 and $3^{4-2\sin 2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P is : [JEE Main-2020 (September)]

(a) 65 (b) 78

(c) 81 (d) 66

55. Suppose that a function $f : R \to R$ satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in R$ and $f(a) = 3$. If $\sum_{i=1}^{n} f(i) = 363$, then n is equal to _____. [JEE Main-2020 (September)]

56. The common difference of the A.P. $b_1, b_2, \ldots, b_m$ is 2 more than the common difference of A.P. $a_1, a_2, \ldots, a_n$. If $a_{40} = -159$, $a_{100} = -399$ and $b_{100} = a_{70}$, then b_1 is equal to: [JEE Main-2020 (September)]

(a) –127 (b) –81

(c) 127 (d) 81

57. Let a, b, c, d and p be any non zero distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) = 0$. Then : [JEE Main-2020 (September)]

(a) a, c, p are in G.P. (b) a, b, c, d are in A. P.

(c) a, c, p are in A.P. (d) a, b, c, d are in G.P.

58. If $0 < \theta, \phi < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \cdot \sin^{2n}\phi$ then: [JEE Main-2021 (February)]

(a) $xy - z = (x + y)z$ (b) $xyz = 4$

(c) $xy + z = (x + y)z$ (d) $xy + yz + zx = z$

59. The sum of the infinite series [JEE Main-2021 (February)]

$1+\frac{2}{3}+\frac{7}{3^2}+\frac{12}{3^3}+\frac{17}{3^4}+\frac{22}{3^5}+\ldots\ldots$ is equal to:

(a) $\frac{9}{4}$ (b) $\frac{13}{4}$

(c) $\frac{15}{4}$ (d) $\frac{11}{4}$

60. In an increasing geometric series, the sum of the second and the sixth term is $\frac{25}{2}$ and the product of the third and fifth term is 25 . Then, the sum of 4th , 6th and 8th terms is equal to: [JEE Main-2021 (February)]

(a) 30 (b) 32

(c) 26 (d) 35

61. The sum of the series [JEE Main-2021 (February)]

$\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$ is equal to :

(a) $\frac{41}{8}e+\frac{19}{8}e^{-1}-10$ (b) $\frac{41}{8}e-\frac{19}{8}e^{-1}-10$

(c) $\frac{41}{8}e+\frac{19}{8}e^{-1}+10$ (d) $-\frac{41}{8}e+\frac{19}{8}e^{-1}-10$

62. If α, β are natural numbers such that

$100^{\alpha} - 199\beta = (100)(100) + (99)(101) + (98)(102) + \ldots\ldots + (1)(199)$

then the slope of the line passing through (α, β) and origin is : [JEE Main-2021 (March)]

(a) 540 (b) 550

(c) 530 (d) 510

63. $\frac{1}{3^2-1}+\frac{1}{5^2-1}+\frac{1}{7^2-1}+\ldots+\frac{1}{(201)^2-1}$ is equal to [JEE Main-2021 (March)]

(a) $\frac{101}{404}$ (b) $\frac{25}{101}$

(c) $\frac{101}{408}$ (d) $\frac{99}{400}$

JEE-ADVANCED PREVIOUS YEARS

1. Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is [IIT JEE-2012]

(a) 22 (b) 23

(c) 24 (d) 25

2. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s)

[JEE Advanced-2013]

(a) 1056 (b) 1088
(c) 1120 (d) 1332

3. A pack contains n card numbered from 1 to n. Two consecutive numbered card are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20 =$ [JEE Advanced-2013]

4. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2 + a - 14}{a+1}$ is

[JEE Advanced-2014]

5. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

[JEE Advanced-2015]

6. Let $b_i > 1$ for $i = 1, 2, ..., 101$. Suppose $\log_e b_1, \log_e b_2, ...\log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$, Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then

[JEE Advanced-2016]

(a) $s > t$ and $a_{101} > b_{101}$ (b) $s > t$ and $a_{101} < b_{101}$

(c) $s < t$ and $a_{101} > b_{101}$ (d) $s < t$ and $a_{101} < b_{101}$

7. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? [JEE Advanced-2017]

8. Let AP(a;d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If $AP(1; 3) \cap AP(2;5) \cap AP(3;7) = AP(a ; d)$ then a + d equals ____ [JEE Advanced -2019]

9. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____.

[JEE(Advanced)-2020]

10. Let $a_1, a_2, a_3, ...$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, ...$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$2(a_1 + a_2 + ...+a_n) = b_1 + b_2 + ... + b_n$

holds for some positive integer n, is _____

[JEE(Advanced)-2020]

11. For any positive integer n, let $S_n : (0,\infty) \to \mathbb{R}$ be defined by $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right)$

Where for any $x \in \mathbb{R}$, $\cot^{-1} x \in (0,\pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE?** [JEE(Advanced)-2021]

(a) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(b) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(c) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0,\infty)$

(d) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

12. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

[JEE(Advanced)-2021]

(a) $k + 21 = 22$ (b) $2k + l = 26$
(c) $2k + 3l = 34$ (d) $3k + 2l = 40$

13. Consider M with $r = \frac{\left(2^{199} - 1\right)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

[JEE(Advanced)-2021]

(a) 198 (b) 199
(c) 200 (d) 201

ANSWER KEY

Topicwise Questions

1. (*b*)	2. (*c*)	3. (*c*)	4. (*b*)	5. (*c*)	6. (*c*)	7. (*c*)	8. (*d*)	9. (*b*)	10. (*c*)	11. (*d*)	12. (*d*)
13. (*b*)	14. (*a*)	15. (*b*)	16. (*d*)	17. (*a*)	18. (*b*)	19. (*a*)	20. (*b*)	21. (*d*)	22. (*c*)	23. (*d*)	24. (*a*)
25. (*d*)	26. (*a*)	27. (*c*)	28. (*d*)	29. (*a*)	30. (*d*)	31. (*a*)	32. (*b*)	33. (*a*)			

Learning Plus

1. (*d*)	2. (*d*)	3. (*b*)	4. (*c*)	5. (*a*)	6. (*d*)	7. (*d*)	8. (*a*)	9. (*b*)	10. (*b*)	11. (*a*)	12. (*b*)
13. (*b*)	14. (*c*)	15. (*d*)	16. (*a*)	17. (*c*)	18. (*a*)	19. (*c*)	20. (*a*)	21. (*a*)	22. (*c*)	23. (*b*)	24. (*b*)
25. (*c*)	26. (*c*)	27. (*b*)									

Advanced Level Multiconcept Questions

1. (*b,d*)	2. (*a*)	3. (*b,c*)	4. (*a,b,c,d*)	5. (*a,b*)	6. (*a,b,c*)	7. (*a,b,c*)	8. (*a,b,c,d*)	9. (*a,c*)	10. (*a*)	11. (*a*)	12. (*d*)
13. (*a*)	14. (*c*)	15. (*d*)	16. (*a*) → (p), (*b*) → (p), (*c*) → (q), (*d*) → (q)				17. (*a*) → (r), (*b*) → (p), (*c*) → (s), (*d*) → (p)				
18. (0002)	19. (0003)	20. (0001)	21. (4)	22. (0)	23. (0.67)	24. (0.5)					

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*b*)	2. (*b*)	3. (*c*)	4. (*b*)	5. (*b*)	6. (*a*)	7. (*d*)	8. (*a*)	9. (*a*)	10. (*b*)	11. (*d*)	12. (*c*)
13. (*b*)	14. (*a*)	15. (*d*)	16. (*c*)	17. (*b*)	18. (*b*)	19. (*b*)	20. (*a*)	21. (*a*)	22. (*b*)	23. (*d*)	24. (*b*)
25. (*a*)	26. (*c*)	27. (*a*)	28. (*b*)	29. (*a*)	30. (*b*)	31. (*b*)	32. (*b*)	33. (*d*)	34. (*d*)	35. (1540)	36. (*c*)
37. (504)	38. (*b*)	39. (*b*)	40. (14)	41. (*a*)	42. (*b*)	43. (*c*)	44. (*b*)	45. (39)	46. (*d*)	47. (4)	48. (*a*)
49. (*c*)	50. (*d*)	51. (*a*)	52. (*d*)	53. (*d*)	54. (*d*)	55. (05.00)	56. (*b*)	57. (*d*)	58. (*c*)	59. (*b*)	60. (*d*)
61. (*b*)	62. (*b*)	63. (*b*)									

JEE-ADVANCED PREVIOUS YEARS

1. (*d*)	2. (*a,d*)	3. (5)	4. (4)	5. (9)	6. (*b*)	7. (6)	8. (157.00)	9. (8.00)	10. (1.00)	11. (*a, b*)
12. (*d*)	13. (*b*)									

6 SEQUENCE AND SERIES

Topicwise Questions

1. (*b*) We have $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+.....$

$=1\sqrt{2}+2\sqrt{2}+3\sqrt{2}+4\sqrt{2}+.....$

$=\sqrt{2}[1+2+3+4+.....\text{upto 24 terms}]$

$=\sqrt{2}\times\frac{24\times25}{2}=300\sqrt{2}$

2. (*c*) Suppose that first term and common difference of A.P.'s are A and D respectively.

Now, p^{th} term $= A+(p-1)D=a$ .....(i)

q^{th} term $= A+(q-1)D=b$ .....(ii)

and r^{th} term $= A+(r-1)D=c$ .....(iii)

So, $a(q-r)+b(r-p)+c(p-q)$

$=a\left\{\frac{b-c}{D}\right\}+b\left\{\frac{c-a}{D}\right\}+c\left\{\frac{a-b}{D}\right\}$

$=\frac{1}{D}(ab-ac+bc-ab+ca-bc)=0$.

3. (*c*) $T_m=a+(m-1)d=\frac{1}{n}$

and $T_n=a+(n-1)d=\frac{1}{m}$

On solving $a=\frac{1}{mn}$ and $d=\frac{1}{mn}$

$\therefore T_{mn}=a+(mn-1)d=\frac{1}{mn}+(mn-1)\frac{1}{mn}=1$

4. (*b*) The given number are in A.P.

$\therefore 2\log_9(3^{1-x}+2)=\log_3(4.3^x-1)+1$

$\Rightarrow 2\log_{3^2}(3^{1-x}+2)=\log_3(4.3^x-1)+\log_3 3$

$\Rightarrow \frac{2}{2}\log_3(3^{1-x}+2)=\log_3[3(4.3^x-1)]$

$\Rightarrow 3^{1-x}+2=3(4.3^x-1)$

$\Rightarrow \frac{3}{y}+2=12y-3$, where $y=3^x$

$\Rightarrow 12y^2-5y-3=0$

$y=\frac{-1}{3}$ or $\frac{3}{4}\Rightarrow 3^x=\frac{-1}{3}$ or $3^x=\frac{3}{4}$

$x=\log_3(3/4)\Rightarrow x=1-\log_3 4$.

5. (*c*) Let S_n and S'_n be the sums of n terms of two A.P.'s and T_{11} and T'_{11} be the respective 11th terms, then

$$\frac{S_n}{S'_n}=\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']}=\frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a+\frac{(n-1)}{2}d}{a'+\frac{(n-1)}{2}d'}=\frac{7n+1}{4n+27}$$

Now put $n=21$,

we get $\frac{a+10d}{a'+10d'}=\frac{T_{11}}{T'_{11}}=\frac{148}{111}=\frac{4}{3}$.

Note: If ratio of sum of n terms of two A.P.'s are given in terms of and ratio of their terms are to be found then put . Here we put .

6. (c) Let $S_{Even}=2+4+6+8+.....\infty$ .....(i)

and $S_{Odd}=1+3+5+7+9+.....\infty$ .....(ii)

Sum $S_E=\frac{n}{2}[4+(n-1)2]=\frac{n}{2}[2n+2]=\frac{n}{2}2(n+1)$

and $S_O=\frac{n}{2}[2+(n-1)2]=\frac{n}{2}(2n)$

Now $\frac{S_E}{S_O}=\frac{(n+1)}{n}$ or $S_E:S_O=(n+1):n$

7. (*c*) $S_{2n} - S_n = \frac{2n}{2}\{2a + (2n-1)d\} - \frac{n}{2}\{2a + (n-1)d\}$

$= \frac{n}{2}\{4a + 4nd - 2d - 2a - nd + d\} = \frac{n}{2}\{2a + (3n-1)d\}$

$= \frac{1}{3}.\frac{3n}{2}\{2a + (3n-1)d\} = \frac{1}{3}S_{3n}$

8. (*d*) Let $S = 1 + 2 + 3 + \ldots\ldots\ldots + 100$

$= \frac{100}{2}(1 + 100) = 50(101) = 5050$

Let $S_1 = 3 + 6 + 9 + 12 + \ldots\ldots\ldots + 99$

$= 3(1 + 2 + 3 + 4 + \ldots\ldots\ldots + 33)$

$= 3.\frac{33}{2}(1 + 33) = 99 \times 17 = 1683$

Let $S_2 = 5 + 10 + 15 + \ldots\ldots\ldots + 100$

$= 5(1 + 2 + 3 + \ldots\ldots\ldots + 20)$

$= 5.\frac{20}{2}(1 + 20) = 50 \times 21 = 1050$

Let $S_3 = 15 + 30 + 45 + \ldots\ldots\ldots + 90$

$= 15(1 + 2 + 3 + \ldots\ldots\ldots + 6)$

$= 15.\frac{6}{2}(1 + 6) = 45 \times 7 = 315$

$\therefore$ Required sum $= S - S_1 - S_2 + S_3$

$= 5050 - 1683 - 1050 + 315 = 2632.$

9. (*b*) $S_{2n} = 3S_n$

$\Rightarrow \frac{2n}{2}\{2a + (2n-1)d\} = 3.\frac{n}{2}\{2a + (n-1)d\}$

$2a = (n+1)d$

Put $2a = (n+1)d$ in $\frac{S_{3n}}{S_n}$, we get its value 6.

10. (*c*) Sum of the integer divided by 2

$= 2 + 4 + \ldots\ldots\ldots + 98 + 100$

$= \frac{50}{2}[2.2 + (50-1)2]$

$= 50[51] = 2550$

Sum of the integer divided by 5

$= 5 + 10 + \ldots\ldots\ldots\ldots\ldots 95 + 100$

$= \frac{20}{2}[5 + 100] = 1050$

Sum of the integer divided by 10

$\frac{10}{2}[10 + 100] = 550$

Sum of the integers divided by 5 or 10

$= 2550 + 1050 - 550 = 3050$

11. (*d*) $x, 2x + 2, 3x + 3 \rightarrow$ G.P.

$\Rightarrow (2x + 2)^2 = x.(3x + 3)$

$4x^2 + 4 + 8x = 3x^2 + 3x$

$x^2 + 5x + 4 = 0$

$x = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = -4$ or -1

$T_1 = a = x = -4$ or -1

$r = \frac{2x+2}{x} = \frac{2.(-4)+2}{(-4)} = \frac{-6}{-4} = \frac{3}{2}$ (if $x = -4$)

if $x = -1$

$r = \frac{2.(-1)+2}{(-1)} = 0$

When $a = -4$, $r = \frac{3}{2}$

$T_4 = ar^3 = (-4).\left(\frac{3}{2}\right)^3 = -\frac{27}{2} = -13.5$

12. (*d*) Given that $x, 2x+2, 3x+3$ are in G.P.

Therefore, $(2x+2)^2 = x(2x+2)^2 = x(3x+3)$

$\Rightarrow x^2 + 5x + 4 = 0$

$\Rightarrow (x+4)(x+1) = 0 \Rightarrow x = -1, -4$

Now first term $a = x$

Second term $ar = 2(x+1) \Rightarrow r = \frac{2(x+1)}{x}$

then 4th term $= ar^3 = x\left[\frac{2(x+1)}{x}\right]^3 = \frac{8}{x^2}(x+1)^3$

Putting $x = -4$

We get $T_4 = \frac{8}{16}(-3)^3 = -\frac{27}{2} = -13.5$

13. (*b*) $T_6 = 32$ and $T_8 = 128 \Rightarrow ar^5 = 32$ .....(i)

and $ar^7 = 128$ .....(ii)

Dividing (ii) by (i), $r^2 = 4 \Rightarrow r = 2$

14. (*a*) Series is a G.P. with $a = 0.9 = \frac{9}{10}$ and $r = \frac{1}{10} = 0.1$

$\therefore S_{100} = a\left(\frac{1-r^{100}}{1-r}\right) = \frac{9}{10}\left(\frac{1-\frac{1}{10^{100}}}{1-\frac{1}{10}}\right) = 1 - \frac{1}{10^{100}}.$

15. (*b*) Given series $6 + 66 + 666 + \ldots\ldots\ldots +$ upto n terms

$= \frac{6}{9}(9 + 99 + 999 + \ldots\ldots$ upto terms)

$= \frac{2}{3}(10 + 10^2 + 10^3 + \ldots\ldots\ldots +$ upto terms)

$= \frac{2}{3}\left(\frac{10(10^n - 1)}{10 - 1} - n\right) = \frac{1}{27}[20(10^n - 1) - 18n]$

$= \frac{2(10^{n+1} - 9n - 10)}{27}$

16. (*d*) Here $\frac{a}{1-r}=4$ and $ar=\frac{3}{4}$. Dividing these.

$$r(1-r)=\frac{3}{16} \text{ or } 16r^2-16r+3=0$$

or $(4r-3)(4r-1)=0$

$r=\frac{1}{4},\frac{3}{4}$ and $a=3,1$ so $(a,r)=\left(3,\frac{1}{4}\right),\left(1,\frac{3}{4}\right)$.

17. (*a*) $\because$ a, b, c are in G.P.

$$\therefore \frac{b}{a}=\frac{c}{b}=r$$

$$\Rightarrow \frac{b^2}{a^2}=\frac{c^2}{b^2}=r^2$$

a^2, b^2, c^2 are in G.P.

18. (*b*) Let the GP be a, ar^2, ar^3, ...

Now, $ar^2=4$

$\therefore a\cdot ar\cdot ar^2\cdot ar^3\cdot ar^4=a^5r^{10}=(ar^2)^5=4^5$

19. (*a*) $S=\frac{p}{1-\frac{1}{p}}=\frac{9}{2}\Rightarrow\frac{p^2}{p-1}=\frac{9}{2}$

$$\Rightarrow 2p^2-9p+9=0$$
$$\Rightarrow 2p^2-6p-3p+9=0$$
$$\Rightarrow (2p-3)(p-3)=0$$
$$p=3/2\ ,3$$

20. (*b*) As given $G=\sqrt{xy}$

$$\therefore \frac{1}{G^2-x^2}+\frac{1}{G^2-y^2}=\frac{1}{xy-x^2}+\frac{1}{xy-y^2}$$

$$=\frac{1}{x-y}\left\{-\frac{1}{x}+\frac{1}{y}\right\}=\frac{1}{xy}=\frac{1}{G^2}.$$

21. (*d*) It is an arithmetico-geometric series

$$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}=\frac{1}{1-\frac{1}{2}}+\frac{2}{\left(1-\frac{1}{2}\right)^2}=\frac{2}{\frac{1}{2}}+\frac{2}{\frac{1}{4}}$$

$$=4+8=12$$

22. (*c*) The sum of the first n terms is

$$S_n=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{2^2}\right)+\left(1-\frac{1}{2^3}\right)+\left(1-\frac{1}{2^4}\right)$$

$$+.....+\left(1-\frac{1}{2^n}\right)=n-\left\{\frac{1}{2}+\frac{1}{2^2}+.....+\frac{1}{2^n}\right\}$$

$$=n-\frac{1}{2}\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)=n-\left(1-\frac{1}{2^n}\right)=n-1+2^{-n}.$$

Trick : Check for $n=1,2$ i.e. $S_1=\frac{1}{2}, S_2=\frac{5}{4}$ and

(3) $\Rightarrow S_1=\frac{1}{2}$ and $S_2=2+2^{-2}-1=\frac{5}{4}$.

23. (*d*) Suppose that x to be added then numbers 13, 15, 19 so that new numbers $x+13, 15+x, 19+x$ will be in H.P.

$$\Rightarrow (15+x)=\frac{2(x+13)(19+x)}{x+13+x+19}$$

$$\Rightarrow x^2+31x+240=x^2+32x+247\Rightarrow x=-7$$

Trick : Such type of questions should be checked with the options.

24. (*a*) Here 5th term of the corresponding A.P. $=a+4d=45$ .....(i)

and 11th term of the corresponding A.P. $=a+10d=69$ .....(ii)

From (i) and (ii), we get $a=29, d=4$

Therefore 16th term of the corresponding A.P. $=a+15d=29+15\times 4=89$.

Hence 16th term of the H.P. is $\frac{1}{89}$.

25. (*d*) Considering corresponding A.P.

$a+6d=10$ and $a+11d=25\Rightarrow d=3, a=-8$

Hence term of the corresponding H.P. is $\frac{1}{49}$

26. (*a*) As given $H=\frac{2pq}{p+q}$

$$\therefore \frac{H}{p}+\frac{H}{q}=\frac{2q}{p+q}+\frac{2p}{p+q}=\frac{2(p+q)}{p+q}=2.$$

27. (*c*) A.M. $=\frac{a+b}{2}=A$ and G.M. $=\sqrt{ab}=G$

On solving a and b are given by the values

$A\pm\sqrt{(A+G)(A-G)}$.

Trick : Let the numbers be 1, 9. Then A=5 and G=3. Now put these values in options.

Here (3) $\Rightarrow 5\pm\sqrt{8\times 2}$ i.e. 9 and 1.

28. (*d*) Let $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$.

Then $G^2 = ab$ .....(i)

and $AH = \left(\frac{a+b}{2}\right).\frac{2ab}{a+b} = ab$ .....(ii)

From (i) and (ii), we have $G^2 = AH$

29. (*a*) Since a^2, b^2, c^2 be in A.P. Then

$b^2 - a^2 = c^2 - b^2$

$\Rightarrow (b-a)(b+a) = (c-b)(c+b)$

$\Rightarrow \frac{b-a}{c+b} = \frac{c-b}{b+a}$

$\Rightarrow \frac{(b-a)(a+b+c)}{(c+a)(b+c)} = \frac{(c-b)(a+b+c)}{(a+b)(c+a)}$

$\Rightarrow \frac{b^2+bc-ac-a^2}{(c+a)(b+c)} = \frac{c^2+ac-ab-b^2}{(a+b)(c+a)}$

$\Rightarrow \frac{b}{c+a} - \frac{a}{b+c} = \frac{c}{a+b} - \frac{b}{c+a}$

Hence $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ be in A.P.

30. (*d*) Here $T_n = \frac{n(n+1)}{2}$

Therefore $S_n = \frac{1}{2}\left\{\Sigma n^2 + \Sigma n\right\} = \frac{n(n+1)(n+2)}{6}$

31. (*a*) $\frac{1^3+2^3+3^3+4^3+.....+12^3}{1^2+2^2+3^2+4^2+.....+12^2}$

$= \frac{\left(\sum_{n=1}^{12} n^3\right)}{\left(\sum_{n=1}^{12} n^2\right)} = \left[\frac{n(n+1)}{2}\right]^2 \times \frac{6}{n(n+1)(2n+1)}$

$= \frac{3}{2}.\frac{n(n+1)}{(2n+1)} = \frac{3}{2}.\frac{12.13}{25} = \frac{234}{25}$

[Putting $n=12$].

32. (*b*) Here, $T_n = 3 + n(n-1) = 3 + n^2 - n$

Now sum $S = \Sigma T_n = \Sigma(3 + n^2 - n)$

$= 3n + \frac{1}{6}n(n+1)(2n+1) - \frac{n(n+1)}{2}$

$= \frac{1}{6}n(n+1)[2n+1-3] + 3n = \frac{n^3+8n}{3}$

33. (*a*) $T_n = \frac{1+2+3+.....+n}{n} = \frac{n(n+1)}{2n} = \frac{1}{2}(n+1)$

Hence, $S = \frac{1}{2}(\Sigma n + n) = \frac{1}{2}\left\{\frac{n(n+1)}{2} + n\right\} = \frac{n(n+3)}{4}$.

Learning Plus

1. (*d*) $S = \frac{2p+1}{2}[2(p^2+1)+2p]$

$= (2p+1)(p^2+1+p)$

$= 2p^3 + 3p^2 + 3p + 1 = p^3 + (p+1)^3$

2. (*d*) $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow 3(a_1 + a_{24}) = 225$(sum of terms equidistant from beginning and end are equal)

$a_1 + a_{24} = 75$

Now $a_1 + a_2 + ... + a_{23} + a_{24}$

$= \frac{24}{2}[a_1 + a_{24}] = 12 \times 75 = 900$

3. (*b*) Let a, a + d, a + 2d ..... are Interior angles

$\because$ sum of interior angles = $(n-2)\pi$, where n is the number of sides

$\therefore$ $a = 120°, d = 5°$

$\Rightarrow \frac{n}{2}[240° + (n-1)5°] = (n-2)180°$

$\Rightarrow n^2 = 25n - 144 \Rightarrow n = 16, 9$ but $n \neq 16$

because if n = 16, then an interior angle will be 180° which is not possible So n = 9

4. (*c*) $y = 2.\overline{357}$

$y = 2.357357357$ .....(1)

$1000y = 2357.357357$ .....(2)

so $999y = 2355$

$y = \frac{2355}{999}$

5. (*a*) $\frac{S_{kx}}{S_x} = \frac{\frac{kx}{2}[2a + (kx-1)d]}{\frac{x}{2}[2a + (x-1)d]} = k\left[\frac{2a + (kx-1)d}{2a + (x-1)d}\right]$

If $2a - d = 0$, then $\frac{S_{kx}}{S_x}$ is independent of x

So $d = 2a$

6. (*d*) $x \in R$

$5^{1+x} + 5^{1-x}$, $a/2$, $5^{x/2} + 5^{-x/2}$ are in A.P.

$\Rightarrow a = (5^{x/2} + 5^{-x/2}) + (5^{1+x/2} + 5^{1-x/2})$

$\Rightarrow a = (5^{x/2} + 5^{-x/2}) + 5(5^{x/2} + 5^{-x/2})$

$= (5^{x/2} + 5^{-x/2})^2 + 2 + 5(5^{x/2} + 5^{-x/2})^2 + 10$

$\Rightarrow a = 12 + (5^{x/2} + 5^{-x/2})^2 + 5(5^{x/2} + 5^{-x/2})^2$

$\Rightarrow a \geq 12$

7. (*d*) $S = \frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + \ldots\ldots + \frac{1}{\log_{2^n} 4}$

$= \frac{1}{2} + \frac{1}{1} + \frac{1}{2/3} + \ldots\ldots + \frac{1}{2/n}$

$= \frac{1}{2} + 1 + \frac{3}{2} + \frac{4}{2} + \ldots\ldots\ldots + \frac{n}{2}$

$= \frac{n(n+1)}{4}$ **Ans**

8. (*a*) Given that

$S = 1^2 - 2^2 + 3^2 - 4^2 + \ldots. 2002^2 + 2003^2$

$= 1 + (3^2 - 2^2) + (5^2 - 4^2) + \ldots. + (2003^2 - 2002^2)$

$= 1 + 2 + 3 + 4 + 5 + \ldots. + 2002 + 2003$

$= \frac{2003}{2}[1 + 2003] = 2003(1002)$

$= (2000 + 3)(1000 + 2) = 2007006$

9. (*b*) Let AP be A, A + d, A + 2d, ...

$T_p = a$, $T_q = b$

$\therefore a = A + (p-1)d$ ...(i)

$b = A + (q-1)d$ ...(ii)

$\Rightarrow$ subtract (i) & (ii) $\frac{a-b}{p-q} = d$

add (i) & (ii) $a + b = 2A + (p+q-1)d - d$

$\Rightarrow 2A + (p+q-1)d = (a+b) + d$

$S_{p+q} = \frac{(p+q)}{2}[2A + (p+q-1)d]$

$= \frac{(p+q)}{2}\left[a + b + \frac{a-b}{p-q}\right]$

10. (*b*) $\frac{(54-3)}{n+1} = d$

$d = \frac{51}{n+1} \Rightarrow \frac{A_8}{A_{n-2}} = \frac{3}{5}$

$\Rightarrow \frac{3 + 8\frac{51}{n+1}}{3 + (n-2)\frac{51}{n+1}} = \frac{3}{5}$

$\Rightarrow \frac{3n + 3 + 408}{3n + 3 + 51n - 102} = \frac{3}{5}$

$\Rightarrow 15n + 2055 = 162n - 297$

$\Rightarrow 147n = 2352$

$n = 16$

11. (*a*) Let $S = \sqrt{2} + \sqrt{6} + \sqrt{18} + \sqrt{54} + \ldots$ (10 terms)

$= \sqrt{2}(1 + \sqrt{3} + \sqrt{9} + \sqrt{27} + \ldots$ (10 terms))

$= \sqrt{2}(1 + 3^{1/2} + 3^1 + 3^{3/2} + \ldots)$

$= \sqrt{2}\left[\frac{(1 - (\sqrt{3})^{10})}{(1 - \sqrt{3})}\right] = \frac{\sqrt{2}((\sqrt{3})^{10} - 1)}{(\sqrt{3} - 1)}$

$= \frac{\sqrt{2}(3^5 - 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{\sqrt{2}}{2} 242(\sqrt{3} + 1)$

$= \sqrt{2}(121)(\sqrt{3} + 1) = 121(\sqrt{6} + \sqrt{2})$

12. (*b*) Let the GP be $a, ar^2, ar^4, \ldots$

We know that sum of G.P. is possible $\Rightarrow |r| < 1$

$S = \frac{a}{1-r} \Rightarrow r = \left(1 - \frac{a}{S}\right)$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a\left(1 - \left(1 - \frac{a}{S}\right)^n\right)}{\frac{a}{S}} = S\left[1 - \left(1 - \frac{a}{S}\right)^n\right]$

13. (*b*) Given,

$a_1 = 2$, & $\frac{a_{n+1}}{a_n} = \frac{1}{3}$

$\Rightarrow r$, $\sum_{r=1}^{20} a_r = \frac{a_1(1-r^{20})}{1-r} = \frac{2\left(1 - \left(\frac{1}{3}\right)^{20}\right)}{\frac{2}{3}} = 3\left(1 - \frac{1}{3^{20}}\right)$

14. (*c*) Given that,

$x^2 - 3x + a = 0 \begin{cases} \alpha \\ \beta \end{cases}$ $x^2 - 12x + b = 0 \begin{cases} \gamma \\ \delta \end{cases}$

Also, $\alpha, \beta, \gamma, \delta$ in increasing G.P.

Let $\alpha, \beta, \gamma, \delta$ be $\alpha, \alpha r, \alpha r^2, \alpha r^3$

$\alpha + \alpha r = 3$ & $\alpha r^2 + \alpha r^3 = 12$

$\alpha(1 + r) = 3$ & $\alpha r^2(1 + r) = 12$

& $r^2 = 3 = 12$

$\therefore \alpha = \frac{3}{1+r}$ & $r^2 = 4 \Rightarrow \alpha = \frac{3}{3} \Rightarrow \alpha = 1$ $\quad r = 2$.

$r = -2$ is rejected since

$\because$ G.P. increasing

$\alpha(\alpha r) = a$ & $(\alpha r^2)(\alpha r^3) = b$

$a = \alpha(\alpha r) = 1(1.2) = 2$

$b = (\alpha r^2)(\alpha r^3) = (1.4)(1.8) = 32$

15. (*d*) Let GP be $a_1, a_2, \ldots, a_k, \ldots$ with first term a & common ratio r.

$a_k = a_{k+1} + a_{k+2} \quad \forall\, a_k > 0$

$\Rightarrow ar^{k-1} = ar^k + ar^{k+1} \Rightarrow r > 0$

$\Rightarrow 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0$

$$\Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} \quad \{r = -\text{ve rejected}\}$$

$$\Rightarrow r = \frac{\sqrt{5}-1}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\sin 18^\circ$$

16. (*a*) Let $S = \frac{1}{(1+p)} - \frac{1-p}{(1+p)^2} + \frac{(1-p)^2}{(1+p)^3} \ldots \infty$

$$-1 < r = -\left(\frac{1-p}{1+p}\right) < 1 \because \; p > 0$$

It is a GP with first term $= \frac{1}{1+p}$ & common ratio r

$$\therefore S = \frac{a}{1-r} = \frac{\left(\frac{1}{1+p}\right)}{1+\frac{1-p}{1+p}} = \frac{1}{1+p+1-p} = \frac{1}{2}$$

17. (*c*) If $x > 0$

$\log_2 x + \log_2(\sqrt{x}) + \log_2 \sqrt[4]{x} + \log_2 \sqrt[8]{x}$

$+ \log_2 \sqrt[16]{x} + \ldots = 4$

$\Rightarrow \log_2 x + \log_2 x^{1/2} + \log_2 x^{1/4} + \ldots = 4$

$$\Rightarrow (\log_2 x)\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right] = 4$$

$$\Rightarrow (\log_2 x)\frac{1}{1-\frac{1}{2}} = 4 \Rightarrow \log_2 x = 2 \Rightarrow x = 4$$

18. (*a*) $3 + \frac{1}{4}(3+d) + \frac{1}{4^2}(3+2d) + \ldots + \infty = 8$

$a = 3, r = \frac{1}{4}$

Sum of AGP upto ∞

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$\Rightarrow 8 = \frac{3}{(3/4)} + \frac{d\left(\frac{1}{4}\right)}{3^2/4^2} \Rightarrow 8 = 4 + \frac{4d}{3^2}$$

$$\Rightarrow 4 = \frac{4d}{3^2} \Rightarrow d = 3^2 \Rightarrow d = 9$$

19. (*c*) $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} \Rightarrow \alpha + \beta = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$$\Rightarrow -\frac{b}{a} = \frac{b^2 - 2ac}{c^2} \Rightarrow -bc^2 = ab^2 - 2a^2c$$

$$\Rightarrow ab^2 + bc^2 = 2a^2c \Rightarrow \frac{b}{c} + \frac{c}{a} = \frac{2a}{b}$$

So $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P. $\Rightarrow \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

20. (*a*) Given that,

$$x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n \quad \begin{cases} |a| < 1 \\ |b| < 1 \\ |c| < 1 \end{cases}$$

$x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$

$\because$ a, b, c are in A.P.

$\Rightarrow 1 - a, 1 - b, 1 - c$ are also in A.P.

$\frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P. $\Rightarrow$ x, y, z in H.P

21. (*a*) a, b, c in A.P. $\Rightarrow 2b = a + c$ ...(i)

p, q, r, in H.P. $\Rightarrow q = \frac{2pr}{p+r}$ ...(ii)

ap, bq, cr in G.P. $\Rightarrow b^2q^2 = acpr$ ...(iii)

From (ii) & (iii), we get

$$\Rightarrow \frac{b^2 \cdot 4(pr)^2}{(p+r)^2} = ac\,pr$$

$$\Rightarrow \frac{(a+c)^2 pr}{(p+r)^2} = ac \qquad \text{(from (i))}$$

$$\Rightarrow \frac{(p+r)^2}{pr} = \frac{(a+c)^2}{ac} \Rightarrow \frac{p^2+r^2}{pr} + 2 = \frac{a^2+c^2}{ac} + 2$$

$$\Rightarrow \frac{p}{r} + \frac{r}{p} = \frac{a}{c} + \frac{c}{a}.$$

22. (*c*) $a^x = b^y = c^z = d^t = k$ and a, b, c, d are in G.P.

a, b, c are in G.P. $\Rightarrow$ So $b^2 = ac$

$$\Rightarrow k^{2/y} = k^{1/x + 1/z} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$\Rightarrow$ x, y, z are in H.P.

$\because$ b, c, d are in GP

then $\frac{2}{z} = \frac{1}{y} + \frac{1}{t} \Rightarrow$ y, z, t are in HP

So x, y, z, t are in H.P.

23. (*b*) $AM = A = \dfrac{a+b+c}{3}$

$GM = G = (abc)^{1/3}$

$HM = H = \dfrac{3abc}{ab+bc+ca} = \dfrac{3G^3}{ab+bc+ca}$

Equation whose roots are a,b,c

$\Rightarrow x^3-(a+b+c)x^2+(\Sigma ab)x-abc=0$

$\Rightarrow x^3-3Ax^2+\dfrac{3G^3}{H}.x-G^3=0$ **Ans**

24. (*b*) Let $S=\sum_{r=2}^{\infty}\dfrac{1}{r^2-1}$

$=\sum_{r=2}^{\infty}\dfrac{1}{(r-1)(r+1)}=\dfrac{1}{2}\sum_{r=2}^{\infty}\left(\dfrac{1}{r-1}-\dfrac{1}{r+1}\right)$

$=\dfrac{1}{2}\left[1-\dfrac{1}{3}+\dfrac{1}{2}-\dfrac{1}{4}+\dfrac{1}{3}-\dfrac{1}{5}+....+\dfrac{1}{n-1}-\dfrac{1}{n+1}+....\right]$

when $n\to\infty \Rightarrow \dfrac{1}{n+1}\to 0$

$\therefore S=\dfrac{1}{2}\left[1+\dfrac{1}{2}\right]=\dfrac{3}{4}.$

25. (*c*) Let $S=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+....\infty=\dfrac{\pi^2}{6}$

Now $S_{even}=\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}+....\infty$

$=\dfrac{1}{2^2}\left[1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+....\infty\right]=\dfrac{1}{2^2}\cdot\dfrac{\pi^2}{6}=\dfrac{\pi^2}{24}$

$S_{odd}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+....\infty$

$=S-S_{even}$

$=\dfrac{\pi^2}{6}-\dfrac{\pi^2}{24}=\dfrac{3\pi^2}{24}=\dfrac{\pi^2}{8}$

26. (*c*) $1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+.....$n terms

$=\dfrac{n(n+1)^2}{2}$, when n is even

$1^2+2.2^2+3^2+.....2.n^2=n\dfrac{(n+1)^2}{2}$

when n is odd $n+1$ is even

$1^2+2.2^2+3^2+.....n^2+2.(n+1)^2$

$=(n+1)\dfrac{(n+2)^2}{2}$

$1^2+2.2^2+3^2+.....n^2=(n+1)\left[\dfrac{(n+2)^2}{2}-2(n+1)\right]$

$=\dfrac{(n+1)n^2}{2}$

27. (*b*) Given that,

$1^2+2^2+...n^2=1015$

$\dfrac{n(n+1)(2n+1)}{6}=1015$

Put $n=15 \Rightarrow \dfrac{15\times16\times31}{6}=1240 \Rightarrow n\neq 15$

Put $n=14 \Rightarrow \dfrac{14\times15\times29}{6}=1015 \Rightarrow n=14$

Advanced Level Multiconcept Questions

1. (*b,d*) (D) $a_1+4a_2+6a_3-4a_4+a_5=0$

$a-4(a+d)+6(a+2d)-4(a+3d)+(a+4d)$

$=0-0=0$

Like wise we can check other options

2. (*a,b*) since $x, |x+1|, |x-1|$ are in A.P.

so $2|x+1|=x+|x-1|$ ....(i)

Case-I If $x<-1$, then (i) becomes $-2(x+1)=x-(x-1)$

$\Rightarrow x=-\dfrac{3}{2}$

then series $-\dfrac{3}{2},\dfrac{1}{2},\dfrac{5}{2},.....$

$\therefore S_{20}=\dfrac{20}{2}[-3+(20-1)2]=350$

Case-II If $-1\le x\le 1$, then (i) becomes

$2(x+1)=x-(x-1)\Rightarrow x=-1/2$

then series $-\dfrac{1}{2},\dfrac{1}{2},\dfrac{3}{2}$

So $S_{20}=\dfrac{20}{2}[-1+(20-1)1]=10\times18=180$

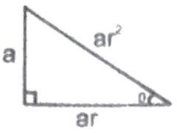
Case-III If $x\ge 1$, then (i) becomes

$2(x+1)=x+x-1$

$2=-1$ impossible.

3. (b,c)

Case - I



$r > 1$

$a^2 + a^2r^2 = a^2r^4$

$\Rightarrow r^4 - r^2 - 1 = 0$

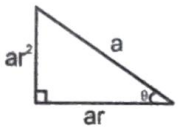
$r^2 = \dfrac{\sqrt{5}+1}{2}$

$r = \sqrt{\dfrac{\sqrt{5}+1}{2}}$ tangent of smallest angle $= \tan\theta = \dfrac{1}{r}$

$= \sqrt{\left(\dfrac{2}{\sqrt{5}+1}\right)}$

Case - II

$0 < r < 1$



$a^2 = a^2r^2 + a^2r^4$

$r^4 + r^2 - 1 = 0$

$r^2 = \dfrac{\sqrt{5}-1}{2}$

$r = \sqrt{\dfrac{\sqrt{5}-1}{2}}$

tangent of smallest angle $= \tan\theta = r = \sqrt{\dfrac{\sqrt{5}-1}{2}}$

4. (a,b,c,d) b_1, b_2, b_3 are in G.P. $\therefore b_3 > 4b_2 - 3b_1$

$\Rightarrow r^2 > 4r - 3$

$\Rightarrow r^2 - 4r + 3 > 0$

$\Rightarrow (r-1)(r-3) > 0$

So $0 < r < 1$ and $r > 3$

5. (a,b) $2b = a + c$

$4b^2 = a^2 + c^2 + 2ac$

$\Rightarrow a^2 + c^2 = 4b^2 - 2ac$ & $b^2 = \dfrac{2a^2c^2}{a^2+c^2}$

$\Rightarrow b^2(4b^2 - 2ac) = 2a^2c^2$

$\Rightarrow b^2(2b^2 - ac) = a^2c^2$

$\Rightarrow 2b^4 - b^2(ac) - (ac)^2 = 0$

$\Rightarrow (b^2 - ac)(2b^2 + ac) = 0$

$\Rightarrow b^2 = ac \Rightarrow$ a, b, c in G.P.

& a, b, c in A.P.

$\Rightarrow a = b = c$

or $2b^2 + ac = 0 \Rightarrow b^2 = -\dfrac{ac}{2}$

$\Rightarrow a, b, \dfrac{-c}{2}$ in G.P.

6. (a,b,c) $\dfrac{a+b}{\frac{2}{\sqrt{ab}}} = \dfrac{2}{1} \Rightarrow \dfrac{a+b}{2\sqrt{ab}} = \dfrac{2}{1}$

use compendendo and dividendo rule

$\Rightarrow \dfrac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \dfrac{3}{1} \Rightarrow \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{3}}{1}$

$\Rightarrow \dfrac{2\sqrt{a}}{2\sqrt{b}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \dfrac{a}{b} = \dfrac{3+1+2\sqrt{3}}{3+1-2\sqrt{3}}$

$\Rightarrow \dfrac{a}{b} = \dfrac{2+\sqrt{3}}{2-\sqrt{3}} = \dfrac{(2+\sqrt{3})(2+\sqrt{3})}{4-3} = 7 + 4\sqrt{3}$ **Ans**

7. (a,b,c) Given, $x = y + 2, a = 5z$

a, x, b in A.P $\Rightarrow 2x = a + b$

a, y, b in G.P $\Rightarrow y^2 = ab$

a, z, b in H.P $\Rightarrow z = \dfrac{2ab}{a+b} \Rightarrow z = \dfrac{2y^2}{2x}$

$\Rightarrow y^2 = zx$

A.M. > G.M. > H.M.

$x > y > z$

$\dfrac{a}{5} = \dfrac{2ab}{a+b}$

$a^2 + ab = 10ab \Rightarrow a(a - 9b) = 0 \Rightarrow a \neq 0$ or $a = 9b$

$y = x - 2$

$\therefore y = \dfrac{2x-4}{2} \Rightarrow y^2 = \dfrac{(a+b-4)^2}{4}$

$\Rightarrow a^2 + b^2 - 2ab - 8a - 8b + 16 = 0$

$\Rightarrow 4b^2 - 5b + 1 = 0 \Rightarrow (b-1)(4b-1) = 0$

$\Rightarrow b = 1, a = 9$

or $b = 1/4, a = 9/4$

8. (a,b,c,d)

$\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$

$\sum_{r=1}^{n} (r^2+r)(2r+3)$

$= \sum_{r=1}^{n} (2r^3 + 5r^2 + 3r)$

$= 2.\dfrac{n^2(n+1)^2}{4} + 5\dfrac{n(n+1)(2n+1)}{6} + 3\dfrac{n(n+1)}{2}$

$$= \frac{n(n+1)}{2}\left[n(n+1)+\frac{5}{3}(2n+1)+3\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{6(n^2+n)+10(2n+1)+18}{6}\right]$$

$$= \frac{n(n+1)}{12}[6n^2+26n+28]$$

$$= \frac{1}{12}[6n^4+26n^3+28n^2+6n^3+26n^2+28n]$$

$$= \frac{1}{12}[6n^4+32n^3+54n^2+28n]$$

$a = \frac{6}{12}, b = \frac{32}{12}, c = \frac{54}{12}, d = \frac{28}{12} = \frac{7}{3}, e = 0$

so $a + c = b + d$

$b - \frac{2}{3} = \frac{32}{12} - \frac{2}{3} = \frac{24}{12}$

$c - 1 = \frac{42}{12}$

so a, b–2/3, c–1 are in A.P

& $\frac{c}{a} = \frac{54}{6} = 9$ is an integer

9. (*a,c*) $\sum_{r=1}^{n} \frac{1}{\sqrt{a+rx}+\sqrt{a+(r-1)x}}$

$$= \sum_{1}^{n} \frac{\sqrt{a+rx}-\sqrt{a+(r-1)x}}{x}$$

$$= \frac{1}{x}\sum_{1}^{n}(\sqrt{a+rx}-\sqrt{a+(r-1)x})$$

$$= \frac{1}{x}[\sqrt{a+nx}-\sqrt{a}]$$

Upon rationalizing

$$= \frac{1}{x}\cdot\frac{a+nx-a}{\sqrt{a+nx}+\sqrt{a}} = \frac{n}{\sqrt{a+nx}+\sqrt{a}}$$

10. (*a*) $g(n) - g(n-1) = 1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 + n^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2) = n^2$

11. (*a*) $g(n) \cdot f(n) = \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2}$

$$= \frac{n(n+1)}{2}\left(\frac{2n+1}{3}-1\right) = \frac{n(n+1)}{2}\cdot\frac{2n-2}{3}$$

$$= \frac{n(n+1)(n-1)}{3} = \frac{(n-1)\,n\,(n+1)}{3}$$

for $n = 2\ \frac{(n-1)\,n\,(n+1)}{3} = \frac{1.2.3}{3}$ which is divisible by 2 but not by 2^2

$\therefore$ greatest even integer which divides $\frac{(n-1)\,n\,(n+1)}{3}$, for every $n \in N, n \geq 2$, is 2

12. (*d*) $f(n) + 3g(n) + h(n) =$

$$\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{2} + \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{n(n+1)}{2}\left(1+2n+1+\frac{n(n+1)}{2}\right) = (1+2+3+\ldots.+n)\left(2n+2+\frac{n(n+1)}{2}\right)$$

$\Rightarrow$ for all $n \in N$

13. (*a*)

14. (*c*)

15. (*d*)

Sol. 13, 14, 15

Let the 3 numbers in strictly increasing GP

are $\frac{a}{r}, a, ar (r > 1)$

According to passage $\frac{a^2}{r^2} + a^2 + a^2r^2 = S^2$

or $a^2\left(\frac{1}{r^2}+1+r^2\right) = S^2$

or $a^2\left(\frac{1}{r}+1+r\right)\left(\frac{1}{r}-1+r\right) = S^2$ ...(i)

and $\frac{a}{r} + a + ar = \alpha S$

or $a\left(\frac{1}{r}+1+r\right) = \alpha S$

$\therefore\ a^2\left(\frac{1}{r}+1+r\right)^2 = \alpha^2 S^2$ ...(ii)

Dividing Eq. (ii) by (i), then $\left(\frac{\frac{1}{r}+1+r}{\frac{1}{r}-1+r}\right)=\alpha^2$

$\Rightarrow (1+r+r^2)=\alpha^2(1-r+r^2)$

$\Rightarrow (\alpha^2-1)r^2-(\alpha^2+1)r+\alpha^2-1=0$ ...(iii)

(1) $r=3 \therefore r=\frac{11}{7}$

(2) Put $\alpha^2=2$ in eq. (iii), then $r^2-3r+1=0$

$\therefore \quad r=\frac{3\pm\sqrt{5}}{2}=\frac{3+\sqrt{5}}{2} \therefore r>2$

(3) Put $r=2$ in eq. (iii), then

$4(\alpha^2-1)-2(\alpha^2+1)+a^2-1=0$

or $3\alpha^2-7=0 \therefore \alpha^2=7/3=a/b \Rightarrow a+b=10$

16. (*a*) $\rightarrow$ (p), (*b*) $\rightarrow$ (p), (*c*) $\rightarrow$ (q), (*d*) $\rightarrow$ (q)

(*a*) $2\log_5(2^x-5)=\log_5 2+\log_5(2^x-7/2)$

$(2^x-5)^2=2\left(2^x-\frac{7}{2}\right)$

$t^2-10t+25=2t-7\{\text{put } 2^x=t\}$

$t^2-12t+32=0$

$\therefore t=8,4$

$\therefore 2^x=4$ or $2^x=8$

$\therefore x=2,3\ 2^x-5>0$

$\therefore 2x=6 \therefore 2^x>5$

so only solution $x=3$

(*b*) $\frac{S_{2n}}{S_n}=\frac{\frac{2n}{2}[2a+(2n-1)d]}{\frac{n}{2}[2a+(n-1)d]}=3$

$=\frac{2a+(2n-1)d}{2a+(n-1)d}=\frac{3}{2}$

$\therefore d=\frac{2a}{n+1}$

Now $\frac{S_{3n}}{S_n}=\frac{\frac{3n}{2}[2a+(3n-1)d]}{\frac{n}{2}[2a+(n-1)d]}$

$=\frac{3\left[2a+(3n-1)\frac{2a}{n+1}\right]}{\left[2a+(n-1)\frac{2a}{n+1}\right]}=\frac{3[(n+1)+(3n-1)]}{(n+1)+(n-1)}$

$=\frac{3.4n}{2n}=6$

(*c*) $S=4+\frac{8}{3}+\frac{12}{3^2}+\frac{16}{3^3}+\ldots\ldots\ldots\frac{4n}{3^{n-1}}$ ...(i)

$\frac{S}{3}=\frac{4}{3}+\frac{8}{3^2}+\frac{12}{3^3}+\ldots\ldots\ldots+\frac{4m}{3^n}$ ...(ii)

(i) – (ii) we get

$\frac{2}{3}S=4+\frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\ldots\ldots\ldots+\frac{4}{3^{n-1}}+\frac{4n}{3^n}$

$=4\left(\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right)-\frac{4n}{3^n}$

$S=\frac{4\cdot3}{2}\cdot\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)-\frac{4n}{3^n}\cdot\frac{3}{2}$

$\therefore S_\infty=9$

(*d*) $\ell, \ell r, \ell r^2$

$\therefore \ell^3 r^3=27$ (volume)

$\ell r=3$

surface area

$2(\ell.\ell r+\ell r.\ell r^2+\ell r^2.\ell)=78$

$\ell^2(r+r^2+r^3)=39 \Rightarrow \ell^2\left(\frac{3}{\ell}+\frac{3^2}{\ell^2}+\frac{3^3}{\ell^3}\right)=39$

$3\ell+3^2+\frac{3^3}{\ell}=39 \Rightarrow \ell+3+\frac{9}{\ell}=13$

$\therefore \quad \ell^2-10\ell+9=0 \Rightarrow \ell=1,9$

$\Rightarrow \quad \ell r=3$ and $\ell>\ell r$

$\therefore \quad r=\frac{1}{3} \therefore \ell=9$

17. (*a*) $\rightarrow$ (r), (*b*) $\rightarrow$ (p), (*c*) $\rightarrow$ (s), (*d*) $\rightarrow$ (p)

(*a*) $2(\log_z x)^2=\log_x y.\log_y z$ GP

$(\log_z x)^2=\log_x z$

$(\log_z x)^3-1=0$

$\log_z x=1,(\log_z x)^2+\log_z x+1=0$ (not possible)

$\therefore x=z$ .....(1)

$2y^3=x^3+z^3 x^3, y^3, z^3$ in A.P

$2y^3=2x^3$

$x=y$ .....(2)

$\therefore x=y=z$

Given $xyz=64, \therefore x=y=z=4$

$\therefore \frac{3x}{y}=3$

(b) $S_x = 2^{1/4}.4^{1/8}.8^{1/16}.....\infty$

$S_x = 2^{1/4}.2^{2/8}.2^{3/16}.....\infty$

$S_x = 2^{1/4+2/8+3/16+.....} = 2^{S'_x}$

Let $S'_x = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+.....$

$S'_n = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}.....\frac{n}{2^{n+1}}$ ...(i)

$\frac{S'_n}{2} = \frac{1}{8}+\frac{2}{16}+.....+\frac{n-1}{2^{n+1}}+\frac{n}{2^{n+2}}$ ...(ii)

(i) – (ii) we get

$\frac{S'_n}{2} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....\frac{1}{2^{n+1}}-\frac{n}{2^{n+2}}$

$\frac{S'_n}{2} = \frac{1}{4}\left(\frac{1-(1/2)^n}{1-1/2}\right)-\frac{n}{2^{n+2}}$

$S'_n = \frac{2.2}{4}\left(1-\left(\frac{1}{2}\right)^n\right)-\frac{2n}{2^{n+2}}$

$S'_\infty = 1$

$\therefore S_\infty = 2^{S'_\infty} = 2$

(c) x,y,z are in A.P

$y = \frac{x+z}{2}$, or $2y = x+z$

$(x+2y-z)(x+z+z-x)(z+x-y) = (x+(x+z)-z)$

$(x+z+z-x)(2y-y) = 2x.2z.y = 4xyz$

$\therefore k = 4$

(d) $d = \frac{31-1}{m+1} = \frac{30}{m+1}$

$\frac{A_7}{A_{m-1}} = \frac{1+7\frac{30}{m+1}}{1+(m-1)\frac{30}{m+1}} = \frac{5}{9} \Rightarrow \frac{m+211}{31m-29} = \frac{5}{9}$

$\Rightarrow 146m = 2044 \Rightarrow m = 14$

$\therefore \frac{m}{7} = 2$

NUMERICAL VALUE BASED

18. (0002) $2^{1/4}\times 4^{1/8}\times 8^{1/16}... = 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+...}$

Now, $S = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+....$

$\Rightarrow \frac{1}{2}S = \frac{1}{8}+\frac{2}{16}+.... \quad \therefore S-\frac{1}{2}S = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+....$

$\Rightarrow \frac{1}{2}S = \frac{\frac{1}{4}}{1-\frac{1}{2}} \qquad \Rightarrow S = 1$

So the given product is 2.

19. (0003) Let common ratio is $\frac{1}{2^b}$

and $S_\infty = \frac{a}{1-r} = \frac{2^a}{1-\frac{1}{2^b}} = \frac{1}{7}$

$\Rightarrow b = 3 \& a = b$

$\Rightarrow b = 3 \& a = b$

Hence, $a = 3$

20. (0001) $T_2 = 3+d, T_{10} = 3+9d, T_{34} = 3+33d$

since $T_2.T_{10}, T_{34}$ are in G.P

$T_{10}^{\ 2} = T_2 T_{34}$

$\Rightarrow (3+9d)^2 = (3+d)(3+33d)$

$\Rightarrow d = 0,1$

hence $d = 1$

21. (4) We can write the given equation as

$\log_2\left(x^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....}\right) = 4$

$\Rightarrow \log_2(x^2) = 4 \Rightarrow x^2 = 2^4 \Rightarrow x = 4$

22. (0) $T_p = AR^{p-1} = x$

$\log x = \log A + (p-1)\log R$

Similary write $\log y, \log z$

Multiply by $q-r, r-p$ and $p-q$ and add we get,

$(q-r)\log x + (r-p)\log y + (p-q)\log z = 0$

23. (0.67) $b_1, b_2, b_3.....$

or a, ar, ar^2,..... ∞

Given $\frac{a}{1-r} = 3$ and $\frac{a^3}{1-r^3} = \frac{108}{13}$

solving $a = 2$ and $r = \frac{1}{3}$ or $r = 3$ (rejected)

$\therefore \quad ar = \frac{2}{3} = b_2$

24. (0.5) $T_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n(2n+2)}[2n+2)-(2n+1)]$

$T_n = \frac{1.3.5...(2n-1)}{2.4.6...2n} - \frac{1.3.5...(2n-1)(2n+1)}{2.4.6...2n(2n+2)}$

$\therefore S_n = \sum T_n = \frac{1}{2} - \frac{1.3.5.....(2n+1)}{2.4.6.....2n(2n+2)}$

Note that $S_\infty = \frac{1}{2}$

JEE-MAIN PREVIOUS YEARS

1. (*b*) Let A be first term & d be com diff.

$T_2 = A + d$, $T_5 = A + 4d$ & $T_9 = A + 8d$

$T_5^2 = T_2 . T_9$

$(A + 4d)^2 = (A + d)(A + 8d)$

$A^2 + 16d^2 + 8Ad = A^2 + 9Ad + 8d^2$

$8d^2 - Ad = 0$

$\Rightarrow d(8d - A) = 0 \Rightarrow A = 8d$

$T_2 = 9d$, $T_5 = 12d$, $T_9 = 16d$

$$\text{Com ratio} = \frac{12d}{9d} = \frac{4}{3}$$

2. (*b*) $\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \left(\frac{24}{5}\right)^2 + \ldots$

$$= \frac{8^2 + 12^2 + 16^2 + \ldots}{25}$$

$$= \frac{4^2\left(2^2 + 3^2 + 4^2 + \ldots + 11^2\right)}{25}$$

$$= \frac{16}{25} \times \left[\frac{11 \times 12 \times 23}{6} - 1\right] = \frac{16}{25} \times 101$$

3. (*c*) $(15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$$\frac{1}{2}\left[(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2\right]$$

It is possible when $15a = 3b = 5c$

$$\therefore b = \frac{5c}{3}, a = \frac{c}{3}$$

$a + b = 2c$

$\Rightarrow$ b, c, a in A.P.

4. (*b*) $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 3$

Now $f(x + y) = f(x) + f(y) = xy$

put $y = 1$

$f(x + 1) = f(x) + f(1) + x$

$f(x + 1) = f(x) + x + 3$

Now

$f(2) = 7$

$f(3) = 12$

Now

$S_n = 3 + 7 + 12 + t_n$ ....(i)

$S_n = 3 + 7 + \ldots.. t_{n-1} + t_n$ ....(ii)

On subtracting (ii) from (i)

$t_n = 3 + 4 + 5 + \ldots..$upto n terms

$$t_n = \frac{(n^2 + 5n)}{2}$$

$$S_n = \sum t_n = \sum \frac{(n^2 + 5n)}{2}$$

$$S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}\right]$$

$S_{10} = 330$

5. (*b*) $\sum_{k=0}^{12} a_{4k+1} = 416$

$$\Rightarrow \frac{13}{2}[2a_1 + 48d] = 416$$

$\Rightarrow a_1 + 24d = 32$ .....(i)

$a_9 + a_{43} = 66 \Rightarrow 2a_1 + 50d = 66$ .....(ii)

$2a_1 + 48d = 64$ by .....(i)

$d = 1$ and $a_1 = 8$

$$\Rightarrow 140m = \sum_{r=1}^{17} a_r^2 = \sum_{r=1}^{17}\left[8 + (r-1).1^2\right]$$

$$\Rightarrow 140m = \sum_{r=1}^{17}(r+7)^2 \Rightarrow 140m = \sum_{r=1}^{24} r^2 - \sum_{r=1}^{7} r^2$$

$$\Rightarrow 140m = \frac{24.25.49}{6} - \frac{7.8.15}{6}$$

$$\Rightarrow 140m = \frac{7.8.5}{6}[105 - 3]$$

$\Rightarrow 140m = 280.17 \Rightarrow m = 34$

6. (*a*) $B - 2A = \sum_{r=1}^{40} T_r - 2\sum_{r=1}^{20} T_r = \sum_{r=21}^{40} T_r - \sum_{r=1}^{20} T_r$

$B - 2A = m(21^2 + 2.22^2 + 23^2 + 2.24^2 + \ldots. + 40^2)$

$- (1^2 + 2.2^2 + 3^2 + 2.4^2 \ldots + 20^2)$

$= 20[22 + 2.24 + 26 + 2.28 + \ldots. + 60]$

$$= 20\left[\underbrace{(22 + 24 + 26 \ldots 60)}_{20\text{terms}} + \underbrace{(24 + 28 + \ldots\ldots + 60)}_{10\text{terms}}\right]$$

$$= 20\left[\frac{20}{2}(22 + 60) + \frac{10}{2}(24 + 60)\right] = 10[20.82 + 10.84]$$

$= 100[164 + 84] = 100.248$

7. (*d*) $a + ar + ar^2 = xar$

since $a \neq 0$ so $\frac{r^2 + r + 1}{r} = x$, $1 + r + \frac{1}{r} = x$

$\because r + \frac{1}{r} \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow x \in (-\infty, -1] \cup [3, \infty)$

8. (*a*) $S = \sum_{r=1}^{30} a_r$, $T = \sum_{r=1}^{15} a_{2r-1}$, $a_5 = 27$, $S - 2T = 75$

Let $a_r = a + (r - 1)D$

$S = a_1 + a_2 + a_3 + \ldots\ldots + a_{30}$

$T = a_1 + a_3 + a_5 + \ldots\ldots + a_{29}$

$\therefore 2T = 2a_1 + 2a_3 + 2a_5 + \ldots\ldots + 2a_{29}$

$S - 2T = (a_2 - a_1) + (a_4 - a_3) + (a_6 - a_5) + \ldots\ldots +$

$(a_{30} - a_{29}) = 75$

$= 15D$

But $S - 2T = 75 \Rightarrow 15D = 75 \Rightarrow D = 5$

Now $a_5 = 27 \Rightarrow a + 4D = 27$

$\therefore a = 27 - 20 \Rightarrow a = 7$

now $a_{10} = a + 9D$

$= 7 + 45 = 52$

9. (*a*) $T_n = \frac{\left(3 + (n-1) \times 3\right)\left(1^2 + 2^2 + \ldots + n^2\right)}{(2n+1)}$

$$T_n = \frac{3.\frac{n^2(n+1)(2n+1)}{6}}{(2n+1)} - \frac{n^2(n+1)}{2}$$

$$S_{15} = \frac{1}{2}\sum_{n=1}^{15}(n^3 + n^2) = \frac{1}{2}$$

$$\left[\left(\frac{15(15+1)}{2}\right)^2 + \frac{15 \times 16 \times 31}{6}\right] = 7820$$

10. (*b*) $a = A + 6d$

$b = A + 10d$

$c = A + 12d$

a, b, c are in G.P.

$\Rightarrow (A + 10d)^2 = (A + 6d)(a + 12d)$

$\Rightarrow \frac{A}{d} = -14$

$$\frac{a}{c} = \frac{A + 6d}{A + 12d} = \frac{6 + \frac{A}{d}}{12 + \frac{A}{d}} = \frac{6 - 14}{12 - 14} = 4$$

11. (*d*) Two digit numbers of the from $7\lambda + 2$ are 16, 23 ....., 93

Two digit numbers of the from $7\lambda + 5$ are 12, 19 ....., 96

Sum of all the above number equals to

$$\frac{12}{2}(16 + 93) + \frac{13}{2}(12 + 96) = 654 + 702 = 1356$$

12. (*c*) $a + 18d = 0 \Rightarrow a = -18d$

$$\frac{t_{49}}{t_{29}} = \frac{a + 48d}{a + 28d} = \frac{-18d + 48d}{-18d + 28d}$$

$$= \frac{30d}{10d} = 3$$

13. (*b*) $\frac{a}{1 - r} = 3$

Cube both sides

$$\frac{a^3}{(1-r)^3} = 27 \quad \text{...(1)}$$

and $\frac{a^3}{1 - r^3} = \frac{27}{19}$ ...(2)

(1)/(2) gives $\frac{1 - r^3}{(1-r)^3} = 19 \quad \Rightarrow \quad r = \frac{2}{3}$

14. (*a*) $\frac{a_3}{a_1} = \frac{a_1 r^2}{a_1} = r^2$

$\Rightarrow r^2 = 25$

Now $\frac{a_9}{a_5} = \frac{a_1 r^8}{a_1 r^4} = r^4 = (25)^2 = 5^4$

15. (*d*) Let the numbers be $\frac{a}{r}$, a, ar

Given $a^3 = 512 \Rightarrow a = 8$

Now given $\frac{8}{r} + 4$, 12, 8r are in A.P.

$\Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow r = \frac{1}{2}$ or 2

Numbers are 4, 8, 16 or 16, 8, 4

Sum of numbers $= 4 + 8 + 16 = 28$

16. (*c*) $S_k = \frac{k+1}{2}$

$$\sum_{k=1}^{10}\left(\frac{k+1}{2}\right)^2 = \frac{5}{12}A$$

$$2^2 + 3^2 + \ldots\ldots 11^2 = \frac{5A}{3}$$

$$\frac{11 \times 12 \times 23}{6} - 1 = \frac{5A}{3}$$

$$505 \times \frac{3}{5} = A$$

$A = 303$

17. (*b*) $S=\left(\frac{3}{4}\right)^3+\left(\frac{6}{4}\right)^3+\left(\frac{9}{4}\right)^3+\left(\frac{12}{4}\right)^3+\ldots.15\text{ term}$

$=\frac{27}{64}\sum_{r=1}^{15} r^3$

$=\frac{27}{64}\cdot\left[\frac{15(15+1)}{2}\right]^2$

$=225\,K$ (Given in question)

$K=27$

18. (*b*) S_A = sum of numbers between 100 & 200 which are divisible by 7.

$\Rightarrow S_A=105+112+\ldots\ldots+196$

$S_A=\frac{14}{2}[105+196]=2107$

S_B = Sum of numbers between 100 & 200 which are divisible by 13.

$S_B=104+117+\ldots\ldots+195=\frac{8}{2}[104+195]=1196$

S_C = Sum of numbers between 100 & 200 which are divisible by both 7 & 13.

$S_C=182$

$\Rightarrow$ H.C.F. $(91,n)>1=S_A+S_B-S_C=3121$

19. (*b*) $S=\sum_{k=1}^{20}\frac{1}{2^k}$

$S=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{3^2}+\ldots+\frac{20}{2^{20}}$

$S\times\frac{1}{2}=\frac{1}{2^2}+\frac{2}{2^3}+\ldots.+\frac{19}{2^{20}}+\frac{20}{2^{21}}$

$\Rightarrow\left(1-\frac{1}{2}\right)S=\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{20}}-\frac{20}{2^{21}}$

$\Rightarrow S=2-\frac{11}{2^{19}}$

20. (*a*) $S_n=50n+\frac{n(n-7)}{2}A$

$T_n=S_n-S_{n-1}$

$=50n+\frac{n(n-7)}{2}A-50(n-1)-\frac{(n-1)(n-8)}{2}A$

$=50+\frac{A}{2}\left[n^2-7n-n^2+9n-8\right]$

$=50+A(n-4)$

$d=T_n-T_{n-1}$

$=50+A(n-4)-50-A(n-5)$

$=A$

$T_{50}=50+46A$

$(d,A_{50})=(A,50+46A)$

21. (*a*) a $-$ d + a + a + d = 33 $\Rightarrow$ a = 11

$a(a^2-d^2)=1155$

$121-d^2=105$

$d^2=16\Rightarrow d=\pm4$

If d = 4 then Ist term = 7

If d = $-$ 4 then Ist term = 15

$T_{11}=7+40=47$

OR $T_{11}=15-40=-25$

22. (*b*) $T_r=r(2r-1)$

$\Sigma=\Sigma2r^2-\Sigma r$

$S=\frac{2.n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$

$S_{11}=\frac{2}{6}\cdot(11)(12)(23)-\frac{11(12)}{2}=(44)(23)-66=946$

23. (*d*) $\text{Sum}=\sum_{n=1}^{15}\frac{1^3+2^3+\ldots..n^3}{1+2+\ldots..+n}-\frac{1}{2}\cdot\frac{15.16}{2}$

$=\sum_{n=1}^{15}\frac{n(n+1)}{2}-60$

$=\sum_{n=1}^{15}\frac{n(n+1)(n+2-(n-1))}{6}-60$

$=\frac{15.16.17}{6}-60=620$

24. (*b*) $b=ar$

$c=ar^2$

3a, 7b and 15 c are in A.P

$\Rightarrow 14b=3a+15c$

$\Rightarrow 14(ar)=3a+15ar^2$

$\Rightarrow 14r=3+15r^2$

$\Rightarrow 15r^2-14r+3=0\quad\Rightarrow(3r-1)(5r-3)=0$ r

$=\frac{1}{3},\frac{3}{5}$

Only acceptable value is $r=\frac{1}{3}$, because

$r\in\left(0,\frac{1}{2}\right]$

$\therefore$ c.d $=7b-3a=7ar-3a=\frac{7}{3}a-3a=-\frac{2}{3}a$

$\therefore 4^{th}\text{ term}=15c-\frac{2}{3}a=\frac{15}{9}a-\frac{2}{3}a=a$

25. (*a*) $T_n=\frac{\left(3+(n-1)\times2\left(1^3+2^3+\ldots+n^3\right)\right)}{\left(1^2+2^2+\ldots+n^2\right)}$

$=\frac{3}{2}n(n+1)=\frac{n(n+1)(n+2)-(n-1)n(n+1)}{2}$

$\Rightarrow S_n=\frac{n(n+1)(n+2)}{2}$

$\Rightarrow S_{10}=660$

$\therefore$ As $G_2^2 = 243 \times 3$

$G_2 = \sqrt{243 \times 3} = 27$

Now, $A_4 = 3 + 4d = 3 + \dfrac{960}{(m+1)}$

As $A_5 = G_2$

$24 = 3 + \dfrac{960}{(m+1)} \quad \therefore m = 39$

46. (*d*) $S_n = 20 + 19\frac{3}{5} + 19\frac{1}{5} + 18\frac{4}{5} + \ldots\ldots$

$\because S_n = 488$

$\dfrac{n}{2}\left\{40 + (n-1).\left(-\dfrac{2}{5}\right)\right\} = 488$

$n\left\{20 - \dfrac{n-1}{5}\right\} = 488$

$n^2 - 101n + 2440 = 0$

$\therefore (n-40)(n-61) = 0$

For negative term $n = 61$

$\therefore$ n^{th} term $= T_{61} = 20 + 60.\left(-\dfrac{2}{5}\right) = -4$

47. (*d*) $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)}$

As sum of GP upt o infinity $= \dfrac{a}{1-r}$

$\therefore \dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^2} + \ldots\ldots\infty = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \dfrac{1}{2}$

$\therefore (0.16)^{\log_{2.5}\left(\frac{1}{2}\right)}$

As $0.16 = \dfrac{16}{100} = \left(\dfrac{4}{10}\right)^2 = \left(\dfrac{10}{4}\right)^{-2} = (2.5)^{-2}$

$\therefore (2.5)^{-2\log_{2.5}\left(\frac{1}{2}\right)} = \left(\dfrac{1}{2}\right)^{-2} = 4$

48. (*a*) Here $a = 3$ and $S_{25} = S_{15}$

$\Rightarrow 2S_{25} = S_{40}$

$\Rightarrow 2 \times \dfrac{25}{2}[6 + 24d] = \dfrac{40}{2}[6 + 39d]$

$\Rightarrow 25[6 + 32d] = 20[6 + 39d]$

$\Rightarrow 150 + 600d = 120 + 780d$

$\Rightarrow 180d = 30$

$\Rightarrow d = \dfrac{1}{6}$

49. (*c*) $a_1 = 1$ and $a_n = 300$ and $d \in Z$

$300 = 1 + (n-1)d$

$\Rightarrow d = \dfrac{299}{(n-1)} = \dfrac{23 \times 13}{(n-1)}$

$\therefore n - 1 = 13$ or 23 (as d is integer)

$\Rightarrow n = 14$ or $24 \Rightarrow n = 24$ and $d = 13$

$a_{20} = 1 + 19.13 = 248$

$S_{20} = 20\dfrac{(248+1)}{2} = 2490$

50. (d) $1 + (1 - 2^2.1) + (1 - 4^2.3) + (1 - 6^2.5) + \ldots\ldots(1 - 20^2.19)$

$S = 1 + \sum_{r=1}^{10}\left[1 - (2r)^2(2r-1)\right]$

$= 1 + \sum_{r=1}^{10}\left(1 - 8r^3 + 4r^2\right) = 1 + 10 - \sum_{r=1}^{10}\left(8r^3 - 4r^2\right)$

$= 11 - 8\left(\dfrac{10 \times 11}{2}\right)^2 + 4 \times \left(\dfrac{10 \times 11 \times 2}{6}\right)$

$= 11 - 2 \times (110)^2 + 4 \times 55 \times 7$

$= 11 - 220(110 - 7)$

$= 11 - 220 \times 103 = \alpha - 220\beta \Rightarrow \alpha = 11$

$\beta = 103$

$\Rightarrow (11, 103)$

51. (*a*) $S = \log_7 x^2 + \log_7 x^3 + \log_7 x^4 + \ldots\ldots$ 20 terms

$\Rightarrow \log_7 x\,(x^2 \cdot x^3 \cdot x^4 \ldots\ldots x^{21}) = 460$ given

$\Rightarrow \log_7 x^{(2+3+4\ldots21)} = 460$

$\Rightarrow (2 + 3 + 4 + \ldots. + 21)\log_7 x = 460$

$\Rightarrow \dfrac{20}{2}(2 + 21)\log_7 x = 460$

$\log_7 x = \dfrac{460}{230} = 2 \qquad \Rightarrow x = 7^2 = 49$

52. (*d*) Let the first term be 'a' and common ratio be 'r'

$\because ar(1 + r + r^2) = 3$ ...(i)

and $ar^5(1 + r + r^2) = 243$ ...(ii)

From (i) and (ii),

$r^4 = 81 \Rightarrow r = 3$ and $a = \dfrac{1}{13}$

$S_{50} = \dfrac{a\left(r^{50} - 1\right)}{r - 1} = \dfrac{3^{50} - 1}{26}$

53. (*d*) LHS is G.P. of common ratio $\dfrac{3}{2}$

$\therefore \dfrac{2^{10}\left(1 - \left(\frac{3}{2}\right)^{11}\right)}{\left(1 - \frac{3}{2}\right)} = S - 2^{11}$

$\Rightarrow 2^{10}\dfrac{\left(\frac{3^{11} - 2^{11}}{2^{11}}\right)}{\frac{1}{2}} = S - 2^{11} \quad \Rightarrow S = 3^{11}$

54. (*d*) Given $3^{2\sin 2\alpha - 1}$, 14, $3^{4 - 2\sin 2\alpha}$ are in A.P.

So $3^{2\sin 2\alpha - 1} + 3^{4 - 2\sin 2\alpha} = 28$

$$\Rightarrow \frac{3^{2\sin 2\alpha}}{3} + \frac{81}{3^{2\sin 2\alpha}} = 28$$

$$\Rightarrow \frac{t}{3} + \frac{81}{t} = 28 \{\text{Put } 3^{2\sin 2\alpha} = t\}$$

$\Rightarrow t^2 - 84t + 243 = 0 \Rightarrow t = 81, t = 3$

$\Rightarrow$ When $t = 81$, when $t = 3$

$\Rightarrow \sin 2\alpha = 2$ (Not possible) $2\sin 2\alpha = 1$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \frac{\pi}{6}$$

$$\alpha = \frac{\pi}{12}$$

So, I term $a = 3^0 = 1$, $d = 14 - 1 = 13$

Now, $T_6 = a + 5d = 1 + 65 = 66$

55. (05.00) $\because f(x + y) = f(x).f(y) \quad \forall x, \in R \quad f(1) = 3.$

$\Rightarrow f(x) = 3^x \Rightarrow f(i) = 3^i$

$$\Rightarrow \sum_{i=1}^{n} f(i) = 363$$

$$\Rightarrow 3 + 3^2 + 3^3 + ... + 3^n = 363$$

$$\frac{3(3^n - 1)}{3 - 1} = 363$$

$$3^n - 1 = \frac{363 \times 2}{3} = 242$$

$3^n = 243 = 3^5$

$n = 5$

56. (*b*) Let common difference of series

$a_1, a_2, a_3, ..., a_n$ be d.

$\because a_{40} = a_1 + 39d = -159$ ...(i)

and $a_{100} = a_1 + 99d = -399$ ...(ii)

From eqn. (ii) and (i)

$d = -4$ and $a_1 = -3$

Common difference of $b_1, b_2, b_3, ...$ be (–2).

$\therefore b_{100} = a_{70}$

$\therefore b_1 + 99(-2) = (-3) + 69(-4)$

$\therefore b_1 = 198 - 279$

$\therefore b_1 = -81$

57. (*d*) $(a^2p^2 + 2abp + b^2) + (b^2p^2 + 2bcp + c^2) + (c^2p^2 + 2cdp + d^2) = 0$

$\Rightarrow (ap + b)^2 + (bp + c)^2 + (cp + d)^2 = 0$

$\therefore ap + b = bp + c = cp + d = 0$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \therefore a, b, c, d \text{ are in G.P.}$$

58. (*c*) $x = 1 + \cos^2\theta + ... \infty$

$$x = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta} .....(1)$$

$y = 1 + \sin^2\phi + ... \infty$

$$y = \frac{1}{1 - \sin^2\phi} = \frac{1}{\cos^2\phi} .....(2)$$

$$z = \frac{1}{1 - \cos^2\theta \sin^2\phi} = \frac{1}{1 - \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)}$$

$$= \frac{xy}{xy - (x-1)(y-1)}$$

$\Rightarrow xz + yz - z = xy$

$\Rightarrow xy + z = (x + y)z$

59. (*b*) $S = 1 + \frac{2}{3} + \frac{7}{3^2} + \frac{12}{3^3} + \frac{17}{3^4} + ...$

$$\frac{S}{3} = \frac{1}{3} + \frac{2}{3^2} + \frac{7}{3^3} + \frac{12}{3^4} + ...$$

$$2S = 1 + \frac{1}{3} + \frac{5}{3^2} + \frac{5}{3^3} + \frac{5}{3^4} + ... + \text{ up to infinite terms}$$

$$\frac{2S}{3} = \frac{4}{3} + \frac{5}{3}\left\{\frac{1/3}{1 - \frac{1}{3}}\right\} = \frac{5}{6} + \frac{4}{3} = \frac{13}{6}$$

$$\Rightarrow S = \frac{13}{4}$$

60. (*d*) $a, ar, ar^2, ...$

$$T_2 + T_6 = \frac{25}{2} \Rightarrow ar\left(1 + r^4\right) = \frac{25}{2}$$

$$a^2r^2\left(1 + r^4\right)^2 = \frac{625}{4} ....(1)$$

$$T_3 \cdot T_5 = 25 \Rightarrow \left(ar^2\right)\left(ar^4\right) = 25$$

$a^2r^6 = 25$.....(2) 0n dividing (1) by (2)

$$\frac{\left(1 + r^4\right)^2}{r^4} = \frac{25}{4} \quad 4r^8 - 14r^4 + 4 = 0$$

$$\left(4r^4 - 1\right)\left(r^4 - 4\right) = 0$$

$r^4 = \frac{1}{4}, 4 \Rightarrow r^4 = 4$(an increasing geometric series

$a^2r^6 = 25 \Rightarrow \left(ar^3\right)^2 = 25$

$T_4 + T_6 + T_8 = ar^3 + ar^5 + ar^7$

$= ar^3\left(1 + r^2 + r^4\right) = 5(1 + 2 + 4) = 35$

61. (*b*) $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$

$Put $2n+1=r$, where $r = 3,5,7,.....$

$$\Rightarrow n = \frac{r-1}{2}$$

$$\frac{n^2+6n+10}{(2n+1)!} = \frac{\left(\frac{r-1}{2}\right)^2+3r-3+10}{r!}$$

$$= \frac{r^2+10r+29}{4r!} = \frac{r(r-1)+11r+29}{4r!}$$

Now, $\sum_{r=3,5,7} \frac{r(r-1)+11r+29}{4r!}$

$$= \frac{1}{4}\sum_{r=3,5,7,.....}\left(\frac{1}{(r-2)!}+\frac{11}{(r-1)!}+\frac{29}{r!}\right)$$

$$= \frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.....\right)+11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+.....\right)+29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+.....\right)\right\}$$

$$= \frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2}+11\left(\frac{e+\frac{1}{e}-2}{2}\right)+29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$$

$$= \frac{1}{8}\left\{e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right\}$$

$$= \frac{1}{8}\left\{41e-\frac{19}{e}-80\right\}$$

62. (*b*) RHS $= \sum_{r=0}^{99}(100-r)(100+r)$

$$= (100)^3 - \frac{99\times100\times199}{6} = (100)^3 - (1650)199$$

LHS $= (100)^{\alpha} - (199)^{\beta}$

So, $\alpha = 3, \beta = 1650$

Slope $= \tan\theta = \frac{\beta}{\alpha}$

$\Rightarrow \tan\theta = 550$

63. (*b*) $S = \sum_{r=1}^{100}\frac{1}{(2n+1)^2-1} = \sum_{r=1}^{100}\frac{1}{(2n+1+1)(2n+1-1)}$

$$= \sum_{r=1}^{100}\frac{1}{2n(2n+2)} = \frac{1}{4}\sum_{r=1}^{100}\frac{1}{n(n+1)}$$

$$= \frac{1}{4}\sum_{r=1}^{100}\frac{(n+1)-n}{n(n+1)}$$

$$= \frac{1}{4}\sum_{r=1}^{100}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$S = \frac{1}{4}\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+.....+\left(\frac{1}{100}-\frac{1}{101}\right)\right)$$

$$\therefore S = \frac{1}{4}\left[\frac{100}{101}\right] = \frac{25}{101}$$

JEE-ADVANCED PREVIOUS YEARS

1. (*d*) Corresponding A.P.

$\frac{1}{5}, ... \frac{1}{25}$ (20th term)

$$\frac{1}{25} = \frac{1}{5} + 19\,d$$

$$\Rightarrow d = \frac{1}{19}\left(\frac{-4}{25}\right) = -\frac{4}{19\times25}$$

$a_n < 0$

$$\frac{1}{5} - \frac{4}{19\times25}\times(n-1) < 0 \Rightarrow \frac{19\times5}{4} < n-1$$

$$\Rightarrow \quad n > 24.75$$

2. (*a,d*) $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$

$$= -1^2-2^2+3^2+4^2-5^2-6^2+7^2+8^2+.....$$
$$= (3^2-1)+(4^2-2^2)+(7^2-5^2)+(8^2-6^2).....$$
$$= 2\underbrace{[4+6+12+14+20+22+.....]}_{2n\ terms}$$
$$= 2[(4+12+20.....)+(6+14+22.....)]$$

n terms n terms

$$= 2\left[\frac{n}{2}(4\times2+(n-1)8)\frac{n}{2}(2\times6+(n+1)8)\right]$$
$$= 2[n(4+4n-4)+n(6+4n-4)]$$
$$= 2(4n^2+(4n+2)n) = 2(8n^2+2n) = 4n(4n+1)$$

(a) $1056 = 32 \times 33$ $\quad n = 8$

(b) $1088 = 32 \times 34$

(c) $1120 = 32 \times 35$

(d) $1332 = 36 \times 37$ $\quad n = 9$

3. (5) Numbers removed are k and k + 1.

$$\text{now } \frac{n(n+1)}{2} - k - (k+1) = 1224$$

$\Rightarrow n^2 + n - 4k = 2450 \Rightarrow n^2 + n - 2450 = 4k$

$\Rightarrow (n+50)(n-49) = 4k \Rightarrow n > 49$

Alternative

$\therefore$ To satisfy this equation n should be of the form of (4p + 1) or (4p + 2) taking $n = 50$

$\Rightarrow 4k = 100$

$\Rightarrow k = 25 \quad \therefore k - 20 = 5$

Now if we take $n = 53$

$\Rightarrow k = 103 \Rightarrow n < k$

so not possible. Hence $n \geq 53$ will not be possible.

4. (d) Let $b = ar, c = ar^2 \Rightarrow r$ is Integers. Also $\frac{a + ar + ar^2}{3}$

$= ar + 2 \Rightarrow a + ar^2 = 2ar + 6$

$\Rightarrow a(r-1)^2 = 6 \Rightarrow r$ must be 2 and $a = 6$. Thus

$$\frac{a^2 + a - 14}{a+1} = \frac{36 + 6 - 14}{7} = 4 \text{ Ans.}$$

5. (9) $\frac{S_7}{S_{11}} = \frac{6}{11}$

$$\frac{\frac{7}{2}[2a + 6d]}{\frac{11}{2}[2a + 10d]} = \frac{6}{11} \text{ Given } 130 < a + 6d < 140$$

$$\frac{7(a+3d)}{11(a+5d)} = \frac{6}{11}$$

$7a + 21d = 6a + 30d \Rightarrow 130 < 15d < 140$

$a = 9d$ Hence $d = 9$

$a = 81$ Hence $d = 9$

Alternative :

Let the AP be a, a + d, a + 2d, ...

where $a, d \in N$

Given $\frac{S_7}{S_{11}} = \frac{6}{11}$ and $130 < a + 6d < 140$..... (2)

$$\Rightarrow \frac{\frac{7}{2}\{2a + 6d\}}{\frac{11}{2}\{2a + 10d\}} = \frac{6}{11}$$

$$\Rightarrow \frac{14a + 42d}{22a + 110d} = \frac{6}{11}$$

$\Rightarrow 154a + 462d = 132a + 660d$

$\Rightarrow 22a = 198d$

$\Rightarrow a = \frac{99d}{11} = 9d$

(2) $\Rightarrow \therefore 130 < 9d + 6d < 140$

$\Rightarrow 8.6 < d < 9.3$

$\therefore d = 9$

6. (b) $\log_e b_1, \log_e b_2, ..., \log_e b_{101}$ are in A.P

$\Rightarrow b_1, b_2, b_3 ..... b_{101}$ are in G.P.

common difference of A.P. is $\log_e 2$,

$\therefore$ common ratio of G.P. is 2

$a_1, a_2 ..... a_{101}$ are in A.P

$a_1 = b_1 = A (Let)$

$$Now, t = \frac{b_1(2^{51} - 1)}{2 - 1} = b_1(2^{51} - 1) = A(2^{51} - 1)$$

$$s = \frac{51}{2}(2a_1 + (51-1)d) = \frac{51}{2}(2A + 50d)$$

$$Now, a_{51} = b_{51} \Rightarrow A + 50d = A2^{50} \Rightarrow 50d = A(2^{50} - 1)$$

$$\therefore s = \frac{51}{2}\left[2A + A(2^{50} - 1)\right]$$

$$\Rightarrow s = \frac{51}{2}A\left[2^{50} + 1\right]$$

$$s = A\left[51.2^{49} + \frac{51}{2}\right] = A\left[4.2^{49} + 47.2^{49} + \frac{51}{2}\right]$$

$$= A\left[2^{51} - 1 + 47.2^{49} + \frac{51}{2} + 1\right]$$

$$= A(2^{51} - 1) + A\left(47.2^{49} + \frac{53}{2}\right)$$

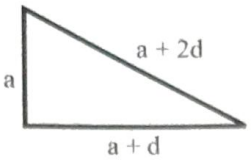
$$\Rightarrow s = t + A\left(47.2^{49} + \frac{53}{2}\right) \Rightarrow s > t$$

$$Now, a_{101} = A + 100d = A + 2A(2^{50} - 1) = A(2^{51} - 1)$$

$b_{101} = A.2^{100}$

$Clearly\ b_{101} > a_{101}$

7. (6)



$$\frac{1}{2}a(a+d)=24 \Rightarrow a(a+d)=48 \quad \ldots\ldots\ldots(1)$$

$$a^2+(a+d)^2=(a+2d)^2 \Rightarrow 3d^2+2ad-a^2=0$$

$$(3d-a)(a+d)=0$$

$$\Rightarrow 3d=a\ (\because a+d\neq 0)$$

$$\Rightarrow d=2$$

$$a=6$$

so smallest side = 6

8. (157.00) We equate the general terms of three respective A.P.'s as $1+3a=2+5b=3+7c$

$\Rightarrow$ 3 divides $1+2b$ and 5 divides $1+2c$

$\Rightarrow 1+2c=5, 15, 25$ etc.

So, first such terms are possible when $1+2c=15$ i.e. $c=7$

Hence, first term $=a=52$

$d=\text{LCM}(3,5,7)=105$

$\Rightarrow a+d=157$

9. (8.00) $\dfrac{3^{y_1}+3^{y_2}+3^{y_3}}{3}\geq\left[3^{(y_1+y_2+y_3)}\right]^{\frac{1}{3}}$

$$\Rightarrow 3^{y_1}+3^{y_2}+3^{y_3}\geq 3^4$$

$$\Rightarrow \log_3\left(3^{y_1}+3^{y_2}+3^{y_3}\right)\geq 4$$

$$\Rightarrow m=4$$

Also, $\dfrac{x_1+x_2+x_3}{3}\geq\sqrt[3]{x_1x_2x_3}$

$$\Rightarrow x_1x_2x_3\leq 27$$

$$\Rightarrow \log_3 x_1+\log_3 x_2+\log_3 x_3\leq 3$$

$$\Rightarrow M=3$$

Thus, $\log_2(m^3)+\log_3(M^2)=6+2$

$=8$

10. (1.00) Given $2(a_1+a_2+\ldots a_n)=b_1+b_2+\ldots b_n$

$$\Rightarrow 2\times\frac{n}{2}\left(2c+(n-2)x_2\right)=c\left(\frac{2^n-1}{2-1}\right)$$

$$\Rightarrow 2n^2-2n=c(2^n-1-2n)$$

$$\Rightarrow c=\frac{2n^2-2n}{2^n-1-2n}\in\mathbb{N}$$

So, $2n^2-2n\geq 2^n-1-2n$

$$\Rightarrow 2n^2+1\geq 2^n \quad \Rightarrow n<7$$

$\Rightarrow$ n can be 1, 2, 3....,

Checking c against these values of n we get c = 12 (when n = 3)

Hence number of such c = 1

11. (a, b)

$$S_n(x)=\sum_{k=1}^{n}\tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$$

$$=\sum_{k=1}^{n}\tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$$

$$S_n(x)=\tan^{-1}(nx+x)-\tan^{-1}x=\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

(a) $S_n(x)=\tan^{-1}(nx+x)-\tan^{-1}x$

$$=\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

(b) $\lim\limits_{n\to\infty}\cot\left(S_n(x)\right)=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}{x}=x(x>0)$

(c) $S_3(x)=\tan^{-1}\dfrac{3x}{1+4x^2}=\dfrac{\pi}{4}\Rightarrow 4x^2-3x+1=0$

$\Rightarrow x\notin\mathbb{R}$

(d) $\tan\left(S_n(x)\right)=\dfrac{nx}{1+(n+1)x^2};\forall n\geq 1;x>0$

We need to check the validity of

$$\frac{nx}{1+(n+1)x^2}\leq\frac{1}{2}\ \forall n\geq 1;x>0;n\in\mathbb{N}$$

$$\Rightarrow 2nx\leq(n+1)x^2+1$$

$$\Rightarrow (n+1)x^2-2nx+1\geq 0\forall n\geq 1;x>0;n\in\mathbb{N}$$

Discriminant of $y=(n+1)x^2-2nx+1$ is

$D=4n^2-4(n+1)$ and $n\in\mathbb{N}$

$D<0$ for $n=1$; true for $x>0$

$D>0$ for $n\geq 2\Rightarrow\exists$ some $x>0$

for which $y<0$ as both roots of $y=0$ will be positive.

$y=(n+1)x^2-2nx+1, n\geq 2$

So, $y\geq 0\forall n\geq 1;\forall x>0;n\in N$ is false.

Paragraph

Let $M = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \le r^2\right\}$

Where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}, n = 1, 2, 3, \ldots$. Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

12. (*d*) $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$

Centre of C_n is $\left(2 - \frac{1}{2^{n-2}}, 0\right)$

and radius of C_n is $\frac{1}{2^{n-1}}$ when $r = \frac{1025}{513} < 2$

C_n will lie inside m when $2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < \frac{1025}{513}$

$\Rightarrow \quad k = 10$ Also $l = 5$

$3k + 2l = 30 + 10 = 40$

13. (*b*) Center of D_n is (S_{n-1}, S_{n-1}) $r = \frac{1}{2^{n-1}}$

D_n will lie inside when $\sqrt{2}(S_{n-1}) < \frac{2^{199} - 1}{2^{198}}\sqrt{2}$

$\Rightarrow \frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}} + \frac{1}{2^{n-1}} \Rightarrow n = 199$

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SAHEEL

5 PERMUTATIONS AND COMBINATIONS

FUNDAMENTAL PRINCIPLE OF COUNTING ADDITIONAL PRINCIPLE

If a work can be done in m different ways and another work which is independent of first can be done in n different ways, then either of the two operations can be preformed in $(m + n)$ ways.

Illustrating the Concepts

There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city?

Number of ways to enter the city from north = 15
Number of ways to enter the city from east = 10
A person can enter the city from northor from east.
Hence, the number of ways to enter the city = 15 + 10 = 25

Multiplication Principle

If one operation (I) can be done in m different ways and another Operation (II) can be performed in n different ways, then total number of ways in which both of these can be performed together is $m \times n$. If there are more than two operations to be done, then the total number of different ways to do all of them together will be $m \times n \times p \times \ldots$

For example: Operation A can be performed in "a, b, c" different ways and operation B can be perfomred in p, q different ways, then possible ways to perform operation A and operation B together are a-p, a-q, b-p, b-q, c-p, c-q i.e., each way of operation A is combined by 2 ways operation B.

i.e., Total ways = 2 + 2 + 2 = 2 added 3 times = 2 × 3 = 6.

In general, total ways = $m \times n$ where m and n are individual ways of performing the operations.

TRAIN YOUR BRAIN

Example 1: How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(*i*) No digit is repeated in any number.

(*ii*) Digits can be repeated.

Sol. (*i*) Number of two digit numbers = 5 × 4 = 20
Number of three digit numbers = 5 × 4 × 3 = 60
Number of four digit numbers = 5 × 4 × 3 × 2 = 120
Total = 200

(*ii*) Number of two digit numbers = 5 × 5 = 25
Number of three digit numbers = 5 × 5 × 5 = 125
Number of four digit numbers = 5 × 5 × 5 × 5 = 625
Total = 775

NOTATION OF FACTORIAL

Notation of factorial and its algebra:

The continued product of first n, natural number is called as "n factorial" and dentoed by $n!$ or $\angle n$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots (n-1)n!$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$3! = 1 \cdot 2 \cdot 3 = 6$
$5! = 120;\ 61 = 720;\ 7! = 5040$

Special Results:

- $0! = 1$ i.e. factorial of zero is 1
- Factorial of negative number is underfined
- $(2n!) = 2^n \cdot n! [1 \cdot 3 \cdot 5 \ldots (2n-1)]$

PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a permutation. It is assumed that

- the given things let us say there are n of them are all distinct, that is, no two are alike.
- the arrangement is placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement, any one thing is used only once. In other words, there is no repetition.

Counting Formulae for Permutation

The meaning of ${}^nP_r$

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ${}^nP_r$. It must be noted that each space has to be filled up with only one letter.

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1)$$

$$= \frac{n!}{(n-r)!} \text{ (using factorial notation)}$$

$n! = n(n-1)\ldots.3.2.1.$ where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time $= {}^nP_n = n!$
- ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$
- ${}^nP_r = n({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots\ldots$ n.

Important Results

Number of permutations under certain conditions:

- Number of permutations of *n* different things, taken *r* at a time, when a particular thing is to be always included in each arrangement is $r.{}^{(n-1)}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m!)(n-m+1)!$.
- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!(n-m+1)!$

Example 2: If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the word QUEST.

Sol. Number of words beginning with $E = {}^4P_4 = 24$

Number of words beginning with $QE = {}^3P_3 = 6$

Number of words beginning with $QS = 6$

Number of words beginning with $QT = 6$.

Next word is 'QUEST'

$\therefore$ its rank is $24 + 6 + 6 + 6 + 1 = 43$.

PERMUTATION OF *n* DISTINCT OBJECT WHEN REPETITION IS ALLOWED

If n distinct things are arrangement at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times \ldots. \times n}_{r \text{ times}} = n^r$.

TRAIN YOUR BRAIN

Example 3: There are m men and n monkeys ($n > m$). If a man can have any number of monkeys, in how many ways may every monkey have a master?

Sol. The first monkey can select his master by m ways, and after that the second monkey can select his master again by m ways, so can the third. And so on, hence all monkeys can select master by $= m \times m \times m \ldots.$ up to n times $= (m)^n$ ways.

Arrangement of *n* things when all are not distinct

- If given *n* things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n - (p + q + r)$ are all distinct is given by

$$\frac{n!}{p!q!r!} \quad (p+q+r \le n)$$

TRAIN YOUR BRAIN

Example 4: Consider word ASSASSINATION, first number of ways of arranging the letters.

(*i*) Number of words using all.

Sol. (*i*) ASSASSINATION contains four *S*, three *A*, two *N* and two *I*.

$$\text{Total ways} = \frac{13!}{(4!)(3!)(2!)(2!)}$$

Exponent of Prime p IN n!

Since, exponent of prime p in n! is denoted by $E_p(n!)$, where p is a prime number and n is a natural number. Then the last integer amongst 1, 2, 3, ..., (n – 1), n which is divisible by p is [n/p] p, where [n/p] denotes the greatest integer. ($\because [x] \le x$)

$$E_p(n!) = E_p(1, 2, 3 \ldots..(n-1).n)$$
$$= E_p(p.2p.3p.\ldots..(n-1)p.[n/p]p)$$

Because the remaining natural numbers from 1 to n are not divisible by p.

Thus, $\left[\frac{n}{p}\right] + E_p\left(1.2.3...\left[\frac{n}{p}\right]\right)$

Now the last integer among 1, 2, 3, ..., $\left[\frac{n}{p}\right]$ which is divisible by *p* is

$$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p.2p.3...\left[\frac{n}{p^2}\right]p\right)$$

Because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by p.

Thus, $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1.2.3...\left[\frac{n}{p^2}\right]\right)$

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots. + \left[\frac{n}{p^s}\right]$$

Where *s* is the largest natural number such that $p^s \le n < p^{s+1}$

Note: $E_{rp}(n!) = E_r(n!)$ (If $r > p$)

Where r and p are prime numbers.

TRAIN YOUR BRAIN

Example 5: Find the number of zeros at the end of 100 ! .

Sol. In terms of prime factors 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$.

Now

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and $E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$

$\therefore \quad 100! = 2^{97} \cdot 3^b \cdot 5^{24} \cdot 7^d \ldots\ldots$

$= 2^{73} \cdot 3^b \cdot (2 \times 5)^{24} \cdot 7^d \ldots$.

$= 2^{73} \cdot 3^b \cdot (10)^{24} \cdot 7^d \ldots$.

Hence, number of zeros at the end of 100 ! is 24.

Exponent of 10 in 100 ! = min (97, 24) = 24.

COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called combinations.

Counting Formulae for Combinations

The number of combinations (selections or groupings) that can be formed from n different objects taken r at a time is denoted by ${}^nC_r$ and its value is equal to

$${}^nC_r = \frac{n!}{(n-r)!r!} \; (0 \le r \le n)$$

as $${}^nC_r = \frac{{}^nP_r}{r!}$$

as in a permutation the arrangement of r selected objects out of n, is done in r! ways and in combination arrangement in a group is not considered.

In particular

- ${}^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or to select all objects out of n distinct objects.
- ${}^nC_1 = n$ There are n ways to select one thing out of n distinct things.
- ${}^nC_r = {}^nC_{n-r}$
  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

Important Results of Combinations (Selections)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.

Some Important Results About Points:

If there are n points in a plane which m ($< n$) are collinear, then

(*a*) The number of different straight lines obtained by joining these n points is
${}^nC_2 - {}^mC_2 + 1$

(*b*) Total number of different triangles formed by joining these n points is
${}^nC_3 - {}^mC_3$

(*c*) Number of diagonals in polygon of n sides is

${}^nC_2 - n$ i.e. $\frac{n(n-3)}{2}$

(*d*) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is

${}^mC_2 \times {}^nC_2$ i.e. $\frac{mn\,(m-1)\,(n-1)}{4}$

(*e*) Number of triangles formed by joining vertices of convex polygon of n sides is ${}^nC_3$ of which

(*i*) Number of triangles having exactly two sides common to the polygon = n

(*ii*) Number of triangles having exactly one side common to the polygon = $n(n-4)$

(*iii*) Number of triangles having no side common to the polygon = $\frac{n(n-4)\,(n-5)}{6}$

TRAIN YOUR BRAIN

Example 6: Fifteen players are there for selection for a cricket match.

(*i*) In how many ways the playing 11 can be selected?

(*ii*) In how many ways the playing 11 can be selected including a particular player?

(*iii*) In how many ways the playing 11 can be selected excluding two particular players?

Sol. (*i*) 11 players are to be selected from 15
Number of ways = ${}^{15}C_{11} = 1365$.

(*ii*) Since one player is already included, we have to select 10 from the remaining 14
Number of ways = $^{14}C_{10} = 1001$.

(*iii*) Since two players are to be excluded, we have to select 11 from the remaining 13.
Number of ways = $^{13}C_{11} = 78$.

Example 7: In how many ways we can select 4 letters from the letters of the word MISSISSIPPI?

Sol. MISSISSIPPI → M, IIII, SSSS, PP
Number of ways of selecting 4 alike letters = $^2C_1 = 2$.
Number of ways of selecting 3 alike and 1 different letters = $^2C_1 \times {}^3C_1 = 6$
Number of ways of selecting 2 alike and 2 alike letters = $^3C_2 = 3$
Number of ways of selecting 2 alike & 2 different = $^3C_1 \times {}^3C_2 = 9$
Number of ways of selecting 4 different = $^4C_4 = 1$
Total number of ways = 2 + 6 + 3 + 9 + 1 = 21

DIVISION OF DISTINCT OBJECT INTO GROUPS

In the case of grouping we have the following. If m + n + p things are divided into 3 groups one containing m, the second n and the third p things; number of groupings is

$$^{(m+n+p)}C_m \cdot {}^{(n+p)}C_n \cdot {}^pC_p$$

$$= \frac{(m+n+p)!}{m!n!p!}$$ where m, n, p are distinct natural numbers.

If m = n = p (say) then the number of groupings (unmindful of the order of grouping) is

$$\frac{3m!}{(m!)^3 3!}$$

Thus if 52 cards be divided into four groups of 13 each, the number of groupings is

$$\frac{52!}{(13!)^4.4!}$$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is

$$\frac{52!}{(13!)^4}$$

As another example, if we consider the division of 52 cards into four groups, three groups containing each 16 and the fourth cards, the number of ways in which this can be done is

$$\frac{52!}{3!(16!)^3 4!}$$

Note the 3 ! factor in the denominator. This is for the reason that there are only 3 equal groups.

In general, the number of ways in which *mn* different things can be divided equally into *m* distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

Whereas when order of groups is not important, then division into m equal group is done in $\frac{(mn)!}{m!(n!)^m}$ ways.

TRAIN YOUR BRAIN

Example 8: In how many ways 10 persons can be divided into 5 pairs?

Sol. We have each group having 2 persons and the qualitative characteristic are same (Since there is no purpose mentioned or names for each pair).

Thus the number of ways = $\frac{10!}{(2!)^5 5!} = 945$.

CIRCULAR PERMUTATIONS

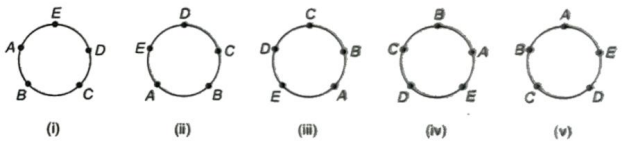
In the event of the given n things arranged in a circular or even elliptical permutation the first and the last thing in the arrangement are indistinguishable.

Arrangements Around a Circular Table

Consider five persons A, B, C, D, E be seated on the circumference of a circular table in order which has no head now, shifting A, B, C, D, E one position in anticlockwise direction we will get arrangements as shown in following figure:

We observe that arrangements in all figures are different.

Thus, the number of circular permutations of n different things taken all at a time is (n – 1)!, if clockwise and anticlockwise orders are taken as different.



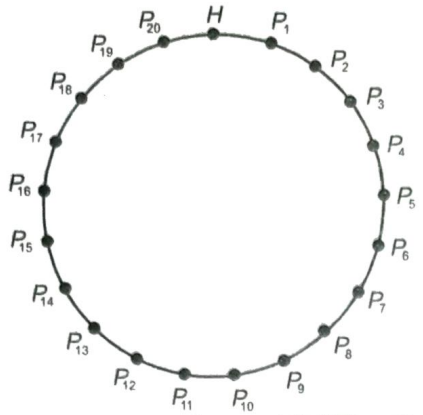
TRAIN YOUR BRAIN

Example 9: 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?

Sol. 1st part: Total persons on the circular table = 20 guest + 1 host = 21 they can be seated in (21 – 1)! = 20! ways.

2nd part: After fixing the places of three persons (1 host + 2 persons). Treating (1 host + 2 person) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be (19 – 1)! = 18! also these two particular person can be seated on either side of the host in 2! ways.

Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host = 18! × 2! = 2 × 18!

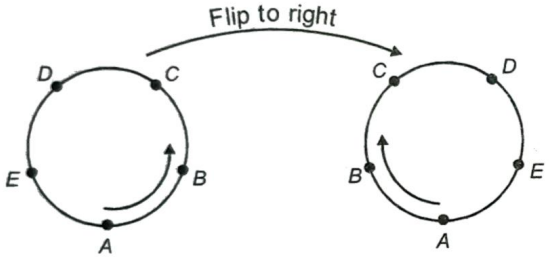


Arrangements of beads or flowers (all different) around a circular necklace or garland

Consider five beads A, B, C, D, E in a necklace or five flowers A, B, C, D, E in a garland etc. If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangement is not different we will get arrangements as follows:

we see that arrangements in above figures are not different.

Then the number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.



TRAIN YOUR BRAIN

Example 10: Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls. Treating 3 specific pearls = 1 units, so we have now 18 pearls + 1 unit = 19 and the number of arrangement will be (19 – 1)! = 18! also, the number of ways of 3 pearls can be arranged between themselves is 3! = 6. Since, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of

arrangements $= \frac{1}{2}\ 18!\ .\ 6 = 3\ (18!)$.

Number of circular permutations of *n* different things taken *r* at a time

CASE I: If clockwise and anticlockwise orders are taken as different, then the required number of circular

permutations $= \frac{{}^{n}P_{r}}{r}$.

CASE II: If clockwise and anticlockwise orders are taken as not different, then the required number of circular

permutations $= \frac{{}^{n}P_{r}}{2r}$

TRAIN YOUR BRAIN

Example 11: In how many ways can 24 persons be seated round a table, if there are 13 seats?

Sol. In case of circular table the clockwise and anticlockwise order are different, then the required number of circular

permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

Example 12: How many necklace of 12 beads each can be made from 18 beads of various colours?

Sol. In the case of necklace there is not distinction between the clockwise and anticlockwise arrangements. then the required

number of circular permutations $= \frac{{}^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24}$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}$$

TOTAL NUMBER OF SELECTION

Selection from Distinct Objects

The number of ways (or combinations) of selection from n distinct objects, taken at least one of them is

$${}^{n}C_{1} + {}^{n}C_{2} + {}^{n}C_{3} + \ldots\ldots + {}^{n}C_{n} = 2^{n} - 1$$

Logically it can be explained in two ways, as one can be selected in ${}^{n}C_{1}$ ways, two in ${}^{n}C_{2}$ ways and so on ……. and by addition principle of counting the total number of ways of doing either of the job is

$${}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots + {}^{n}C_{n}$$

Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected = $2^n - 1$.

TRAIN YOUR BRAIN

Example 13: Given five different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green, one blue dye?

Sol. Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included = $2^5 - 1 = 31$

(1 is subtracted to correspond to the case when none of the green dyes is chosen.)

A similar argument may be advanced in respect of other two colours also.

Number of combinations = $(2^5 - 1)(2^4 - 1)(2^3)$.

$= 31 \times 15 \times 8 = 3720$

Selection from Identical Objects

1. The number of selections of r ($r \le n$) objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2 \ldots\ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and so on ……..a_n are alike of nth kind, is $(a_1 + 1)(a_2 + 1) \ldots\ldots (a_n + 1) - 1$.
4. The number of ways of selections of atleast one out of $a_1 + a_2 + a_3 + \ldots\ldots + a_n + k$ objects, where a_1 are alike of one kind, ……..a_n are alike of nth kind and k are distinct is $(a_1 + 1)(a_2 + 1).....(a_n + 1)\,2^k - 1$.

TRAIN YOUR BRAIN

Example 14: Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when it is essential to take

(*i*) at least one fruit

(*ii*) one fruit of each kind.

Sol. Here 5 oranges are alike of one kind, 4 mangoes are alike of second kind and 3 bananas are alike of third kind

(*i*) The required number of combinations (when at least one fruit)
$= (5 + 1)(4 + 1)(3 + 1)\,2^0 - 1 = 120 - 1 = 119$

(*ii*) The required number of combinations (when one fruit of each kind)
$= {}^5C_1 \times {}^4C_1 \times {}^3C_1 = 5 \times 4 \times 3 = 60.$

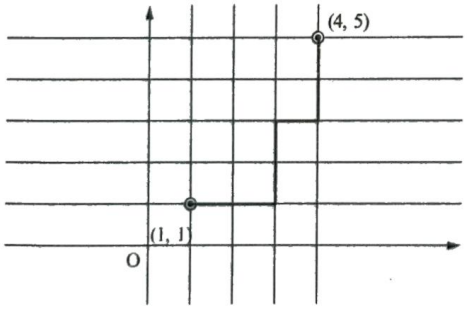
GRID PROBLEM

Complete cartesian plane is partitioned by drawing line $\parallel$ to x and y-axis equidistant apart like the lines on a chees board. Then the

Example 15: Number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Sol. Whatever may be the mode of travel of the ant; it has to traverse 3H (Horizontal) and 4V (Vertical) paths.

Hence required number of ways $= \dfrac{7!}{4!3!} = {}^7C_3$



Note: If there are n vertical and m horizontal lines then there will be $(n-1)$ horizontal and $(m-1)$ vertical paths.

DIVISORS OF A GIVEN NATURAL NUMBER

Let $n \in N$ and $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots\ldots p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_k$ are natural numbers then:

- the total number of divisors of N including 1 and n is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots\ldots (\alpha_k + 1)$
- the total number of divisors of n excluding 1 and n is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots\ldots (\alpha_k + 1) - 2$
- the sum of these divisors is

$= (p_1^0 + p_1^1 + p_1^2 + \ldots\ldots + p_1^{\alpha_1})\ (p_2^0 + p_2^1 + p_2^2 + \ldots\ldots + p_2^{\alpha_2})$

$(p_k^0 + p_k^1 + p_k^2 + \ldots\ldots + p_k^{\alpha k})$

(Use sum of G.P. in each bracket)

- the number of ways in which n can be resolved as a product of two factors is

$\dfrac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) \ldots\ldots (\alpha_k + 1)$, if n is not a perfect square

$\dfrac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) \ldots\ldots (\alpha_k + 1) + 1]$, if n is a perfect square

- The number of ways in which composite number n can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{k-1} where k is the number of different factors (or different primes) in n.

TRAIN YOUR BRAIN

Example 16: If $n = 10800$, then find the

(*a*) total number of divisors of n

(*b*) the number of even divisors

(*c*) the number of divisors of the form $4m + 2$

(*d*) the number of divisors which are multiples of 15

Sol. $n = 10800 = 2^4 \times 3^3 \times 5^2$

Any divisor of n will be of the form $2^a \times 3^b \times 5^c$ where $0 \le a \le 4,\ 0 \le b \le 3,\ 0 \le c \le 2$.

For any distinct choices of a, b and c, we get a divisor of n

(*a*) total number of divisors $= (4+1)(3+1)(2+1) = 60$

(*b*) for a divisor to be even, a should be at least one. So total number of even divisors $= 4(3 + 1)(2 + 1) = 48$.

(*c*) $4m + 2 = 2(2m + 1)$. In any divisor of the form $4m + 2$, a should be exactly 1. So number of divisors of the form $4m + 2 = 1(3 + 1)(2 + 1) = 12$.

(*d*) A divisor of n will be a multiple of 15 if b is at least one and c is at least one. So number of such divisors $= (4+1) \times 3 \times 2 = 30$.

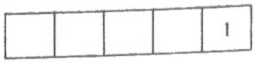
SUMMATION OF NUMBERS (3 DIFFERENT WAYS)

(*a*) Sum of all the numbers greater then 10000 formed by the digits 1, 3, 5, 7, 9 if no digit being repeated.

Method 1: All possible numbers = 5! = 120

If one occupies the units place then total number = 24.

Hence 1 enjoys units place 24 times



Similarly 1 enjoys each place 24 times

Sum due to 1 = $1 \times 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

Similarly

Sum due to the digit 3 = $3 \times 24(1 + 10 + 10^2 + 10^3 + 10^4)$

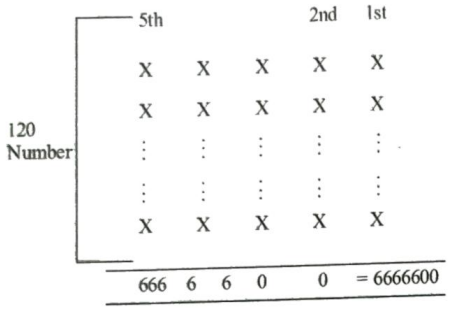
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Required total sum = $24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

$(1 + 3 + 5 + 7 + 9)$

Method 2: In 1st column there are twenty four 1's, Twenty four 3's and so on and their sum is = $24 \times 25 = 600$

Hence add. in vertical column normally we get = 6666600



Method 3: Applicable only if the digits used are such that they have the same common difference. (valid even if the digits are repeating) Writing all the numbers in ascending order of magnitude

S = (13579 + 13597 + ... + 97513 + 97531)

S = (13579 + 99531) + (13597 + 97513) + ...

= (111110) 60 time = 6666600 Ans.

$S = \frac{n}{2}(l + L)$ where n = number of numbers, l = smallest, L = Largest

ARRANGEMENT IN GROUPS

The number of ways of distribution and arrangement of n distinct things into r different groups is $n!\,{}^{n+r-1}C_{r-1}$ or $n!\,{}^{n-1}C_{r-1}$ according as empty groups are allowed or not allowed.

TRAIN YOUR BRAIN

Example 17: In how many ways can three balls of different colours be put in 4 glass cylinders of equal width such that any glass cylinder may have either 0, 1, 2 or 3 balls?

Sol. There are four glass cylinders. Consider additionally (4 – 1) = 3 things; and, the number of ways is ${}^6C_3$ (corresponding to ${}^{(n+r-1)}C_{r-1}$) multiplied by 3! = 120.

METHOD OF INCLUSION EXCLUSION

If $A_1, A_2, \ldots\ldots, A_m$ are finite sets and $A = A_1 \cup A_2 \cup \ldots \cup A_m$ then

$$n(A) = a_1 - a_2 + a_3 - a_4 + \ldots\ldots + (-1)^{m+1} a_m$$

where $a_1 = n(A_1) + n(A_2) + \ldots\ldots + n(A_m)$

$$a_2 = \sum_{1 \le i < j < m} n(A_i \cap A_j)$$

$$a_3 = \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k)$$

and so on.

Corollary (Sieve-Formula)

If $A_1, A_2, \ldots\ldots A_m$ are m subsets of a set A containing N elements, then $n(A_1' \cap A_2' \cap A_m')$

$$= N - \sum_i n(A_i) + \sum_{1 \le i < j \le m} n(A_i \cap A_j)$$

$$- \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k)$$

$$+ \ldots\ldots + (-1)^m n\,(A_1 \cap A_2 \cap \ldots\ldots \cap A_m)$$

DEARRANGEMENTS

It is rearrangement of objects such that no one goes to its original place.

If n things are arranged in a row, the number of ways in which they can be rearranged so that none of them occupies its original placed is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} \ldots\ldots + (-1)^n \frac{1}{n!}\right)$$

It is denoted by $D_n = n!\sum_{r=0}^{n} (-1)^r \frac{1}{r!}$

TRAIN YOUR BRAIN

Example 18: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) at least two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (i) The number of ways in which at least two of them are in the wrong envelopes

$$= \sum_{r=2}^{6} {}^nC_{n-r} D_r$$

$$= {}^nC_{n-2} D_2 + {}^nC_{n-3} D_3 + {}^nC_{n-4} D_4 + {}^nC_{n-5} D_5 + {}^nC_{n-6} D_6$$

Here n = 6

$$= {}^6C_4 .2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 .3i\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$

$$+ {}^6C_2 .4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+{}^6C_1.5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$+{}^6C_0.6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$= 15+40+135+264+265 = 719.$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$$

$= 360-120+30-6+1 = 265.$

TRAIN YOUR BRAIN

Example 19: Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number?

Sol. Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of dearrangements of 4 objects $= 4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 9$

Thus the required number of ways. = 24 – 9 = 15

USE OF MULTIONOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of
  $(1 + x + x^2 + x^3 + ... x^l)\ (1 + x + x^2 + ... + x^m)\ (1+x+x^2+.....+x^n)$
  Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of
  $(x+x^2+x^3+....x^l)(x+x^2+x^3+.....+x^m)(x+x^2+x^3+..... +x^n)$
- If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....+\frac{x^\ell}{\ell!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....+\frac{x^m}{m!}\right)$$

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....+\frac{x^n}{n!}\right).....$$

Note: For use in problems of the above type the following Binomial expansions may be noted

$$\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+...+x^r+...$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1+2x+3x^2+...+(r+1)x^r+...$$

$$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1+3x+6x^2+...+\frac{(r+1)(r+2)}{1.2}x^r+....$$

$$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1+4x+10x^2+...+\frac{(r+1)(r+2)(r+3)}{1.2.3}x^r+.....$$

and more generally $\frac{1}{(1-x)^p} = (1-x)^{-p} = 1+px+...$

$$+\frac{(r+1)(r+2)...(r+\overline{p-1})}{1.2.3...(p-1)}x^r+...$$

and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$

TRAIN YOUR BRAIN

Example 20: Let us consider the more general problem of distributing n identical things given among r persons, each one whom, can receive 0, 1, 2 or more things (≤ n).

Sol. Consider r brackets corresponding to the r persons.

In each bracket take an expression given by $1 + x + x^2 + ... + x^n$ (the various powers of x; namely, 0, 1, 2, ..., n correspond to the number of things each person can have in the distribution)

In the continued product $(1 + x + x^2 + ... + x^n)\ (\ )\ (\ )$ ... repeated r times, collect the coefficient of x^n. This coefficient gives the required number of ways of distribution.

∴ the number of ways

= Coefficient of x^n in $(1+x+x^2+...+x^n)\ (\ )\ (\ )$ ... repeated r times

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^r$$

$= \text{Coefficient of } x^n \text{ in } (1-x^{n+1})^r (1-x)^{-r}$

$= \text{Coefficient of } x^n \text{ in } (1-x)^{-r}$

$$= \frac{(r+1)(r+2)....(r+n-1)}{1.2.....(r-1)} = {}^{(n+r-1)}C_{r-1}$$

An Alternative Method for the General Problem

In this case, we consider additionally (r – 1) things (this number is one less than r, the number of persons). It may be seen that the number of ways of dividing the n things among the r persons as per the condition of the problem is ${}^{(n+r-1)}C_{(r-1)}$.

TRAIN YOUR BRAIN

Example 21: An unlimited number of Red, White, Blue and Green balls are given. Ten balls are drawn. Find the number of ways of selection.

Sol. In this the expression in x to be considered is $1 + x + x^2 + \ldots$ to correspond to the unlimited number available in each colour. There are four such colours.

The required number of ways

$$= \text{Coefficient of } x^{10} \text{ in } [1 + x + x^2 + \ldots]^4$$

$$= \text{Coefficient of } x^{10} \text{ in } \left(\frac{1}{1-x}\right)^4 = \frac{11 \times 12 \times 13}{6} = 286$$

Use of Multinomials in Solving Linear Equation

Multinomials can be used to solve linear equations. The method is illustrated with the help of following illustrations.

***e.g.* Find the number of non negative integral solutions of $x + y + z + w = 20$.**

Any one of the four variables can take from the value zero to the value 20; and hence we construct a polynomial in a variable (say x) with x raised to different powers which would constitute the values that any one variable can take when the equation is solved in the manner indicated. We, thus, consider the product expression,

$(1 + x + x^2 + \ldots + x^{20})\ (1 + x + .. + x^{20})\ (1 + x + \ldots + x^{20})\ (1 + x + \ldots + x^{20})$

four factors since there are four variables. If we take x^4 in the first factor; x^5 in the second; x^8 in the third; then we take the term x^3 in the fourth so that the sum of the powers $(4 + 5 + 8 + 3 = 20)$ is 20. It is then we say that there is a solution corresponding to $x = 4; y = 5; z = 8; w = 3$. Hence the number of solutions in the manner required is the coefficient of x^{20} in

$(1 + x + \ldots + x^{20})^4$

i.e. in $\left(\frac{1-x^{21}}{1-x}\right)^4$

i.e. in $(1 - x^{21})^4 (1 - x)^{-4}$

i.e. in $(1 - x)^{-4}$ and this is $= {}^{23}C_3$

(**Note:** In $(1 - x)^{-4}$ coefficient of x^n is ${}^{(n+3)}C_3$

Method of Fictious Partition:

Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.

TRAIN YOUR BRAIN

Example 22: Find the number of ways in which 11 identical apples can be distributed among 6 children, so that each child receives at least one apple.

Sol. First give one apple to each child. There are remaining 5 apples to be distributed among 6 children (so that each may receive any number of apples not exceeding five).

The number of ways required

$$= {}^{(11-1)}C_{6-1} = {}^{10}C_5 = 252$$

Alternatively, the number of ways of distribution is given by the coefficient of x^{11} in $(x + x^2 + \ldots + x^{11})^6$.

Example 23: Find the number of solutions of the equation $x + y + z = 6$, where $x, y, z \in W$.

Sol. Number of solutions = coefficient of x^6 in $(1 + x + x^2 + \ldots\ldots x^6)^3$

$$= \text{coefficient of } x^6 \text{ in } (1 - x^7)^3 (1 - x)^{-3}$$
$$= \text{coefficient of } x^6 \text{ in } (1 - x)^{-3}$$
$$= {}^{3+6-1}C_6 = {}^8C_2 = 28.$$

Topicwise Questions

Fundamental Principle of Counting

1. The number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number, is

(*a*) 1024 (*b*) 2048

(*c*) 512 (*d*) 64

2. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is:

(*a*) 144 (*b*) 72

(*c*) 288 (*d*) 720

3. Number of positive integers which have no two digits having the same value with sum of their digits being 45, is

(*a*) 10! (*b*) 9!

(*c*) $9 \cdot 9!$ (*d*) $17 \cdot 8!$

4. The number of signals that can be made with 3 flags each of different colour by hoisting 1 or 2 or 3 above the other, is:

(*a*) 3 (*b*) 7

(*c*) 15 (*d*) 16

Permutation and Combination under Conditions

5. Number of cyphers at the end of $^{2002}C_{1001}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 200

6. If all the letters of the word "QUEUE" are arranged in all possible manner as they are in a dictionary, then the rank of the word QUEUE is:

(*a*) 15th (*b*) 16th

(*c*) 17th (*d*) 18th

7. 8 chairs are numbered from 1 to 8. Two women & 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, then the men select the chairs from among the remaining. The number of possible arrangements is:

(*a*) $^6C_3 . {}^4C_4$ (*b*) $^6P_2 . {}^4P_3$

(*c*) $^4C_3 . {}^4P_3$ (*d*) $^4P_2 . {}^6P_3$

8. Number of words that can be made with the letters of the word "GENIUS" if each word neither begins with G nor ends in S, is:

(*a*) 24 (*b*) 240

(*c*) 480 (*d*) 504

9. 5 boys & 3 girls are sitting in a row of 8 seats. Number of ways in which they can be seated so that not all the girls sit side by side, is:

(*a*) 36000 (*b*) 9080

(*c*) 3960 (*d*) 11600

10. 10 different letters of an alphabet are given. Words with 5 letters are formed from these given letters, then the number of words which have atleast one letter repeated is:

(*a*) 69760 (*b*) 30240

(*c*) 99748 (*d*) None

11. In a conference 10 speakers are present. If S_1 wants to speak before S_2 & S_2 wants to speak after S_3, then the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number is:

(*a*) $^{10}C_3$ (*b*) $^{10}P_8$

(*c*) $^{10}P_3$ (*d*) $\frac{10!}{3!}$

12. 20 Passengers are to travel by a double decked bus which can accommodate 13 in the upper deck and 7 in the lower deck. The number of ways that they can be divided if 5 refuse to sit in the upper deck and 8 refuse to sit in the lower deck, is

(*a*) 25 (*b*) 21

(*c*) 18 (*d*) 15

13. How many nine digit numbers can be formed using the digits 2, 2, 3, 3, 5, 5, 8, 8, 8 so that the odd digits occupy even positions?

(*a*) 7560 (*b*) 180

(*c*) 16 (*d*) 60

14. There are five different peaches and three different apples. Number of ways they can be divided into two packs of four fruits if each pack must contain atleast one apple, is

(*a*) 95 (*b*) 65

(*c*) 60 (*d*) 30

Grouping

15. 9 students of class 12th of whom 2 from section A, 3 from section B and 4 from section C are available for inter school debate competition. Number of ways in which 4 students can be selected so that there is atleast 1 from each section, is

(*a*) 144 (*b*) 120

(*c*) 72 (*d*) 71

16. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(*a*) 91 (*b*) 182

(*c*) 126 (*d*) 3920

Circular Permutation

17. The number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together, is

(*a*) 2170 (*b*) 2165

(*c*) 2160 (*d*) 2155

18. The number of ways in which 5 beads, chosen from 8 different beads be threaded on to a ring, is:

(*a*) 672 (*b*) 1344

(*c*) 336 (*d*) None

Selection and Permutation of Alike Objects

19. The number of words that can be formed by using the letters of the word 'MATHEMATICS' that start as well as end with T, is

(*a*) 80720 (*b*) 90720

(*c*) 20860 (*d*) 37528

20. The number of permutations that can be formed by arranging all the letters of the word 'NINETEEN' in which no two E's occur together. is

(*a*) $\frac{8!}{3!3!}$ (*b*) $\frac{5!}{3!\times {}^6C_2}$

(*c*) $\frac{5!}{3!}\times {}^6C_3$ (*d*) $\frac{8!}{5!}\times {}^6C_3$.

21. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour, is:

(*a*) $6(7!-4!)$ (*b*) $7(6!-4!)$

(*c*) $8!-5!$ (*d*) None

22. Number of 7 digit numbers the sum of whose digits is 61 is

(*a*) 12 (*b*) 24

(*c*) 28 (*d*) None

Distribution of Alike Objects

23. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is:

(*a*) 126 (*b*) 252

(*c*) 378 (*d*) None of these

24. The number of terms in a complete homogeneous expression of degree n in x, y and z is

(*a*) n (*b*) $\frac{n(n+1)}{2}$

(*c*) $\frac{(n+1)(n+2)}{2}$ (*d*) None

25. Number of ways of selecting 5 coins from coins three each of Rs. 1, Rs. 2 and Rs. 5 if coins of the same denomination are alike, is:

(*a*) 9 (*b*) 12

(*c*) 21 (*d*) None

Multinomial, De-arrangement and other Concepts

26. The sum of all the numbers which can be formed by using the digits 1, 3, 5, 7 all at a time and which have no digit repeated, is

(*a*) $16\times 4!$ (*b*) $1111\times 3!$

(*c*) $16\times 1111\times 3!$ (*d*) $16\times 1111\times 4!$.

27. The sum of all the four digit numbers which can be formed using the digits 6,7,8,9 (repetition is allowed)

(*a*) 2133120 (*b*) 2133140

(*c*) 2133150 (*d*) 2133122

28. The number of proper divisors of $a^pb^qc^rd^s$ where a, b, c, d are primes & p, q, r, s $\in$ N, is

(*a*) pqrs

(*b*) $(p+1)(q+1)(r+1)(s+1)-4$

(*c*) $pqrs-2$

(*d*) $(p+1)(q+1)(r+1)(s+1)-2$

29. N is a least natural number having 24 divisors. Then the number of ways N can be resolved into two factors is

(*a*) 12 (*b*) 24

(*c*) 6 (*d*) None of these

30. How many divisors of 21600 are divisible by 10 but not by 15?

(*a*) 10 (*b*) 30

(*c*) 40 (*d*) None

Learning Plus

1. The number of 4 digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contain digit 1 is (Repetition is not allowed)
   (a) 1225 (b) 1252
   (c) 750 (d) 480
2. A polygon has 35 diagonals, then the number of its sides is
   (a) 8 (b) 9
   (c) 10 (d) 11
3. In how many ways can 10 balls be divided between two boys, one receiving two and the other eight balls
   (a) 45 (b) 75
   (c) 90 (d) 65
4. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are together
   (a) $5! \times 3!$ (b) ${}^4P_3 \times 5!$
   (c) ${}^6P_3 \times 5!$ (d) ${}^5P_3 \times 3!$
5. In how many ways can a girl and a boy be selected from a group of 15 boys and 8 girls
   (a) 15×8 (b) $15 + 8$
   (c) ${}^{23}P_2$ (d) ${}^{23}C_2$
6. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, then the value of r is
   (a) 3 (b) 4
   (c) 5 (d) 8
7. In a football championship, there were played 153 matches. Every team played one match with each other. The number of teams participating in the championship is
   (a) 17 (b) 18
   (c) 9 (d) 13
8. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded
   (a) 2020 (b) 2002
   (c) 2008 (d) 8002
9. In how many ways can a committee consisting of one or more members be formed out of 12 members of the Municipal Corporation
   (a) 4095 (b) 5095
   (c) 4905 (d) 4090
10. The number of ways in which we can select three numbers from 1 to 30 so as to exclude every selection of all even numbers is
    (a) 4060 (b) 3605
    (c) 455 (d) 446
11. A man has 10 friends. In how many ways he can invite one or more of them to a party
    (a) 10! (b) 2^{10}
    (c) 10! - 1 (d) $2^{10} - 1$
12. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
    (a) 6 (b) 18
    (c) 12 (d) 9
13. The greatest possible number of points of intersection of 8 straight lines and 4 circles is
    (a) 32 (b) 64
    (c) 76 (d) 104
14. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals
    (a) 5 (b) 7
    (c) 6 (d) 4
15. Out of 10 points in a plane 6 are in a straight line. The number of triangles formed by joining these points are
    (a) 100 (b) 150
    (c) 120 (d) 140
16. There are n points in a plane of which p points are collinear. How many lines can be formed from these points
    (a) ${}^{(n-p)}C_2$ (b) ${}^nC_2 - {}^pC_2$
    (c) ${}^nC_2 - {}^pC_2 + 1$ (d) ${}^nC_2 - {}^pC_2 - 1$
17. Out of 6 books, in how many ways can a set of one or more books be chosen
    (a) 64 (b) 63
    (c) 62 (d) 65
18. The number of ways of dividing 52 cards amongst four players equally, are
    (a) $\frac{52!}{(13!)^4}$ (b) $\frac{52!}{(13!)^2\,4!}$
    (c) $\frac{52!}{(12!)^4\,(4!)}$ (d) $\frac{52!}{(12!)^2\,4!}$
19. If among the 13 cricket players 4 are bowlers, then in how many ways can we form a cricket team of 11 players in which at least 2 bowlers are included
    (a) 55 (b) 72
    (c) 78 (d) 75

20. To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, then the number of ways in which the selection can be made
(a) ${}^{20}C_7 + 5 \times {}^{20}C_8 + 10 \times {}^{20}C_9$
(b) ${}^{22}C_9 - {}^{5}C_3$
(c) ${}^{22}C_3 + {}^{5}C_3$
(d) ${}^{5}C_3 - {}^{22}C_9$

21. In how many ways can 5 boys and 5 girls sit in a circle so that no two boys sit together
(a) $5! \times 5!$ (b) $4! \times 5!$
(c) $\frac{5! \times 5!}{2}$ (d) 9!

22. In how many ways a garland can be made from exactly 10 flowers?
(a) 10! (b) 9!
(c) 2(9!) (d) $\frac{9!}{2}$

23. The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two female are not seated together is
(a) 480 (b) 600
(c) 720 (d) 840

24. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by
(a) $6! \times 5!$ (b) 30
(c) $5! \times 4!$ (d) $7! \times 5!$

25. The number of ways in which four letters of the word 'MATHEMATICS' can be arranged is given by
(a) 136 (b) 192
(c) 1680 (d) 2454

26. Number of divisors of n = 38808 (except 1 and n) is
(a) 70 (b) 68
(c) 72 (d) 74

27. The total number of ways of selecting six coins out of 20 one rupee coins, 10 fifty paise coins and 7 twenty five paise coins is
(a) 28 (b) 56
(c) ${}^{37}C_6$ (d) ${}^{37}P_6$

28. The number of positive integral solutions of abc = 30 is
(a) 30 (b) 27
(c) 8 (d) 16

29. In how many ways 3 letters can be posted in 4 letter-boxes, if all the letters are not posted in the same letter-box
(a) 63 (b) 60
(c) 77 (d) 81

30. The exponent of 3 in 100! is
(a) 33 (b) 44
(c) 48 (d) 52

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. The kindergarten teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by:
(a) ${}^{25}C_5 - {}^{24}C_4$ (b) ${}^{24}C_5$
(c) ${}^{25}C_5 - {}^{24}C_5$ (d) ${}^{24}C_4$

2. Consider the word 'MULTIPLE' then in how many other ways can the letters of the word 'MULTIPLE' be arranged;
(a) without changing the order of the vowels equals 3359
(b) keeping the position of each vowel fixed equals 59
(c) without changing the relative order/position of vowels & consonants is 359
(d) Total number of words using all the letters equals $4 \cdot 7!$

3. Which of the following statements are correct?
(a) Number of words that can be formed with 6 of the letters of the word "CENTRIFUGAL" if each word must contain all the vowels is $3 \cdot 7!$
(b) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls can be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same colour to be alike.
(c) There are 12 things, 4 alike of one kind, 5 alike of another kind and the rest are all different. The total number of combinations is 240.
(d) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

4. The number of ways in which 10 students can be divided into three teams, one containing 4 and others 3 each, is

(a) $\frac{10!}{4!3!3!}$ (b) 2100

(c) ${}^{10}C_4 \cdot {}^{5}C_3$ (d) $\frac{10!}{6!3!3!} \cdot \frac{1}{2}$

5. Consider the word D = F R E E W H E E L. Which of the following statement(s) is/are correct
   (a) Number of other ways in which the letters of the word D can be arranged is ${}^{9}P_5$.
   (b) Number of ways is which the letters of the word D can be an arranged in a circle is ${}^{8}P_4$ distisguishing between clockwise and anticlockwise.
   (c) Number of ways in which the letters of the word D can be arranged if vowels and consonants both are in alphabetical order is $\frac{1}{2}{}^{10}C_5$
   (d) If as many more words are formed as possible using the letters of the word D the number of words which contain the word FEEL is ${}^{6}P_4$.
6. The number of ways of arranging the letters AAAAA, BBB, CCC, D, EE & F in a row if the letter C are separated from one another is:

(a) ${}^{13}C_3 \cdot \frac{12!}{5!3!2!}$ (b) $\frac{13!}{5!3!3!2!}$

(c) $\frac{14!}{3!3!2!}$ (d) 11. $\frac{13!}{6!}$

7. Number of ways in which the letters of the word "TOMATO" can be arranged if two alike vowels are separated, is not equal to
   (a) number of ways in which letters of the word "RAITHATHA" can be arranged if no two alike letters are separated.
   (b) number of 3 digit numbers with atleast one 3 and atleast one 2.
   (c) number of ways in which Ram and Rama can exchange their maps if Ram has 3 and Rama has 7 maps, all maps being different, maintaining their original number of maps at the end.
   (d) number of ways in which 2 alike apples and 4 alike oranges can be distributed in three children if each child get none, one or more fruits.
8. The number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \leq n$ (where n is a positive integer) is
   (a) ${}^{n+3}C_3$ (b) ${}^{n+4}C_4$
   (c) ${}^{n+5}C_5$ (d) ${}^{n+4}C_n$
9. Number of ways in which 6 different toys can be distributed equally among 3 children is also equal to
   (a) number of ways in which all the letters of the word KUMKUM can be arranged in a definite order.
   (b) number of different signals which can be obtained from 10 different flags taken two at a time keeping one above the other.
   (c) number of ways in which five different marbles can be placed in three different boxes so that no box remains empty.
   (d) number of ways in which all the letters of the word CALCULUS can be arranged without changing the order and position of non alike letters.
10. Coefficient of $x^2 y^3 z^4$ in the expansion of $(x + y + z)^9$ is equal to
    (a) the number of ways in which 9 things of which 2 alike of one kind, 3 alike of 2nd kind, and 4 alike of 3rd kind can be arranged.
    (b) the number of ways in which 9 identical things can be distributed in 3 persons each receiving atleast two things.
    (c) the number of ways in which 9 identical things can be distributed in 3 persons each receiving none one or more.
    (d) the number of ways in which 9 different books can be tied up in to three bundles one containing 2, other 3 and third containing 4 books.

Comprehension – 01 (11 and 13)

There are 8 official and 4 non-official members, out of these 12 members a committee of 5 members is to be formed, then answer the following questions.

11. Number of committees consisting of 3 official and 2 non-official members, are
    (a) 363 (b) 336
    (c) 236 (d) 326
12. Number of committees consisting of at least two non-official members, are
    (a) 456 (b) 546
    (c) 654 (d) 466
13. Number of committees in which a particular official member is never included, are
    (a) 264 (b) 642
    (c) 266 (d) 462

Comprehenssion – 02 (No. 14 and 15)

Let n be the number of ways in which the letters of the word "RESONANCE" can be arranged so that vowels appear at the even places and m be the number of ways in which "RESONANCE" can be arrange so that letters R, S, O, A, appear in the order same as in the word RESONANCE, then answer the following questions.

14. The value of n is

(*a*) 360 (*b*) 720

(*c*) 240 (*d*) 840

15. The value of m is

(*a*) 3780 (*b*) 3870

(*c*) 3670 (*d*) 3760

16. Match the column

Column – I	**Column – II**
(*a*) The total number of selections of fruits which can be made from, 3 bananas, 4 apples and 2 oranges is, it is given that fruits of one kind are identical	(P) 120
(*b*) There are 10 true-false statements in a question paper. How many sequences of answers are possible in which exactly three are correct?	(Q) 286
(*c*) The number of ways of selecting 10 balls from unlimited number of red, black, white and green balls is, it is given that balls of same colours are identical	(R) 59
(*d*) The number of words which can be made from the letters of the word 'MATHEMATICS' so that consonants occur together?	(S) 75600

17. Match the column

Column-I	**Column-II**
(*a*) There are 12 points in a plane of which 5 are collinear. The number of distinct convex quadrilaterals which can be formed with vertices at these points is:	(P) 185
(*b*) If 7 points out of 12 are in the same straight line, then the number of triangles formed is	(Q) 420
(*c*) If AB and AC be two line segemets and there are 5, 4 points on AB and AC (other than A), then the number of quadrilateral, with vertices on these points equals	(R) 126
(*d*) The maximum number of points of intersection of 8 unequal circles and 4 straight lines.	(S) 60

INTEGER TYPE

18. A man has 7 relatives, 4 of them are ladies & 3 gentlemen; his wife has also 7 relatives, 3 of them are ladies & 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies & 3 gentlemen so that there are 3 of the man's relative & 3 of the wife's relatives?

19. The exponent of 7 in 50! is

20. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is:

21. The number of ways in which 5 beads, chosen from 8 different beads be threaded on to a ring, is:

22. Number of five digits numbers divisible by 3 that can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if, each digit is to be used atmost one is N where 'N' is three digit number then sum of the digit of 'N'

23. The number of permutations which can be formed out of the letters of the word "SERIES" taking three letters together, is:

24. A box contains 6 balls which may be all of different colours or three each of two colours or two each of three different colours. The number of ways of selecting 3 balls from the box (if ball of same colour are identical), is N then sum of the digits in the number 'N' equals

25. Sum of all the numbers that can be formed using all the digits 2, 3, 3, 4, 4, 4, is N then sum of the digit of the number 'N' equals

26. Six married couple are sitting in a room. Number of ways in which 4 people can be selected so that there is exactly one married couple among the four is N then number of divisors of N equals

27. The number of ways selecting 8 books from a library which has 10 books each of Mathematics, Physics, Chemistry and English, if books of the same subject are alike, is N then number of divisors of N. l

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is: [*JEE Main-2016*]
(*a*) 46th (*b*) 59th
(*c*) 52nd (*d*) 58th

2. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is: [*JEE (Main) 2017*]
(*a*) 415 (*b*) 435
(*c*) 475 (*d*) 485

3. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in row on a shelf so that the dictionary is always in the middle. The number of such arrangements is- [*JEE Main-2018*]
(*a*) less than 500
(*b*) at least 500 but less than 750
(*c*) at least 750 but less than 1000
(*d*) at least 1000

4. Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team is: [*JEE Main–2019 (January)*]
(*a*) 500 (*b*) 200
(*c*) 300 (*d*) 350

5. The number of natural number less than 7, 000 which can be formed by using the digits 0, 1, 3, 7, 9 (repetition of digits allowed) is equal to: [*JEE Main–2019 (January)*]
(*a*) 250 (*b*) 374
(*c*) 372 (*d*) 375

6. Consider three boxes, each containing 10 balls labelled 1, 2,....., 10. Suppose one ball is randomly drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the ith box, (i = 1, 2, 3). Then, the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is: [*JEE Main–2019 (January)*]
(*a*) 120 (*b*) 82
(*c*) 240 (*d*) 164

7. There are m men and two women participating in a chess tournament. Each participant plays two games with every other participant. If the number of games played by the men between themselves exceeds the number of games played between the men and the women by 84, then the value of m is: [*JEE Main-2019 (January)*]
(*a*) 12 (*b*) 11
(*c*) 9 (*d*) 7

8. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time.The number of such numbers in which the odd digits occupy even places is: [*JEE Main-2019 (April)*]
(*a*) 175 (*b*) 162
(*c*) 160 (*d*) 180

9. The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0,1,2,3,4,5 (repetition of digits is allowed) is: [*JEE Main-2019 (April)*]
(*a*) 288 (*b*) 306
(*c*) 360 (*d*) 310

10. A committee of 11 members is to be formed from 8 males and 5 females. If m is the number of ways the committee is formed with at least 6 males and n is the number of ways the committee is formed with at least 3 females, then: [*JEE Main-2019 (April)*]
(*a*) m = n = 78 (*b*) n = m –8
(*c*) m + n = 68 (*d*) m = n = 68

11. Some identical balls are arranged in rows to form an equilateral traingle. The first row consists of one ball, the second row cosists of two balls and so on. If 99 more identical balls are added to the total number of balls used in forming the equilaterial triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than the number of balls each side of the triangle contains. Then the number of balls used to form the equialteral triangle is: [*JEE Main-2019 (April)*]
(*a*) 190 (*b*) 262
(*c*) 225 (*d*) 157

12. The number of 6 digit numbers that can be formed using the digits 0, 1, 2, 5, 7 and 9 which are divisible by 11 and no digit is repeated, is: [*JEE Main-2019 (April)*]
(*a*) 36 (*b*) 60
(*c*) 48 (*d*) 72

13. Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number beams is: [JEE Main-2019 (April)]

(a) 210 (b) 190

(c) 170 (d) 180

14. The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is: [JEE Main-2019 (April)]

(a) 2^{20} (b) $2^{20}-1$

(c) $2^{20}+1$ (d) 2^{21}

15. A group of students comprises of 5 boys and n girls. If the number of ways, in which a team of 3 students can randomly be selected from this group such that there is at least one boy and at least one girl in each team, is 1750, then n is equal to: [JEE Main-2019 (April)]

(a) 25 (b) 28

(c) 27 (d) 24

16. A person throws two fair dice. He wins Rs. 15 for throwing a doublet (same numbers on the two dice), wins Rs. 12 when the throw results in the sum of 9, and loses Rs. 6 for any other outcome on the throw. Then the expected gain/ loss (in Rs.) of the person is: [JEE Main-2019 (April)]

(a) 2 gain (b) $\frac{1}{2}$ loss

(c) $\frac{1}{4}$ loss (d) $\frac{1}{2}$ gain

17. Total number of 6-digit numbers in which only one digit is repeated and all the five digits 1, 3, 5, 7 and 9 appear, is [JEE Main-2020 (January)]

(a) 5^6 (b) $\frac{1}{2}6!$

(c) 6! (d) $\frac{5}{2}(6!)$

18. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is _____. [JEE Main-2020 (January)]

19. The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word **'EXAMINATION'** is [JEE Main-2020 (January)]

20. If the number of five digit numbers with distinct digits and 2 at the 10th place is 336 k, then k is equal to: [JEE Main-2020 (January)]

(a) 7 (b) 4

(c) 6 (d) 8

21. Let $n > 2$ be an integer. Suppose that there are n Metro stations in a city located along a circular path. Each pair of stations is connected by a straight track only. Further, each pair of nearest stations is connected by blueline, whereas all remaining pairs of stations are connected by redline. If the number of red lines is 99 times the number of blue lines, then the value of n is: [JEE Main-2020 (September)]

(a) 199 (b) 201

(c) 101 (d) 200

22. If the letters of the word 'MOTHER' be permuted and all the words so formed (with or without meaning) be listed as in a dictionary, then the position of the word 'MOTHER' is _____ [JEE Main-2020 (September)]

23. A test consists of 6 multiple choice questions, each having 4 alternative answers of which only one is correct. The number of ways, in which a candidate answers all six questions such that exactly four of the answers are correct, is _____. [JEE Main-2020 (September)]

24. There are 3 sections in a question paper and each section contains 5 questions. A candidate has to answer a total of 5 questions, choosing at least one question from each section. Then the number of ways, in which the candidate can choose the questions, is: [JEE Main-2020 (September)]

25. The number of words, with or without meaning, that can be formed by taking 4 letters at a time from the letters of the word 'SYLLABUS' such that two letters are distinct and two letters are alike, is _____. [JEE Main-2020 (September)]

26. The number of words (with or without meaning) that can be formed from all the letters of the word "LETTER" in which vowels never come together is _____. [JEE Main-2020 (September)]

27. Two families with three members each and one family with four members are to be seated in a row. In how many ways can they be seated so that the same family members are not separated? [JEE Main-2020 (September)]

(a) 2! 3! 4! (b) $(3!)^3(4!)$

(c) $(3!)^2(4!)$ (d) $3!(4!)^3$

28. A scientific committee is to formed from 6 Indians and 8 foreigners, which includes at least 2 Indians and double the number of foreigners as Indians. Then the number of ways, the committee can be formed is: [JEE Main-2021 (February)]

(a) 560 (b) 1050

(c) 1625 (d) 575

29. The students $S_1, S_2, \ldots, S_{10}$ are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is [JEE Main-2021 (February)]

30. The total number of positive integral solutions (x, y, z) such that $xyz = 24$ is [*JEE Main-2021 (February)*]
(*a*) 36 (*b*) 45
(*c*) 24 (*d*) 30

31. The total number of numbers, lying between 100 and 1000 that can be formed with the digits 1, 2, 3, 4, 5, if the repetition of digits is not allowed and numbers are divisible by either 3 or 5 is [*JEE Main-2021 (February)*]

32. The total number of two digit numbers 'n', such that $3^n + 7^n$ is a multiple of 10, is [*JEE Main-2021 (February)*]

33. The number of seven digit integers with sum of the digits equal to 10 and formed by using the digits 1,2 and 3 only is [*JEE Main-2021 (February)*]
(*a*) 77 (*b*) 42
(*c*) 35 (*d*) 82

34. A natural number has prime factorization given by $n = 2^x3^y5^z$, where y and z are such that $y + z = 5$ and $y^{-1} + z^{-1} = \frac{5}{6}$, $y > z$. Then the numberof odd divisors of n, including 1, is: [*JEE Main-2021 (February)*]
(*a*) 11 (*b*) 6x
(*c*) 12 (*d*) 6

35. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is
[*JEE Main-2021 (February)*]

36. Consider a rectangle $ABCD$ having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the numberof triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to:
[*JEE Main-2021 (March)*]
(*a*) 795 (*b*) 1173
(*c*) 1890 (*d*) 717

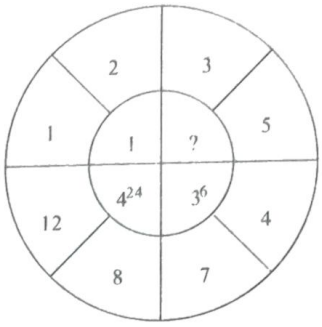
37. Team 'A' consists of 7 boys and girls and Team 'B' has 4 boys and 6 girls. If a total of 52 single matches can be arranged between these two teams when a boy plays against a boy and a girl plays against a girl, then n is equal to : [*JEE Main-2021 (March)*]
(*a*) 5 (*b*) 2
(*c*) 4 (*d*) 6

38. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to : [*JEE Main-2021 (March)*]
(*a*) 364 (*b*) 240
(*c*) 333 (*d*) 360

39. The sum of all the 4 -digit distinct numbers that can be formed with the digits 1,2,2 and 3 is:
[*JEE Main-2021 (March)*]
(*a*) 26664 (*b*) 122664
(*c*) 122234 (*d*) 22264

40. The number of times the digit 3 will be written when listing the integers from 1 to 1000 is [*JEE Main-2021 (March)*]

41. The missing value in the following figure is



Use the logic which gives answer in single digit
[*JEE Main-2021 (March)*]

JEE-ADVANCED PREVIOUS YEARS

1. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is [*IIT JEE 2012*]
(*a*) 75 (*b*) 150
(*c*) 210 (*d*) 243

Paragraph for Question Nos. 2 & 3

Let a_n denote the number of all n-digit positive integers formed by the digits 0,1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

2. Which of the following is correct? [*IIT JEE 2012*]
(*a*) $a_{17} = a_{16} + a_{15}$ (*b*) $c_{17} \neq c_{16} + c_{15}$
(*c*) $b_{17} \neq b_{16} + c_{16}$ (*d*) $a_{17} = c_{17} + b_{16}$

3. The value of b_6 is
(*a*) 7 (*b*) 8
(*c*) 9 (*d*) 11

4. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positivie integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is
[*JEE Advanced-2014*]

5. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is [*JEE Advanced-2014*]

6. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is [*JEE Advanced-2014*]
(*a*) 264 (*b*) 265
(*c*) 53 (*d*) 67

7. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of m/n is

[*JEE Advanced-2015*]

8. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

[*JEE Advanced-2016*]

(*a*) 380 (*b*) 320

(*c*) 260 (*d*) 95

9. Let $S = \{1, 2, 3, ...,9\}$. For $k = 1, 2, ...5$, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ [*JEE Advanced-2017*]

(*a*) 210 (*b*) 252

(*c*) 126 (*d*) 125

10. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$ [*JEE Advanced-2017*]

11. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is _____ [*JEE Advanced-2018*]

12. Let X be a set with exactly 5 element and Y be a set with exactly 7 elements. If α is the number of one-one functions from X and Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is .....

[*JEE Advanced-2018*]

13. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.

(*i*) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.

(*ii*) Let a_2 be the total number of ways in whcih the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(*iii*) Let α_3 be te total number of ways in which the committe can be formed such that the committee has 5 members, at elst 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committeee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

[*JEE Advanced-2018*]

LIST-I	**LIST-II**
(P) The value of α_1 is	(*a*) 136
(Q) The value of α_2 is	(*b*) 189
(R) The value of α_3 is	(*c*) 192
(S) The value of α_4 is	(*d*) 200

(5) 381

(6) 461

The correct option is:-

(*a*) P → 4; Q → 6; R → 2; S → 1

(*b*) P → 1; Q → 4; R → 2; S → 3

(*c*) P → 4; Q → 6; R → 5; S → 2

(*d*) P → 4; Q → 2; R → 3; S → 1

14. Five person A,B,C,D and E are seated in a ciruclar arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is

[*JEE Advanced-2019*]

15. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is _____ [*JEE(Advanced)-2020*]

16. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____

[*JEE(Advanced)-2020*]

ANSWER KEY

Topicwise Questions

1. (*b*)	2. (*d*)	3. (*a*)	4. (*c*)	5. (*b*)	6. (*c*)	7. (*d*)	8. (*d*)	9. (*a*)	10. (*a*)	11. (*d*)	12. (*b*)
13. (*d*)	14. (*d*)	15. (*c*)	16. (*c*)	17. (*c*)	18. (*a*)	19. (*b*)	20. (*c*)	21. (*a*)	22. (*c*)	23. (*a*)	24. (*c*)
25. (*b*)	26. (*c*)	27. (*a*)	28. (*d*)	29. (*a*)	30. (*a*)						

Learning Plus

1. (*c*)	2. (*c*)	3. (*c*)	4. (*c*)	5. (*a*)	6. (*a*)	7. (*b*)	8. (*b*)	9. (*a*)	10. (*b*)	11. (*d*)	12. (*b*)
13. (*d*)	14. (*b*)	15. (*a*)	16. (*c*)	17. (*b*)	18. (*a*)	19. (*c*)	20. (*a*)	21. (*b*)	22. (*d*)	23. (*a*)	24. (*a*)
25. (*d*)	26. (*a*)	27. (*a*)	28. (*b*)	29. (*b*)	30. (*c*)						

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *b*)	2. (*a*,*b*,*c*,*d*)	3. (*a*, *b*, *c*, *d*)	4. (*b*,*c*)	5. (*b*,*c*,*d*)	6. (*a*,*d*)
7. (*b*,*c*,*d*)	8. (*b*,*d*)	9. (*a*,*b*,*d*)	10. (*a*,*d*)	11. (*b*)	12. (*a*)
13. (*d*)	14. (*b*)	15. (*a*)	16. (*a*)→(R), (*b*)→(P),(*c*)→(Q), (*d*)→(S)		
17. (*a*) - (Q); (*b*) - (P); (*c*) - (S); (*d*) -(R)		18. [0485]	19. [0008]	20. [0126]	21. [0672]
22. [0015]	23. [42]	24. [4]	25. [12]	26. [20]	27. [8]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. (*d*)	2. (*d*)	3. (*d*)	4. (*c*)	5. (*b*)	6. (*a*)	7. (*a*)	8. (*d*)	9. (*d*)	10. (*a*)	11. (*a*)	12. (*b*)
13. (*c*)	14. (*a*)	15. (*a*)	16. (*b*)	17. (*d*)	18. [490]	19. [2454]	20. (*d*)	21. (*b*)	22. [309]	23. [135]	24. [2250]
25. [240]	26. [120]	27. (*b*)	28. (*c*)	29. (31650)		30. (*d*)	31. (32)	32. (45)	33. (*a*)	34. (*c*)	35. (1000)
36. (*d*)	37. (*c*)	38. (*c*)	39. (*a*)	40. (300)	41. (4)						

JEE-ADVANCED
PREVIOUS YEARS

1. (*b*)	2. (*a*)	3. (*b*)	4. [7]	5. [5]	6. (*c*)	7. [5]	8. (*a*)	9. (*c*)	10. (5)	11. [625]	12. [119]
13. (*c*)	14. [30.00]		15. [495.00]		16. [1080.00]					

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SAHEEL

5 PERMUTATIONS AND COMBINATIONS

Topicwise Questions

1. (*b*) As per the given condition, digit 1 should occur at alternate places of the number and at the remaining 5 places either 2, 3, 5 or 7 should appear. Now when the number starts with 1, number of numbers = 4^5 and when the number starts with either 2, 3, 5 or 7, number of numbers = 4^5

So, total number = $2 \times 4^5 = 2048$ **Ans.**

2. (*d*) 1 . ② . 3 . ④ . 5 . ⑥ . 7 | T | | | | | T |

$(3 \times 2) \times (5 \times 4 \times 3 \times 2) = 6 \times 120 = 720$]

3. (*a*) $1 + 2 + 3 + \ldots\ldots\ldots + 9 = 45 = 0 + 1 + 2 + 3 + \ldots\ldots\ldots + 9$

All 9 digit such numbers = 9 !

All 10 digit such numbers when '0' included = 10 ! – 9 !

So, total = 9 ! + (10 ! – 9 !) = (10) ! **Ans.**

4. (*c*) Total number of signals can be made from 3 flags each of different colour by hoisting 1 or 2 or 3 above.

i.e. $3 + 3 \times 2 + 3 \times 2 \times 1 = 3 + 6 + 6 = 15$

5. (*b*) $^{2002}C_{1001} = \dfrac{(2002)!}{(1001)!(1001)!}$

no. of zeros in (2002)! are

$400 + 80 + 16 + 3 = 499$

no. of zeroes in $(1001!)^2 = 2(200 + 40 + 8 + 1) = 498$

Hence no. of zeroes is $\dfrac{(2002)!}{(1001!)^2} = 1$

6. (*c*) Word QUEUE .

E → 2, Q , U - 2

| E | | | | | = 12

$\dfrac{4!}{2!}$

| Q | E | | | | = 3

$\dfrac{3!}{2!}$

| Q | U | E | E | U | = 1

| Q | U | E | U | E | = 1

17th rank

7. (*d*) Total number of possible arrangements is $^4P_2 \times {}^6P_3$.

8. (*d*) First we have to find all the arrangements of the word 'GENIUS' is

6 ! = 720

number of arrangement which in either started with G ends with S is

$(5! + 5! - 4!) = (120 + 120 - 24) = 216$

Hence total number of arrangement which is neither started with G nor ends with S is.

$(720 - 216) = 504$

9. (*a*) Total no. of arrangement if all the girls do not sit side by side is = [all arrangement – girls seat side by side]

$= 8! - (6! \times 3!) = 6!\,(56 - 6) = 6! \times 50 = 720 \times 50$

$= 36000$

10. (*a*) Number of words which have at least one letter repeated = total words – number of words which have no letter repeated = $10^5 - 10 \times 9 \times 8 \times 7 \times 6 = 69760$

11. (*d*) First we select 3 speaker out of 10 speaker and put in any way and rest are no restriction i.e. total number of ways = $^{10}C_3 \cdot 7! = \dfrac{10!}{3!}$

12. (*b*) upperdeck - 13 seats → 8 in upper deck.

lowerdeck - 7 seats → 5 in lower deck

Remains passengers = 7

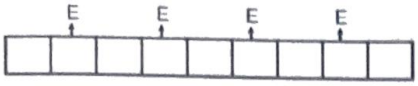
Now Remains 5 seats in upper deck and 2 seats in lower deck

for upper deck number of ways = $^7C_5$

for lower deck number of ways = $^2C_2$

So total number of ways = $^7C_5 \times {}^2C_2 = \dfrac{7.6}{2} = 21$

13. (*d*) Even place



There are four even places and four odd digit number so total number of filling is $\frac{4!}{2!.2!}$ rest are also occupy is $\frac{5!}{3!.2!}$ ways

Hence total number of ways = $\frac{4!}{2!.2!}\times\frac{5!}{3!.2!} = 60$

14. (*d*) Peaches 5 p_1, p_2, p_3, p_4, p_5
Apples 3 a_1, a_2, a_3
Hence number of ways = ${}^3C_1 \times {}^5C_3 = 30$ **Ans.**

15. (*c*) 9 → 2 Section A, 3 Section B, 4 Section C — 4 are selected

required number of ways

A	*B*	*C*
2	1	1
1	1	2
1	2	1

Hence ${}^2C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^2C_1 \cdot {}^3C_2 \cdot {}^4C_1 + {}^2C_2 \cdot {}^3C_1 \cdot {}^4C_1$
$= 36 + 24 + 12 = 72$ **Ans.**

16. (*c*) They can sit in groups of either 5 and 3 or 4 and 4

required number = $\frac{8!}{5!\times 3!}\times 1 + \frac{8!\times 2!}{4!\times 4!\times 2!} = 126$

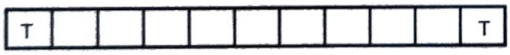
17. (*c*) Total number of ways is

$\frac{6!\times 3!}{2!} = 720\times 3 = 2160$

18. (*a*) First we select 5 beads from 8 different beads to ${}^8C_5$
Now total number of arrangement is

${}^8C_5 \times \frac{4!}{2!} = 672$

19. (*b*)



Total arrangement is $\frac{9!}{2!.2!} = 90720$

20. (*c*) NINETEEN
$\Rightarrow$ N → 3 : I, T
E → 3
First we arrange the word of N, N, N, I and T
then the number of ways = $\frac{5!}{3!}$.
Now total 6 number of place which are arrange E is ${}^6C_3$
Hence total number of ways = $\frac{5!}{3!}\cdot {}^6C_3$

21. (*a*) Total number of ways of arranging 2 identical white balls,
3 identical red balls and 4 green balls of different shades = $\frac{9!}{2!3!} = 6.7!$

Number of ways when balls of same colour are together = $3!\times 4! = 6.4!$
$\therefore$ Number of ways of arranging the balls when atleast one ball is separated from the balls of the same colour $= 6.7! - 6.4! = 6(7! - 4!)$

22. (*c*) only 7, 8 and 9 can be used

Aliter: $9,9,9,9,9,9,7 \rightarrow \frac{7!}{6!} = 7$

$9,9,9,9,9,8,8 \rightarrow \frac{7!}{2!5!} = 21$

Total = 28 **Ans.**

23. (*a*) Coefficient x^{10} in $(x + x^2 .....x^5)^6$ = coefficient of x^4 in $(x^0 + x^1 .....x^4)^6 = {}^{6+4-1}C_4 = {}^9C_4 = 126$
Alternatively: Give one apple to each child and then for rest 4 apples = ${}^{4+6-1}C_{6-1} = 126$

24. (*c*) $(x + y + z)^n \rightarrow$ use beggar] ${}^{n+2}C_2$

25. (*b*) 3 A + 2 O. A. = 3 . 2 = 6 ; 3 A + 2 diff = 3 ;
2 A + 2 O. A. + 1 D = 3 $\Rightarrow$ 12

26. (*c*) If 1 be unit digit then total no. of number is 3! = 6
Similarly so on if 3, 5, or 7 be unit digit number then total no. of no. is 3! = 6
Hence sum of all unit digit no. is = 6× (1+3+5+7) = 6× 16 = 96
Hence total sum is = $96\times 10^3 + 96\times 10^2 + 96\times 10^1 + 96\times 10^0$
$= 96000 + 9600 + 960 + 96 = 106656 = 16\times 1111\times 3!$

27. (*a*) $(1 + 10 + 10^2 + 10^3)\times 4^3\times (6 + 7 + 8 + 9) = (1111)\times 64\times 30 = 2133120$

28. (*d*) Total number of proper divisors is
$(p + 1)(q + 1)(r + 1)(s + 1) - 2$ (Number and 1 are not proper divisor)

29. (*a*) $N = 2^\alpha .3^\beta 5^\gamma = 2^3 .3^2 .5$
$(\alpha + 1)(\beta + 1)(\gamma + 1) = 4.3.2$
$N = 360 = 2^3 . 3^2 .5$
$\frac{4.3.2}{2} = 12$

30. (*a*) Here $21600 = 2^5 . 3^3 . 5^2$
$\Rightarrow (2\times 5)\times 2^4\times 3^3\times 5^1$
Now numbers which are divisible by
$10 = (4 + 1)(3 + 1)(1 + 1) = 40$
$(2\times 3\times 5)\times (2^4\times 3^2\times 5^1)$ now numbers which are divisible by both 10 and 15
$= (4+1)(2+1)(1+1) = 30$
So the numbers which are divisible by only
$40 - 30 = 10$

1. (c) $7\times7\times6\times5-6\times6\times5\times4=750$

2. (c) Since ${}^{n}C_2-n=35 \Rightarrow \frac{n!}{2!\,(n-2)!}-n=35$

$\Rightarrow n(n-1)-2n=70 \Rightarrow n^2-3n=70$

$\Rightarrow n^2-3n-70=0 \Rightarrow (n+7)(n-10)=0$

$\Rightarrow n=10$

3. (c) A gets 2, B gets 8; $\frac{10!}{2!8!}=45$

A gets 8, B gets 2; $\frac{10!}{8!2!}=45$

$45+45=90$

4. (c) Since the 5 boys can sit in 5 ! ways. In this case there are 6 places are vacant in which the girls can sit in ${}^{6}P_3$ ways. Therefore required number of ways are ${}^{6}P_3\times5!$

5. (a) Required number of ways

$={}^{15}C_1\times{}^{8}C_1=15\times8$

6. (a) ${}^{15}C_{3r}={}^{15}C_{r+3}\Rightarrow{}^{15}C_{15-3r}={}^{15}C_{r+3}$

$\Rightarrow 15-3r=r+3\Rightarrow r=3$

7. (b) ${}^{n}C_2=153\Rightarrow\frac{n(n-1)}{2}=153\Rightarrow n=18$

8. (b) Since 5 are always to be excluded and 6 always to be included, therefore 5 players to be chosen from 14.

Hence required number of ways are ${}^{14}C_5=2002$.

9. (a) Required number of ways

$={}^{12}C_1+{}^{12}C_2+{}^{12}C_3+.....+{}^{12}C_{12}=2^{12}-1$

$=4096-1=4095$

10. (b) Required number of ways $={}^{30}C_3-{}^{15}C_3=3605$

11. (d) Required number of ways $=2^{10}-1$

(Since the case that no friend be invited *i.e.*, ${}^{10}C_0$ is excluded).

12. (b) Required number of ways $={}^{4}C_2\times{}^{3}C_2=18$

13. (d) The required number of points

$={}^{8}C_2\times1+{}^{4}C_2\times2+\left({}^{8}C_1\times{}^{4}C_1\right)\times2$

$=28+12+32\times2=104$

14. (b) Clearly, ${}^{n}C_3=T_n$.

So, ${}^{n+1}C_3-{}^{n}C_3=21\Rightarrow({}^{n}C_3+{}^{n}C_2)-{}^{n}C_3=21$

$\therefore {}^{n}C_2=21$ or $n(n-1)=42=7.6\ \therefore n=7$

15. (a) Number of triangles $={}^{10}C_3-{}^{6}C_3=120-20=100$.

16. (c) Given, total number of points = n and number of collinear points = p. We know that one line has two end points. Therefore total number of lines = ${}^{n}C_2$. Since p points are collinear, therefore total number of lines drawn from collinear points = ${}^{p}C_2$. We also know that, corresponding to the line of collinearity, one will also be added.

Therefore number of lines = ${}^{n}C_2-{}^{p}C_2+1$.

17. (b) Required number of ways

$={}^{6}C_1+{}^{6}C_2+{}^{6}C_3+{}^{6}C_4+{}^{6}C_5+{}^{6}C_6=2^6-1=63$

18. (a) Required number of ways

$={}^{52}C_{13}\times{}^{39}C_{13}\times{}^{26}C_{13}\times{}^{13}C_{13}$

$=\frac{52!}{39!\times13!}\times\frac{39!}{26!\times13!}\times\frac{26!}{13!\times13!}\times\frac{13!}{13!}=\frac{52!}{(13!)^4}$

19. (c) The number of ways can be given as follows

2 bowlers and 9 other players $={}^{4}C_2\times{}^{9}C_9$

3 bowlers and 8 other players $={}^{4}C_3\times{}^{9}C_8$

4 bowlers and 7 other players $={}^{4}C_4\times{}^{9}C_7$

Hence required number of ways

$=6\times1+4\times9+1\times36=78$

20. (a) The selection can be made in

${}^{20}C_7\times{}^{5}C_5+{}^{20}C_8\times{}^{5}C_4+{}^{5}C_3\times{}^{20}C_9$

21. (b) Since total number of ways in which boys can occupy any place is $(5-1)!=4!$ and the 5 girls can be sit accordingly in 5 ! ways.

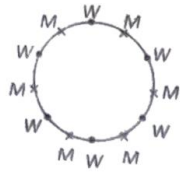
Hence required number of ways are $4!\times5!$

22. (d) A garland can be made from 10 flowers in $\frac{1}{2}(9!)$ ways.

{$\because$ n flowers' garland can be made in $\frac{1}{2}(n-1)!$ ways}

23. (a) Fix up a male and the remaining 4 male can be seated in 4! ways. Now no two female are to sit together and as such the 2 femaleare to be arranged in five empty seats between two consecutive male and number of arrangement will be ${}^{5}P_2$. Hence by fundamental theorem the total number of ways is $=4!\times{}^{5}P_2=24\times20=480$ ways.

24. (*a*)



No. of ways in which 6 men can be arranged at a round table = (6 – 1)!

Now women can be arranged in 6! ways.

Total Number of ways = 6! × 5!

25. (*d*) Word 'MATHEMATICS' has 2*M*, 2*T*, 2*A*, *H*, *E*, *I*, *C*, *S*. Therefore 4 letters can be chosen in the following ways.

Case I : 2 alike of one kind and 2 alike of second kind

i.e., ${}^3C_2 \Rightarrow$ No. of words $= {}^3C_2 \frac{4!}{2!2!} = 18$

Case II : 2 alike of one kind and 2 different

i.e., ${}^3C_1 \times {}^7C_2 \Rightarrow$ No.of

words $= {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$

Case III : All are different

i.e. ${}^8C_4 \Rightarrow$ No. of words ${}^8C_4 \times 4! = 1680$

Hence total number of words are 2454.

26. (*a*) Since, $38808 = 8 \times 4851$

$= 8 \times 9 \times 539 = 8 \times 9 \times 7 \times 7 \times 11 = 2^3 \times 3^2 \times 7^2 \times 11$

So, number of divisors

$= (3+1)(2+1)(2+1)(1+1) = 72.$

This includes two divisors 1 and 38808. Hence, the required number of divisors = 72 – 2 = 70.

27. (*a*) Since the total number of selections of r things from n things where each thing can be repeated as many times as one can, is ${}^{n+r-1}C_r$

Therefore the required number $= {}^{3+6-1}C_6 = 28$

28. (*b*) We have, $30 = 2 \times 3 \times 5$. So, 2 can be assigned to either *a* or *b* or *c i.e.* 2 can be assigned in 3 ways. Similarly, each of 3 and 5 can be assigned in 3 ways. Thus, the no. of solutions are $3 \times 3 \times 3 = 27$.

29. (*b*) Three letters can be posted in 4 letter boxes in $4^3 = 64$ ways but it consists the 4 ways that all letters may be posted in same box. Hence required ways = 60.

30. (*c*) Let E(n) denote the exponent of 3 in n. The greatest integer less than 100 divisible by 3 is 99.

We have $E(100!) = E(1.2.3.4....99.100)$

$= E(3.6.9....99) = E[(3.1)(3.2)(3.3).....(3.33)]$

$= 33 + E(1.2.3.....33)$

Now $E(1.2.3.....33) = E(3.6.9....33)$

$= E[(3.1)(3.2)(3.3).....(3.11)]$

$= 11 + E(1.2.3.....11)$

and

$E(1.2.3....11) = E(3.6.9) = E[(3.1)(3.2)(3.3)]$

$3 + E(1.2.3) = 3 + 1 = 4$

Thus $E(100!) = 33 + 11 + 4 = 48$.

Advanced Level Multiconcept Questions

1. (*a*, *b*)

Total no. of visits that a teacher goes is $= {}^{25}C_5$ (selection of 5 different kids each time & teacher goes every time)

Number of visits of a boy = select one particular boy & 4 from rest 24 $= {}^{24}C_4$

So extra visits of a teacher from a boy is

$= {}^{25}C_5 - {}^{24}C_4 = {}^{24}C_5$

2. (*a*,*b*,*c*,*d*)

(*a*) Without changing the order of the vowels of MULTIPLE

So we choose the first three place in ${}^8C_3$ ways and the rest are arranged is

$\frac{8!}{3!5!} \times \frac{5!}{2!} = \frac{8!}{3!2!} = 3360$

Hence required no. is 3360 – 1 = 3359

(*b*) Keeping the position of each vowel fixed M_LT_PL_

Number of ways $= \frac{5!}{2} = 60$ other ways = 60–1 = 59

(*c*) without changing the relative order/position of vowels & consonants

so number of ways is $= \frac{5!}{2!} \times 3! = 60 \times 6 = 360$

Hence required number is = 360–1 = 359

(*d*) Total $\frac{8!}{2!}$

3. (*a, b, c, d*)

(*a*) $11 \begin{cases} 4v \\ 7c \end{cases} \Rightarrow {}^{7}C_{2} \cdot 6! = 3 \cdot 7 \cdot 6! = 3 \cdot 7!$

(*b*) No. of ways $= \dfrac{15!}{r!(15-r)!} = 15c_r$ $\underbrace{W\,W.....W}_{r}$

$\underbrace{B\,B\,B.....B}_{15-r}$

This is maximum if $r = 7$ or 8

(*c*) $12 \begin{cases} 4A \\ 5A \\ 3 \text{ all different} \end{cases} \Rightarrow$ Total no. of combinations

$= 5 \cdot 6 \cdot 2^3 = 240$

(*d*) 2 alike + 2 other alike + 2 other a like = 1

2 alike + 2 other alike + 2 different $= {}^{3}C_{2} \cdot {}^{4}C_{2} = 18$

2 alike + 4 different $= {}^{3}C_{1} \cdot {}^{5}C_{1} = 15$

All 6 different = 1

$= 35$ Ans.

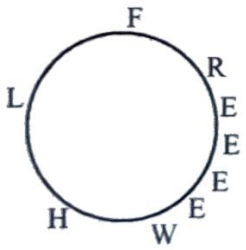
4. (*b,c*)

Total required number of teams is

$= {}^{10}C_{4} \cdot {}^{6}C_{3} \cdot {}^{3}C_{3} \cdot \dfrac{1}{2!} = 2100 = {}^{10}C_{4} \cdot {}^{5}C_{2} = 2100$

5. (*b,c,d*)

(*a*) False as is should be ${}^{9}P_{5} - 1$



(*b*) $x \cdot 4! = 8!$

$\therefore\ x = \dfrac{8!}{4!} = {}^{8}P_{4}$

(*c*) Vowels E E E E select 4 places in ${}^{9}C_{4}$ ways arrange consonants alphabatically only us one ways.

$\therefore\ {}^{9}C_{4} = 126 = \dfrac{1}{2} \cdot 256 = \dfrac{1}{2} \cdot {}^{10}C_{5}$

(*d*) True

$\therefore$ correct answer are (*b*), (*c*) and (*d*)

6. (*a,d*)

We have arrange all the letter except 'CCC' is $\dfrac{12!}{5!3!2!}$

now there are 13 place where 'C' can be placed ${}^{13}C_{3}$

Hence required number of ways is $= \dfrac{12!}{5!3!2!}\ {}^{13}C_{3}$

$= 11 \cdot \dfrac{13!}{6!}$

7. (*b,c,d*)

|T|M|A|T|

Number of ways $= {}^{5}C_{2} \times \dfrac{4!}{2!} = 10 \times 12 = 120.$

(*a*) RAITHATHA

A = 3, H = 2, T = 2, R = 1, I = 1

$\boxed{A\,A\,A}$ $\boxed{H\,H}$ $\boxed{T\,T}$ R I

$\therefore$ Number of ways $= 5! = 120$

(*b*) $\begin{cases} 2, 3, 3 \\ 3, 2, 2 \\ 2, 3, \text{non-zero digit} \\ 2, 3, 0 \end{cases} = \dfrac{3!}{2!} + \dfrac{3!}{2!} + ({}^{7}C_{1}$

$\times 3!) + (2 \times 2) = 3 + 3 + 42 + 4 = 52.$

(*c*) $M_1, M_2, ..., M_7 \,|\, N_1, N_2, N_3$

Total ways $= 3({}^{7}C_{1}) + 3({}^{7}C_{2}) + 1({}^{7}C_{3}) = 21 + 63 + 35$

$= 119.$

Alternatively:

Number of ways $= {}^{10}C_{3} - 1 = 120 - 1 = 119$ **Ans.**

(*d*) Number of ways $= {}^{4}C_{2} \times {}^{6}C_{2} = 6 \times 15 = 90.$ **Ans.**

8. (*b,d*)

$x_1 + x_2 + x_3 + x_4 \le n \Rightarrow x_1 + x_2 + x_3 + x_4 + y = n$ (where y is known as pseudo variable)

Total no. of required solution is $= {}^{n+5-1}C_{n} = {}^{n+4}C_{n}$

or ${}^{n+4}C_{4}$

9. (*a,b,d*)

$\left(\dfrac{6!}{2!\ 2!\ 2!\ 3!}\right) \times 3! = \dfrac{6!}{2!2!2!} = 90.$

(*a*) $\dfrac{6!}{2!2!2!} \Rightarrow$ **(a) is correct.**

(*b*) ${}^{10}C_{2} \cdot 2! \Rightarrow$ **(b) is correct.**

(*c*) $5 \begin{cases} 1, 2, 2 \\ 1, 1, 3 \end{cases} = \left(\dfrac{5!}{1!1!3!}\dfrac{1}{2!} + \dfrac{5!}{2!2!1!}\dfrac{1}{2!}\right) 3! = 150 \Rightarrow$

(c) is incorrect. (Using Division and Distribution)

(*d*) A and S remains in 2nd and 8th position

Hence, number of ways $= \dfrac{6!}{2!2!2!} \Rightarrow$ **(d) is correct.]**

10. (*a,d*)
Consider $(x+y+z)^9 = {}^9C_r x^{9-r}(y+z)^r$
$= {}^9C_r x^{9-r} \cdot {}^rC_p y^p z^{r-p}[x+(y+z)]^9$
put $r=7$; $p=4$
or ${}^9C_2 \cdot {}^7C_3 \cdot {}^4C_4 = \dfrac{9!}{2!\,3!\,4!}$

11. (*b*) 3 official out of 8 can be selected by ${}^8C_3 = 56$ ways
2 non-official out of 4 can be selected in ${}^4C_2 = 6$ ways
$\therefore$ required number of committees are $56 \times 6 = 336$.

12. (*a*) Two non-officials and 3 officials i.e.
${}^4C_2 \times {}^8C_3 = 6 \times 56 = 336$.
Three non-official and 2 officials
${}^4C_3 \times {}^8C_2 = 4 \times 28 = 112$.
Four non-officials and 1 official
${}^4C_4 \times {}^8C_1 = 1 \times 8 = 8$
Total $336 + 112 + 8 = 456$.

13. (*d*) Required no. of ways
$= {}^{12}C_5 = {}^{11}C_5 = 462$

14. (*b*)

15. (*a*) In the word RESONANCE there are 9 letters.
Consonants (5), 1R, 1S, 1C and 2N
Vowels (*d*), 2E, 1O, 1A
total even places 4 ;
No. of ways arranging vowels in even places is $\dfrac{4!}{2!} = 12$
No. of ways arranging consonants in remaining odd places is $\dfrac{5!}{2!} = 60$
required number of arrangement $= 12 \times 60 = 720 = n$
Required number of arrangements are $\dfrac{9!}{2!\,2!\,4!} = 3780$

16. (*a*) $\rightarrow$ (R), (*b*) $\rightarrow$ (P), (*c*) $\rightarrow$ (Q), (*d*) $\rightarrow$ (S)
(*a*) Required number of ways $= (2+1)(3+1)(4+1)-1 = 59$.
(*b*) ${}^{10}C_3 = 120$
(*c*) Requied number of ways
= Coefficient of x^{10} in $(1+x+x^2+.....)^4$
= Coefficient of x^{10} in $(1-x)^{-4} = {}^{10+4-1}C_{4-1} = {}^{13}C_3 = 286$
(*d*) The word 'MATHEMATICS' consists of 11 letters of which 7 are consonants namely M,M, T, T,H,C,S and 4 vowels and a group of consonants can be arranged in $\dfrac{{}^5P_5}{2!} = \dfrac{5!}{2}$ ways. ($\because$ A is repeated twice)
In any such arrangement, seven consonants can be reshuffled among themselves in
$\dfrac{{}^7P_7}{2!\,2!} = \dfrac{7!}{2!\,2!}$ ways.
($\because$ Either of M and T is repeated twice)
Hence, the required number of ways $= \dfrac{5!}{2!} \times \dfrac{7!}{2!2!}$
$= \dfrac{120 \times 5040}{8} = 75600$.

17. (*a*) - (Q) ; (*b*) - (P) ; (*c*) - (S) ; (*d*) - (R)
A) ${}^8C_4 + {}^8C_3 \times {}^5C_1 + {}^8C_2 \times {}^5C_2$
(*b*) The number of ways of selecting 3 points out of 12 points is ${}^{12}C_3$.
Three points out of 7 collinear points can be selected in ${}^7C_3$ ways.
Hence, the number of triangles formed is ${}^{12}C_3 - {}^7C_3 = 185$.
(*c*) ${}^mC_2 \times {}^nC_2$
(*d*) Two circles intersect in 2 points.
$\therefore$ Maximum number of points of intersection of two circles = 2 × number of selections of two circles from 8 circles.
$= 2 \times {}^8C_2 = 2 \times 28 = 56$
$\therefore$ Maximum number of points of intersection of two straight line = 1 × number of selections of two straight line from 4 straight line $= {}^4C_2 = 6$
$\therefore$ Maximum number of points of intersection of one straight line and one circle = 2 × number of selections of one straight line from 4 straight line and number of selections of one circles from 8 circles
$= {}^4C_1 \cdot {}^8C_1 \cdot 2 = 64$

NUMERICAL VALUE BASED

18 [0485]

Man – 7 $\begin{cases} 4L \\ 3G \end{cases}$; Wife – 7 $\begin{cases} 3L \\ 4G \end{cases}$

3L and 3G are to be invited.

Man's	Wife's	Total Ways
3L	3G	${}^4C_3 \cdot {}^4C_3 = 16$
3G	3L	${}^3C_3 \cdot {}^3C_3 = 1$
2L+1G	1L+2G	$({}^4C_2 \cdot {}^3C_1) \times ({}^3C_1 \cdot {}^4C_2) = 324$
1L+2G	2L+1G	$({}^4C_1 \cdot {}^3C_2) \times ({}^3C_2 \cdot {}^4C_1) = 144$

Total = 485.

19. [0008]

$\left[\dfrac{50}{7}\right] + \left[\dfrac{50}{7^2}\right] = 7 + 1 = 8$

20. [0126]
Coefficient x^{10} in $(x + x^2 .....x^5)^6$ = coefficient of x^4 in $(x^0 + x^1 .....x^4)^6 = {}^{6+4-1}C_4 = {}^9C_4 = 126$
Alternatively: Give one apple to each child and then for rest 4 apples $= {}^{4+6-1}C_{6-1} = 126$

21. [0672]

First we select 5 beads from 8 different beads to ${}^8C_5$

Now total number of arrangement is

$${}^8C_5 \times \frac{4!}{2!} = 672$$

22. [0015]

Number divisible by 3 if sum of digits divisible

case-I If $1+2+3+4+8=18$

Number of ways = 120

case-II If $1+2+3+7+8=21$

Number of ways = 120

case-III If $2+3+4+7+8=24$

Number of ways = 120

case-IV If $1+2+0+4+8=15$

Number of ways = 96

case-V If $1+2+0+7+8=18$

Number of ways = 96

case-VI If $2+0+4+7+8=21$

Number of ways = 96

case-VII If $0+1+3+4+7=15$

Number of ways = 96

Total number 744

23. [42]

SERIES S - 2, E - 2, R, I

case-I when all letter distinct is

${}^4C_3 \times 3! = 4 \times 6 = 24$

case-II when 2 letters are same the

$${}^2C_1 \cdot {}^3C_1 \times \frac{3!}{2!} = 2 \cdot 3 \cdot 3 = 18$$

total number is $24 + 18 = 42$

24. [4]

Case -I If all are different then no. of ways is $= {}^6C_3 = 20$

Case-II If three each of two colours, then combination is

3 0 → 2!

2 1 → $2! = 2! + 2! = 4$ ways

Case-III If two each of three colours, then combination is

2 1 0 → 3!

1 1 1 → $1! = 3! + 1! = 7$ ways

Hence required no.is $= 20 + 7 + 4 = 31$

25. [12]

$$\square\square\square\square\square\boxed{2} = \frac{5!}{2!3!} = 10$$

$$\square\square\square\square\square\boxed{3} = \frac{5!}{3!} = 20$$

$$\square\square\square\square\square\boxed{4} = \frac{5!}{2!2!} = 30$$

Hence sum of unit places is

$2 \times 10 + 3 \times 20 + 4 \times 30 = 200$

Hence required sum is

$= 200 \times (10^5 + 10^4 + 10^3 + 10^2 + 10^1 + 10^0) = 200 \times (111111) = 22222200$

26. [20]

First we select one married couple out of 6 married couple i.e. ${}^6C_1$ ways total number of required case

${}^6C_1 \times {}^5C_1 \times {}^4C_1 \times 2 = 6 \times 5 \times 4 \times 2 = 240$

$N = 240$

27. [8]

Using multinomial theorem total number of required selection is

${}^{8+3}C_8 = {}^{11}C_8 = {}^{11}C_3 = 165$

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. *(d)* SMALL Total words $= \frac{\lfloor 5}{\lfloor 2} = 60$

$$\boxed{A}\square\square\square\square \rightarrow \frac{\lfloor 4}{\lfloor 2} = 12$$

$$\boxed{L}\square\square\square\square \rightarrow \lfloor 4 = 24$$

$$\boxed{M}\square\square\square\square \rightarrow \frac{\lfloor 4}{\lfloor 2} = 12$$

$$\boxed{S}\boxed{A}\square\square\square \rightarrow \frac{\lfloor 3}{\lfloor 2} = 3$$

$$\boxed{S}\boxed{L}\square\square\square \rightarrow \lfloor 3 = 6$$

$$\boxed{S}\boxed{M}\boxed{A}\boxed{L}\boxed{L} \rightarrow 1$$

Rank of SMALL is 58th

2. (*d*) x → 4 Ladies, 3 Men; y → 3 Ladies, 4 Men

Total number of ways

$^4C_0 \cdot {}^3C_3 \cdot {}^3C_3 \cdot {}^4C_0 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_3 \cdot {}^3C_0 \cdot {}^3C_0 \cdot {}^4C_3 = 485$

3. (*d*) Number of ways $= \binom{6}{4}\binom{3}{4}4!$

$= 15 \times 3 \times 24 = 1080$

4. (*c*) Number of ways = Total number of ways without restriction – When two specific boys are in team without any restriction, total number of ways of forming team is $^7C_3 \times {}^5C_2 = 350$. If two specific boys B_1, B_2 are in same team then total number of ways of forming team equals to $^5C_1 \times {}^5C_2 = 50$ ways total ways $= 350 - 50 = 300$ ways

5. (*b*) $\boxed{a_1}\boxed{a_2}\boxed{a_3}$

Number of numbers $= 5^3 - 1$

$\boxed{a_4}\boxed{a_1}\boxed{a_2}\boxed{a_3}$

2way for a_4

Numbers of numbers $= 2 \times 5^3$

R equired number $0020 = 5^3 + 2 \times 5^3 - 1$

$= 374$

6. (*a*) $^{10}C_3 = 120$

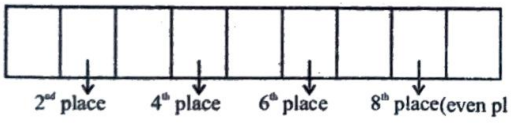
7. (*a*) Let m–men, 2–women

$^mC_2 \times 2 = {}^mC_1 \, {}^2C_1 . 2 + 84$

$m^2 - 5m - 84 = 0 \Rightarrow (m-12)(m+7) = 0$

$m = 12$

8. (*d*)



Number of such numbers $= {}^4C_3 \times \frac{3!}{2!} \times \frac{6!}{2!4!} = 180$

9. (*d*) (*a*) The number of four digit numbers starting with 5 is equal to $6^3 = 216$

(*b*) Starting with 44 and 55 is equal to $36 \times 2 = 72$

(*c*) Starting with 433, 434 and 435 is equal to $6 \times 3 = 18$

(*d*) Remaining numbers are

4322, 4323, 4324, 4325 is equal to 4 so total numbers are

$216 + 72 + 18 + 4 = 310$

10. (*a*) Since there are 8 males and 5 females. Out of these 13, if we select 11 persons, then there will be at least 6 males and at least 3 females in the selection.

$$m = n = \binom{13}{11} = \binom{13}{2} = \frac{13 \times 12}{2} = 78$$

11. (*a*) $\frac{n(n+1)}{2} + 99 = (n-2)^2$

$n^2 + n + 198 = 2(n^2 + 4 - 4n)$

$n^2 - 9n - 190 = 0$

$n^2 - 19n + 10n - 190 = 0$

$n(n-19) + 10(n-19) = 0$

$n = 19$

12. (*b*) sum of given digits 0, 1, 2, 5, 7, 9 is 24. Let the six digit number be abcdef and to be divisible by 11

so $|(a + c + e) - (b + d + f)|$ is multiple of 11.

Hence only possibility is $a + c + e = 12 = b + d + f$

Case-I {a, c, e} = {9, 2, 1} & {b, d, f} = {7, 5, 0}

So Number of numbers $= 3! \times 3! = 36$

Case-II {a,c,e} = {7,5,0} and {b,d,f} = {9,2,1}

So, Number of numbers $2 \times 2! \times 3! = 24$

Total = 60

13. (*c*) Total cases = number of diagonals

$= {}^{20}C_2 - 20 = 170$

14. (*a*) 10 Identical 21 Distinct 1 0

Object

0	10	$^{21}C_{10} \times 1$
1	9	$^{21}C_9 \times 1$
⋮	⋮	⋮
10	0	$^{21}C_0 \times 1$

$^{21}C_0 + ... + {}^{21}C_{10} + {}^{21}C_1 + ... + {}^{21}C_0 = 2^{21}$

$(^{21}C_0 + ... + {}^{21}C_{10}) = 2^{20}$

15. (*a*) $^5C_1 \cdot {}^nC_2 + {}^5C_2 \cdot {}^nC_1 = 1750$

$n^2 + 3n = 700$

$\therefore n = 25$

16. (*b*) win Rs. 15 → number of cases = 6

win Rs. 12 → number of cases = 4

loss Rs. 6 → number of cases = 26

p(expected gain/loss)

$= 15 \times \frac{6}{36} + 12 \times \frac{4}{36} - 6 \times \frac{26}{36} = -\frac{1}{2}$

17. (*d*) 1, 3, 5, 7, 9

For digit to repeat we have $^5C_1$ choice

and six-digit can be arrange in $\frac{6!}{2!}$ ways

Hence, total such numbers $= \frac{5(6!)}{2!}$.

18. [490]

0 Red, 1 Red, 2 Red, 3 Red

Number of ways $= {}^7C_4 + {}^5C_1 \cdot {}^7C_3 + {}^5C_2 \cdot {}^7C_2 + {}^5C_3 \cdot {}^7C_1$

$= 35 + 175 + 210 + 70 = 490.$

19. [2454]

EXAMINATION

2N, 2A, 2I, E, X, M, T, O

Case I

All are different so ${}^8P_4 = \dfrac{8!}{4!} = 8.7.6.5 = 1680$

Case II

2 same and 2 different so ${}^3C_1 \cdot {}^7C_2 \cdot \dfrac{4!}{2!} = 3.21.12 = 756$

Case III

2 same and 2 same so ${}^3C_2 \cdot \dfrac{4!}{2!.2!} = 3.6 = 18$

$\therefore$ Total $= 1680 + 756 + 18 = 2454$

20. (*d*)

			2		

Number of numbers $= 8 \times 8 \times 7 \times 6 = 2688 = 336\,k$

$\Rightarrow \quad k = 8$

21. (*b*) Number of two consecutive stations = n

Number of two non-consecutive stations $= n_{C_2} - n$

Now, According to the question,

$\Rightarrow n_{C_2} - n = 99n$

$\Rightarrow \dfrac{n(n-1)}{\lfloor 2} - 100n = 0$

$\Rightarrow n - 1 - 200 = 0$

$\Rightarrow n = 201$

22. [309]

EHMORT in alphabetical order

E.....5!

H.....5!

M E.....4!

M H.....4!

M O E.....3!

M O H.....3!

M O R.....3!

M O T E.....2!

M O T H E R.....1!

Rank $= 2 \times 5! + 2 \times 4! + 3 \times 3! + 2! + 1 = 309$

23. [135]

Select any 4 questions in ${}^6C_4$ ways which are correct

Number of ways of answering wrong question = 3

$\therefore$ Required number of ways $= {}^6C_4 \times 3^2 = 135$

24. [2250]

Each section has 5 questions.

$\therefore$ Total number of selection of 5 questions

$3 \times {}^5C_1 \times {}^5C_1 \times {}^5C_3 + 3 \times {}^5C_1 \times {}^5C_2 \times {}^5C_2$

$= 3 \times 5 \times 5 \times 10 + 3 \times 5 \times 10 \times 10 = 750 + 1500 = 2250$

25. [240]

LLSSYABU

For two alike and two distinct letters, select any one pair from LL, SS in ${}^2C_1$ ways

Now from rest, select any 2 in ${}^5C_2$ ways and they can be arranged in $\dfrac{4!}{2!}$ ways

$\therefore$ Required number of ways $= {}^2C_1 \times {}^5C_2 \times \dfrac{4!}{2!} = 240$

26. [120]

For vowels not together

1st arrange L,T,T, R in $\dfrac{4!}{2!}$ ways

Then put both E in 5 gaps formed in ${}^5C_2$ ways of ways $\therefore$

No. of ways $\dfrac{4!}{2!}\,{}^5C_2 = 120$

27. (*b*) No. of arrangement $= (\lfloor 3 \times \lfloor 3 \times \lfloor 4) \times \lfloor 3 = (\lfloor 3)^3 \lfloor 4$

28. (*c*) (2I, 4F) + (3I, 6F) + (4I, 8F)

$= {}^6C_2\,{}^8C_4 + {}^6C_3\,{}^8C_6 + {}^6C_4\,{}^8C_8$

$= 15 \times 70 + 20 \times 28 + 15 \times 1 = 1050 + 560 + 15 = 1625$

29. (31650)

$C \to 1 \qquad 9 \begin{cases} A \\ \\ B \end{cases}$

$C \to 2 \qquad 8 \begin{cases} A \\ \\ B \end{cases}$

$C \to 3 \qquad 7 \begin{cases} A \\ \\ B \end{cases}$

$= {}^{10}C_1[2^9 - 2] + {}^{10}C_2[2^8 - 2] + {}^{10}C_3[2^7 - 2]$

$= 2^7[{}^{10}C_1 \times 4 + {}^{10}C_2 \times 2 + {}^{10}C_3] - 20 - 90 - 240$

$= 128[40 + 90 + 120] - 350 = (128 \times 250) - 350$

$= 10[3165] = 31650$

30. (*d*) $x \cdot y \cdot z = 24$

$x \cdot y \cdot z = 2^3 \cdot 3^1$

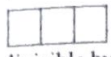
Now using beggars method. 3 things to be distributed among 3 persons Each may receive none, one or more

$\therefore {}^5C_2$ ways

Similarly for '1' $\therefore {}^3C_2$ ways

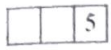
Total ways $= {}^5C_2 \cdot {}^3C_2 = 30$ ways

31. (32)



divisible by $\rightarrow 3$

divisible by 5



$= 12$

4×3

$12 \rightarrow 3, 4, 5 \rightarrow 3! = 6$

$15 \rightarrow 2, 3, 4 \rightarrow 3! = 6$

$24 \rightarrow 1, 3, 5 \rightarrow 3! = 6$

$42 \rightarrow 1, 2, 3 \rightarrow 3! = 6$

Required number $= 24 + 12 - 4 = 32$

32. (45)

$\because 7^n = (10-3)^n = 10K + (-3)^n$

$\therefore 7^n + 3^n = 10K + (-3)^n + 3^n$ → $10K$ if n = odd; → $10K + 2.3^n$ if n = even

Let $n = 2t; t \in N$

$\therefore 3^n = 3^{2t} = (10-1)^t$

$= 10p + (-1)^t$

$= 10p \pm 1$

$\therefore$ If n = even then $7^n + 3^n$ will be multiply of 10

So if n is odd then only $7^n + 3^n$ will be multiply of 10

$\therefore n = 11, 13, 15, \ldots, 99$

$\therefore$ Ans. 45

33. (*a*) Case-I: 1, 1, 1, 1, 1, 2, 3

$$\text{Ways} = \frac{7!}{5!} = 42$$

Case-II: 1, 1, 1, 1, 2, 2, 2

$$\text{Ways} = \frac{7!}{4!3!} = 35$$

Total ways $= 42 + 35 = 77$

34. (*c*) $y + z = 5$ ...(1)

$$\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$$

$$\Rightarrow \frac{y+z}{y^z} = \frac{5}{6}$$

$$\Rightarrow \frac{5}{y^z} = \frac{5}{6}$$

$\Rightarrow yz = 6$

Also $(y-z)^2 = (y+z)^2 - 4yz$

$\Rightarrow (y-z)^2 = (y+z)^2 - 4yz$

$\Rightarrow (y-z)^2 = 25 - 4(6) = 1$

$\Rightarrow y - z = 1$

from (1) and (2), $y = 3$ and $z = 2$ for calculating odd divisor of $p = 2^x \cdot 3^y \cdot 5^z$ x must be zero $P = 2^0 \cdot 3^3 \cdot 5^2$

$\therefore$ total odd divisors must be $(3+1)(2+1) = 12$

35. (1000)

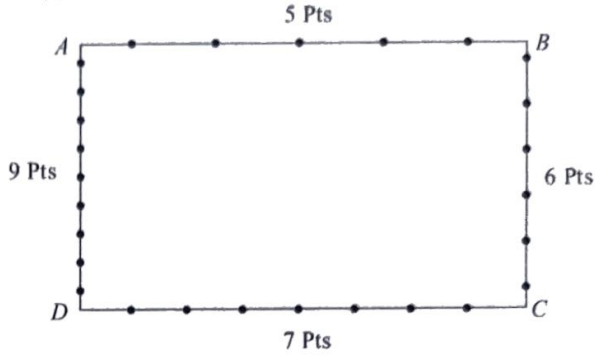
Since, required number has G.C.D with 18 as 3. It must be odd multiple of '3' but not a multiple of '9'.

(*i*) Now, 4-digit number which are odd multiple of '3' are, 1005, 1011, 1017,..... $.9999 \rightarrow 1499$

(*ii*) 4-digit number which are odd multiple of 9 are, 1017, 1035, ..., $9999 \rightarrow 499$

$\because$ Required numbers $= 1499 - 499 = 1000$

36. (*d*)



$\alpha =$ Number of triangles

$\alpha = 5 \cdot 6 \cdot 7 + 5 \cdot 7 \cdot 9 + 5 \cdot 6 \cdot 9 + 6 \cdot 7 \cdot 9$

$= 210 + 315 + 270 + 378$

$= 1173$

$\beta =$ Number of Quadrilateral

$\beta = 5 \cdot 6 \cdot 7 \cdot 9 = 1890$

$\beta = \alpha = 1890 - 1173 = 717$

37. (*c*) Total matches between boys of both team

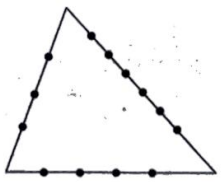
$= {}^7C_1 \times {}^4C_1 = 28$

Total matches between girls of both team $= {}^nC_1 \, {}^6C_1 = 6n$

Now, $28 + 6n = 52$

$\Rightarrow n = 4$

38. (*c*)



Total number of triangles formed

$= {}^{14}C_3 - {}^3C_3 - {}^5C_3 - {}^6C_3 = 333$

39. (*a*) Digits are 1, 2, 2, 3 total distinct numbers $\frac{4!}{2!} = 12$.

Total numbers when 1 at unit place is 3.

2 at unit place is 6, 3 at unit place is 3.

So, sum $= (3 + 12 + 9)(10^3 + 10^2 + 10 + 1)$

$= (1111) \times 24 = 26664$

40. (300)

$3 = 10 \times 10 = 100$

$3 = 10 \times 10 = 100$

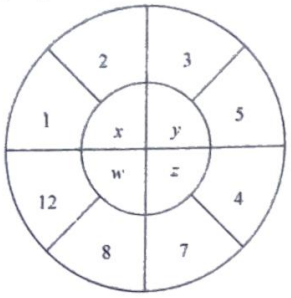
$3 = 10 \times 10 = \dfrac{100}{300}$

41. (4) $x = (2-1)^{11} = 1$

$w = (12-8)^{47} = 4^{24}$

$z = (7-4)^{31} = 3^6$

hence $y = (5-3)^{21} = 2^2$



JEE-ADVANCED PREVIOUS YEARS

1. (*b*)

	B_1	B_2	B_3
Case-1:	1	1	3
Case-2:	2	2	1

Ways of distribution $= \dfrac{5!}{1!1!3!2!} \cdot 3! + \dfrac{5!}{2!2!1!2!} \cdot 3!$

$= 150$

2. (*a*) $\underline{1}$----------$\underline{1}\#a_{n-1}$

----------$\underline{1}\,\underline{0}\#a_{n-2}$

So $a_n = a_{n-1} + a_{n-2}$

So A choice is correct

consider B choice $c_{17} \neq c_{16} + c_{15}$

$c_{15} \neq c_{14} + c_{13}$ is not true

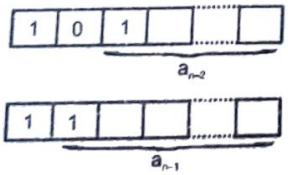
consider C choice $b_{17} \neq b_{16} + c_{16}$

$a_{16} \neq a_{15} + a_{14}$ is not true

consider D choice $a_{17} = c_{17} + b_{16}$

$a_{17} = a_{15} + a_{15}$ which is not true

Aliter



using the Recursion formula $a_n = a_{n-1} + a_{n-2}$

Similarly $b_n = b_{n-1} + b_{n-2}$ and $c_n = c_{n-1} + c_{n-2}$ $\forall$ $n \geq 3$

and $a_n = b_n + c_n \; \forall \; n \geq 1$

so $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8$.....

$b_1 = 1, b_2 = 1, b_3 = 2, b_4 = 3, b_5 = 5, b_6 = 8$ .....

$c_1 = 0, c_2 = 1, c_3 = 1, c_4 = 2, c_5 = 3, c_6 = 5$ .....

using this $b_{n-1} = c_n \; \forall \; n \geq 2$

3. (*b*) $b_6 = a_5$

$a_5 = \underline{1}\text{---}\underline{1} \quad \underline{1}\text{---}\underline{0}$

${}^3C_0 + {}^3C_1 + 1 + {}^2C_1 + 1$

$1 + 3 + 1 + 2 + 1$

$4 + 4 = 8$

4. (7) $n_2 = n_1 + t_1 + 1$

$n_3 = n_2 + t_2 + 1$

$n_4 = n_3 + t_3 + 1$

$n_5 = n_4 + t_4 + 1$

The given equation becomes

$5n_1 + 4t_1 + 3t_2 + 2t_3 + t_4 = 10$

where $n_1 \geq 1 \; ; t_1 \geq 0$

$n_1 = t_0 + 1 \Rightarrow 5t_0 + 4t_1 + 3t_2 + 2t_3 + t_4 = 5$

$t_0 = 1$ will yield only 1 solution.

so $t_0 = 0$,

$4t_1 + 3t_2 + 2t_3 + t_4 = 5$.

$t_1 = 0 = t_2$. there will be 3 solution

$t_1 = 0, t_2 = 1$ will yield 2 solution.

$t_1 = 1$, t_2 must be zero 1 solution.

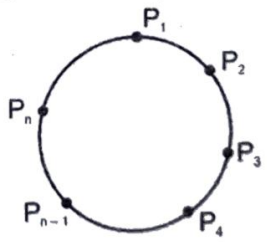
Hence in total there will be 7 solution.

Alternative:

n_1	n_2	n_3	n_4	n_5
1	2	3	4	10
1	2	3	5	9
1	2	3	6	8
1	2	4	5	7
1	2	4	6	8
1	3	4	6	7
2	3	4	5	6

5. (5) Number of adjacent lines = n

Number of line segment joining non-adjacent points is ${}^nC_2 - n$.



Now, $n = ({}^nC_2 - n) \Rightarrow 2n = \dfrac{n(n-1)}{2} \Rightarrow n = 0, 5$

But $n \geq 2$. so, $n = 5$.

6. (c)

Cards	Envelopes
1	1
2	2
3	3
4	4
5	5
6	6

If '2' goes in '1' then it is dearrangement of 4 things which can be done in $4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=9$ ways.

If '2' doen't go in 1, it is dearrangement of 5 things which can be done in 44 ways. Hence total 53 ways.

7. [5] $n = 5! \times 6!$

$m = 5! \times {}^6C_2 \times {}^5C_4 .2!.4!$

$\frac{m}{n} = \frac{5! \times 15 \times 2 \times 5!}{5! \times 6!} = 5.$

8. (a) If there is no boy then, no. of ways $= {}^6C_4 \times {}^4C_1 = 60$

If one boy is there, then, no. of ways $= {}^4C_1 \times {}^6C_3 \times ({}^1C_1 + {}^3C_1) = 320$

Hence, total no. of ways = 380

Hence, (a)

9. (c) $N_1 = {}^5C_1 . {}^4C_4 = 5$

$N_2 = {}^5C_2 . {}^4C_3 = 40$

$N_3 = {}^5C_3 . {}^4C_2 = 60$

$N_4 = {}^5C_4 . {}^4C_1 = 20$

$N_5 = {}^5C_5 . {}^4C_0 = 1$ Total = 126

10. (5) A, B, C, D, E, F, G, H, I, J

$x = 10!$

$y = {}^{10}C_1 . {}^{10}C_2 . 8! \, {}^9C_8$

$\frac{y}{9x} = \frac{{}^{10}C_1 . {}^{10}C_2 . 8! \times 9}{9 \times 10!} = \frac{10! \times 45}{9 \times 10!} = 5$

11. (625)

Option for last two digits are

(12), (24), (32), (44) are (52).

$\therefore$ Total No. of digits $= 5 \times 5 \times 5 \times 5 = 625$

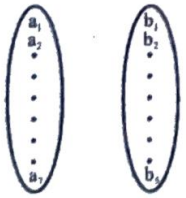
12. (119)

$n(X) = 5$

$n(Y) = 7$

$\alpha \rightarrow$ Number of one-one function ${}^7C_5 \times 5!$

$\beta \rightarrow$ Number of onto function Y to X



1, 1, 1, 1, 3 1, 1, 1, 2, 2,

$\frac{7!}{3!4!} \times 5! + \frac{7}{(2!)^3 3!} \times 5! = ({}^7C_3 + 3.{}^7C_3)5! = 4 \times {}^7C_3 \times 5!$

$\frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$

13. (c)

(a) $\alpha_1 = \binom{6}{3}\binom{5}{2} = 200$

So $P \rightarrow 4$

(b) $\alpha_2 = \binom{6}{1}\binom{5}{1} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{3} + \binom{6}{4}\binom{5}{4} + \binom{6}{5}\binom{5}{5}$

$= \binom{11}{5} - 1$

$= 46!$

So $Q \rightarrow 6$

(c) $\alpha_3 = \binom{5}{2}\binom{6}{3} + \binom{5}{3}\binom{6}{2} + \binom{5}{4}\binom{6}{1} + \binom{5}{5}\binom{6}{0}$

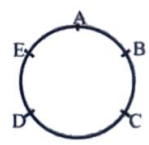
$= 381$

So $R \rightarrow 5$

(d) $\alpha_2 = \binom{5}{2}\binom{6}{2} - \binom{4}{1}\binom{5}{1} + \binom{5}{3}\binom{6}{1} - \binom{4}{2}\binom{1}{1} + \binom{5}{4} = 189$

So $R \rightarrow 2$

14. (30.00)



When 1R, 2B, 2G

$5C_1 \times 2 = 10$

Other possibilities

1B, 2R, 2G

or 1G, 2R, 2B

So total no. of ways $= 3 \times 10 = 30$

15. [495.00]

Selection of 4 days out of 15 days such that no two of them are consecutive

$= {}^{15-4+1}C_4 = {}^{12}C_4$

$\frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2} = 11 \times 5 \times 9 = 495$

16. (1080.00)

required ways $= \frac{6!}{2!2!1!1!2!2!} \times 4! = 1080$

4 BINOMIAL THEOREM

PRAYAS 1.0 : t.me/realprayaas
2.0 : t.me/realprayaas2

Properties Related to ${}^nC_r$:

(*i*) ${}^nC_r = {}^nC_{n-r}$

Note: If ${}^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(*ii*) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(*iii*) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(*iv*) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = \ldots\ldots$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-(r-1))}{r\,(r-1)(r-2)\ldots\ldots 2.1}$$

(*v*) If n and r are relatively prime, then ${}^nC_r$ is divisible by n. But converse is not necessarily true.

BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power. In general for a positive integer n

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + \ldots\ldots + {}^nC_nx^0y^n$, where

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

for $r = 0, 1, 2, \ldots\ldots, n$ is called binomial coefficient.

Observations:

(*a*) The number of terms in the expansion is $(n+1)$ i.e. one more than the index.

(*b*) The sum of the indices of x & y in each term is n.

(*c*) The binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$...) equidistant from the beginning and the end are equal.

GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x+y)^n$ is $(r+1)^{th}$ term, given by $T_{r+1} = {}^nC_r\,x^{n-r}\,y^r$ where $r = 0, 1, 2 \ldots\ldots n$.

- Every term in the expansion is of nth degree in variables x and y.
- The total number of terms in the expansion is $n+1$.
- Binomial expansion can also be expressed as $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\,x^{n-r}\,y^r$.
- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- **Term independent of x:** Term independent of x contains no x; Hence find the value of r for which the exponent of x is zero.

TRAIN YOUR BRAIN

Example 1: Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

Sol. The general term $= {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(\frac{-1}{3x}\right)^r$

$$= (-1)^r\ {}^9C_r\,\frac{3^{9-2r}}{2^{9-r}}\,x^{18-3r}$$

The term independent of x, (or the constant term) corresponds to x^{18-3r} being

x^0 or $18-3r=0 \quad \Rightarrow r=6$

$\therefore$ the term independent of x is the 7th term and its value is

$$(-1)^6\ {}^9C_6\,\frac{3^{9-12}}{2^{9-6}} = {}^9C_3\,\frac{3^{-3}}{2^3} = \frac{9.8.7}{3.2.1}\cdot\frac{1}{(6)^3} = \frac{7}{18}$$

Example 2: Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Sol. The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\,3^{\frac{1000-r}{2}}\,2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers. i.e. $\dfrac{1000-r}{2}$ and $\dfrac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ...1000}. Hence, number of rational terms is 501

MIDDLE TERMS OF THE EXPANSION

In the binomial expansion of $(x+y)^n$

WHEN n IS ODD

There are $(n+1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \text{ for } r = \frac{n-1}{2}$$

and $$t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}} \text{ for } r = \frac{n+1}{2}$$

WHEN n IS EVEN

There are odd terms in the expansion and hence, only one middle term is given by:

$$t_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2} \text{ for } r = \frac{n}{2}$$

Note: The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

Example 3: Find the middle term in the expression of $(1-2x+x^2)^n$.

Sol. $(1-2x+x^2)^n = [(1-x)^2]^n = (1-x)^{2n}$

Here, $2n$ is even integer, therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$

i.e. $(n+1)^{\text{th}}$ term will be the middle term.

Now $(n+1)^{\text{th}}$ term in $(1-x)^{2n} = {}^{2n}C_n (1)^{2n-n}(-x)^n$

$= {}^{2n}C_n (-x)^n = \frac{(2n)!}{n!n!}(-x)^n$

Example 4: Find the middle term in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$

Sol. The number of terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+3}{2}\right)^{\text{th}}$ two middle terms. They are given by T_5 and T_6

$$\therefore\ T_5 = T_{4+1} = {}^9C_4 (3x)^5 \left(-\frac{x^3}{6}\right)^4$$

$${}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4} = \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{3^5}{2^4 3^4} x^{17} = \frac{189}{8} x^{17}$$

and $T_6 = T_{5+1} = {}^9C_5 (3x)^4 \left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4 . x^4 \frac{x^{15}}{6^5}$

$$= \frac{-9.8.7.6}{1.2.3.4} \cdot \frac{3^4}{2^5 . 3^5} x^{19} = -\frac{21}{16} x^{19}$$

GREATEST TERM IN THE EXPANSION

The numerically greatest term (absolute value) in the expansion of $(1+x)^n$ is determined by the following process:

Let t_{r+1} be the numerically greatest term, then $|t_r| \le |t_{r+1}| \ge |t_{r+2}|$

Considering, $\left|\frac{t_{r+1}}{t_r}\right| \ge 1 \Rightarrow \left|\frac{n-r+1}{r} x\right| \ge 1$

$$\Rightarrow \frac{n-r+1}{r}|x| \ge 1 \Rightarrow (n-r+1)|x| \ge r$$

$$\Rightarrow r \le (n+1)\frac{|x|}{1+|x|} \quad \ldots(i)$$

Also, $\left|\frac{t_{r+1}}{t_{r+2}}\right| \ge 1 \Rightarrow \left|\frac{r+1}{n-r}\frac{1}{x}\right| \ge 1 \Rightarrow r+1 \ge (n-r)|x|$

$$\Rightarrow r \ge \frac{n|x|-1}{1+|x|}$$

$$\Rightarrow r \ge (n+1)\frac{|x|}{1+|x|} - 1 \quad \ldots(ii)$$

Thus, from (i) and (ii) for t_{r+1} to be the greatest term

$r \le m$ and $r \ge m-1$ where, $m = (n+1)\frac{|x|}{1+|x|}$

For expansion of $(a+b)^n$

Case - I When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 \ldots, m-1)$
i.e., $T_2 > T_1, T_3 > T_2, \ldots\ldots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when
$r = m$
i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when
$r > m$ $(r = m+1, m+2, \ldots\ldots n)$
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots T_{n+1} < T_n$

Conclusion:

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude).

Case - II

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(*i*) $T_{r+1} > T_r$ when $r < \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ $(r = 1, 2, 3, \ldots, m-1, m)$

i.e. $T_2 > T_1, T_3 > T_2, \ldots\ldots\ldots\ldots, T_{m+1} > T_m$

(*ii*) $T_{r+1} < T_r$ when $r > \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ $(r = m+1, m+2, \ldots\ldots n)$

i.e., $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots\ldots\ldots\ldots, T_{n+1} < T_n$

Conclusion:

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note: (*i*) In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a+b)^n$

If *n*	**No. of greatest binomial coeff.**	**Greatest binomial coefficient**
Even →	1	${}^nC_{n/2}$
Odd →	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal).

(*ii*) In order to obtain the term having numerically greatest coefficient, put $a = b = 1$, and proceed as discussed above.

In general, for the expansion $(a+x)^n$ or $a^n\left(1+\dfrac{x}{a}\right)^n$ we can consider $m = (n+1)\dfrac{|x/a|}{1+|x/a|}$.

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Example 5: Find the greatest term in the expansion of $(4+3x)^7$ when $x = \dfrac{2}{3}$.

Sol. Here the greatest term means the numerically greatest term.

$$\left|\frac{t_{r+1}}{t_r}\right| = \frac{{}^7C_r\,4^{7-r}(3x)^r}{{}^7C_{r-1}\,4^{8-r}(3x)^{r-1}} = \frac{8-r}{r}\frac{3x}{4} = \frac{8-r}{2r} \text{ since } x = 2/3$$

Now $|t_{r+1}| \geq |t_r|$ if $8-r \geq 2r$ or $\dfrac{8}{3} \geq r$

This inequality is valid only for $r = 1$ or 2

Thus, for $r = 1, 2$; $|t_{r+1}| > |t_r|$ and

for $r = 3, 4$; $|t_{r+1}| < |t_r|$

$\therefore \quad |t_1| < |t_2| < |t_3| > |t_4| > |t_5| > \ldots\ldots\ldots$

greatest term $= |t_3| = {}^7C_2\,4^5\cdot(3x)^2$ where $x = \dfrac{2}{3}$

$= 21 \times 4^5 \times 2^2 = 86016$

SUMMATION OF SERIES INCLUDING BINOMIAL COEFFICIENTS

In the binomial expansion of $(1+x)^n$, let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots\ldots, {}^nC_n$ by $C_0, C_1, C_2, \ldots\ldots, C_r, \ldots\ldots, C_n$ respectively.

- **The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n :**

  $\because \quad (1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_nx^n$

- **The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.**

  $\therefore \quad C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

  sum of coefficients of odd terms = sum of coefficients of even terms $= 2^{n-1}$

Some Results on Binomial Coefficients

(*a*) $C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_n = 2^n$

(*b*) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(*c*) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

(*d*) $C_0 \cdot C_r + C_1 \cdot C_{r+1} + C_2 \cdot C_{r+2} + \ldots + C_{n-r}$

$$C_n = \frac{(2n)!}{(n+r)!(n-r)!} = {}^{2n}C_{n-r}$$

Remember: $(2n)! = 2^n \cdot n!\,[1.3.5\ldots\ldots\ldots(2n-1)]$

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Example 6: Prove that the sum of the coefficients in the expansion of $(1+x-3x^2)^{2163}$ is -1.

Sol. Putting $x = 1$ in $(1+x-3x^2)^{2163}$, the required sum of coefficients

$$= (1+1-3)^{2163} = (-1)^{2163} = -1.$$

- **Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form $k \cdot {}^nC_r$.**

Example 7: Show that $3 \cdot C_0 + 7 \cdot C_1 + 11 \cdot C_2 + \ldots\ldots + (4n+3)\,C_n = (2n+3)\,2^n$.

Sol. This problem can be done be differentiating the expansion of $x^3(1+x^4)^n$ and putting $x = 1$.

$$x^3(1+x^4)^n = x^3(C_0 + C_1x^4 + C_2x^8 + \ldots\ldots + C_nx^{4n})$$
$$= C_0x^3 + C_1x^7 + C_2x^{11} + \ldots\ldots + C_nx^{4n+3}$$

Differentiating we get,

$$\Rightarrow \quad 3x^2(1+x^4)^n + x^3n(1+x^4)^{n-1}4x^3$$
$$= 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + \ldots\ldots + (4n+3)x^{4n+2}C_n$$

Now substituting $x = 1$ in both sides.

$$\Rightarrow \quad 3C_0 + 7C_1 + 11C_2 + \ldots\ldots + (4n+3)\,C_n$$
$$= 3(2^n) + 4n\,(2)^{n-1} = (3+2n)\,2^n$$

- **Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\frac{{}^nC_r}{k}$.**

Example 8: Show that

$$2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

Sol. Integrating the expansion of $(1 + x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n\,dx = \int_0^2 (C_0 + C_1x + ... + C_nx^n)\,dx$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1}\Big|_0^2 = C_0x + C_1\frac{x^2}{2} + ... + C_n\frac{x^{n+1}}{n+1}\Big|_0^2$$

$$\Rightarrow 2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

2ND Method: L.H.S.

$$= \frac{1}{n+1}\left\{{}^{(n+1)}C_1.2 + {}^{(n+1)}C_2.2^2 + {}^{(n+1)}C_3.2^3 + ... + {}^{(n+1)}C_{n+1}.2^{(n+1)}\right\}$$

$$= \frac{1}{n+1}\left\{1 + {}^{(n+1)}C_1.2 + {}^{(n+1)}C_2.2^2 + ... + {}^{(n+1)}C_{n+1}.2^{(n+1)} - 1\right\}$$

$$= \frac{1}{(n+1)}\left\{(1+2)^{n+1} - 1\right\} = \frac{3^{n+1}-1}{n+1}$$

- **Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.**

Example 9: If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+C_nx^n$ then prove that $\sum\sum_{0\le i<j\le n} (C_i + C_j)^2 = (n-1)^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum\sum_{0\le i<j\le n} (C_i + C_j)^2 = (C_0 + C_1)^2 + (C_0 + C_2)^2 + ...$

$$+ (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + ... + (C_1 + C_n)^2$$
$$+ (C_2 + C_3)^2 + (C_2 + C_4)^2 + ... + (C_2 + C_n)^2 + ... + (C_{n-1} + C_n)^2$$
$$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2\sum\sum_{0\le i<j\le n} C_iC_j$$

$$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \frac{2n!}{2.n!n!}\right\}$$

$$= n\,.\,{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1)\,.\,{}^{2n}C_n + 2^{2n} = \text{R.H.S.}$$

Important Results

If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integers, n being odd, and $0 \le f < 1$, then show that $(I + f)f = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof. Given $\sqrt{P} - Q < 1$

$\therefore \quad 0 < (\sqrt{P} - Q)^n < 1$

Now let $\left(\sqrt{P} - Q\right)^n = f'$ where $0 < f' < 1$

$\therefore \quad I + f - f' = (\sqrt{P} + Q)^n - \left(\sqrt{P} - Q\right)^n$

$\because$ RHS contains even powers of $\sqrt{P}$ ($\because n$ is odd)

$\Rightarrow$ RHS is an integer

Since RHS and I are integers,

$\therefore \quad f - f'$ is also integer.

$\therefore \quad \Rightarrow f - f' = 0$

$\because \quad -1 < f - f' < 1$

or $\quad f = f'$

$\therefore (I + f)f = (I + f)f' = (\sqrt{P} + Q)^n(\sqrt{P} - Q)^n$

$$= (P - Q^2)^n = k^n$$

If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integer, n being even, and $0 \le f < 1$, then show that $(I + f)(1 - f) = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof: If n is an even integer then

$$(\sqrt{P} + Q)^n + (\sqrt{P} - Q)^n = I + f + f'$$

Hence, LHS and I are integer.

$\therefore \quad f + f'$ is also integer.

$\Rightarrow \quad f + f' = 1 \qquad \because 0 < f + f' < 2$

$\therefore \quad f' = (1 - f)$

Hence

$$(I + f)(1 - f) = (I + f)f' = (\sqrt{P} + Q)^n(\sqrt{P} - Q)^n$$
$$= (P - Q^2)^n = k^n.$$

TRAIN YOUR BRAIN

Example 10: If $(2+\sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that

(*i*) I is an odd integer and

(*ii*) $(I + f)(1 - f) = 1$.

Sol. (*i*) Now $0 < 2 - \sqrt{3} < 1$, since $2 - \sqrt{3} = 0.268$ (approx.)

$\therefore \quad 0 < (2 - \sqrt{3})^n < 1$;

we can take $(2 - \sqrt{3})^n$ as f'.

Now $(2 + \sqrt{3})^n + (2 - \sqrt{3})^n = I + f + f'$

But L.H.S.

$= 2\{2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + ...\}$

= an integer

(in fact an even integer)

$\therefore$ RHS. $= I + f + f'$ = an even integer

Also $f + f' = 1$, since f and f' are both positive proper fractions.

$\therefore$ I = an even integer -1 = an odd integer.

(*ii*) $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n.(2-\sqrt{3})^n$

$= (4-3)^n = 1^n = 1$

DIVISIBILITY PROBLEMS

Working Rule:

1. First of all write down the given expression in such a way that there is a term containing n^{th} power of an integer a.
2. If an occurs, then go on subtracting 1, 2, 3, ... from a and find integer r such that some power of $(a-r)$ is divisible by the number k from which the given expression is to be shown to be divisible.
3. Now write $a_n = [r + (a-r)]^n$ and expand and then collect the terms containing $(a-r)^m$ and higher powers of $(a-r)$ in one bracket if $(a-r)^m$.

TRAIN YOUR BRAIN

Example 11: Show that $3^{2n+2} - 8n - 9$ is divisible by 64 if $n \in N$.

Sol. $3^{2n+2} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= [1 + (n+1)8 + {}^{n+1}C_2 8^2 + ...] - 8n - 9$

$= {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + {}^{n+1}C_4 8^4 + ...$

$= 64[{}^{n+1}C_2 + {}^{n+1}C_3 8 + {}^{n+1}C_4 8^2 + ...]$

which is clearly divisible by 64.

MULTINOMIAL EXPANSION

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + ... + x_k)^n$ is

$$\frac{n!}{a_1!a_2!a_3!.....a_k!} x_1^{a_1} . x_2^{a_2} . x_3^{a_3} .... x_k^{a_k}$$

where $a_1 + a_2 + a_3 + ... + a_k = n$

and $0 \le a_i \le n, i = 1, 2, 3, ... k.$

and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

The greatest coefficient in the expansion of

$$(a_1 + a_2 + a_3 + ... + a_m)^n \text{ is } \frac{n!}{(q!)^{m-r}((q+1)!)^r}$$

where q is the quotient and r is the remainder when n is divided by m.

TRAIN YOUR BRAIN

Example 12: Find the co-efficient of $a^2b^3c^5$ in the expansion of $(a+b+c)^{10}$.

Sol. General term is $\frac{10!}{a_1!a_2!a_3!}(a)^{a_1}(b)^{a_2}(c)^{a_3}$

$\therefore$ $a_1 = 2, a_2 = 3, a_3 = 5$

$\therefore$ Co-efficient of at $a^2b^3c^5 = \frac{10!}{2!3!5!}$.

BINOMIAL THEOREM FOR ANY INDEX

If $n \in R, -1 < x < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r + ...\infty$$

Note:

(*i*) In the expansion if $n \in R$ and $n > 0$ then $-1 < x < 1$.

(*ii*) ${}^nC_r$ can not be used because it is defined only for natural number.

(*iii*) If x be so small then its square and higher powers may be neglected, then approximate value of $(1+x)^n$ $1 + nx$.

Important Results:

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$
- $(1+x)^{-2} = 1 - {}^2C_1 x + {}^3C_2 x^2 - {}^4C_3 x^3 + ...+ (-1)^r {}^{r+1}C_r x^r + ...$
- $(1-x)^{-2} = 1 + {}^2C_1 x + {}^3C_2 x^2 + {}^4C_3 x^3 + ... + {}^{r+1}C_r x^r + ...$
- $(1+x)^{-3} = 1 - {}^3C_1 x + {}^4C_2 x^2 - {}^5C_3 x^3 + ...+ (-1)^r {}^{r+2}C_r x^r + ...$
- $(1-x)^{-3} = 1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + ...+ {}^{r+2}C_r x^r + ...$

In general, the coefficient of x^n in $(1-x)^{-k}$ is ${}^{n+k-1}C_{k-1}$

APPROXIMATIONS

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately,

This is an approximate value of $(1+x)^n$

TRAIN YOUR BRAIN

Example 13: If x is so small such that its square and higher powers may be neglected then find the approximate value of

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$$

Sol. $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \dfrac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$$

EXPONENTIAL SERIES

(*a*) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(*b*) Logarithms to the base '*e*' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(*c*) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\infty$; where x may be any real or complex number & $e = \underset{n\to\infty}{\text{Lim}}\left(1+\frac{1}{n}\right)^n$

(*d*) $a^x = 1+\frac{x}{1!}\ell na+\frac{x^2}{2!}\ell n^2 a+\frac{x^3}{3!}\ell n^3 a+\ldots\infty$
where $a>0$

(*e*) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\infty$

LOGARITHMIC SERIES

(*a*) $\ell n(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ where $-1<x\le 1$

(*b*) $\ell n(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ where $-1\le x<1$

Remember:

(*i*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\infty = \ell n\ 2$

(*ii*) $e^{\ell n x} = x$

(*iii*) $\ell n 2 = 0.693$

(*iv*) $\ell n\ 10 = 2.303$

Topicwise Questions

General Term

1. 6th term in expansion of $\left(2x^2 - \frac{1}{3x^2}\right)^{10}$ is

(a) $\frac{4580}{17}$ (b) $-\frac{896}{27}$

(c) $\frac{5580}{17}$ (d) None of these

2. If the coefficients of rth term and $(r+4)^{th}$ term are equal in the expansion of $(1+x)^{20}$, then the value of r will be

(a) 7 (b) 8

(c) 9 (d) 10

3. 16th term in the expansion of $(\sqrt{x} - \sqrt{y})^{17}$ is

(a) $136xy^7$ (b) $136\,xy$

(c) $-136\,xy^{15/2}$ (d) $-136\,xy^2$

4. In $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7th term from the beginning to the 7th term from the end is 1/6, then n is equal to

(a) 7 (b) 8

(c) 9 (d) 5

5. If x^4 occurs in the rth term in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$, then $r =$

(a) 7 (b) 8

(c) 9 (d) 10

6. If the third term in the binomial expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is

(a) 2 (b) 1/2

(c) 3 (d) 4

7. Coefficient of x in the expansion of $\left(x^2 + \frac{a}{x}\right)^5$ is

(a) $9a^2$ (b) $10a^3$

(c) $10a^2$ (d) $10a$

8. In the expansion of $\left(x - \frac{1}{x}\right)^6$, the constant term is

(a) – 20 (b) 20

(c) 30 (d) –30

9. In the expansion of $(x^2 - 2x)^{10}$, the coefficient of x^{16} is

(a) –1680 (b) 1680

(c) 3360 (d) 6720

10. The coefficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is

(a) ${}^{15}C_5$ (b) ${}^{15}C_6$

(c) ${}^{15}C_4$ (d) ${}^{15}C_7$

11. If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55

(c) 45 (d) 15

12. In the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, the term independent of x is

(a) ${}^9C_3 \cdot \frac{1}{6^3}$ (b) ${}^9C_3\left(\frac{3}{2}\right)^3$

(c) ${}^9C_3$ (d) ${}^9P_2\left(\frac{3}{2}\right)^3$

13. The term independent of x in the expansion of $\left(x^2 - \frac{3\sqrt{3}}{x^3}\right)^{10}$ is

(a) 153090 (b) 150000

(c) 150090 (d) 153180

14. The term independent of x in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ is

(a) $C_0^2 + 2C_1^2 + ... + (n+1)C_n^2$

(b) $(C_0 + C_1 + ... + C_n)^2$

(c) $C_0^2 + C_1^2 + ... + C_n^2$

(d) $C_0^2 + 2C_1^2 + ... + nC_n^2$

15. In the expansion of $(1 + 3x + 2x^2)^6$ the coefficient of x^{11} is

(a) 144 (b) 288

(c) 216 (d) 576

Middle Term/Numerically Greatest term and Algebrically Greatest & Least Term

16. If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924\,x^6$, then $n =$

(a) 10 (b) 12

(c) 14 (d) 16

17. The middle term in the expansion of $(1+x)^{2n}$ is

(a) $\frac{(2n)!}{n!}x^2$ (b) $\frac{(2n)!}{n!(n-1)!}x^{n+1}$

(c) $\frac{(2n)!}{(n!)^2}x^n$ (d) $\frac{(2n)!}{(n+1)!(n-1)!}x^n$

18. The middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$, is

(a) $\frac{1.3.5....(2n-3)}{n!}$ (b) $\frac{1.3.5....(2n-1)}{n!}$

(c) $\frac{1.3.5....(2n+1)}{n!}$ (d) $\frac{1.3.5....(2n+3)}{n!}$

Properties of Binomial Coefficient

19. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n =$

(a) $\frac{(2n)!}{(n-r)!(n+r)!}$ (b) $\frac{n!}{(-r)!(n+r)!}$

(c) $\frac{n!}{(n-r)!}$ (d) $\frac{2n!}{(n-r)!}$

20. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+....+\frac{C_n}{n+1}=$

(a) $\frac{2^n}{n+1}$ (b) $\frac{2^n-1}{n+1}$

(c) $\frac{2^{n+1}-1}{n+1}$ (d) $\frac{2^{n+1}+1}{n+1}$

21. $\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+.... =$

(a) $\frac{2^n}{n!}$; for all even values of n

(b) $\frac{2^{n-1}}{n!}$; for all values of n *i.e.*, all even odd values

(c) 0

(d) $\frac{2n}{n!}$; for all odd value of n

22. The sum of all the coefficients in the binomial expansion of $(x^2+x-3)^{119}$ is

(a) 1 (b) 2

(c) -1 (d) 0

23. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

(a) 256 (b) 128

(c) 512 (d) 64

24. The value of ${}^{15}C_0^2 - {}^{15}C_1^2 + {}^{15}C_2^2 - .... - {}^{15}C_{15}^2$ is

(a) 15 (b) -15

(c) 0 (d) 51

Application of Binomial Theorem

25. The greatest integer which divides the number $101^{100}-1$, is

(a) 100 (b) 1000

(c) 10000 (d) 100000

26. The last digit in 7^{300} is

(a) 7 (b) 9

(c) 1 (d) 3

27. The number 111.....1 (91 times) is

(a) Not a prime (b) An even number

(c) Not an odd number (d) Prime Number

Multinomial theorem and other index

28. In the expansion of $\left(\frac{1+x}{1-x}\right)^2$, the coefficient of x^n will be [where $|x|<1$]

(a) $4n$ (b) $4n-3$

(c) $4n+1$ (d) $4n-1$

29. The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal, if

(a) n is any integer (b) n is an odd integer

(c) n is an even integer (d) n is prime number

Learning Plus

1. The $(m+1)^{th}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:

(a) independent of x

(b) a constant

(c) depends on the ratio x/y and m

(d) none of these

2. The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification is :

(a) 50

(b) 202

(c) 51

(d) none of these

3. If the 6th term in the expansion of the binomial $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then $x =$

(a) 10 (b) 8
(c) 11 (d) 9

4. If the second term of the expansion $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:

(a) 4 (b) 3
(c) 12 (d) 6

5. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$ is :

(a) 56 (b) 65
(c) 154 (d) 62

6. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:

(a) 730 (b) 729
(c) 725 (d) 750

7. The term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is:

(a) −3 (b) 0
(c) 1 (d) 3

8. The value of m, for which the coefficients of the $(2m + 1)^{th}$ and $(4m + 5)^{th}$ terms in the expansion of $(1 + x)^{10}$ are equal, is

(a) 3 (b) 1
(c) 5 (d) 8

9. If the coefficients of x^7 & x^8 in the expansion of $\left[2 + \frac{x}{3}\right]^n$ are equal, then the value of n is

(a) 15 (b) 45
(c) 55 (d) 56

10. Number of rational terms in the expansion of $(\sqrt{2} + \sqrt[4]{3})^{100}$ is

(a) 25 (b) 26
(c) 27 (d) 28

11. If $n \in N$ & n is even, then

$$\frac{1}{1.(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \ldots + \frac{1}{(n-1)!1!}$$ equals

(a) 2^n (b) $\frac{2^{n-1}}{n!}$
(c) $2^n\, n!$ (d) None of these

12. If $k \in R$ and the middle term of $\left(\frac{k}{2} + 2\right)^8$ is 1120, then value of k is:

(a) 3 (b) 2
(c) −3 (d) −4

13. The sum of the binomial coefficients of $\left[2x + \frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is

(a) 1120 (b) 2110
(c) 1210 (d) None of these

14. The sum of the co-efficients in the expansion of $(1 - 2x + 5x^2)^n$ is 'a' and the sum of the co-efficients in the expansion of $(1 + x)^{2n}$ is b. Then

(a) $a = b$ (b) $a = b^2$
(c) $a^2 = b$ (d) $ab = 1$

15. The binomial expansion of $\left(x^k + \frac{1}{x^{2k}}\right)^{3n}$, $n \in N$ contains a term independent of x

(a) only if k is an integer
(b) only if k is a natural number
(c) only if k is rational
(d) for any real k

16. If the expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, then x lies in the interval.

(a) $(-2/3, 2/3)$ (b) $(-3/2, 3/2)$
(c) $(-1/3, 2/3)$ (d) $(-\infty, -3/2) \cup (3/2, \infty)$

17. Coefficient of α^t in the expansion of $(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha+q) + (\alpha+p)^{m-3}(\alpha+q)^2 + \ldots (\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is

(a) $\frac{{}^mC_t(p^t - q^t)}{p-q}$ (b) $\frac{{}^mC_t(p^{m-t} - q^{m-t})}{p-q}$

(c) $\frac{{}^mC_t(p^t + q^t)}{p-q}$ (d) $\frac{{}^mC_t(p^{m-t} + q^{m-t})}{p-q}$

18. The greatest terms of the expansion $(2x + 5y)^{13}$ when $x = 10$, $y = 2$ is

(a) ${}^{13}C_5 . 20^8 . 10^5$ (b) ${}^{13}C_6 . 20^7 . 10^4$
(c) ${}^{13}C_4 . 20^9 . 10^4$ (d) none of these

19. Find numerically greatest term in the expansion of $(2 + 3x)^9$, when $x = 3/2$.

(a) ${}^9C_6 . 2^9 . (3/2)^{12}$ (b) ${}^9C_3 . 2^9 . (3/2)^6$
(c) ${}^9C_5 . 2^9 . (3/2)^{10}$ (d) ${}^9C_4 . 2^9 . (3/2)^8$

20. The numerically greatest term in the expansion of $(2x + 5y)^{34}$, when $x = 3$ & $y = 2$ is :

(a) T_{21} (b) T_{22}
(c) T_{23} (d) T_{24}

21. $\sum_{r=0}^{n-1} \frac{{}^{n}C_r}{{}^{n}C_r + {}^{n}C_{r+1}} =$

(a) $\frac{n}{2}$ (b) $\frac{n+1}{2}$

(c) $(n+1)\frac{n}{2}$ (d) $\frac{n(n-1)}{2(n+1)}$

22. Set of values of r for which, ${}^{18}C_{r-2} + 2.{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains

(a) 4 elements (b) 5 elements

(c) 7 elements (d) 10 elements

23. $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_{10}}{11} = (n = 10)$

(a) $\frac{2^{11}}{11}$ (b) $\frac{2^{11}-1}{11}$

(c) $\frac{3^{11}}{11}$ (d) $\frac{3^{11}-1}{11}$

24. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to:

(a) ${}^{47}C_5$ (b) ${}^{52}C_5$

(c) ${}^{52}C_4$ (d) ${}^{49}C_4$

25. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots + \binom{50}{49}\binom{50}{50}$ is, where ${}^{n}C_r = \binom{n}{r}$

(a) $\binom{100}{50}$ (b) $\binom{100}{51}$

(c) $\binom{50}{25}$ (d) $\binom{50}{25}^2$

26. The remainder when 2^{2003} is divided by 17 is :

(a) 1

(b) 2

(c) 8

(d) None of these

27. The last two digits of the number 3^{400} are:

(a) 81 (b) 43

(c) 29 (d) 01

28. The greatest integer less than or equal to $(\sqrt{2}+1)^6$ is

(a) 196 (b) 197

(c) 198 (d) 199

29. Let $\left(5 + 2\sqrt{6}\right)^n = p + f$, where $n \in N$ and $p \in N$ and $0 < f < 1$, then the value of $f^2 - f + pf - p$ is :

(a) a natural number

(b) a negative integer

(c) a prime number

(d) an irrational number

30. The co-efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is

(a) ${}^{12}C_3$

(b) ${}^{13}C_3$

(c) ${}^{14}C_4$

(d) ${}^{12}C_3 + 3\,{}^{13}C_3 + {}^{14}C_4$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$.

(a) the number of irrational terms is 19

(b) middle term is irrational

(c) the number of rational terms is 2

(d) 9th term is rational

2. In the expansion of $\left(x^{2/3} - \frac{1}{\sqrt{x}}\right)^{30}$, a term containing the power x^{13}

(a) does not exist

(b) exists & the co-efficient is divisible by 29

(c) exists & the co-efficient is divisible by 63

(d) exists & the co-efficient is divisible by 65

3. In the expansion of $\left(x^3 + 3\cdot 2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$

(a) there appears a term with the power x^2

(b) there does not appear a term with the power x^2

(c) there appears a term with the power x^{-3}

(d) the ratio of the co-efficient of x^3 to that of x^{-3} is $\frac{1}{3}$

4. If the 6th term in the expansion of $\left(\frac{3}{2} + \frac{x}{3}\right)^n$ when $x = 3$ is numerically greatest then the possible integral value(s) of n can be

(a) 11 (b) 12

(c) 13 (d) 14

5. Let $(1+x^2)^2(1+x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots$ If A_0, A_1, A_2 are in $A.P.$ then the value of n is

(a) 2 (b) 3

(c) 5 (d) 7

6. If $(9+\sqrt{80})^n = I + f$, where I, n are positive integers and $0 < f < 1$, then:

(a) I is an odd integer

(b) I is an even integer

(c) $(I+f)(1-f) = 1$

(d) $1 - f = (9-\sqrt{80})^n$

7. The number $101^{100} - 1$ is divisible by

(a) 100 (b) 1000

(c) 10000 (d) 100000

8. $7^9 + 9^7$ is divisible by:

(a) 16 (b) 24

(c) 64 (d) 72

9. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2 - 2)^2$

$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ ..... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is

(a) 4 (b) 2

(c) 16 (d) a perfect square

10. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then:

(a) $a_1 = 20$

(b) $a_2 = 210$

(c) $a_4 = 8085$

(d) $a_{20} = 2^2 \cdot 3^7 \cdot 7$

Comprehension – 1 (No. 11 to 13)

If m, n, r are positive integers and if $r < m$, $r < n$, then

${}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$

= Coefficient of x^r in $(1+x)^m (1+x)^n$

= Coefficient of x^r in $(1+x)^{m+n}$

$= {}^{m+n}C_r$

On the basis of the above information, answer the following questions.

11. The value of ${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + \ldots + {}^nC_n \cdot {}^nC_0$ is

(a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_n$

(c) ${}^{2n}C_{n+1}$ (d) ${}^{2n}C_2$

12. The value of r for which ${}^{30}C_r \cdot {}^{20}C_0 + {}^{30}C_{r-1} \cdot {}^{20}C_1 + \ldots + {}^{30}C_0 \cdot {}^{20}C_r$ is maximum, is

(a) 10 (b) 15

(c) 20 (d) 25

13. The value of $r (0 \le r \le 30)$ for which ${}^{20}C_r \cdot {}^{10}C_0 + {}^{20}C_{r-1} \cdot {}^{10}C_1 + \ldots + {}^{20}C_0 \cdot {}^{10}C_r$ is minimum, is

(a) 0 (b) 1

(c) 5 (d) 15

Comprehension – 2 (No. 14 to 16)

We know that if ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_n$ be binomial coefficients, then $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$. Various relations among binomial coefficients can be derived by putting $x = 1, -1, i, \omega$ $\left(\text{where } i = \sqrt{-1},\ \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$.

14. The value of ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots\ldots$ must be

(a) $2^{n/2} \cos \frac{n\pi}{2}$

(b) $2^{n/2} \sin \frac{n\pi}{2}$

(c) $2^{n/2} \cos \frac{n\pi}{4}$

(d) $2^{n/2} \sin \frac{n\pi}{4}$

15. The value of expression $({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots\ldots)^2 + ({}^nC_1 - {}^nC_3 + {}^nC_5 \ldots\ldots)^2$ must be

(a) 2^{2n} (b) 2^n

(c) 2^{n^2} (d) None of these

16. The value of ${}^nC_0 + {}^nC_3 + {}^nC_6 + \ldots\ldots$ must be

(a) $\frac{2^n}{3}$

(b) $\frac{1}{3}\left(2^n + \cos\frac{n\pi}{3}\right)$

(c) $\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$

(d) $\frac{1}{3}\left(2^n + 2\sin\frac{n\pi}{3}\right)$

17. Match the Column:

Column – I	Column – II
(a) If $(r+1)^{th}$ term is the first negative term in the expansion of $(1+x)^{7/2}$, then the value of r (where $\|x\|<1$) is	(p) divisible by 2
(b) The coefficient of y in the expansion of $(y^2+1/y)^5$ is	(q) divisible by 5
(c) ${}^nC_r$ is divisible by n, $(1<r<n)$ if n is	(r) divisible by 10
(d) The coefficient of x^4 in the expansion of $(1+2x+3x^2+4x^3+.....\text{up to}\ \infty)^{1/2}$ is c, $(c \in N)$, then $c+1$ (where $\|x\|<1$) is	(s) a prime number

18. Match the Column:

Column - I	Column - II
(a) If $x=\left(7+4\sqrt{3}\right)^{2n}=[x]+f$, then $x(1-f)=$	(p) 6
(b) If second, third and fourth terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, then n is equal to	(q) 1
(c) Value of ${}^4C_0{}^4C_4-{}^4C_1{}^4C_3+{}^4C_2{}^4C_2-{}^4C_3{}^4C_1+{}^4C_4{}^4C_0$ is	(r) 2
(d) If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}}=1+\frac{y^2}{kx^2}$	(s) 5

INTEGER TYPE

19. The value of p, for which coefficient of x^{50} in the expression

$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+.....+1001x^{1000}$ is equal to ${}^{1002}C_p$, is :

20. The index 'n' of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if for $x=1$ the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$, is :

21. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1}-{}^{39}C_{r^2}={}^{39}C_{r^2-1}-{}^{39}C_{3r}$ is:

22. 2^{60} when divided by 7 leaves the remainder equal to

23. The last digit of the number 3^{400} is equal to ...

24. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q}=0$ if $p<q$) is maximum when m is

25. How many terms in the expansion of $\left(y^{\frac{1}{3}}+x^{\frac{1}{10}}\right)^{55}$ are free from radical sign?

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x\neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is: [*JEE Main-2016*]

(a) 64 (b) 2187
(c) 243 (d) 729

2. The value of $({}^{21}C_1-{}^{10}C_1)+({}^{21}C_2-{}^{10}C_2)+({}^{21}C_3-{}^{10}C_3)+({}^{21}C_4-{}^{10}C_4)+.....+({}^{21}C_{10}-{}^{10}C_{10})$ is [*JEE Main-2017*]

(a) $2^{20}-2^{10}$ (b) $2^{21}-2^{11}$
(c) $2^{21}-2^{10}$ (d) $2^{20}-2^9$

3. The sum of the co-efficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$, $(x>1)$ is [*JEE Main-2018*]

(a) 0 (b) 1
(c) 2 (d) –1

4. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to:

[JEE Main-2019 (January)]

(a) 6 (b) 8

(c) 4 (d) 14

5. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is

[JEE Main-2019 (January)]

(a) 12 (b) 15

(c) 10 (d) 14

6. If $\sum_{i=1}^{20}\left(\frac{^{20}C_{i-1}}{^{20}C_i + {}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals:

[JEE Main-2019 (January)]

(a) 400 (b) 50

(c) 200 (d) 100

7. If the third term in the binomial expansions of $(1 + x^{\log_2 x})^5$ equals 2560, then a possible value of x is:

[JEE Main-2019 (January)]

(a) $\frac{1}{4}$ (b) $4\sqrt{2}$

(c) $\frac{1}{8}$ (d) $2\sqrt{2}$

8. The positive value of λ for which the co-efficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is :

[JEE Main - 2019 (January)]

(a) 4 (b) $2\sqrt{2}$

(c) $\sqrt{5}$ (d) 3

9. If $\sum_{r=0}^{25}\left\{^{50}C_r \cdot {}^{50-r}C_{25-r}\right\} = K\left(^{50}C_{25}\right)$, then K is equal to

[JEE Main - 2019 (January)]

(a) $(25)^2$ (b) $2^{25} - 1$

(c) 2^{24} (d) 2^{25}

10. Let $S_n = 1 + q + q^2 + ...+q^n$ and $T_n = 1+ \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 +...+ \left(\frac{q+1}{2}\right)^n$ where q is a real number and $q \neq 1$. If $^{101}C_1 + {}^{101}C_2 . S_1 + ...+{}^{101}C_{101} . S_{100} = \alpha T_{100}$ then α is equal to:

[JEE Main - 2019 (January)]

(a) 2^{99} (b) 202

(c) 200 (d) 2^{100}

11. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1 x + a_2 x^2 + ... + a_{50} x^{50}$, for $x \in R$: then $\frac{a_2}{a_0}$ is equal to:

[JEE Main-2019 (January)]

(a) 12.50 (b) 12.00

(c) 12.25 (d) 12.75

12. The value of r for which $^{20}C_r {}^{20}C_0 + {}^{20}C_{r-1} {}^{20}C_1 + {}^{20}C_{r-2} {}^{20}C_2 + ... {}^{20}C_0 {}^{20}C_r$ is maximum is :

[JEE Main-2019 (January)]

(a) 15 (b) 20

(c) 11 (d) 10

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is:

[JEE Main-2019 (January)]

(a) 0 (b) 6

(c) 4 (d) 8

14. A ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{\frac{1}{3}}+\frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is :

[JEE Main - 2019 (January)]

(a) $1:2(6)^{\frac{1}{3}}$ (b) $1:4(16)^{\frac{1}{3}}$

(c) $4(36)^{\frac{1}{3}}:1$ (d) $2(36)^{\frac{1}{3}}:1$

15. The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is:

[JEE Main - 2019 (January)]

(a) 55 (b) 49

(c) 48 (d) 54

16. If $^nC_4, {}^nC_5, and\ {}^nC_6$ are in $A.P.$, then n can be :

[JEE Main - 2019 (January)]

(a) 9 (b) 14

(c) 11 (d) 12

17. The sum of the series $2 .{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + ... + 62 .{}^{20}C_{20}$ is equal to :

[JEE Main-2019 (April)]

(a) 2^{24} (b) 2^{25}

(c) 2^{26} (d) 2^{23}

18. The sum of the co-efficients of all even degree terms in x in the expansion of $(x+\sqrt{x^3-1})^6 + (x-\sqrt{x^3-1})^6, (x>1)$ is equal to :

[JEE Main-2019 (April)]

(a) 32 (b) 26

(c) 29 (d) 24

19. If the fourth term in the binomial expansion of $\left(\sqrt{\frac{1}{x^{1+\log_{10} x}}}+x^{\frac{1}{12}}\right)^6$ is equal to 200, and $x > 1$, then the value of x is : [JEE Main-2019 (April)]

(a) 10^3 (b) 100
(c) 10^4 (d) 10

20. If the fourth term in the binomial expansion of $\left(\frac{2}{x}+x^{\log_8 x}\right)^6 (x>0)$ is 20×8^7, then a value of x [JEE Main-2019 (April)]

(a) 8 (b) 8^2
(c) 8^{-2} (d) 8^3

21. If coefficient of some three consecutive terms in the binomial expansion of $(x+1)^n$ are in the ratio 2 : 15 : 70, then the average of these three coefficient is: [JEE Main-2019 (April)]

(a) 964 (b) 625
(c) 227 (d) 232

22. The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is : [JEE Main-2019 (April)]

(a) 35 (b) 38
(c) 23 (d) 58

23. If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1+ax+bx^2)(1-3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to : [JEE Main-2019 (April)]

(a) (28, 315) (b) (–54, 315)
(c) (–21, 714) (d) (24, 861)

24. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1+x+x^2)^9$ is: [JEE Main-2019 (April)]

(a) –84 (b) 84
(c) 126 (d) –126

25. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to: [JEE Main-2019 (April)]

(a) 36 (b) –108
(c) –72 (d) –36

26. If the sum of the coefficients of all even powers of x in the product $(1+x+x^2+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n})$ is 61, then n is equal to [JEE Main-2020 (January)]

27. The number of ordered pairs (r, k) for which $6\,{}^{35}C_r = (k^2-3)\cdot{}^{36}C_{r+1}$, where k is an integer, is: [JEE Main-2020 (January)]

(a) 3 (b) 6
(c) 4 (d) 2

28. The coefficient of x^7 in the expression $(1+x)^{10}+x(1+x)^9+x^2(1+x)^8+\ldots\ldots+x^{10}$ is:

(a) 210 (b) 330
(c) 420 (d) 120

29. If a, b and c are the greatest values of ${}^{19}C_p$, ${}^{20}C_q$ and ${}^{21}C_r$ respectively, then [JEE Main-2020 (January)]

(a) $\frac{a}{10}=\frac{b}{11}=\frac{c}{42}$ (b) $\frac{a}{10}=\frac{b}{11}=\frac{c}{21}$
(c) $\frac{a}{11}=\frac{b}{22}=\frac{c}{42}$ (d) $\frac{a}{11}=\frac{b}{22}=\frac{c}{21}$

30. The coefficient of x^4 in the expansion of $(1+x+x^2)^{10}$ is..... [JEE Main-2020 (January)]

31. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if I_1 is the least value of the term independent of x when $\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and I_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio $I_2 : I_1$ is equal to : [JEE Main-2020 (January)]

(a) 1 : 8 (b) 1 : 16
(c) 8 : 1 (d) 16 : 1

32. If $C_r = {}^{25}C_r$ and $C_0+5.C_1+9.C_2+\ldots.+(101).C_{25}=2^{25}.k$, then k is equal to ..... [JEE Main-2020 (January)]

33. For a positive integer n, $\left(1+\frac{1}{x}\right)^n$ is expanded in increasing powers of x. If three consecutive coef ficients in this expansion are in the ratio, 2 : 5 : 12, then n is equal to ____. [JEE Main-2020 (September)]

34. Let $\alpha>0, \beta>0$ be such that $\alpha^3+\beta^2=4$. If the maximum value of the term independent of x in the binomial expansion of $(\alpha x^{1/9}+\beta x^{-1/6})^{10}$ is 10 K, then K is equal to: [JEE Main-2020 (September)]

(a) 84 (b) 176
(c) 336 (d) 352

35. If the term independent of x in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is k, then $18k$ is equal to :

(a) 9 (b) 11
(c) 5 (d) 7

36. If the number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^n$ is exactly 33, then the least value of n is:

[*JEE Main-2020 (September)*]

(a) 264 (b) 128

(c) 256 (d) 248

37. The value of $(2.{}^1P_0 - 3.{}^2P_1 + 4.{}^3P_2 - \ldots$ up to 51th term) + $(1! - 2! + 3! - \ldots$ up to 51th term) is equal to :

[*JEE Main-2020 (September)*]

(a) 1 (b) $1 + (52)!$

(c) $1 - 51(51)!$ (d) $1 + (51)!$

38. If for some positive integer n, the coefficients of three consecutive terms in the binomial expansion of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14, then the largest coefficient in this expansion is :

[*JEE Main-2020 (September)*]

(a) 252 (b) 462

(c) 792 (d) 330

39. The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to :

[*JEE Main-2020 (September)*]

(a) ${}^{51}C_7 - {}^{30}C_7$ (b) ${}^{50}C_7 - {}^{30}C_7$

(c) ${}^{51}C_7 + {}^{30}C_7$ (d) ${}^{50}C_6 - {}^{30}C_6$

40. The coef ficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^6$ in powers of x, is _____. [*JEE Main-2020 (September)*]

41. The natural number m, for which the coefficient of x in the binomial expansion of $\left(x^m + \frac{1}{x^2}\right)^{22}$ is 1540, is _____. [*JEE Main-2020 (September)*]

42. If the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $|k|$ equals:

[*JEE Main-2020 (September)*]

(a) 2 (b) 3

(c) 9 (d) 1

43. The value of $-{}^{15}C_1 + 2\cdot{}^{15}C_2 - 3\cdot{}^{15}C_3 + \ldots - 15\cdot{}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11}$ is

[*JEE Main-2021 (February)*]

(a) 2^{14} (b) $2^{13} - 13$

(c) $2^{16} - 1$ (d) $2^{13} - 14$

44. If $n \geq 2$ is a positive integer, then the sum of the series ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2)$ is:

[*JEE Main-2021 (February)*]

(a) $\frac{n(n+1)^2(n+2)}{12}$ (b) $\frac{n(n-1)(2n+1)}{6}$

(c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(2n+1)(3n+1)}{6}$

45. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^nC_r, & \text{if } n \geq r \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The maximum value of k for which the sum $\sum_{i=0}^{k} \binom{10}{i}\binom{15}{k-i} + \sum_{i=0}^{k} \binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to

Note: NTA has dropped this question in the final official answer key. [*JEE Main-2021 (February)*]

46. If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is

[*JEE Main-2021 (February)*]

47. The maximum value of the term independent of 't' in the expansion of $\left(tx^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{10}$ where $x \in (0, 1)$ is:

[*JEE Main-2021 (February)*]

(a) $\frac{10!}{\sqrt{3}(5!)^2}$ (b) $\frac{2.10!}{3(5!)^2}$

(c) $\frac{10!}{3(5!)^2}$ (d) $\frac{2.10!}{3\sqrt{3}(5!)^2}$

48. Let $m, n \in N$ and $\gcd(2, n) = 1$, If $30\binom{30}{0} + 29\binom{30}{1} + \ldots + 2\binom{30}{28} + 1\binom{30}{29} - n \cdot 2^m$, then $n + m$ is equal to

[*JEE Main-2021 (February)*]

49. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then $(n - 1)$ is divisible by:

[*JEE Main-2021 (March)*]

(a) 26 (b) 30

(c) 8 (d) 7

50. Let $[x]$ denote greatest integer less than or equal to x. If for $n \in \mathbb{N}$, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j} + 1$ is equal to: [*JEE Main-2021 (March)*]

(a) 2 (b) 2^{n+1}

(c) 1 (d) n

51. Let n be a positive integer. Let

$$A = \sum_{k=0}^{n} (-1)^k {}^nC_k \left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to _____

[*JEE Main-2021 (March)*]

52. If the fourth term in the expansion of $(x + x^{\log_2 x})^7$ is 4480, then the value of x where $x \in N$ is equal to:

[JEE Main-2021 (March)]

(a) 2 (b) 4
(c) 3 (d) 1

53. If $(2021)^{3762}$ is divided by 17, then the remainder is ____.

[JEE Main-2021 (March)]

54. The value of $\sum_{r=0}^{6} ({}^6C_r \cdot {}^6C_{6-r})$ is equal to

[JEE Main-2021 (March)]

(a) 1124 (b) 1324
(c) 1024 (d) 924

55. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12 : 8 : 3. Then the rearest integer to the term independent of x in the expansion, is equal to _____ .[JEE Main-2021 (March)]

56. Let $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{40}x^{40}$ then $a_1 + a_3 + a_5 + ... + a_{37}$ is equal to

[JEE Main-2021 (March)]

(a) $2^{20}(2^{20} - 21)$ (b) $2^{19}(2^{20} - 21)$
(c) $2^{19}(2^{20} + 21)$ (d) $2^{20}(2^{20} + 21)$

57. The term independent of x in the expansion of

$$\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right]^{10}, x \neq 1, \text{ is equal to } \_\_\_\_$$

[JEE Main-2021 (March)]

58. Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$.

If $\sum_{k=0}^{10} (2^2 + 2k)^n C_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$, $\alpha, \beta \in R$, then $\alpha + \beta$ is equal to ____

[JEE Main-2021 (March)]

JEE-ADVANCED PREVIOUS YEARS

1. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE Advanced-2013]

2. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is

[JEE Advanced-2014]

(a) 1051 (b) 1106
(c) 1113 (d) 1120

3. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) .....(1 + x^{100})$ is

[JEE Advanced-2015]

4. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2 + (1+x)^3 + ... + (1+x)^{49} + (1+mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is

[JEE Advanced-2016]

5. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$, where ${}^{10}C_r, r \in \{1, 2, ..., 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430} X$ is ....

[JEE Advanced-2018]

6. Suppose $\det \begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE Advanced -2019]

7. For non-negative integers s and r, let

$$\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } r \leq s, \\ 0 & \text{if } r > s. \end{cases}$$

For positive integers m and n, let $(m,n) \sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$ where for any nonnegative integer p,

$$f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i} \binom{n+i}{p} \binom{p+n}{p-i}$$

[JEE(Advanced)-2020]

Then which of the following statements is/are TRUE?

(a) $g(m, n) = g(n, m)$ for all positive integers m, n
(b) $g(m, n+1) = g(m+1, n)$ for all positive integers m, n
(c) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n
(d) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

ANSWER KEY

Topicwise Questions

1. (*b*) **2.** (*c*) **3.** (*c*) **4.** (*c*) **5.** (*c*) **6.** (*b*) **7.** (*b*) **8.** (*a*) **9.** (*c*) **10.** (*c*) **11.** (*b*) **12.** (*a*)
13. (*a*) **14.** (*c*) **15.** (*d*) **16.** (*b*) **17.** (*c*) **18.** (*b*) **19.** (*a*) **20.** (*c*) **21.** (*b*) **22.** (*c*) **23.** (*c*) **24.** (*c*)
25. (*c*) **26.** (*c*) **27.** (*a*) **28.** (*a*) **29.** (*b*)

Learning Plus

1. (*c*) **2.** (*c*) **3.** (*a*) **4.** (*a*) **5.** (*c*) **6.** (*a*) **7.** (*b*) **8.** (*b*) **9.** (*c*) **10.** (*b*) **11.** (*b*) **12.** (*b*)
13. (*a*) **14.** (*a*) **15.** (*d*) **16.** (*a*) **17.** (*b*) **18.** (*c*) **19.** (*a*) **20.** (*b*) **21.** (*a*) **22.** (*c*) **23.** (*b*) **24.** (*c*)
25. (*b*) **26.** (*c*) **27.** (*d*) **28.** (*b*) **29.** (*b*) **30.** (*d*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*,*b*,*c*,*d*) **2.** (*b*,*c*,*d*) **3.** (*b*,*c*,*d*) **4.** (*b*,*c*,*d*) **5.** (*a*,*b*) **6.** (*a*,*c*,*d*)
7. (*a*,*b*,*c*) **8.** (*a*,*c*) **9.** (*a*,*d*) **10.** (*a*,*b*,*c*) **11.** (*b*) **12.** (*d*)
13. (*a*) **14.** (*c*) **15.** (*b*) **16.** (*c*)
17. (*a*) → (q, s), (*b*) → (p, q, r), (*c*) → (s), (*d*) → (p, s) **18.** (*a*) → (q), (*b*) → (s), (*c*) → (p), (*d*) → (r)
19. [50] **20.** [n = 12] **21.** [2] **22.** [0001] **23.** [0001] **24.** [0015]
25. [0006]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. (Bonus) **2.** (*a*) **3.** (*c*) **4.** (*b*) **5.** (*b*) **6.** (*d*) **7.** (*a*) **8.** (*a*) **9.** (*d*) **10.** (*d*) **11.** (*c*) **12.** (*b*)
13. (*a*) **14.** (*c*) **15.** (*d*) **16.** (*b*) **17.** (*b*) **18.** (*d*) **19.** (Bonus) **20.** (*b*) **21.** (*d*) **22.** (*b*) **23.** (*a*) **24.** (*b*)
25. (*d*) **26.** [30] **27.** (*c*) **28.** (*b*) **29.** (*c*) **30.** [615] **31.** (*d*) **32.** [51] **33.** [118] **34.** (*c*) **35.** (*d*) **36.** (*c*)
37. (*b*) **38.** (*b*) **39.** (*a*) **40.** [120] **41.** [13] **42.** (*b*) **43.** (*d*) **44.** (*c*) **45.** (12) **46.** (1) **47.** (*d*) **48.** (45)
49. (*a*) **50.** (*c*) **51.** (6) **52.** (*a*) **53.** (4) **54.** (*d*) **55.** (4) **56.** (*b*) **57.** (210) **58.** (19)

JEE-ADVANCED
PREVIOUS YEARS

1. [6] **2.** (*c*) **3.** [8] **4.** [5] **5.** [646] **6.** [6.20] **7.** (*a*,*b*,*d*)

PRAYAS 1.0 t.me/realprayasians
PRAYAS 2.0 t.me/realprayasians2

3 COMPLEX NUMBERS

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

Every complex number z can be written as $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. x is called real part of z and y is the imaginary part of z.

Integral Powers of i :

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

or $i^{4n+1} = i$, $i^{4n+2} = -1$ for any $n \in I$,

$i^{4n+3} = -i$, $i^{4n} = 1$

Thus any integral power of i can be expressed as ± 1 or $\pm i$.

TRAIN YOUR BRAIN

Example 1: If $x = -5 + 2\sqrt{-4}$
find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Sol. $x = -5 + 4i$ $(i = \sqrt{-1})$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $(x^2 - x + 4)(x^2 + 10x + 41) - 160$

and $x^2 + 10x + 41 = 0$

Hence given expression $= 0 - 160 = -160$

Note:

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) For $z = x + iy$, if $y = 0$, z is purely real and if, $x = 0$ then z is purely imaginary.

(c) Zero is purely real as well as purely imaginary.

(d) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(e) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non-negative.

(f) If $z = a + ib$, then $a - ib$ is called complex conjugate of z and written as $\bar{z} = a - ib$

(g) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e., $i > 0$, $3 + i < 2$ are meaningless.

Algebraic Operations:

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. **Addition:**
$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2. **Subtraction:**
$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3. **Multiplication:**
$(a + bi)(c + di) = ac + adi + bci + bdi^2$
$= (ac - bd) + (ad + bc)i$
4. **Division:**

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$

$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g., $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$, however in complex numbers.

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

TRAIN YOUR BRAIN

Example 2: Find the sum and product of the two complex numbers $Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$

Sol. $Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8i$

$Z_1 Z_2 = (2 + 3i)(-1 + 5i) = -2 + 15i^2 - 3i + 10i$

$= -17 + 7i$ $(i^2 = -1)$

Equality in Complex Number:

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively.

i.e., $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{I}_\text{m}(z_1) = \text{I}_\text{m}(z_2)$.

TRAIN YOUR BRAIN

Example 3: Find the value of x and y for which $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$ where $x, y \in R$.

Sol. $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$

$\Rightarrow \quad 2x^2 - 3y = 2x - 3y \quad \Rightarrow \quad x^2 - x = 0$

$\Rightarrow \quad x = 0, 1$ and $3x^2 + 2y = 5$

$\Rightarrow \quad$ if $x = 0, y = \dfrac{5}{2}$ and if $x = 1, y = 1$

$\therefore \quad x = 0, y = \dfrac{5}{2}$ and $x = 1, y = 1$

are two solutions of the given equation which can also be represented as $\left(0, \dfrac{5}{2}\right)$ & $(1, 1)$

GEOMETRICAL REPRESENTATION

It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

It is therefore, that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane.

ARGAND DIAGRAM

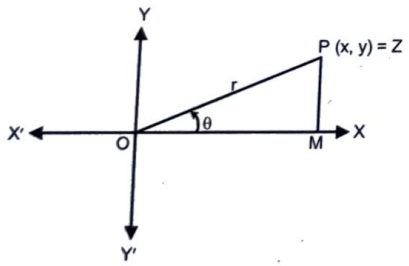
The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane.

POLAR REPRESENTATION

Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axis of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$



Thus $z = (x, y) = x + iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

where $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ by Euler's formula

Thus $z = r(\cos\theta + i\sin\theta)$ can be written as

$z = re^{i\theta}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2 + y^2}$, a non-negative number. $|z| = 0$ for the only number $(0, 0)$.

Euler's Form:

$$z = re^{i\theta}, \ |z| = r, \ \arg z = \theta$$

$$\bar{z} = re^{-i\theta}$$

Note: If θ is real then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

VECTORIAL REPRESENTATION

Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

TRAIN YOUR BRAIN

Example 4: Represent the given complex numbers in polar form:

(*i*) $(1 + i\sqrt{3})^2 / 4i(1 - i\sqrt{3})$

(*ii*) $\sin\alpha - i\cos\alpha$ (α acute)

(*iii*) $1 + \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

Sol. (*i*) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1 + i\sqrt{3})^2}{4i(1 - i\sqrt{3})} = \frac{(1 + i\sqrt{3})^2}{4(\sqrt{3} + i)} = \frac{-2 + 2i\sqrt{3}}{4(\sqrt{3} + i)}$$

$$= \frac{(-1 + i\sqrt{3})(\sqrt{3} - i)}{2(\sqrt{3} + i)(\sqrt{3} - i)}$$

$$= \frac{-\sqrt{3} + \sqrt{3} + 4i}{2(3 + 1)} = \frac{i}{2}$$

and $\dfrac{i}{2} = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$.

Hence $\dfrac{(1 + i\sqrt{3})^2}{4i(1 - i\sqrt{3})} = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = \dfrac{1}{2}e^{i\pi/2}$

(*ii*) Real part > 0, Imaginary part < 0 argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle.

$$\therefore \ \sin\alpha - i\cos\alpha = \cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)$$

$$= e^{i\left(\alpha - \frac{\pi}{2}\right)}$$

(iii) $1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=2\cos^2\frac{\pi}{6}+i\cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

$=2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\cos\frac{\pi}{6}e^{i\pi/6}$

CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z=(a,b)=a+ib$ and $\bar{z}=(a,-b)=a-ib$, where a and b are real numbers, $i=\sqrt{-1}$ and $b\neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

Note that, sum $=(a+ib)+(a-ib)=2a$ which is real and product $=(a+ib)(a-ib)=a^2-(ib)^2=a^2-i^2b^2=a^2-(-1)b^2=a^2+b^2$ which is real.

Properties of conjugate

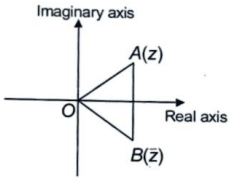
(i) $\overline{(\bar{Z})}=Z$

(ii) $Z=\bar{Z}\Leftrightarrow z$ is real

(iii) $Z=-\bar{Z}\Leftrightarrow z$ is purely imaginary

(iv) $\mathrm{Re}(z)=\mathrm{Re}(\bar{z})=\frac{z+\bar{z}}{2}$

(v) $\mathrm{Im}(z)=\frac{z-\bar{z}}{2i}$



(vi) $\overline{Z_1+Z_2}=\overline{Z_1}+\overline{Z_2}$

(vii) $\overline{Z_1-Z_2}=\overline{Z_1}-\overline{Z_2}$

(viii) $\overline{\left(\frac{Z_1}{Z_2}\right)}=\frac{\overline{Z_1}}{\overline{Z_2}}\ (Z_2\neq 0)$

(ix) $Z_1\overline{Z_2}+\overline{Z_1}Z_2=2\,\mathrm{Re}(\overline{Z_1}Z_2)=2\,\mathrm{Re}(Z_1\overline{Z_2})$

(x) $\overline{Z^n}=(\bar{Z})^n$

(xi) If $z=f(z_1)$, then $\bar{Z}=f(\overline{Z_1})$

Modulus of a Complex Number

Modulus of a complex number $z=x+iy$ is a real number given by $|z|=\sqrt{x^2+y^2}$. It is always non-negative and $|z|=0$ only for $z=0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

Properties of Modulus

(i) $|z|\geq 0\Rightarrow |z|=0$ iff $z=0$ and $|z|>0$ iff $z\neq 0$.

(ii) $-|z|\leq \mathrm{Re}(z)\leq |z|$ and $-|z|\leq |z|$.

(iii) $|z|=|\bar{z}|=|-z|=|-\bar{z}|$

(iv) $z\bar{z}=|z|^2$

(v) $|z_1z_2|=|z_1||z_2|$

In general $|z_1z_2z_3\ldots\ldots z_n|=|z_1||z_2||z_3|\ldots..|z_n|$

(vi) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\ (z_2\neq 0)$

(vii) $|z_1\pm z_2|\leq |z_1|+|z_2|$

In particular, if $|z_1+z_2|=|z_1|+|z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.

(viii) $|z_1\pm z_2|\geq||z_1|-|z_2||$

In particular, if $|z_1-z_2|=||z_1|-|z_2||$, then origin, z_1 and z_2 are collinear with origin at one of the ends.

(ix) $|z^n|=|z|^n$

(x) $||z_1|-|z_2||\leq|z_1|+|z_2|$

Thus $|z_1|+|z_2|$ is the greatest possible value of $|z_1+z_2|$ and $||z_1|-|z_2||$ is the least possible value of $|z_1+z_2|$

(xi) $|z_1\pm z_2|^2=(z_1\pm z_2)(\bar{z}_1\pm\bar{z}_2)$

$=|z_1|^2+|z_2|^2\pm(z_1\bar{z}_2+\bar{z}_1z_2)$

or $|z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\overline{z_2})$

(xii) $z_1\overline{z_2}+\overline{z_1}z_2=2|z_1||z_2|\cos(\theta_1-\theta_2)$

where $\theta_1=\arg(z_1)$ and $\theta_2=\arg(z_2)$.

(xiii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1+z_2|^2=|z_1-z_2|^2=2\{|z_1|^2+|z_2|^2\}$

$|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b^2)(|z_1|^2+|z_2|^2)$

where $a, b\in R$.

Unimodular i.e., unit modulus

If z is unimodular then $|z|=1$. A unimodular complex number can always be expressed as $\cos\theta+i\sin\theta$, $\theta\in R$.

Note: $\frac{z}{|z|}$ is always a unimodular complex number if $z\neq 0$.

Note:

(a) $||z_1|-|z_2||=|z_1+z_2|$, $|z_1-z_2|=|z_1|+|z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(b) $|z_1+z_2|=|z_1|+|z_2|$, $||z_1|-|z_2||=|z_1-z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

TRAIN YOUR BRAIN

Example 5: If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

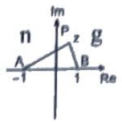
Sol. We have $9 = |z - (5 + 7i)|$ = distance between z and $5 + 7i$.

Thus locus of z is the circle of radius 9 and centre at $5 + 7i$. For such a z (on the circle), we have to find its greatest and least distance as from $2 + 3i$, which obviously 14 and 4.

Example 6: Find the minimum value of $|1 + z| + |1 - z|$.

Sol. $|1 + z| + |1 - z| \geq |1 + z + 1 - z|$

(triangle inequality)



$\Rightarrow |1 + z| + |1 - z| \geq 2$

$\therefore$ minimum value of $(|1 + z| + |1 - z|) = 2$

Geometrically $|z + 1| + |1 - z| = |z + 1| + |z - 1|$

which represents sum of distances of z from 1 and -1

it can be seen easily that minimum $(PA + PB) = AB = 2$

Example 7: $\left|z - \frac{2}{z}\right| = 1$ then find the maximum and minimum value of $|z|$

Sol. $\left|z - \frac{2}{z}\right| = 1$

$$\left|\,|z| - \left|\frac{2}{z}\right|\,\right| \leq \left|z - \frac{2}{z}\right| \leq |z| + \left|-\frac{2}{z}\right|$$

Let $|z| = r$

$$\Rightarrow \quad \left|r - \frac{2}{r}\right| \leq 1 \leq r + \frac{2}{r}$$

$$r + \frac{2}{r} \geq 1 \Rightarrow r \in R^+ \qquad ...(i)$$

and $\left|r - \frac{2}{r}\right| \leq 1 \Rightarrow -1 \leq r - \frac{2}{r} \leq 1$

$\Rightarrow \quad r \in [1, 2]$ ...(ii)

$\therefore$ from (i) and (ii) $r \in [1, 2]$

Ans. $|z|_{max} = 2, |z|_{min} = 1$

ARGUMENTS OF A COMPLEX NUMBER

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$.

θ is called the argument of Z or amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \leq \pi$ is defined as the principal argument of Z and is denoted by $\arg z$.

Properties of Arguments

(i) $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$
$(k = 0$ or 1 or $-1)$

In general $\text{Arg}(z_1z_2z_3 \ldots z_n) = \text{Arg}(z_1) + \text{Arg}(z_2) + \text{Arg}(z_3) + \ldots + \text{Arg}(z_n) + 2k\pi$ (where $k = I$)

(ii) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}\,z_1 - \text{Arg}\,z_2 + 2k\pi$ $(k = 0$ or 1 or $-1)$

(iii) $\text{Arg}\left(\frac{z}{\bar{z}}\right) = 2\,\text{Arg}\,z + 2k\pi$ $(k = 0$ or 1 or $-1)$

(iv) $\text{Arg}(z^n) = n\,\text{Arg}\,z + 2k\pi$ $(k \in I)$

(v) If $\text{Arg}\left(\frac{z_2}{z_1}\right) = \theta$, then $\text{Arg}\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k = I$

(vi) $\text{Arg}\,\bar{z} = -\text{Arg}\,z$

(vii) If $\arg(z) = 0 \Rightarrow z$ is real.

Note: Proper value of k must be chosen so, that R.H.S. of (i), (ii), (iii), (iv) lies in $[-\pi, \pi]$.

All the above formulae are written on the basis of principal argument.

Vector Representation of a Complex Number

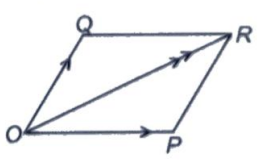
In the Argand's diagram any complex number $Z = x + iy$ can be represented by a point P with coordinates (x, y). The vector $\overrightarrow{OP}$ can also be used to represent Z. The length of the vector $\overrightarrow{OP}$, i.e., OP is the modulus of Z and the angle θ that OP makes with the positive X-axis is the amplitude of Z.

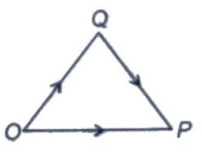
Representation of an algebraic operation on complex numbers

Sum: If two complex numbers Z_1 and Z_2 be represented by the points P and Q or by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the sum $Z_1 + Z_2$ is represented by R or $\overrightarrow{OR}$, where $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ and OR is the diagonal of the parallelogram with OP and OQ as adjacent sides.

Difference: $Z_1 - Z_2$ will be represented by $\overrightarrow{QP}$ since $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$.

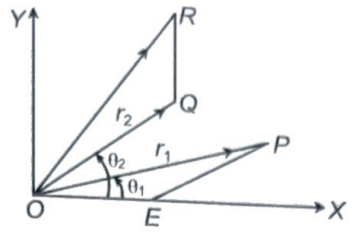
$Z_2 - Z_1$ will be represented by $\overrightarrow{PQ}$.





Multiplication: If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $Z_1 Z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$.

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent Z_1 and Z_2, construct ΔOQR similar to ΔOEP where $OE = 1$.



$\angle XOR = \angle XOQ + \angle QOR = \angle XOR + \angle EOP = \theta_2 + \theta_1$

and $\dfrac{OR}{OQ} = \dfrac{OP}{OE}$,

$\therefore \quad OR = OP \cdot OQ = r_1 r_2$ { as $OE = 1$}

Hence $\overrightarrow{OR}$ represents the product $Z_1 Z_2$.

Division $\dfrac{Z_1}{Z_2} = \left(\dfrac{r_1}{r_2}\right)\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 - \theta_2)\}$

Construct ΔORP similar to ΔOEQ

Now $\dfrac{OR}{OE} = \dfrac{OP}{OQ} \Rightarrow OR = \dfrac{r_1}{r_2}$

and $\angle ROX = \angle ROP - \angle EOP = \angle EOQ - \angle EOP = \theta_2 - \theta_1$

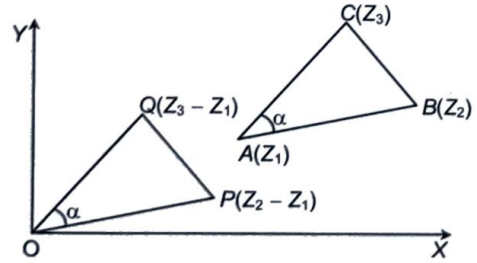
$\therefore \quad \angle XOR = \theta_1 - \theta_2$

Hence, $\overrightarrow{OR}$ represents $\dfrac{Z_1}{Z_2}$.

Corollary 1

If Z_1, Z_2, Z_3 are the vertices of a triangle ABC described in the counter-clockwise direction, then

$\dfrac{Z_3 - Z_1}{Z_3 - Z_1} = \dfrac{CA}{BA}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$



Let P and Q be the points representing $Z_2 - Z_1$ and $Z_3 - Z_1$.

Then the triangles POQ and BAC are congruent.

$\therefore \dfrac{CA}{BA} = \dfrac{OQ}{OP}$ and $\angle QOP = \angle BAC = \alpha$

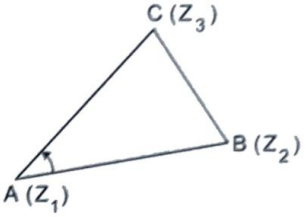
Now $\dfrac{Z_3 - Z_1}{Z_2 - Z_1}$ has modulus $\dfrac{OQ}{OP} = \dfrac{CA}{BA}$ and argument $\angle POQ = \alpha$

Hence $\dfrac{Z_3 - Z_1}{Z_2 - Z_1} = \left(\dfrac{CA}{BA}\right)(\cos\alpha + i\sin\alpha)$

In particular, if $\alpha = 90°$ and $AB = AC$, then

$\dfrac{Z_3 - Z_1}{Z_2 - Z_1} = i$ or $(Z_3 - Z_1) = i(Z_2 - Z_1)$

Corollary 2: If Z_1, Z_2, Z_3 are represented by A, B, C, then



$\arg\left(\dfrac{Z_3 - Z_1}{Z_2 - Z_1}\right) = \angle BAC$

$\arg\left(\dfrac{Z_2 - Z_3}{Z_1 - Z_3}\right) = \angle ACB$ and

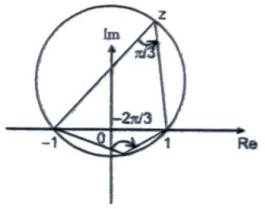
$\arg\left(\dfrac{Z_1 - Z_2}{Z_3 - Z_2}\right) = \angle CBA$

TRAIN YOUR BRAIN

Example 8: If $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$ then interpret the locus.

Sol. $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$

$\Rightarrow \arg\left(\dfrac{1-z}{-1-z}\right) = \dfrac{\pi}{3}$



Here $\arg\left(\dfrac{1-z}{-1-z}\right)$ represents the angle between lines joining -1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

De moivre's Theorem

Case I

Statement: If n is any integer then

(*i*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(*ii*) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)(\cos\theta_4 + i\sin\theta_4)$ .....$(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + ... \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$

Case II

Statement: If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, \ldots\ldots, q-1$

Note: Continued product of the roots of a complex quantity should be determined using theory of equations.

TRAIN YOUR BRAIN

Example 9: If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, prove the following:

(*i*) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

(*ii*) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta + n\phi)$

Sol. (*i*) Given $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2\cos\theta + 1 = 0$.

Solving this, $x = \cos\theta \pm i\sin\theta$

In fact if $x = \cos\theta + i\sin\theta$; $\frac{1}{x} = \cos\theta - i\sin\theta$.

It may be noted also that $x + \frac{1}{x} = 2\cos\theta$ is symmetrical w.r.t. $\frac{1}{x}$ and hence if one root is the value for x, the other root is $\frac{1}{x}$ and vice-versa.

Similarly, given that $2\cos\phi = y + \frac{1}{y}$, we have

$y = \cos\phi + i\sin\phi$

$\therefore \quad x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

and $y^n = (\cos\theta + i\sin\theta)^n = \cos n\phi + i\sin n\phi$

$x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

and $\frac{1}{x^m y^n} = \cos(m\theta + n\theta) - i\sin(m\theta + n\theta)$

Adding we get

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$$

(*ii*) By similar reasoning $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(I\theta + n\phi)$

Cube Root of Unity

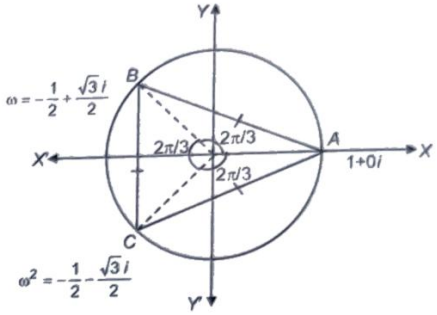
Consider the cubic (3rd degree) equation

$x^3 = 1 = \cos 0 + i\sin 0$

$= \cos 2k\pi + i\sin 2k\pi$

$\therefore \quad x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$

$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$



To get the three roots of the cubic equation we give

$k = 0$, giving the real root, $\cos 0 + i\sin 0 = 1$

$k = 1$, giving one imaginary root,

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$$

$k = 2$, giving the other imaginary root,

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \omega^2$$

It is said that 1, ω, ω^2 are the three cubic roots of unity satisfying

(*i*) $1 + \omega + \omega^2 = 0$

(*ii*) $\omega^3 = 1$

(*iii*) 1, ω, ω^2 are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle of sides $\sqrt{3}$.

(*iv*) The following factorisation should be remembered:
($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b)$;

$x^2 + x + 1 = (x-\omega)(x-\omega^2)$;

$a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b)$;

$a^2 + ab + b^2 = (a-b\omega)(a-b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)$

$(a+\omega^2 b+\omega c)$

TRAIN YOUR BRAIN

Example 10: Find the value of $\omega^{192} + \omega^{194}$

Sol. $\omega^{192} + \omega^{194} = 1 + \omega^2 = -\omega$

Example 11: If 1, ω, ω^2 are cube roots of unity, then prove that

(*i*) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

(*ii*) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

(*iii*) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ ..... to $2n$ factors $= 2^{2n}$

Sol. (*i*) $(1-\omega+\omega^2)(1+\omega-\omega^2)=(-2\omega)(-2\omega^2)=4$

(*ii*) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$

$=(1+\omega^2-\omega)^5+(1+\omega-\omega^2)^5$

$=(-\omega-\omega)^5+(-\omega^2-\omega^2)^5=(-2\omega)^5+(-2\omega^2)^5$

$=-32\omega^5-32\omega^{10}=-32(\omega^2)(\omega^3)-32(\omega^3)^3\cdot\omega$

$=-32\omega^2-32\omega=-32(\omega^2+\omega)-32(-1)=32$

(*iii*) $\omega^4=\omega, \omega^8=\omega^2$ etc. 3rd, 5th, 7th factors are each equal to 1st and 4th, 6th, 8th factors are each equal to 2nd.

L.H.S. $=(-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)\ldots$ to $2n$ factors

$=(2^2\omega^3)(2^2\omega^3)\ldots$ to n factors $=(2^2)^n=2^{2n}$

nth ROOTS OF UNITY

More generally the nth degree equation $x^n=1$ has 'n' nth roots of unity given by

$$\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}+i\sin\frac{4\pi}{n},\ldots,\cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)\pi}{n}$$

i.e. $1, \alpha, \alpha^2 \ldots. \alpha^{n-1}$ satisfying

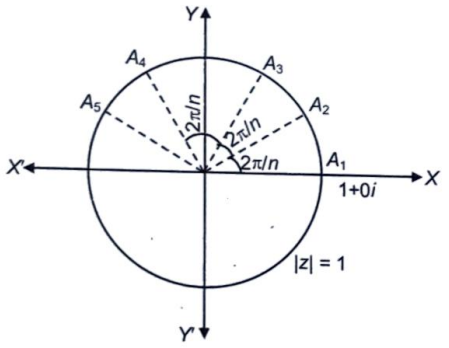
(*i*) $1+\alpha+\alpha^2+\ldots.+\alpha^{n-1}=0$

(*ii*) $\alpha^n=1$

(*iii*) $1, \alpha, \ldots..., \alpha^{n-1}$ represent n points in the Argand's plane situated on the unit circle $|Z|=1$ and forming the corners of a regular n sides polygon.

As sum of nth roots of unity $=0$

$$\Rightarrow \sum_{k=0}^{n-1}\alpha^k=0 \quad \Rightarrow \quad \sum_{k=0}^{n-1}\left(\cos\left(\frac{2k\pi}{n}\right)+i\left(\sin\frac{2k\pi}{n}\right)\right)$$



$$\Rightarrow \sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right)=0 \quad \text{and} \quad \sum_{k=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right)=0$$

More general equation like $x^n=a+ib$ can be solved by using this method.

First write $a+ib=r[\cos\theta+i\sin\theta]=r[\cos(\theta+2k\pi)+i\sin(\theta+2k\pi)]$ and hence the n nth roots of $x^n=a+ib$ are

$$\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right);\ k=0,1,2\ldots,(n-1)$$

TRAIN YOUR BRAIN

Example 12: Find the roots of the equation $z^6+64=0$ where real part is positive.

Sol. $z^6=-64$

$z^6=2^6\cdot e^{i(2n+1)\pi}$

$n=0,1,2,3,4,5$

$$\Rightarrow \quad z=2e^{i\frac{(2n+1)\pi}{6}}$$

$$\therefore \quad z=2e^{i\frac{\pi}{6}},\ 2e^{i\frac{\pi}{2}},\ 2e^{i\frac{5\pi}{6}},\ 2e^{i\frac{7\pi}{6}},\ 2e^{i\frac{3\pi}{2}},\ 2e^{i\frac{11\pi}{6}}$$

$\therefore$ roots with +ve real part are $=2e^{i\frac{\pi}{6}},\ 2e^{i\frac{11\pi}{6}}$

Example 13: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-\cos\frac{2\pi k}{7}\right)$

Sol. $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right)-\sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right)$

$=\sum_{k=0}^{6}\sin\frac{2\pi k}{7}-\sum_{k=0}^{6}\cos\frac{2\pi k}{7}+1$

$=\sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) $+1=0-0+1=1$

The sum of the following series should be remembered :

(*i*) $\cos\theta+\cos2\theta+\cos3\theta+\ldots..+\cos n\theta$

$$=\frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta.$$

(*ii*) $\sin\theta+\sin2\theta+\sin3\theta+\ldots..+\sin n\theta$

$$=\frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta.$$

Note : If $\theta=(2\pi/n)$ then the sum of the above series vanishes.

IMPORTANT RESULTS:

(*i*) **Section formula:** If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio $m:n$ then affix z of C is given by

$$z=\frac{mz_2+nz_1}{m+n} \text{ where } m,n>0$$

If C divides PQ in the ratio $m:n$ externally then

$$z=\frac{mz_2-nz_1}{m-n}$$

Note: If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a+b+c=0$ and a, b, c are not all simultaneously zero, then the complex numbers z_1, z_2 and z_3 are collinear.

(ii) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

- Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$:
- Ortho centre of the

$$\Delta ABC = \frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

or $$\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

- Incentre of the $\Delta ABC = (az_1 + bz_2 + cz_3) \div (a+b+c)$.
- Circumcentre of the $\Delta ABC = (Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(iii) $\text{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the positive x–axis.

(iv) $|z-a| = |z-b|$ is the perpendicular bisector of the line joining a to b.

(v) The equation of a line joining z_1 and z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(vi) $z = z_1(1+it)$ where t is a real parameter is a line through the point z_1 and perpendicular to the line joining z_1 to the origin.

(vii) The equation of a line passing through z_1 and z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear. The above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

Note: If we replace z by $ze^{i\theta}$ and $\bar{z}$ by $\bar{z}e^{-i\theta}$ then we get equation of a straight line which makes an angle θ with the given straight line.

(viii) The equation of circle having centre z_0 and radius ρ is: $|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form $z\bar{z} + \bar{\alpha}z + a\bar{z} + k = 0, k$ is real. Centre is $-\alpha$ and radius $= \sqrt{|\alpha|^2 - k}$.

Circle will be real if $|a|^2 - k \geq 0$.

(ix) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg \dfrac{z - z_2}{z - z_1} = \pm\dfrac{\pi}{2}$

or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(x) Condition for four given points z_1, z_2, z_3 and z_4 to be concyclic is the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 and z_3 can be taken as $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real

$$\Rightarrow \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}.$$

(xi) $\text{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ represent

(i) a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$

(iii) a part of circle, if $0 < \theta < \pi$.

(xii) Area of triangle formed by the points z_1, z_2 and z_3 is

$$\left| \frac{1}{4i} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right|.$$

(xiii) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\dfrac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(xiv) (i) Complex slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is

$$\omega = -\frac{\alpha}{\bar{\alpha}}.$$

(ii) Complex slope of a line joining by the points z_1 and z_2 is $\omega = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(iii) Complex slope of a line making θ angle with real axis $\omega = e^{2i\theta}$

(xv) Dot and cross product: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The dot product [also called the scalar product] of z_1 and z_2 is defined by

$$z_1 \cdot z_2 = |z_1||z_2| \cos\theta = x_1x_2 + y_1y_2 = \text{Re}\{\bar{z}_1 z_2\}$$

$$= \frac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π

If vectors z_1, z_2 are perpendicular then

$$z_1 \cdot z_2 = 0 \Rightarrow \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0.$$

i.e. Sum of complex slopes = 0

The cross product of z_1 and z_2 is defined by

$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = x_1y_2 - y_1x_2 = \text{Im}\{\bar{z}_1 z_2\}$

$= \frac{1}{2i}\{\bar{z}_1 z_2 - z_1\bar{z}_2\}$

If vectors z_1, z_2 are parallel then

$z_1 \times z_2 = 0 \Rightarrow \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}$.

i.e. Complex slopes are equal.

Note: ω_1 and ω_2 are the compelx slopes of two lines.

(*i*) If lines are parallel then $\omega_1 = \omega_2$

(*ii*) If lines are perpendicular then $\omega_1 + \omega_2 = 0$

(*xvi*) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 and z_2

(*xvii*) If $|z - z_0| = \left|\frac{\bar{\alpha}z + \alpha\bar{z} + r}{2|\alpha|}\right|$ then locus of z is parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ (Provided $\bar{\alpha}z_0 + \alpha\bar{z}_0 + r \neq 0$)

(*xviii*) If $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

(*xix*) If $||z - z_1| - |z - z_2|| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 and z_2.

(*xx*) Reflection points for a straight line: Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 and z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(*xxi*) Inverse points w.r.t. a circle: Two points P and Q are said to be inverse with respect to a circle with centre 'O' and radius ρ, if:

(*i*) the point O, P, Q are collinear and P, Q are on the same side of O.

(*ii*) $OP \cdot OQ = \rho^2$.

Note: That the two points z_1 and z_2 will be the inverse points w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3||z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

LOGARITHM OF A COMPLEX QUANTITY

$$\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

where $n \in I$.

TRAIN YOUR BRAIN

Example 14: Find the value of

(*i*) $\log(1 + \sqrt{3}\,i)$ (*ii*) $\log(-1)$

(*iii*) 2^i (*iv*) i^i

(*v*) $|(1 + i)^i|$ (*vi*) $\arg((1 + i)^i)$

Ans. (i) $\log 2 + i(2n\pi + \frac{\pi}{3})$

(*ii*) $i\pi$

(*iii*) $\cos(\ln 2) + i\sin(\ln 2) = e^{i(\ln 2)}$

(*iv*) $e^{-(4n+1)\cdot\frac{\pi}{2}}$

(*v*) $e^{-(8n+1)\cdot\frac{\pi}{4}}$

(*vi*) $\frac{1}{2}\ln(2)$.

Sol. (*i*) $\log(1 + \sqrt{3}\,i) = \log\left(2e^{i\left(\frac{\pi}{3} + 2n\pi\right)}\right)$

$= \log 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

(*iii*) $2^i = e^{i\ln 2} = [\cos(\ln 2) + i\sin(\ln 2)]$

Example 15: If z_1 and z_2 are two complex numbers and $c > 0$, then prove that $|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

Sol. We have to prove :

$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^3$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le c|z_1|^2 + c^{-1}|z_2|^2$

or $c|z_1|^2 + \frac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$

(using $\text{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{c}\,|z_1| - \frac{1}{\sqrt{c}}|z_2|\right)^2 \ge 0$ which is always true.

Example 16: If $\theta_i \in [\pi/6, \pi/3]$, $i = 1, 2, 3, 4, 5$, and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \frac{3}{4}$

Sol. Given that

$\cos\theta_1 \cdot z^4 + \cos\theta_2 \cdot z^3 + \cos\theta_3 \cdot z^2 + \cos\theta_4 \cdot z + \cos\theta_5 = 2\sqrt{3}$

or $|\cos\theta_1 \cdot z^4 + \cos\theta_2 \cdot z^3 + \cos\theta_3 \cdot z^2 + \cos\theta_4 \cdot z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1 \cdot z^4| + |\cos\theta_2 \cdot z^3| + |\cos\theta_3 \cdot z^2| + \cos\theta_4 \cdot z| + |\cos\theta_5|$

$\because$ $\theta_i \in [\pi/6, \pi/3]$

$\therefore$ $\frac{1}{2} \le \cos\theta_i \le \frac{\sqrt{3}}{2}$

$$2\sqrt{3} \le \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$$

$$3 \le |z|^4 + |z|^3 + |z|^2 + |z|$$

Case I : If $|z| \ge 1$, then above result is automatically true

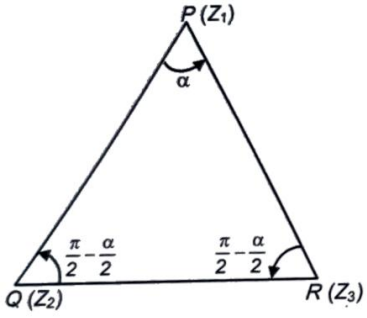
Case II : If $|z| < 1$, then

$$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \ldots\ldots\infty$$

$$3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \frac{3}{4}$$

Hence by both cases, $|z| > \frac{3}{4}$

Example 17: The points P, Q and R represent the complex numbers Z_1, Z_2 and Z_3 respectively and the angles of the triangle PQR at Q and R are both $\frac{\pi}{2} - \frac{\alpha}{2}$, Prove that $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\left(\frac{\alpha}{2}\right)$



Sol. QP is obtained from QR by a rotation counter clockwise the length PQ is different from the length of QR

$$\therefore \quad Z_1 - Z_2 = \frac{PQ}{QR}(Z_3 - Z_2)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

through an angle $\frac{\pi}{2} - \frac{\alpha}{2}$; of course

Similarly

$$\therefore \quad Z_1 - Z_3 = \frac{PQ}{QR}(Z_3 - Z_2)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Multiplying the two

$$\left[(Z_1 - Z_2)(Z_1 - Z_3) = \frac{PQ \cdot PR}{QR^2}(Z_3 - Z_2)(Z_2 - Z_1)\left\{\cos^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \sin^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}\right]$$

Now $\frac{QR}{\sin\alpha} = \frac{PQ}{\cos\frac{\alpha}{2}} = \frac{PR}{\cos\frac{\alpha}{2}}$

$$\therefore \quad \frac{PQ \cdot PR}{QR^2} = \frac{\cos^2\frac{\alpha}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}$$

$$\therefore \quad (Z_1 - Z_2)(Z_1 - Z_3)4\sin^2\frac{\alpha}{2} = (Z_3 - Z_2)(Z_2 - Z_3)$$

i.e., $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\frac{\alpha}{2}$.

Topicwise Questions

IOTA AND BASIC PROPERTIES

1. If n is a positive integer, then which of the following relations is false

(a) $i^{4n}=1$ (b) $i^{4n-1}=i$

(c) $i^{4n+1}=i$ (d) $i^{-4n}=1$

2. The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1=$

(a) -1 (b) -2

(c) -3 (d) -4

3. The real part of $\frac{1}{1-\cos\theta+i\sin\theta}$ is equal to

(a) 1/4 (b) 1/2

(c) $\tan\theta/2$ (d) $1/1-\cos\theta$

REPRESENTATION OF COMPLEX NUMBER

4. If $(-7-24i)^{1/2}=x-iy$, then $x^2+y^2=$

(a) 15 (b) 25

(c) -25 (d) -15

5. The square root of $3-4i$ is

(a) $\pm(2+i)$ (b) $\pm(2-i)$

(c) $\pm(1-2i)$ (d) $\pm(1+2i)$

ALGEBRAIC OPERATION, MODULUS ARGUMENT AND CONJUGATE

6. The conjugate of $\frac{(2+i)^2}{3+i}$, in the form of $a+ib$, is

(a) $\frac{13}{2}+i\left(\frac{15}{2}\right)$ (b) $\frac{13}{10}+i\left(\frac{-15}{2}\right)$

(c) $\frac{13}{10}+i\left(\frac{-9}{10}\right)$ (d) $\frac{13}{10}+i\left(\frac{9}{10}\right)$

7. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1+z_2}{z_1-z_2}\right|=1$, then $\frac{z_1}{z_2}$ is a number which is

(a) Positive real
(b) Negative real
(c) Zero or purely imaginary
(d) Both (a) and (b)

8. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is

(a) 1 (b) 1/2

(c) 2 (d) $\sqrt{2}$

9. Let z_1 be a complex number with $|z_1|=1$ and z_2 be any complex number, then $\left|\frac{z_1-z_2}{1-z_1\bar{z}_2}\right|=$

(a) 0 (b) 1
(c) -1 (d) 2

10. If $|z|=4$ and $\arg z=\frac{5\pi}{6}$, then $z=$

(a) $2\sqrt{3}-2i$ (b) $2\sqrt{3}+2i$

(c) $-2\sqrt{3}+2i$ (d) $-\sqrt{3}+i$

11. The amplitude of $\frac{1+\sqrt{3}i}{\sqrt{3}+1}$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

12. Let z be a purely imaginary number such that $\text{Im}(z)<0$. Then $\arg(z)$ is equal to

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) $-\frac{\pi}{2}$

13. If z is a purely real number such that $\text{Re}(z)<0$, then $\arg(z)$ is equal to

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) $-\frac{\pi}{2}$

14. The argument of the complex number $\frac{13-5i}{4-9i}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

15. If z and ω are two non–zero complex numbers such that $|z\omega|=1$ and $\arg(z)-\arg(\omega)=\frac{\pi}{2}$, then $\bar{z}\omega$ is equal to

(a) 1 (b) -1
(c) i (d) $-i$

16. The modulus and amplitude of $\frac{1+2i}{1-(1-i)^2}$ are

(a) $\sqrt{2}$ and $\frac{\pi}{6}$ (b) 1 and 0

(c) 1 and $\frac{\pi}{3}$ (d) 1 and $\frac{\pi}{4}$

17. If $z_1 = 1 + 2i$ and $z_2 = 3 + 5i$, and then $\text{Re}\left(\frac{\bar{z}_2 z_1}{z_2}\right)$ is equal to

(a) $\frac{-31}{17}$ (b) $\frac{17}{22}$

(c) $\frac{-17}{31}$ (d) $\frac{22}{17}$

ROTATION THEOREM

18. The vector $z = -4 + 5i$ is turned counter clockwise through an angle of 180° & stretched 1.5 times. The complex number corresponding to the newly obtained vector is :

(a) $6 - \frac{15}{2}i$ (b) $-6 + \frac{15}{2}i$

(c) $6 + \frac{15}{2}i$ (d) $-6 - \frac{15}{2}i$

DE MOIVER'S THEOREM & ROOTS OF UNITY

19. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(a) $\text{Re}(z) = 0$ (b) $\text{Im}(z) = 0$

(c) $\text{Re}(z) > 0, \text{Im}(z) > 0$ (d) $\text{Re}(z) > 0, \text{Im}(z) < 0$

20. If $z = \frac{\sqrt{3} + i}{2}$, then the value of z^{69} is

(a) $-i$ (b) i

(c) 1 (d) -1

21. The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)^6$, where ω, ω^2 are cube roots of unity

(a) 128ω (b) $-128\omega^2$

(c) -128ω (d) $128\omega^2$

22. If 1, ω, ω^2 are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$ is equal to-

(a) 0 (b) 1

(c) ω (d) ω^2

Geometry and Complex No

23. If the vertices of a quadrilateral be $A = 1 - 2i$, $B = 3 - i$, $C = -2 - 3i$ and $D = 2 - 2i$, then the quadrilateral is

(a) Parallelogram

(b) Rectangle

(c) Square

(d) Rhombus

24. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0 (b) 2

(c) 7 (d) 17

25. If $z = x + iy$, then area of the triangle whose vertices are points z, iz and $z + iz$ is

(a) $2|z|^2$ (b) $\frac{1}{2}|z|^2$

(c) $|z|^2$ (d) $\frac{3}{2}|z|^2$

26. $R(z^2) = 1$ is represented by

(a) The parabola $x^2 + y^2 = 1$

(b) The hyperbola $x^2 - y^2 = 1$

(c) Parabola or a circle

(d) Straight line

27. If $\arg(z - a) = \frac{\pi}{4}$, where $a \in R$, then the locus of $z \in C$ is a

(a) Hyperbola

(b) Parabola

(c) Ellipse

(d) Straight line

28. If $|8 + z| + |z - 8| = 16$ where z is a complex number, then the point z will lie on

(a) A circle

(b) An ellipse

(c) A straight line

(d) A parabola

Learning Plus

1. The least value of $n(n \in N)$, for which $\left(\frac{1+i}{1-i}\right)^n$ is real, is

(a) 1 (b) 2
(c) 3 (d) 4

2. If $a^2 + b^2 = 1$, then $\frac{1+b+ia}{1+b-ia} =$

(a) 1 (b) 2
(c) $b + ia$ (d) $a + ib$

3. Total number of complex numbers 'z', satisfying $\text{Re}(z^2) = 0$, $|z| = \sqrt{3}$, is equal to

(a) 2 (b) 4
(c) 6 (d) None of these

4. If $(a + ib)^5 = \alpha + i\beta$, then $(b + ia)^5$ is equal to

(a) $\beta + i\alpha$ (b) $\alpha - i\beta$
(c) $\beta - i\alpha$ (d) $-\alpha - i\beta$

5. The number of solutions of the system of equations $\text{Re}(z^2) = 0, |z| = 2$ is

(a) 4 (b) 3
(c) 2 (d) 1

6. The complex number $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on :

(a) the x-axis
(b) the straight line $y = 5$
(c) a circle passing through the origin
(d) the y-axis

7. If $z = (3 + 7i)(p + iq)$, where $p, q \in I - \{0\}$, is purely imaginary, then minimum value of $|z|^2$ is

(a) 0 (b) 58
(c) $\frac{3364}{3}$ (d) 3364

8. If $(2 + i)(2 + 2i)(2 + 3i) \ldots\ldots (2 + ni) = x + iy$, then the value of $5.8.13. \ldots\ldots (4 + n^2)$

(a) $(x^2 + y^2)$ (b) $\sqrt{(x^2 + y^2)}$
(c) $2(x^2 + y^2)$ (d) $(x + y)$

9. Let Z is complex number satisfying the equation $z^2 - (3 + i)I + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root.
The additive inverse of non real root, is

(a) $1 - i$ (b) $1 + i$
(c) $-1 - i$ (d) -2

10. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to

(a) 1 (b) 3
(c) 0 (d) 2

11. In G.P. the first term & common ratio are both $\frac{1}{2}(\sqrt{3} + i)$, then the modulus of nth term is:

(a) 1 (b) 2^n
(c) 4^n (d) 3^n

12. If z_1 and $\overline{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin and if $\frac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to

(a) 8 (b) 12
(c) 16 (d) 24

13. If $z = x + iy$ and $z^{1/3} = a - ib$ then $\frac{x}{a} - \frac{y}{b} = k\,(a^2 - b^2)$ where $k =$

(a) 1 (b) 2
(c) 3 (d) 4

14. If $|z| \le 4$, then the maximum value of $|iz + 3 - 4i|$ is equal to

(a) 2 (b) 4
(3) 3 (d) 9

15. The set of values of $a \in R$ for which $x^2 + i(a - 1)x + 5 = 0$ will have a pair of conjugate imaginary roots is

(a) R (b) $\{1\}$
(c) $\{a : a^2 - 2a + 21 > 0\}$ (d) $\{0\}$

16. If $z = x + iy$ satisfies $\text{amp}(z - 1) = \text{amp}(z + 3)$ then the value of $(x - 1) : y$ is equal to

(a) 2 : 1 (b) 1 : 3
(b) $-1 : 3$ (d) does not exist

17. The minimum value of $|z - 1 + 2i| + |4i - 3 - z|$ is

(a) $\sqrt{5}$ (b) 5
(c) $2\sqrt{13}$ (d) $\sqrt{15}$

18. If $(1 + i)z = (1 - i)\overline{z}$ then z is

(a) $t(1 - i), I \in R$ (b) $t(1 + i), t \in R$
(c) $\frac{t}{1+i}, t \in R^+$ (d) $\frac{t}{1-i}, t \in R^+$

19. If z_1 and z_2 are two non-zero complex numbers such that $\left|\frac{z_1}{z_2}\right| = 2$ and $\arg(z_1 z_2) = \frac{3\pi}{2}$, then $\frac{\overline{z}_1}{z_2}$ is equal to

(a) 2 (b) -2
(c) $-2i$ (d) $2i$

20. If $|z^2-1|=|z|^2+1$, then z lies on :

(a) the real axis (b) the imaginary axis

(c) a circle (4) an ellipse

21. The inequality $|z-4|<|z-2|$ represents :

(a) $\text{Re}(z)>0$ (b) $\text{Re}(z)<0$

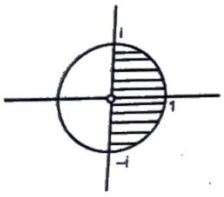
(c) $\text{Re}(z)>2$ (d) $\text{Re}(z)>3$

22. The system of equations $\left.\begin{matrix}|z+1-i| & =2\\ \text{Re}(z) & \geq 1\end{matrix}\right\}$, where z is a complex number has :

(a) no solution (b) exactly one solution

(c) two distinct solutions (d) infinite solution

23. The locus of z which lies in shaded region is best represented by



(a) $|z|\leq 1, \frac{-\pi}{2}\leq \arg z\leq \frac{\pi}{2}$

(b) $|z|=1, \frac{-\pi}{2}\leq \arg z\leq 0$

(c) $|z|\geq 0,\ 0\leq \arg z\leq \frac{\pi}{2}$

(d) $|z|\leq 1, \frac{\pi}{2}\leq \arg z\leq \pi$

24. The equation $|z-1|^2+|z+1|^2=2$ represents

(a) a circle of radius '1'

(b) a straight line

(c) the ordered pair (0, 0)

(d) a circle of radius '$\sqrt{2}$'

25. The region of Argand diagram defined by $|z-1|+|z+1|\leq 4$ is :

(a) interior of an ellipse

(b) exterior of a circle

(c) interior and boundary of an ellipse

(d) exterior of ellipse

26. If $|z|=1$ and $\omega=\frac{z-1}{z+1}$ (where $z\neq -1$), the $\text{Re}(\omega)$ is

(a) 0

(b) $-\frac{1}{|z+1|^2}$

(c) $\left|\frac{z}{z+1}\right|, \frac{1}{|z+1|^2}$

(d) $\frac{\sqrt{2}}{|z+1|^2}$

27. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if:

(a) $z_1+z_4=z_2+z_3$

(b) $z_1+z_3=z_2+z_4$

(c) $z_1+z_2=z_3+z_4$

(d) $z_1z_3=z_2z_4$

28. The expression $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n-\frac{1+i\tan n\alpha}{1-i\tan n\alpha}$ when simpli-fied reduces to :

(a) zero

(b) $2\sin n\alpha$

(c) $2\cos n\alpha$

(d) none of these

29. If $\alpha=-1+i\sqrt{3}$ and n is a positive integer which is not a multiple of 3, then $\alpha^{2n}+2^n\alpha^n+2^{2n}=$

(a) 1 (b) -1

(c) 0 (d) α^2

30. Let z and ω be two non-zero complex numbers such that $|z|=|\omega|$ and $\arg z=\pi-\arg\omega$, then z equals

(a) ω

(b) $-\omega$

(c) $\overline{\omega}$

(d) $-\overline{\omega}$

31. If α is non real and $\alpha=\sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to

(a) 4 (b) 2

(c) 1 (d) 8

Advanced Level Multiconcept Questions

MCQ COMPREHENSION / MATCHING / NUMERICAL

1. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is:

(a) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (b) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$

(c) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (d) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

2. If amp $(z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$, then

(a) $z_1 + z_2 = 0$ (b) $z_1 z_2 = 1$

(c) $z_1 = \bar{z}_2$ (d) $z_1 = z_2$

3. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then

(a) $\frac{z_1}{z_2}$ is purely real (b) $\frac{z_1}{z_2}$ is purely imaginary

(c) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ (d) amp $\frac{z_1}{z_2}$ may be equal to $\frac{\pi}{2}$

4. For complex numbers z and w, if $|z|^2 w - |w|^2 z = z - w$. Which of the following are true:

(a) $z = w$ (b) $z\bar{w} = 1$

(c) $z = w + 2$ (d) $\bar{z}\, w = 1$

5. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0, z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then which of the following are true.

(a) $|z_{2050}| = \sqrt{3}$ (b) $|z_{2017}| = \sqrt{2}$

(c) $|z_{2016}| = 1$ (d) $|z_{2111}| = \sqrt{2}$

6. If z is a complex number then the equation $z^2 + z|z| + |z^2| = 0$ is satisfied by (ω and ω^2 are imaginary cube roots of unity)

(a) $z = k\omega$ where $k \in R$

(b) $z = k\omega^2$ where k is non negative real

(c) $z = k\omega$ where k is positive real

(d) $z = k\omega^2$ where $k \in R$.

7. If z is a complex number which simultaneously satisfies the equations $3|z-12| = 5|z-8i|$ and $|z-4| = |z-8|$ then the Im(z) can be

(a) 15 (b) 16

(c) 17 (d) 8

8. If $|z_1| = |z_2| = |z_3| = 1$ and z_1, z_2, z_3 are represented by the vertices of an equilateral triangle then

(a) $z_1 + z_2 + z_3 = 0$ (b) $z_1 z_2 z_3 = 1$

(c) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$ (d) $z_2^3 + z_3^3 = 2z_1^3$

9. Let z_1, z_2, z_3 be three distinct complex numbers satisfying, $|z_1 - 1| = |z_2 - 1| = |z_3 - 1| = 1$. Let A, B & C be the points representing vertices of equilateral triangle in the Argand plane corresponding to z_1, z_2 and z_3 respectively. Which of the following are true

(a) $z_1 + z_2 + z_3 = 3$

(b) $z_1^2 + z_2^2 + z_3^2 = 3$

(c) area of triangle $= \frac{3\sqrt{3}}{4}$

(d) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 1$

10. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ ($a, b, c, d \in R$) has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$.

(a) $b = 51$ (b) $a = -6$

(c) $c = -70$ (d) $d = 170$

11. a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$. If two roots of the equation $P(z) = 0$ are 2 and i. Then which of the following are true.

(a) $a = -\frac{11}{2}$ (b) $b = 5$

(c) $c = -\frac{11}{2}$ (d) $a = -11$

12. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, then

(a) $x^n + \frac{1}{x^n} = 2\cos(n\theta), n \in z$

(b) $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \varphi)$

(c) $xy + \frac{1}{xy} = 2\cos(\theta + \varphi)$

(d) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\varphi), m, n \in z$

Comprehension – 1 (Q. No. 13 to 15)

Consider a complex number $w=\dfrac{z-i}{2z+1}$ where $z=x+iy$, where $x, y \in R$.

13. If the complex number w is purely imaginary then locus of z is

(*a*) a straight line

(*b*) a circle with centre $\left(-\dfrac{1}{4},\dfrac{1}{2}\right)$ and radius $\dfrac{\sqrt{5}}{4}$.

(*c*) a circle with centre $\left(\dfrac{1}{4},-\dfrac{1}{2}\right)$ and passing through origin.

(*d*) neither a circle nor a straight line.

14. If the complex number w is purely real then locus of z is

(*a*) a straight line passing through origin

(*b*) a straight line with gradient 3 and y intercept (–1)

(*c*) a straight line with gradient 2 and y intercept 1.

(*d*) None of these

15. If $|w|=1$ then the locus of P is

(*a*) a point circle (*b*) an imaginary circle

(*c*) a real circle (*d*) not a circle.

Comprehension – 2 (Q. No. 16 to 18)

Let a, b and c be three real numbers satisfying

$$[a \;\; b \;\; c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix}=[0 \;\; 0 \;\; 0] \qquad ...(E)$$

16. If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x+y+z=1$, then the value of $7a+b+c$, is

(*a*) 0 (*b*) 12

(*c*) 7 (*d*) 6

17. Let ω be a solution of $x^3-1=0$ with $\text{Im}(\omega)>0$. If $a-2$ with b and c satisfying (E), then the value of

$\dfrac{3}{\omega^a}+\dfrac{1}{\omega^b}+\dfrac{3}{\omega^c}$ is equal to

(*a*) –2 (*b*) 2

(*c*) 3 (*d*) –3

18. Let $b=6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then

$\sum_{n=0}^{\infty}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)^n$, is

(*a*) 6 (*b*) 7

(*c*) 6/7 (*d*) ∞

19. Match the items in Column-I with those in Column-II

Column-I	Column-II
(*a*) If $\lvert z-(3+4i)\rvert=2$ then the maximum value of $\lvert z\rvert$ is	(*P*) 10
(*b*) If $\lvert z-i\rvert<1$, then the value of $\lvert z+12-6i\rvert$ is less than	(*Q*) 14
(*c*) If $\lvert z_1\rvert=1$ and $\lvert z_2\rvert=2$, then $\lvert z_1+z_2\rvert^2+\lvert z_1-z_2\rvert^2$ is equal to	(*R*) 1
(*d*) If $z=1+i$, then $4(z^4-4z^3+7z^2-6z+3)$ is equal to	(*S*) 7
	(*T*) 5

20. Match the Column

Column-I	Column-II
(*a*) If $\text{Re}\left(\dfrac{iz+1}{iz-1}\right)=2$, then z lies on the curve	(*P*) $4x^2+4y^2+x-6y+2=0$
(*b*) $z_1=6+i, z_2=4-3i$ and z is a complex number such that $\arg\left(\dfrac{z-z_1}{z_2-z}\right)=\dfrac{\pi}{2}$, then z lies on	(*Q*) $x^2+y^2+4y+3=0$
	(*R*) $3(x^2+y^2)-2x-4y=0$
(*c*) If $\text{Im}\left(\dfrac{2z+1}{1+iz}\right)=2$, then z lies on	(*S*) $x^2+y^2-x+2y-1=0$
(*d*) If $\left\lvert\dfrac{2z-i}{z+1}\right\rvert=1$, then z lies on	(*T*) $(x-5)^2+(y+1)^2=5$

NUMERICAL VALUE BASED

21. Total number of complex numbers 'z', satisfying $\text{Re}(z^2) = 0$, $|z| = \sqrt{3}$, is equal to

22. Find the number of distinct complex numbers z such that $|z| = 1$ and z^3 is purely imaginary.

23. If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer then find the value of $\frac{N}{2}$

24. Let z is a complex number satisfying the equation, $z^3 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root α, then find the value of $\alpha^4 + m^4$

25. $z_1, z_2 \in C$ and $z_1^2 + z_2^2 \in R$,
$z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$
If $z_1^2 + z_2^2 = \lambda$ then determine λ^2

26. If z_1 and z_2 both satisfy the relation, $z + \bar{z} = 2|z - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find the imaginary part of $(z_1 + z_2)$.

27. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$, then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

28. Let $z = \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{i}{2}\right)$. Then the smallest positive integer n such that $(z^{95} + i^{67})^{94} = z^n$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. A value of θ for which $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, is

[JEE (Main) 2016]

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. Let $A = \left\{\theta \in \left(-\frac{\pi}{2}, \pi\right) : \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ purely imaginary}\right\}$. Then the sum of the elements in A is:

[JEE Main–2019 (January)]

(1) $\frac{5\pi}{6}$ (2) π

(3) $\frac{3\pi}{4}$ (4) $\frac{2\pi}{3}$

3. Let Z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$. If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then arg z is equal to

[JEE Main–2019 (January)]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) 0 (d) $\frac{\pi}{6}$

4. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$ then;

[JEE Main-2019 (January)]

(a) Re $(z) = 0$ (b) $|z| = \sqrt{\frac{5}{2}}$

(c) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (d) Im$(z) = 0$

5. Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$). Then $|Z|$ is equal to:

[JEE Main–2019 (January)]

(a) $\frac{\sqrt{34}}{3}$ (b) $\frac{5}{3}$

(c) $\frac{\sqrt{41}}{4}$ (d) $\frac{5}{4}$

6. Let $\left(-2 - \frac{1}{3}i\right)^3 = \frac{x + iy}{27}$ $(i = \sqrt{-1})$, where x and y are real numbers, then $y - x$ equals:

[JEE Main-2019 (January)]

(a) 91 (b) -85
(c) 85 (d) -91

7. If $\frac{z-\alpha}{z+\alpha}$ ($\alpha \in R$) is a purely imaginary number and $|z| = 2$, then a value of α is: [JEE Main-2019 (January)]

(a) 2 (b) 1

(c) 1/2 (d) $\sqrt{2}$

8. All the points in the set $S = \left\{\frac{\alpha+i}{\alpha-i} : \alpha \in R\right\}$ ($i = \sqrt{-1}$) lie on a [JEE Main-2019 (April)]

(a) circle whose radius is 1

(b) straight line whose slope is 1.

(c) straight line whose slope is –1

(d) circle whose radius is $\sqrt{2}$

9. Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then: [JEE Main-2019 (April)]

(a) $5\,\text{Im}(\omega) < 1$ (b) $4\,\text{Im}(\omega) > 5$

(c) $5\,\text{Re}(\omega) > 1$ (d) $5\,\text{Re}(\omega) > 4$

10. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$ then: [JEE Main-2019 (April)]

(a) $\bar{z}w = i$ (b) $\bar{z}w = -i$

(c) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (d) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

11. If $a > 0$ and $z = \frac{(1+i)^2}{a-i}$, has magnitude $\sqrt{\frac{2}{5}}$, then $\bar{z}$ is equal to: [JEE Main-2019 (April)]

(1) $-\frac{3}{5} - \frac{1}{5}i$ (2) $-\frac{1}{5} + \frac{3}{5}i$

(3) $-\frac{1}{5} - \frac{3}{5}i$ (4) $\frac{1}{5} - \frac{3}{5}i$

12. The equation $|z-i| = |z-1|$, $i = \sqrt{-1}$, represents: [JEE Main-2019 (April)]

(a) the line through the origin with slope –1.

(b) a circle of radius 1.

(c) a circle of radius 1/2.

(d) the line through the origin with slope 1.

13. Let $z \in C$ with $\text{Im}(z) = 10$ and it satisfies $\frac{2z-n}{2z+n} = 2i-1$ for some natural number n. Then : [JEE Main-2019 (April)]

(a) $n = 20$ and $\text{Re}(z) = -10$ (b) $n = 20$ and $\text{Re}(z) = 10$

(c) $n = 40$ and $\text{Re}(z) = -10$ (d) $n = 40$ and $\text{Re}(z) = 10$

14. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ ($i = \sqrt{-1}$), then $(1 + iz + z^5 + iz^8)^9$ is equal to [JEE Main-2019 (April)]

(a) –1 (b) 1

(c) 0 (d) $(-1+2i)^9$

15. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x, y) lies on a [JEE Main-2020 (January)]

(a) straight line whose slope is $\frac{-2}{3}$

(b) straight line whose slope is $\frac{3}{2}$

(c) circle whose diameter is $\frac{\sqrt{5}}{2}$

(d) circle whose centre is at $\left(\frac{-1}{2}, \frac{-3}{2}\right)$

16. If $\frac{3+i\sin\theta}{4-i\cos\theta}, \theta \in [0, 2\pi]$ is a real number, then an argument of $\sin\theta + i\cos\theta$ is: [JEE Main-2020 (January)]

(a) $\pi - \tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$

(c) $\pi - \tan^{-1}\left(\frac{4}{3}\right)$ (d) $-\tan^{-1}\left(\frac{3}{4}\right)$

17. If the equation, $x^2 + bx + 45 = 0$ ($b \in R$) has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then [JEE Main-2020 (January)]

(a) $b^2 - b = 30$ (b) $b^2 + b = 72$

(c) $b^2 - b = 42$ (d) $b^2 + b = 12$

18. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$ If $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation: [JEE Main-2020 (January)]

(a) $x^2 + 101x + 100 = 0$

(b) $x^2 - 102x + 101 = 0$

(c) $x^2 - 101x + 100 = 0$

(d) $x^2 + 102x + 101 = 0$

19. Let z be a complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z+3i|$ is: [JEE Main-2020 (January)]

(a) $\sqrt{10}$ (b) $2\sqrt{3}$

(c) $\frac{7}{2}$ (d) $\frac{15}{4}$

20. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be: [JEE Main-2020 (January)]

(a) $\sqrt{10}$ (b) $\sqrt{8}$

(c) $\sqrt{7}$ (d) $\frac{\sqrt{17}}{2}$

21. The imaginary part of

$(3+2\sqrt{-54})^{1/2}-(3-2\sqrt{-54})^{1/2}$ can be:

[JEE Main-2020 (September)]

(a) $\sqrt{6}$ (b) $-\sqrt{6}$

(c) $-2\sqrt{6}$ (d) 6

22. The value of $\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is:

[JEE Main-2020 (September)]

(a) $-\frac{1}{2}(1-i\sqrt{3})$ (b) $\frac{1}{2}(1-i\sqrt{3})$

(c) $\frac{1}{2}(\sqrt{3}-i)$ (d) $-\frac{1}{2}(\sqrt{3}-i)$

23. If z_1, z_2 are complex numbers such that $\mathrm{Re}(z_1)=|z_1-1|$, $\mathrm{Re}(z_2)=|z_2-1|$, and $\arg(z_1-z_2)=\frac{\pi}{6}$, then $\mathrm{Im}(z_1+z_2)$ is equal to [JEE Main-2020 (September)]

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{2}{\sqrt{3}}$ (d) $2\sqrt{3}$

24. If $\left(\frac{1+i}{1-i}\right)^{m/2}=\left(\frac{1+i}{1-i}\right)^{n/3}=1\,(m,n\in N)$ then the greatest common divisor of the least values of m and n is_____.

[JEE Main-2020 (September)]

25. If a and b are real numbers such that $(2+\alpha)^4=a+b\alpha$, where $\alpha=\frac{-1+i\sqrt{3}}{2}$, then $a+b$ is equal to

[JEE Main-2020 (September)]

(a) 33 (b) 9

(c) 24 (d) 57

26. Let $u=\frac{2z+i}{z-ki}$, $z=x+iy$ and $k>0$. If the curve represented by $\mathrm{Re}(u)+\mathrm{Im}(u)=1$ intersects the y-axis at the point P and Q where $PQ=5$, then the value of K is:

[JEE Main-2020 (September)]

(a) 1/2 (b) 3/2

(c) 2 (d) 4

27. The value of $\left(\frac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is :

[JEE Main-2020 (September)]

(a) $-2^{15}i$ (b) -2^{15}

(c) $2^{15}i$ (d) 6^5

28. If the four complex numbers $z, \bar{z}, \bar{z}-2\mathrm{Re}(\bar{z})$ and $z-2\mathrm{Re}(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to:

[JEE Main-2020 (September)]

(a) $4\sqrt{2}$ (b) 2

(c) $2\sqrt{2}$ (d) 4

29. Let $z=x+iy$ be a non-zero complex number such that $z^2=i|z|^2$, where $i=\sqrt{-1}$, then z lies on the:

[JEE Main-2020 (September)]

(a) line, $y=x$ (b) imaginary axis

(c) real axis (d) line, $y=-x$

30. The region represented by $\{z=x+iy\in C:|z|-\mathrm{Re}(z)\le 1\}$ is also given by the inequality:

[JEE Main-2020 (September)]

(a) $y^2\le x+\frac{1}{2}$ (b) $y^2\le 2\left(x+\frac{1}{2}\right)$

(c) $y^2\ge x+1$ (d) $y^2\ge 2(x+1)$

31. Let the lines $(2-i)z=(2+i)\bar{z}$ and $(2+i)z+(i-2)\bar{z}-4i=0$, (here $i^2=-1$) be normal to a circle C. If the line $iz+\bar{z}+1+i=0$ is tangent to this circle C, then its radius is : [JEE Main-2021 (February)]

(a) $\frac{3}{2\sqrt{2}}$ (b) $3\sqrt{2}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{3}{\sqrt{2}}$

32. If $\alpha,\beta\in\mathrm{R}$ are such that $1-2i$ (here $i^2=-1$) is a root of $z^2+\alpha z+\beta=0$, then $(\alpha-\beta)$ is equal to :

[JEE Main-2021 (February)]

(a) –7, (b) 7

(c) 3 (d) –3

33. Let a complex number $z, |z|\neq 1$, satisfy

$\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right)\le 2$. Then, the largest value of $|z|$ is equal to [JEE Main-2021 (March)]

(a) 5 (b) 8

(c) 6 (d) 7

34. The least value of $|z|$ where z is complex number which satisfies the inequality

$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\log_e 2\right)\ge\log_{\sqrt{2}}|5\sqrt{7}+9i|, i=\sqrt{-1}$,

is equal to: [JEE Main-2021 (March)]

(a) 8 (b) 3

(c) 2 (d) $\sqrt{5}$

JEE-ADVANCED PREVIOUS YEARS

1. Let z be a complex number such that the imaginary part of z is non zero and $a = z^2 + z + 1$ is real. Then a cannot take the value [IIT JEE 2012]

(a) -1 (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

2. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lies on circles $(x-x_0)^2 + (y-y_0)^2 = r^2$ and $(x-x_0)^2 + (y-y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ [JEE Advanced-2013]

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

3. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re } z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re } z < -\frac{1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$ [JEE Advanced-2013]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

4. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$ [JEE Advanced-2013]

(a) 57 (b) 55

(c) 58 (d) 56

Comprehension – 1 (Q. No. 5 to 6)

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : |z| < 4\}$,

$S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and

$S_3 : \{z \in C : \text{Re } z > 0\}$. [JEE Advanced-2013]

5. Area of $S =$

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$

(c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

6. $\min_{z \in S} |1 - 3i - z| =$

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$

(c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

7. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, ...9$. [JEE Advanced-2014]

List I

P. For each z_k there exists a z_j such that $z_k \cdot z_j = 1$

Q. There exists a $k \in \{1,2,....,9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.

R. $\frac{|1-z_1||1-z_2|.....|1-z_9|}{10}$ equals

S. $1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals

$z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, ...9$.

List II

1. True

2. False

3. 1

4. 2

	P	Q	R	S
(a)	1	2	4	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	2	1	4	3

8. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$$ is

[JEE Advanced-2015]

9. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

[JEE Advanced-2016]

10. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose

$$S = \left\{ z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0 \right\}$$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on

[JEE Advanced-2016]

(a) The circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(b) The circle with radius $\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) The x-axis for $a \neq 0, b = 0$

(d) The y-axis for $a = 0, b \neq 0$

11. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is (are) possible value(s) of x?

[JEE Advanced-2017]

(a) $1 - \sqrt{1+y^2}$ (b) $-1 - \sqrt{1-y^2}$

(c) $1 + \sqrt{1+y^2}$ (d) $-1 + \sqrt{1-y^2}$

12. For a non-zero complex number z, let $\arg(z)$ denotes the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) **False**?

[JEE Advanced-2018]

(a) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(b) The function $f: R \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$

(c) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(d) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a stright line

13. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) True?

[JEE Advanced-2018]

(a) If L has exactly one element, then $|s| \neq |t|$

(b) If $|s| = |t|$, then L has infinitely many elements

(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

14. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are True?

[JEE (Advanced)-2020]

(a) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$

(b) $|z| \leq 2$ for all $z \in S$

(c) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

15. For a complex number z, let $\text{Re}(z)$ denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_1) < 0$, is .....

[JEE (Advanced)-2020]

16. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \ldots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1} e^{i\theta_k}$ for $k = 2, 3, \ldots, 10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below:

$$P: |z_2 - z_1| + |z_3 - z_2| + \ldots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$$

$$Q: |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \ldots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$$

Then,

[JEE (Advanced)-2021]

(a) P is **TRUE** and Q is **FALSE**

(b) Q is **TRUE** and P is **FALSE**

(c) both P and Q are **TRUE**

(d) both P and Q are **FALSE**

17. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle $x^2 + y^2 + 5x - 3y + 4 = 0$

Then which of the following statements is (are) TRUE?

(a) $\alpha = -1$ (b) $\alpha\beta = 4$

(c) $\alpha\beta = -4$ (d) $\beta = 4$

[JEE (Advanced)-2021]

ANSWER KEY

Topicwise Questions

1. (b)	2. (b)	3. (b)	4. (b)	5. (c)	6. (c)	7. (c)	8. (a)	9. (b)	10. (c)	11. (a)	12. (d)
13. (a)	14. (b)	15. (d)	16. (b)	17. (d)	18. (a)	19. (b)	20. (a)	21. (c)	22. (a)	23. (c)	24. (b)
25. (b)	26. (b)	27. (d)	28. (c)								

Learning Plus

1. (c)	2. (c)	3. (a)	4. (a)	5. (c)	6. (a)	7. (b)	8. (b)	9. (c)	10. (b)	11. (b)	12. (b)
13. (a)	14. (a)	15. (d)	16. (a)	17. (b)	18. (c)	19. (d)	20. (b)	21. (d)	22. (b)	23. (a)	24. (c)
25. (c)	26. (a)	27. (b)	28. (a)	29. (c)	30. (d)	31. (a)					

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (bd)	2. (b, c)	3. (b, c, d)	4. (a, b, d)	5. (b, c, d)	6. (b, c)
7. (c, d)	8. (a, c, d)	9. (a, b, c)	10. (a, b, c, d)	11. (a, b, c)	12. (a, b, c, d)
13. (b)	14. (c)	15. (c)	16. (d)	17. (a)	18. (b)

19. (a) (s), (b) (q), (c) (p), (d) (No option match)

20. (a) → (q), (b) → (t), (c) → (p), (d) → (r)

21. [0004]	22. [0006]	23. [99]	24. [32]	25. [25]	26. [2]
27. [0000]	28. [0010]			

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (d)	2. (d)	3. (a)	4. (None options is correct)	5. (b)	6. (a)	7. (a)	8. (a)	9. (c)	10. (b)	
11. (c)	12. (d)	13. (c)	14. (a)	15. (c)	16. (c)	17. (a)	18. (b)	19. (c)	20. (c)	21. (c)	22. (d)
23. (d)	24. (4)	25. (b)	26. (c)	27. (a)	28. (c)	29. (a)	30. (b)	31. (a)	32. (a)	33. (d)	34. (b)

JEE-ADVANCED PREVIOUS YEARS

1. (d)	2. (c)	3. (c, d)	4. (b, c, d)	5. (b)	6. (c)	7. (c)	8. [4]	9. [1]	10. (a, c, d)
11. (b, d)	12. (a, b, d)		13. (a, c, d)	14. (b, c)	15. [8]	16. (b)	17. (b, d)		

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/

8 DETERMINANTS

PRAYAS 1.0 time/real programmers

PRAYAS 2.0 time/real programmers

DETERMINANT OF A SQUARE MATRIX

Let $A = [a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times1}$ $\quad |A| = -3$

MINORS & COFACTORS

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}.$$

$$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay.$$

Determinant of Any Order

Let $A = [a_{ij}]_n$ be a square matrix ($n > 1$). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

(using second column)

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Note that a determinant of order 3 will have 9 minors each minor will be determinant of order 2 and a determinant of a order 4 will have 16 minors each minor will be determinant of order 3.

Note :

(*i*) Sum of product of element of any row (column) with their corresponding cofactors is equal to the value of DETERMINANT.

i.e. $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \sum_{j=1}^{3} a_{ij}C_{ij}$, $i = 1$

for $i = 1, 2, 3$, for $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(*ii*) Sum of product of elements of any row (column) with cofactors of corresponding elements of any other row (column) is ZERO.

TRANSPOSE OF A DETERMINANT

The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Rightarrow \quad D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Determinant

1. $|A| = |A'|$ for any square matrix A.
   i.e.the value of a determinant remains unaltered, if the rows & columns are inter changed,
   i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

2. If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.
   e.g. Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$
   Then $D_2 = -D_1$

3. Let λ be a scalar. Than $\lambda|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } E=\begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $E = KD$

4. $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

5. A skew-symmetric matrix of odd order has deteminant value zero.

6. If a determinant has all the elements zero in any row or column, then its value is zero,

$$\text{i.e. } D=\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

7. If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

$$\text{i.e. } D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

8. If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

9. The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

$$\text{i.e., } D_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and } D_2=\begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}.$$

Then $D_2 = D_1$

10. Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

TRAIN YOUR BRAIN

Example 1: Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$$= (a+b+c)\,((b-c)(a-b)-(c-a)^2)$$
$$= (a+b+c)\,(ab+bc-ca-b^2-c^2+2ca-a^2)$$
$$= (a+b+c)\,(ab+bc+ca-a^2-b^2-c^2) \equiv 3abc-a^3-b^3-c^3$$

Example 2: Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Sol. Since detereminant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\,[ab^2+abc+ac^2+b^3+b^2c+bc^2-a^2b - a^2c-ab^2-abc-b^3-b^2c]$$
$$= (a-b)(b-c)\,[c(ab+bc+ca)-a(ab+bc+ca)]$$
$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

SOME SPEICAL DETERMINANTS

(a) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(b) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

(c) $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(d) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Note : If a, b, c > 0, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} < 0$

(e) **For the determinants where $a_{ij} = -a_{ji}$ for all i & j**

(i) Diagonal elements are zero.

$a_{ii} = -a_{ii}$ (for diagonal elements)

$\Rightarrow \quad 2a_{ii} = 0 \quad \Rightarrow \quad a_{ii} = 0$

(*ii*) If the order of above determinant is ODD then its value is ZERO e.g. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

FACTOR THEOREM

Use of factor theorem to find the value of determinant. If by putting x = a the value of a determinant vanishes then (x – a) is a factor of the determinant.

TRAIN YOUR BRAIN

Example 3: Prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$ by using factor theorem.

Sol. Let $a = b$

$\Rightarrow D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ac & ab \end{vmatrix} = 0$

Hence $(a-b)$ is a factor of determinant

Similarly,

let $b - c, D = 0$

$c - a, D = 0$

Hence, $(a-b)(b-c)(c-a)$ is factor of determinant.

$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)\{\lambda(a^2+b^2+c^2) + \mu(ab+bc+ca)\}$

Since this is an identity so in order to find the values of λ and μ.

Let

$a = 0, b = 1, c = -1$

$-2 = (2)(2\lambda - \mu)$

$(2\lambda - \mu) = -1.$ ...(i)

Let

$a = 1, b = 2, c = 0$

$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (-1)2(-1)(5\lambda + 2\mu)$

$\Rightarrow \quad 5\lambda + 2\mu = 2$ ...(ii)

from (*i*) and (*ii*) $\lambda = 0$ and $\mu = 1$

Hence $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$

MULTIPLICATION OF TWO DETERMINANTS

If a and b are two square matrices of same order, then |ab| = |a| |b|.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix}$

$= \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$

Note: As $|A| = |A'|$, we have $|A||B| = |AB'|$ (row - row method)

$|A||B| = |A'B|$ (column - column method)

$|A||B| = |A'B'|$ (column - row method)

TRAIN YOUR BRAIN

Example 4: Find the value of $\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix} \times \begin{vmatrix}3 & 0\\ -1 & 4\end{vmatrix}$ and prove that it is equal to $\begin{vmatrix}1 & 8\\ -6 & 12\end{vmatrix}$.

Sol. $\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix} \times \begin{vmatrix}3 & 0\\ -1 & 4\end{vmatrix}$

$$= \begin{vmatrix}1\times3-2\times1 & 1\times0+2\times4\\ -1\times3+3\times(-1) & -1\times0+3\times4\end{vmatrix} = \begin{vmatrix}1 & 8\\ -6 & 12\end{vmatrix} = 60$$

Example 5: Prove that

$$\begin{vmatrix}a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3\\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3\\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3\end{vmatrix}=0$$

Sol. Given determinant can be splitted into product of two determinants

i.e. $\begin{vmatrix}a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3\\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3\\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3\end{vmatrix}$

$$= \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} \times \begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ 0 & 0 & 0\end{vmatrix} = 0$$

Example 6: Prove that $\begin{vmatrix}(a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2\\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2\\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2\end{vmatrix}$

$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$.

Sol. $\begin{vmatrix}(a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2\\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2\\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2\end{vmatrix}$

$$= \begin{vmatrix}a_1^2+b_1^2-2a_1b_1 & a_1^2+b_2^2-2a_1b_2 & a_1^2+b_3^2-2a_1b_3\\ a_2^2+b_1^2-2a_2b_1 & a_2^2+b_2^2-2a_2b_2 & a_2^2+b_3^2-2a_2b_3\\ a_3^2+b_1^2-2a_3b_1 & a_3^2+b_2^2-2a_3b_2 & a_3^2+b_3^2-2a_3b_3\end{vmatrix}$$

$$= \begin{vmatrix}a_1^2 & 1 & -2a_1\\ a_2^2 & 1 & -2a_2\\ a_3^2 & 1 & -2a_3\end{vmatrix} \times \begin{vmatrix}1 & 1 & 1\\ b_1^2 & b_2^2 & b_3^2\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$= 2\begin{vmatrix}1 & a_1^2 & a_1\\ 1 & a_2^2 & a_2\\ 1 & a_3^2 & a_3\end{vmatrix} \times \begin{vmatrix}1 & b_1^2 & b_1\\ 1 & b_2^2 & b_2\\ 1 & b_3^2 & b_3\end{vmatrix}$$

$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$

Note: The above problem can also be solved using factor theorem method.

SUMMATION OF DETERMINANTS

Let $\Delta(r) = \begin{vmatrix}f(r) & g(r) & h(r)\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants indepedent of r, then

$$\sum_{r=1}^{n}\Delta(r) = \begin{vmatrix}\sum_{r=1}^{n}f(r) & \sum_{r=1}^{n}g(r) & \sum_{r=1}^{n}h(r)\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here the functions of r can be the elements of only one row or column. None of the elements other then that row or column should be dependent on r. If more than one column or row have elements dependent on r then first expand the determinant and then find the summation.

TRAIN YOUR BRAIN

Example 7: Evaluate $\sum_{r=1}^{n}\begin{vmatrix}2r-1 & {}^nC_r & 2^r\\ x & \cos^2\theta & y\\ n^2 & 2^n-1 & 2^{n+1}-2\end{vmatrix}$

Sol: $\sum_{r=1}^{n}D_r = \begin{vmatrix}\sum_{r=1}^{n}(2r-1) & \sum_{r=1}^{n}{}^nC_r & \sum_{r=1}^{n}2^r\\ x & \cos^2\theta & y\\ n^2 & 2^n-1 & 2^{n+1}-2\end{vmatrix} \begin{vmatrix}\sum_{r=1}^{n}(2r-1) & \sum_{r=1}^{n}n_{C_r} & \sum_{r=1}^{n}2^r\\ x & \cos^2\theta & y\\ n^2 & 2^n-1 & 2^{n+1}-2\end{vmatrix}$

$$= \begin{vmatrix}n^2 & 2^n-1 & 2^{n+1}-2\\ x & \cos^2\theta & y\\ n^2 & 2^n-1 & 2^{n+1}-2\end{vmatrix} = 0$$

Example 8: If $\Delta_r = \begin{vmatrix}r-1 & 1 & 0\\ 2 & r & 3+r\\ r+1 & -1 & -2\end{vmatrix}$, find $\sum_{r=1}^{n}\Delta_r$

Sol. On expansion of determinent, we get

$$D_r = (r-1)(3-r)+7+r^2+4r = 8r+4$$

$$\Rightarrow \sum_{r=1}^{n}\Delta_r = 4n(n+2)$$

DIFFERENTIATION OF DETERMINANT

Let $\Delta(x) = \begin{vmatrix}f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$

then $\Delta'(x) = \begin{vmatrix}f_1'(x) & f_2'(x) & f_3'(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$

$$+ \begin{vmatrix}f_1(x) & f_2(x) & f_3(x)\\ g_1'(x) & g_2'(x) & g_3'(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix} + \begin{vmatrix}f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1'(x) & h_2'(x) & h_3'(x)\end{vmatrix}$$

Note: We can differentiate a determinant columnwise also.

TRAIN YOUR BRAIN

Example 9: If $f(x)=\begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & a & a^2 \end{vmatrix}$, then find the value of $f'(a)$.

Sol: $f'(x)=\begin{vmatrix} 3 & 2 & 1 \\ 12x & 6x^2 & 4x^3 \\ 1 & a & a^2 \end{vmatrix}$

$$f''(x)=\begin{vmatrix} 3 & 2 & 1 \\ 12 & 12x & 12x^2 \\ 1 & a & a^2 \end{vmatrix}$$

$$\Rightarrow f''(a)=12\begin{vmatrix} 3 & 2 & 1 \\ 1 & a & a^2 \\ 1 & a & a^2 \end{vmatrix}=0.$$

Example 10: Let α be a repeated root of quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomial of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ divisible by $f(x)$.

Sol. Let $g(x)=\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$$\Rightarrow g'(x)=\begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Since $g(\alpha)=g'(\alpha)=0$

$\Rightarrow g(x)=(x-\alpha)^2 h(x)$ i.e. α is the repeated root of $g(x)$ and $h(x)$ is any polynomial expression of degree 3. Also $f(x) = 0$ have repeated root α. So $g(x)$ is divisible by $f(x)$.

INTEGRATION OF A DETERMINANT

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ where a_1, b_1, c_1, a_2, b_2, c_2 are constants independent of x. Hence

$$\int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Note: If more than one row or one column are function of x then first expand the determinant and then integrate it.

TRAIN YOUR BRAIN

Example 11: If $f(x)=\begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then find $\int_0^{\pi/2} f(x)\,dx$

Sol. Here $f(x) = \cos x\,(4\cos^2 x - 1) - 2\cos x = 4\cos^3 x - 3\cos x = \cos 3x$ so $\int_0^{\pi/2} \cos 3x\,dx = \frac{\sin 3x}{3}\Big]_0^{\pi/2} = -\frac{1}{3}$

Example 12: If $\Delta=\begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ x & x^2 & x^3 \end{vmatrix}$, then find $\int_0^1 \Delta(x)\,dx$

Sol. $\int_0^1 \Delta(x)\,dx=\begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ \int_0^1 x\,dx & \int_0^1 x^2\,dx & \int_0^1 x^3\,dx \end{vmatrix}$

$$=\begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{vmatrix}$$

$$=\frac{1}{12}\begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ 6 & 4 & 3 \end{vmatrix}=0$$

CRAMER'S RULE: SYSTEM OF LINEAR EQUATIONS

(i) Two Variables

Let $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ then:

$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}\Rightarrow$ Given equations are inconsistent &

$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\Rightarrow$ Given equations are consistent

(ii) Three Variables

consider the system

$a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Then, $D.x = D_1$, $D.y = D_2$, $D.z = D_3$

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$;

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

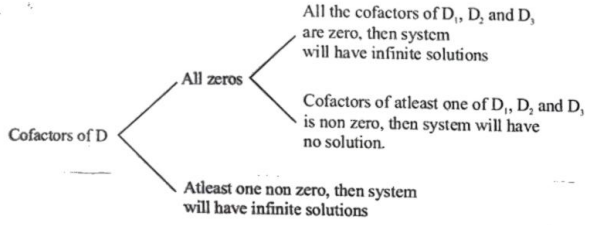
Consistency of a System of Equations

(a) If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$ then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations have either infinite solutions or no solution. (For 2×2 system, $D = 0 = D_1 = D_2 \Leftrightarrow$ system has infinitely many solutions).

(d) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.



Homogeneous System

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

(x, y, z) = (0, 0, 0) is always a solution of this system. This solution is called as the trivial solution (or zero solution) of this system.

$D \neq 0 \Rightarrow$ this system has only the trivial solution.

$D = 0 \Rightarrow$ this system has nontrivial solutions (infinitely many solutions).

Three Equation in Two Variables

If x and y are not zero, then condition for $a_1x+b_1y+c_1=0$; $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ to be consistent in x and y is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

TRAIN YOUR BRAIN

Example 13: Find the nature of solution for the given system of equations.

$x + 2y + 3z = 1$
$2x + 3y + 4z = 3$
$3x + 4y + 5z = 0$

Sol. Let $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$D = \begin{vmatrix} -1 & -1 & 3 \\ -1 & -1 & 4 \\ -1 & -1 & 5 \end{vmatrix} = 0$

$D = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix}$

$C_3 \to C_3 - C_2$

$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 0 & 4 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$D_1 = \begin{vmatrix} -2 & -1 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 5$

$D = 0$ But $D_1 \neq 0$ Hence no solution

Example 14: Consider the following system of equations

$x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

Find values of λ and μ if such that sets of equation have

(i) unique solution
(ii) infinite solution
(iii) no solution

Sol. $x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}$

Here for $\lambda = 3$ second and third rows are identical hence $D = 0$ for $\lambda = 3$.

$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix}$ $D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix}$

$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix}$

If $\lambda = 3$ then $D_1 = D_2 = D_3 = 0$ for $\mu = 10$

(i) For unique solution $D \neq 0$

i.e. $\lambda \neq 3$

(ii) For infinite solutions

$D = 0 \Rightarrow \lambda = 3$

$D_1 = D_2 = D_3 = 0 \Rightarrow \mu = 10.$

(iii) For no solution

$D = 0 \Rightarrow \lambda = 3$

Atleast one of D_1, D_2 or D_3 is non zero

$\Rightarrow \mu \neq 10.$

APPLICATION OF DETERMINANTS

Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is:

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If $D = 0$ then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines: $a_1x + b_1y + c_1 = 0$ ...(1)

$a_2x + b_2y + c_2 = 0$ ...(2)

$a_3x + b_3y + c_3 = 0$ ...(3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Topicwise Questions

ELEMENTRY

Minor Cofactors & Expansion of Determinant

1. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} =$

(*a*) $a^2+b^2+c^2$

(*b*) $(a+b)(b+c)(c+a)$

(*c*) $(a-b)(b-c)(c-a)$

(*d*) $a+b+c$

2. The determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$ is not equal to

(*a*) $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 3 & 6 \end{vmatrix}$ (*b*) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{vmatrix}$

(*c*) $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 1 & 9 & 6 \end{vmatrix}$ (*d*) $\begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 10 & 3 & 6 \end{vmatrix}$

3. If ω is the cube root of unity, then $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} =$

(*a*) 1 (*b*) 0

(*c*) ω (*d*) ω^2

4. If $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$, then $x =$

(*a*) 1, 9 (*b*) –1, 9

(*c*) –1, –9 (*d*) 1, –9

5. $\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix} =$

(*a*) $a^2+b^2+c^2-3abc$ (*b*) 3ab

(*c*) 3a + 5b (*d*) 0

6. The cofactor of the element '4' in the determinant $\begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 3 & 4 & 2 \\ 8 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$ is

(*a*) 4 (*b*) 10

(*c*) – 10 (*d*) –4

7. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 denote the co-factors of a_1, b_1, c_1 respectively, then the value of the determinant $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is

(*a*) Δ (*b*) Δ^2

(*c*) Δ^3 (*d*) 0

8. If $A = \begin{vmatrix} 5 & 6 & 3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$, then cofactors of the elements of 2nd row are

(*a*) 39, –3, 11 (*b*) –39, 3, 11

(*c*) –39, 27, 11 (*d*) –39, –3, 11

9. The minors of – 4 and 9 and the co-factors of – 4 and 9 in determinant $\begin{vmatrix} -1 & -2 & 3 \\ -4 & -5 & -6 \\ -7 & 8 & 9 \end{vmatrix}$ are respectively

(*a*) 42, 3 ; – 42, 3

(*b*) – 42, – 3 ; 42, –3

(*c*) 42, 3 ; – 42, - 3

(*d*) 42, 3; 42, 3

Properties of Determinant

10. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} =$

(*a*) abc (*b*) 4abc

(*c*) $4a^2b^2c^2$ (*d*) $a^2b^2c^2$

11. If $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = k(x+y+z)(x-z)^2$, then k =

(a) 2xyz (b) 1
(c) xyz (d) $x^2y^2z^2$

12. If $a^{-1}+b^{-1}+c^{-1}=0$ such that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \lambda$.

then the value of λ is

(a) 0 (b) abc
(c) –abc (d) 2abc

13. If $a^2+b^2+c^2=-2$

and $f(x)=\begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$ then f(x) is a polynomial of degree

(a) 3 (b) 2
(c) 1 (d) 0

14. The determinant $\begin{vmatrix} 4+x^2 & -6 & -2 \\ -6 & 9+x^2 & 3 \\ -2 & 3 & 1+x^2 \end{vmatrix}$ is not divisible by

(a) x (b) x^3
(c) $14+x^2$ (d) x^5

15. The value of the determinant

$\begin{vmatrix} 0 & b^3-a^3 & c^3-a^3 \\ a^3-b^3 & 0 & c^3-b^3 \\ a^3-c^3 & b^3-c^3 & 0 \end{vmatrix}$ is equal to

(a) $a^3+b^3+c^3$ (b) $a^3-b^3-c^3$
(c) 0 (d) $-a^3+b^3+c^3$

16. $\begin{vmatrix} 1+\sin^2\theta & \sin^2\theta & \sin^2\theta \\ \cos^2\theta & 1+\cos^2\theta & \cos^2\theta \\ 4\sin 4\theta & 4\sin 4\theta & 1+4\sin 4\theta \end{vmatrix} = 0$

then $\sin 4\theta$ equal to

(a) 1/2 (b) 1
(c) –1/2 (d) –1

17. If $f(x)=\begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$ then

$f(1).f(3)+f(3).f(5)+f(5).f(1) =$

(a) f(1) (b) f(3)
(c) f(1)+f(3) (d) f(1)+f(5)

18. If $\begin{vmatrix} y+z & x-z & x-y \\ y-z & z-x & y-x \\ z-y & z-x & x+y \end{vmatrix} = k\,xyz$. then the value of k is

(a) 2 (b) 4
(c) 6 (d) 8

Application of determinant

19. $x+ky-z=0, 3x-ky-z=0$ and $x-3y+z=0$ has non-zero solution for k =

(a) –1 (b) 0
(c) 1 (d) 2

20. The number of solutions of equations $x+4y-z=0$, $3x-4y-z=0, x-3y+z=0$ is

(a) 0 (b) 1
(c) 2 (d) Infinite

21. If the system of equations, $x+2y-3z=1$, $(k+3)z=3$, $(2k+1)x+z=0$ is inconsistent, then the value of k is

(a) –3 (b) 1/2
(c) 0 (d) 2

22. If the system of equation

$3x-2y+z=0$, $\lambda x-14y+15z=0$, $x+2y+3z=0$

have a non-trivial solution, then λ =

(a) 5 (b) –5
(c) –29 (d) 29

23. The existence of the unique solution of the system $x+y+z=\lambda$, $5x-y+\mu z=10$, $2x+3y-z=6$ depends on

(a) μ only (b) λ only
(c) λ and μ both (d) neither λ nor μ

24. The system of equations $x+y+z=2$, $3x-y+2z=6$ and $3x+y+z=-18$ has

(a) A unique solution
(b) No solutions
(c) An infinite number of solutions
(d) Zero solution as the only solution

25. If the system of equations $x+ay=0$, $az+y=0$ and $ax+z=0$ has infinite solutions, then the value of a is

(a) –1 (b) 1
(c) 0 (d) No real values

1. Let $\Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$ and let Δ_1 denote the determinant formed by the cofactors of elements of Δ_0 and Δ_2 denote the determinant formed by the cofactor at Δ_1 similarly Δ_n denotes the determinant formed by the cofactors at Δ_{n-1} then the determinant value of Δ_n is

(*a*) Δ_0^{2n} (*b*) $\Delta_0^{2^n}$

(*c*) $\Delta_0^{n^2}$ (*d*) Δ_0^2

2. Three distinct points $P(3u^2, 2u^3)$; $Q(3v^2, 2v^3)$ and $R(3w^2, 2w^3)$ are collinear then

(*a*) $uv + vw + wu = 0$ (*b*) $uv + vw + wu = 3$

(*c*) $uv + vw + wu = 2$ (*d*) $uv + ww + wu = 1$

3. Let a determinant is given by $A = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ and suppose that det. $A = 6$. If $B = \begin{vmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{vmatrix}$ then

(*a*) det. $B = 6$ (*b*) det. $B = -6$

(*c*) det. $B = 12$ (*d*) det. $B = -12$

4. The absolute value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is:

(*a*) $16\sqrt{2}$ (*b*) $8\sqrt{2}$

(*c*) 8 (*d*) $4\sqrt{2}$

5. If α, β & γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} =$

(*a*) p (*b*) q

(*c*) $p^2 - 2q$ (*d*) 0

6. If a, b, c > 0 & x, y, z ∈ R, then the determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix} =$

(*a*) $a^x b^y c^z$ (*b*) $a^{-x} b^{-y} c^{-z}$

(*c*) $a^{2x} b^{2y} c^{2z}$ (*d*) zero

7. The determinant $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} =$

(*a*) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (*b*) $2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(*c*) $3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (*d*) $4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

8. If x, y, z ∈ R & $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} =$ – 16 then value of x is

(*a*) –2 (*b*) –3

(*c*) 2 (*d*) 3

9. The determinant $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} =$

(*a*) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (*b*) $2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(*c*) $3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (*d*) $4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

10. If x, y, z ∈ R & $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} =$ – 16 then value of x is

(*a*) –2 (*b*) –3

(*c*) 2 (*d*) 3

11. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is:

(*a*) 0

(*b*) independent of θ

(*c*) independent of ϕ

(*d*) independent of θ & ϕ both

12. Let $\Delta = \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{vmatrix}$, then

(*a*) Δ is independent of θ

(*b*) Δ is indepedent of ϕ

(*c*) Δ is a constant

(*d*) Δ is dependent of ϕ

13. The determinant $\begin{vmatrix} \cos(x-y) & \cos(y-z) & \cos(z-x) \\ \cos(x+y) & \cos(y+z) & \cos(z+x) \\ \sin(x+y) & \sin(y+z) & \sin(z+x) \end{vmatrix} =$

(*a*) $2\sin(x-y)\sin(y-z)\sin(z-x)$

(*b*) $-2\sin(x-y)\sin(y-z)\sin(z-x)$

(*c*) $2\cos(x-y)\cos(y-z)\cos(z-x)$

(*d*) $-2\cos(x-y)\cos(y-z)\cos(z-x)$

14. $\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ is

equal to

(*a*) $(a-b)^2(b-c)^2(c-a)^2$

(*b*) $2(a-b)(b-c)(c-a)$

(*c*) $4(a-b)(b-c)(c-a)$

(*d*) $(a+b+c)^3$

15. The values of θ, λ for which the following equations $\sin\theta x - \cos\theta y + (\lambda+1)z = 0$; $\cos\theta x + \sin\theta y - \lambda z = 0$; $\lambda x + (\lambda+1)y + \cos\theta z = 0$ have non trivial solution, is

(*a*) $\theta = n\pi, \lambda \in R - \{0\}$

(*b*) $\theta = 2n\pi$, λ is any rational number

(*c*) $\theta = (2n+1)\pi, \lambda \in R^+, n \in I$

(*d*) $\theta = (2n+1)\frac{\pi}{2}, \lambda \in R, n \in I$

16. The system of equations

$x - y\cos\theta + z\cos 2\theta = 0$

$-x\cos\theta + y - z\cos\theta = 0$

$x\cos 2\theta - y\cos\theta + z = 0$

has non trivial solution for θ equals

(*a*) $n\pi$ only, $n \in I$ (*b*) $n\pi + \frac{\pi}{4}$ only, $n \in I$

(*c*) $(2n-1)\frac{\pi}{2}$ only, $n \in I$ (*d*) all value of θ

17 If a, b, c are non zeros, then the system of equations

$(\alpha+a)x + \alpha y + \alpha z = 0$

$\alpha x + (\alpha+b)y + \alpha z = 0$

$\alpha x + \alpha y + (\alpha+c)z = 0$

has a non-trivial solution if

(*a*) $\alpha^{-1} = -(a^{-1}+b^{-1}+c^{-1})$

(*b*) $\alpha^{-1} = a+b+c$

(*c*) $\alpha + a + b + c = 1$

(*d*) none of these

18. The system of equation $-2x + y + z = 1$, $x - 2y + z = -2$, $x + y + \lambda z = 4$ will have no solution if

(*a*) $\lambda = -2$ (*b*) $\lambda = -1$

(*c*) $\lambda = 3$ (*d*) none of these

19. If the system of equations $x - 2y + z = a$; $2x + y - 2z = b$ and $x + 3y - 3z = c$ have atleast one solution, then the relationship between a, b and c is

(*a*) $a+b+c=0$ (*b*) $a-b+c=0$

(*c*) $-a+b+c=0$ (*d*) $a+b-c=0$

20. If the system of linear equations

$x + 2ay + az = 0$

$x + 3by + bz = 0$

$x + 4cy + cz = 0$

has a non-zero solution, then a, b, c

(*a*) are in G.P. (*b*) are in H.P.

(*c*) satisfy $a + 2b + 3c = 0$ (*d*) are in A.P.

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/ NUMERICAL

1. If $f(x) = \begin{vmatrix} a^{-x} & e^{x\ell na} & x^2 \\ a^{-3x} & e^{3x\ell na} & x^4 \\ a^{-5x} & e^{5x\ell na} & 1 \end{vmatrix}$, then

(*a*) $f^2(x) - f^2(-x) = 0$ (*b*) $f(x) \cdot f(-x) = 0$

(*c*) $f(x) + f(-x) = 0$ (*d*) $f(x) - f(-x) = 0$

2. Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$, then

(*a*) $1 - x^3$ is a factor of Δ

(*b*) $(1-x^3)^2$ is factor of Δ

(*c*) $\Delta(x) = 0$ has 4 real roots

(*d*) $\Delta'(1) = 0$

3. If p, q, r, s are in A.P. and

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$ such that

$\int_0^2 f(x)dx = -4$ then the common difference of the A.P. can be

(*a*) -1 (*b*) 1/2

(*c*) 1 (*d*) 2

4. If $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & 2x-2y-z \end{vmatrix}$, then

(*a*) $x-y$ is a factor of Δ

(*b*) $(x-y)^2$ is a factor of Δ

(*c*) $(x-y)^3$ is a factor of Δ

(*d*) Δ is independent of z

5. Let $D_1 = \begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix}$ and

$D_2 = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

The divisor which is common to both D_1 and D_2 is

(*a*) $(a-b)$ (*b*) $(ab+bc+ca)$

(*c*) $a+b+c$ (*d*) $(c-a)$

6. Suppose a_1, a_2, a_3 are in A.P. and b_1, b_2, b_3 are in H.P. and let

$\Delta = \begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2 & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3 \end{vmatrix}$, then

(*a*) Δ is independent of a_1, a_2, a_3,

(*b*) $a_1-\Delta, a_2-2\Delta, a_3-3\Delta$ are in A.P.

(*c*) $b_1+\Delta, b_2+\Delta^2, b_3+\Delta$ are in H.P.

(*d*) Δ is independent of b_1, b_2, b_3

7. Let $a, b > 0$ and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then

(*a*) $a+b-x$ is a factor of Δ

(*b*) $x^2+(a+b)x+a^2+b^2-ab$ is a factor of Δ

(*c*) $\Delta = 0$ has three real roots if $a = b$

(*d*) $a+b+x$ is a factor of Δ

8. The determinent $\Delta = \begin{vmatrix} b & c & b\alpha+c \\ c & d & c\alpha+d \\ b\alpha+c & c\alpha+d & a\alpha^3-c\alpha \end{vmatrix}$ is equal to zero if

(*a*) b, c, d are in A.P.

(*b*) b, c, d are in G.P.

(*c*) b, c, d are in H.P.

(*d*) α is a root of $ax^3-bx^2-3cx-d=0$

9. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by

(*a*) $x+3$ (*b*) $(1+x)^2$

(*c*) x^2 (*d*) x^2+1

10. If D is a determinant of order three and Δ is a determinant formed by the cofactors of determinant D; then

(*a*) $\Delta = D^2$

(*b*) $D = 0$ implies $\Delta = 0$

(*c*) if $D = 27$, then Δ is perfect cube

(*d*) if $D = 27$, then Δ is perfect square

Comprehension – 1 (No. 11 & 13)

For $\alpha, \beta, \gamma, \theta \in R$. Let

$$A_\theta(\alpha,\beta,\gamma) = \begin{vmatrix} \cos(\alpha+\theta) & \sin(\alpha+\theta) & 1 \\ \cos(\beta+\theta) & \sin(\beta+\theta) & 1 \\ \cos(\gamma+\theta) & \sin(\gamma+\theta) & 1 \end{vmatrix}$$

11. If $a = A_{\pi/2}(\alpha,\beta,\gamma)$, $b = A_{\pi/3}(\alpha,\beta,\gamma)$. Which of the following is true

(*a*) $a = b$ (*b*) $a < b$

(*c*) $a > b$ (*d*) $2a = b$

12. $A_\theta^2 + A_\phi^2 - 2(A_{\theta+\phi})^2$ equals

(*a*) $-2A_\theta A_\phi$ (*b*) $A_\theta + A_\phi$

(*c*) $A_\theta - A_\phi$ (*d*) None of these

13. If α, β, γ are fixed, then $y = A_x(\alpha,\beta,\gamma)$ represents

(*a*) a straight line parallel to x-axis

(*b*) a straight line through the origin

(*c*) a parabola with vertex at origin

(*d*) None of these

Comprehension –2 (No. 14 & 15)

Let α, β, γ be the roots of the cubic $x^3+ax^2+bx+c=0$, which (taken in given order) are in G.P. If α and β are such that $\begin{vmatrix} 2 & 1 & 2 \\ 1+\alpha & \alpha & \beta \\ 4-\beta & 3-\beta & \alpha+1 \end{vmatrix} = 0$, then

14. The value of $a+b+c$ equals

(*a*) 0 (*b*) 1

(*c*) -1 (*d*) 2

15. If $S = \sum_{r=1}^{100}\left(\left(\frac{\alpha}{\beta}\right)^r + \left(\frac{a}{b}\right)^r\right)$, then S equals

(*a*) $\frac{1}{3}\left(1-\frac{1}{2^{100}}\right)$ (*b*) $\frac{4}{3}\left(1-\frac{1}{2^{100}}\right)$

(*c*) $\frac{8}{3}\left(1-\frac{1}{2^{100}}\right)$ (*d*) $\frac{2}{3}\left(1-\frac{1}{2^{100}}\right)$

16. Consider a square matrix A of order 2 which has its elements as 0,1,2 and 4.Let N denote the number of such matrices, all elements of which are distinct.

Column-I	**Column-II**
(*a*) Possible non-negative value of det(A) is	(p) 2
(*b*) Sum of values of determinants corresponding to N matrices is	(q) 4
(*c*) If absolute value of (det(A)) is least, then possible value of adj(adj(adj A)) |	(r) –2
(*d*) If det (A) is algebraically least, then possible value of $\det(4A^{-1})$ is	(s) 0
	(t) 8

17. Consider a system of linear equations $3x+y-z=0, x-\frac{py}{4}+z=2$ and $2x-y+2z=q$ where $p, q \in I$ and $p, q \in [1, 10]$, then identify the correct statement(s).

List-I	**List-II**
(P) Number of ordered pairs (p, q) for which system of equation has unique solution is	(1) 1
(Q) Number of ordered pairs (p, q) for which system of equation has no solution is	(2) 9
(R) Number of ordered pairs (p, q) for which system of equation has infinite solutions is	(3) 10
(S) Number of ordered pairs (p, q) for which system of equation has atleast one solution is	(4) 90
	(5) 91

(*a*) $P \to 4; Q \to 2; R \to 1; S \to 5$

(*b*) $P \to 4; Q \to 3; R \to 1; S \to 4$

(*c*) $P \to 4; Q \to 3; R \to 2; S \to 5$

(*d*) $P \to 4; Q \to 2; R \to 2; S \to 4$

NUMERICAL VALUE BASED

18. $\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^4 & 1 \end{vmatrix} =$

19. If $a^2+b^2+c^2=-2$ and

$$f(x)=\begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree

20. The system of equations

$\alpha x+y+z=\alpha-1$
$x+\alpha y+z=\alpha-1$
$x+y+\alpha z=\alpha-1$

has no solution , if absolute value of α is

21. If $a_1, a_2, a_3, \ldots\ldots a_n \ldots\ldots$ are in G.P, then the determinant

$$\Delta=\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$ is equal to

22. The number of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}=0 \text{ in the interval } 0 \le x \le \frac{\pi}{4} \text{ is}$$

23. The value of $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}-\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is

24. For what value of 2k/33 the equations

$x+ky+3z=0, 3x+ky-2z=0, 2x+3y-4z=0$

possess a nontrival solution over the set of rationals ?

25. If $2s=a+b+c$ and

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}=ks^3(s-a)(s-b)(s-c)$$

then the numerical quantity k should be

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The system of linear equation

$x+\lambda y-z=0\ ;\ \lambda x-y-z=0\ ;\ x+y-\lambda z=0$

has a non-trivial solution for: [*JEE Main-2016*]

(*a*) infinitely many values of λ

(*b*) exactly one value of λ

(*c*) exactly two values of λ

(*d*) exactly three values of λ

2. If S is the set of distinct values of 'b' for which the following system of linear equations.

$x+y+z=1,\ \ x+ay+z=1,ax+by+z=0$

has no. solution, then S is : [*JEE Main-2017*]

(*a*) a singleton

(*b*) an empty set

(*c*) an infinite set

(*d*) a finite set containing two or more elements.

3. Let ω be a complex number such that $2\omega+1=z$ where $z=\sqrt{-3}$. If $\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=3k$, then K is equal to:

[*JEE Main-2017*]

(*a*) 1 (*b*) $-z$

(*c*) z (*d*) -1

4. If the system of linear equations $x+ky+3z=0\ ;\ 3x+ky-2z=0\ ;\ 2x+4y-3z=0$ has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to: [*JEE Main-2018*]

(*a*) 10 (*b*) -30

(*c*) 30 (*d*) -10

5. Let $d\in R$, and $A=\begin{bmatrix}-2 & 4+d & (\sin\theta)-2\\ 1 & (\sin\theta)+2 & d\\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d\end{bmatrix}$, $\theta\in[0,2\pi]$. If the minimum value of det (A) is 8, then a value of d is: [*JEE Main-2019 (January)*]

(*a*) -5 (*b*) -7

(*c*) $2\left(\sqrt{2}+1\right)$ (*d*) $2\left(\sqrt{2}+2\right)$

6. The system of linear equation $x+y+z=2,\ 2x+3y+2z=5,\ 2x+3y+(a^2-1)z=a+1$ then

[*JEE Main-2019 (January)*]

(*a*) is inconsistent when a=4

(*b*) has a unique solution for $|a|=\sqrt{3}$

(*c*) has infinitely many solutions for $a=4$

(*d*) inconsistent when $|a|=\sqrt{3}$

7. If the system of linear equation $x-4y+7z=g,\ 3y-5z=h,\ -2x+5y-9z=k$ is consistent, then :

[*JEE Main-2019 (January)*]

(*a*) $g+h+k=0$ (*b*) $2g+h+k=0$

(*c*) $g+h+2k=0$ (*d*) $g+2h+k=0$

8. If $A=\begin{bmatrix}e^t & e^{-t}\cos t & e^{-t}\sin t\\ e^t & -e^{-t}\cos t-e^{-t}\sin t & -e^{-t}\sin t+e^{-t}\cos t\\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t\end{bmatrix}$

Then A is: [*JEE Main-2019 (January)*]

(*a*) invertible only if $t=\frac{\pi}{2}$

(*b*) not invertible for any $t\in R$

(*c*) invertible for all $t\in R$

(*d*) invertible only if $t=\pi$

9. If the system of equations

$x+y+z=5$

$x+2y+3z=9$

$x+3y+\alpha z=\beta$

has infinitely many solutions, then $\beta-\alpha$ equals: [*JEE Main-2019 (January)*]

(*a*) 21 (*b*) 8

(*c*) 18 (*d*) 5

10. The number of values of $\theta\in(0,\pi)$ for which the system of linear equations

$x+3y+7z=0$

$-x+4y+7z=0$

$x\sin3\theta+y\cos2\theta+2z=0$ has a non-trivial solution, is

[*JEE Main-2019 (January)*]

(*a*) three (*b*) two

(*c*) four (*d*) one

11. Let $A=\begin{bmatrix}2 & b & 1\\ b & b^2+1 & b\\ 1 & b & 2\end{bmatrix}$ where $b>0$. Then the minimum value of $\frac{\det(A)}{b}$ is:

[*JEE Main-2019 (January)*]

(*a*) $2\sqrt{3}$ (*b*) $-2\sqrt{3}$

(*c*) $-\sqrt{3}$ (*d*) $\sqrt{3}$

12. Let $a_1, a_2, a_3, \ldots a_{10}$ be in G.P. with $a_i > 0$ for $i = 1, 2, \ldots 10$ and S be the set of pairs (r, k), $r, k \in N$(the set of natural numbers) for which

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^k a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^k a_6^k & \log_e a_6^k a_7^k \\ \log_e a_7^k a_8^k & \log_e a_8^k a_9^k & \log_e a_9^k a_{10}^k \end{vmatrix} = 0$$

Then the number of elements in S, is:

[*JEE Main-2019 (January)*]

(*a*) 4 (*b*) infinitely many
(*c*) 2 (*d*) 10

13. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$
$= (a+b+c)(x+a+b+c)^2$, and $a+b+c \neq 0$, then x is equal to:

[*JEE Main-2019 (January)*]

(*a*) abc (*b*) $-(a+b+c)$
(*c*) $2(a+b+c)$ (*d*) $-2(a+b+c)$

14. An ordered pair (α, β) for which the system of linear equations
$(1+\alpha)x + \beta y + z = 2$
$\alpha x + (1+\beta)y + z = 3$
$\alpha x + \beta y + 2z = 2$
has a unique solution, is: [*JEE Main-2019 (January)*]

(*a*) (2,4) (*b*) (–3, 1)
(*c*) (–4,2) (*d*) (1,–3)

15. If $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$; then for all $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, det (A) lies in the interval:

[*JEE Main-2019 (January)*]

(*a*) $\left(1, \frac{5}{2}\right]$ (*b*) $\left[\frac{5}{2}, 4\right)$

(*c*) $\left(0, \frac{3}{2}\right]$ (*d*) $\left(\frac{3}{2}, 3\right]$

16. The set of all values of λ for which the system of linear equations
$x - 2y - 2z = \lambda x$
$x + 2y + z = \lambda y$
$-x - y = \lambda z$ [*JEE Main-2019 (January)*]

(*a*) is a singleton
(*b*) contains exactly two elements
(*c*) is an empty set
(*d*) contains more than two elements

17. The greatest value of $c \in R$ for which the system of linear equations
$x - cy - cz = 0$
$cx - y + cz = 0$
$cx + cy - z = 0$
has a non-trivial solution, is: [*JEE Main-2019 (April)*]

(*a*) 1/2 (*b*) –1
(*c*) 0 (*d*) 2

18. Let the number 2, b, c be in an A.P. and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$.

If $\det(A) \in [2,16]$, then c lies in the interval:

[*JEE Main-2019 (April)*]

(*a*) [2, 3) (*b*) $(2+2^{3/4}, 4)$
(*c*) $[3, 2+2^{3/4}]$ (*d*) [4, 6]

19. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then for $y \neq 0$ in R,

$\begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$ is equal to [*JEE Main-2019 (April)*]

(*a*) y^3 (*b*) $y^3 - 1$
(*c*) $y(y^2 - 1)$ (*d*) $y(y^2 - 3)$

20. If the system of equations $2x + 3y - z = 0$, $x + ky - 2z = 0$ and $2x - y + z = 0$ has a non-trival solution (x, y, z), then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to:

[*JEE Main-2019(April)*]

(*a*) $\frac{3}{4}$ (*b*) –4

(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{4}$

21. If $\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}$, $x \neq 0$;

then for all $\theta \in \left(0, \frac{\pi}{2}\right)$: [*JEE Main-2019 (April)*]

(*a*) $\Delta_1 - \Delta_2 = x(\cos 2\theta - \cos 4\theta)$
(*b*) $\Delta_1 + \Delta_2 = -2x^3$
(*c*) $\Delta_1 - \Delta_2 = -2x^3$
(*d*) $\Delta_1 + \Delta_2 = -2(x^3 + x - 1)$

22. Let λ be a real number for which the system of linear equations
$x + y + z = 6$
$4x + \lambda y - \lambda z = \lambda - 2$
$3x + 2y - 4z = -5$
has infinitely many solutions. Then λ is a root of the quadratic equation. [*JEE Main-2019 (April)*]

(*a*) $\lambda^2 - 3\lambda - 4 = 0$ (*b*) $\lambda^2 - \lambda - 6 = 0$
(*c*) $\lambda^2 + 3\lambda - 4 = 0$ (*d*) $\lambda^2 + \lambda - 6 = 0$

23. The sum of the real roots of the equation

$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, is equal to:

[*JEE Main-2019 (April)*]

(*a*) 6 (*b*) 1
(*c*) 0 (*d*) –4

24. If $B=\begin{bmatrix}5 & 2\alpha & 1\\ 0 & 2 & 1\\ \alpha & 3 & -1\end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which $\det(A)+1=0$, is:

[JEE Main-2019 (April)]

(a) 0 (b) 2
(c) 1 (d) -1

25. A value of $\theta \in (0, \pi/3)$, for which

$$\begin{vmatrix}1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta\\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta\\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta\end{vmatrix}=0, \text{ is :}$$

[JEE Main-2019 (April)]

(a) $\frac{7\pi}{24}$ (b) $\frac{\pi}{18}$
(c) $\frac{\pi}{9}$ (d) $\frac{7\pi}{36}$

26. If the system of linear equations
$x-2y+kz=1$
$2x+y+z=2$
$3x-y-kz=3$
has a solution (x, y, z), $z\neq 0$, then (x, y) lies on the straight line whose equation is: [JEE Main-2019 (April)]

(a) $3x-4y-1=0$ (b) $3x-4y-4=0$
(c) $4x-3y-4=0$ (d) $4x-3y-1=0$

27. If the system of linear equations
$2x+2ay+az=0$
$2x+3by+bz=0$
$2x+4cy+cz=0$
where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution, then [JEE Main-2020 (January)]

(a) a, b, c are in A.P. (b) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
(c) $a+b+c=0$ (d) a, b, c are in G.P.

28. Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two 3×3 real matrices such that $b_{ij}=(c)^{(i+j-2)}a_{ji}$, where $i, j=1, 2, 3$. If the determinant of B is 81, then the determinant of A is:

[JEE Main-2020 (January)]

(a) 1/9 (b) 1/81
(c) 3 (d) 1/3

29. For which of the following ordered pairs (μ, δ), the system of linear equations [JEE Main-2020 (January)]
$x+2y+3z=1$
$3x+4y+5z=\mu$
$4x+4y+4z=\delta$
is inconsistent?

(a) (4, 6) (b) (3, 4)
(c) (1, 0) (d) (4, 3)

30. The system of linear equations

[JEE Main-2020 (January)]

$\lambda x+2y+2z=5$
$2\lambda x+3y+5z=8$
$4x+\lambda y+6z=10$ has

(a) no solution when $\lambda=8$
(b) a unique solution when $\lambda=-8$
(c) no solution when $\lambda=2$
(d) infinitely many solution when $\lambda=2$

31. If for some α and β in R, the intersection of the following three planes
$x+4y-2z=1$
$x+7y-5z=\beta$
$x+5y+\alpha z=5$
is a line in R^3, then $\alpha+\beta$ is equal to :

[JEE Main-2020 (January)]

(a) 0 (b) -10
(c) 10 (d) 2

32. Let $a-2b+c=1$.

If $f(x)=\begin{vmatrix}x+a & x+2 & x+1\\ x+b & x+3 & x+2\\ x+c & x+4 & x+3\end{vmatrix}$, then :

[JEE Main-2020 (January)]

(a) $f(-50)=-1$ (b) $f(50)=1$
(c) $f(50)=-501$ (d) $f(-50)=501$

33. The following system of linear equations
$7x+6y-2z=0$
$3x+4y+2z=0$
$x-2y-6z=0$, has [JEE Main-2020 (January)]

(a) infinitely many solution, (x, y, z) satisfying $y=2z$
(b) infinitely many solution, (x, y, z) satisfying $x=2z$
(c) Only the trivial solution
(d) no solution

34. The values of λ and μ for which the system of linear equations
$x+y+z=2$
$x+2y+3z=5$
$x+3y+\lambda z=\mu$
has infinitely many solutions are, respectively:

[JEE Main-2020 (September)]

(a) 5 and 7 (b) 6 and 8
(c) 4 and 9 (d) 5 and 8

35. Consider the system of linear equations

$-x+y+2z=0$

$3x-ay+5z=1$

$2x-2y-az=7$

Let S_1 be the set of all $a\in R$ for which the system is inconsistent and S_2 be the set of all $a\in R$ for which the system has infinitely many solutions. If $n(S_1)$ and $n(S_2)$ denote the number of elements in S_1 and S_2 respectively, then [JEE Main-2021 (September)]

(a) $n(S_1)=2, n(S_2)=2$ (b) $n(S_1)=1, n(S_2)=0$
(c) $n(S_1)=2, n(S_2)=0$ (d) $n(S_1)=0, n(S_2)=2$

36. If $\alpha+\beta+\gamma=2\pi$, then the system of equations

$x+(\cos\gamma)y+(\cos\beta)z=0$

$(\cos\gamma)x+y+(\cos\alpha)z=0$

$(\cos\beta)x+(\cos\alpha)y+z=0$

has: [JEE Main-2021 (August)]

(a) no solution
(b) infinitely many solution
(c) exactly two solutions
(d) a unique solution

37. If $a_r = \cos\frac{2r\pi}{9} + i\sin\frac{2r\pi}{9}, r = 1,2,3,\ldots,i = \sqrt{-1}$, then the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ is equal to :

[JEE Main-2021 (August)]

(a) $a_2a_6 - a_4a_8$ (b) a_8

(c) $a_1a_9 - a_3a_7$ (d) a_5

JEE-ADVANCED PREVIOUS YEARS

1. The number of all possible values of Q, where $0 < Q < P$, for which the system of equations
$(y+z)\cos 3\theta = (xyz)\sin 3\theta$
$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$
$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is [IIT JEE-2012]
2. Let M and N be two 3 × 3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then [JEE Advanced-2014]
(a) determinant of $(M^2 + MN^2)$ is 0
(b) there is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(c) determinant of $(M^2 + MN^2) \geq 1$
(d) for a 3 × 3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix
3. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$ [JEE Advanced-2015]
(a) –4 (b) 9
(c) –9 (d) 4
4. The total number of distinct $x \in \mathbb{R}$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is [JEE Advanced-2016]
5. Let p be a matric of order 3 × 3 such that all the entries in P are from the set (–1,0,1). Then, the maximum possible value of the determinant of P is [JEE Advanced-2018]

ANSWER KEY

Topicwise Questions

1. (c)	2. (a)	3. (b)	4. (d)	5. (d)	6. (b)	7. (b)	8. (c)	9. (b)	10. (c)
11. (b)	12. (b)	13. (b)	14. (d)	15. (c)	16. (c)	17. (b)	18. (d)	19. (c)	20. (b)
21. (a)	22. (d)	23. (a)	24. (a)	25. (a)					

Learning Plus

1. (b)	2. (a)	3. (c)	4. (a)	5. (d)	6. (d)	7. (b)	8. (c)	9. (b)	10. (c)
11. (b)	12. (b)	13. (b)	14. (a)	15. (d)	16. (d)	17. (a)	18. (a)	19. (b)	20. (b)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/NUMERICAL

1. (a, c)	2. (a, b, d)	3. (a, c)	4. (a, b)	5. (c, d)	6. (a, b, c, d)	7. (a, b, c)	8. (b, d)	9. (a, c)
10. (a,b,c,d)	11. (a)	12. (d)	13. (a)	14. (c)	15. (d)	16. (a) → P, Q, T ; (b) → S; (c) → P, R ; (d) → R		
17. (a)	18. (0003)	19. (0002)	20. (2)	21. (0000)	22. (0001)	23. (0000)	24. (0001)	25. (0002)

JEE Mains & Advanced Past Years Questions

JEE MAIN PREVIOUS YEARS

1. (d)	2. (a)	3. (b)	4. (a)	5. (a)	6. (d)	7. (b)	8. (c)	9. (b)	10. (b)
11. (a)	12. (b)	13. (b)	14. (a)	15. (d)	16. (a)	17. (a)	18. (d)	19. (a)	20. (c)
21. (b)	22. (b)	23. (c)	24. (c)	25. (c)	26. (c)	27. (b)	28. (a)	29. (d)	30. (c)
31. (c)	32. (b)	33. (b)	34. (d)	35. (c)	36. (b)	37. (c)			

JEE-ADVANCED PREVIOUS YEARS

1. [3] 2. [a, b] 3. [b, c] 4. [2] 5. [4]

PRAYAS 1.0 t.me/realprayasians
PRAYAS 2.0 t.me/realprayasians2

7 MATRICES

PRAYAS 1.0 time./realprayasions

Prayas 2.0 t.me./realprasins2

INTRODUCTION

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is written as

$m \times n$ and we call it as order m by n

The general m × n matrix is A =

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of ith row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$.

If any matrix A contains 'm' rows and 'n' columns then $m \times n$ is termed as order of matrix.

Order is generally written as suffix of the array.

Now any matrix of order $m \times n$ will have the notation $[a_{ij}]_{m \times n}$.

i.e. $A = \left[a_{ij}\right]_{m \times n}$ or $\left(a_{ij}\right)_{m \times n}$ or $\| a_{ij} \|_{m \times n}$

it is obvious that $1 \le i \le m$ and $1 \le j \le n$

Note:

(*i*) Generally Capital letters of English alphabets are used to denote matrices.

(*ii*) Order of a matrix : If a matrix has m rows and n columns, then we say that its order is "m by n", written as "$m \times n$".

TRAIN YOUR BRAIN

Example 1: In the inter sports meet of local colleges the games to be played are T.T., Hockey, Badminton, Tennis, and B. Ball. The three colleges of Meerut sent the following number of players.

Meerut College (*M.C.*) – 35 players ; 5(T. T.),

11 (Hockey), 5(Badminton), 6 (Tennis) and 8(B. Ball).

Nanak Chand College (N. A. S.) – 22 players ; 3(T. T), 13 (Hockey), 2 (Badminton), 4 (Tennis) and none for (B. Ball).

Dev Nagri College (D. N.) – 31 players ; 2(T. T.),

15 (Hockey) 3(Bad), 5 (Tennis) and 6 (B. Ball). Put this information in matrix form.

Sol. The above information can be put in tabular form as under.

Colleges	Number of players				
	T.T.	Hockey	Badminton	Tennis	B.Ball
M.C. (35)	5	11	5	6	8
N.A.S. (22)	3	13	2	4	0
D.N. (31)	2	15	3	5	6

The above can be put in rectangular array form as

$$\begin{bmatrix} 5 & 11 & 5 & 6 & 8 \\ 3 & 13 & 2 & 4 & 0 \\ 2 & 15 & 3 & 5 & 6 \end{bmatrix}$$

Above is a 3 × 5 matrix, 3 represents the number of rows (number of colleges) participating and 5 represents the number of games being played in the meet.

SPECIAL TYPES OF MATRIX

Row matrix

A matrix having only one row is called as row matrix (or row vector). General form of row matrix is

$A = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]$

This is a matrix of order "$1 \times n$" (or a row matrix of order n)

Column matrix

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}$

This is a matrix of order "$m \times 1$" (or a column matrix of order m)

Square matrix

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix}$$ which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "$n \times n$" (or a square matrix of order n)

Leading Element

The element lying in the first row and first column is called leading element (or leading entry) of a matrix.

Diagonal Elements

An element of a matrix $A = [a_{ij}]$ is said to be diagonal element if $i = j$. Thus an element whose row suffix equals to the column suffix is a diagonal element. e.g. $a_{11}, a_{22}\, a_{33}$... are all diagonal element.

Principal Diagonal

The line along which the diagonal elements lie is called the principal diagonal or simply the diagonal of the matrix.

Zero matrix

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0\ \forall\ i\ \&\ j$.

e.g. : (*i*) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Upper triangular matrix

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

Lower Triangular Matrix

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above the diagonal elements are zero.)

e.g. : $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$

Diagonal matrix

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, \ldots\ldots a_{nn})$.

e.g. : (*i*) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (*ii*) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$

Scalar Matrix

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (*i*) $a_{ij} = 0$ for $i \neq j$ and (*ii*) $a_{ij} = k$ for $i = j$.

e.g. : (*i*) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (*ii*) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

Unit matrix (identity matrix)

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(*i*) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots$ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as tr(A)

Comparable Matrices

Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (*i*) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ &

$B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ are comparable

e.g. : (*ii*) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ &

$D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ are not comparable

EQUALITY OF MATRICES

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

$A = B$ iff (*i*) $m = p$, $n = q$

(*ii*) $a_{ij} = b_{ij}\ \forall\ i\ \&\ j$.

TRAIN YOUR BRAIN

Example 2: Let $A=\begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B=\begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$.

Find θ so that $A = B$.

Sol. By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta=\frac{1}{\sqrt{2}}$, $\cos\theta=-\frac{1}{\sqrt{2}}$ & $\tan\theta=-1$

$\Rightarrow \qquad \theta=(2n+1)\pi-\frac{\pi}{4}$.

ALGEBRA OF MATRIX

Multiplication of Matrix by Scalar

Let λ be a scalar (real or complex number) & $A=[a_{ij}]_{m\times n}$ be a matrix. Thus the product λA is defined as $\lambda A=[b_{ij}]_{m\times n}$ where $b_{ij}=\lambda a_{ij}\ \forall$ i & j.

e.g.: $A=\begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ &

$$-3A\equiv(-3)A=\begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Note:

(*i*) If A is a scalar matrix, then $A=\lambda I$, where λ is a diagonal entry of A.

(*ii*) Let $A=[a_{ij}]_{m\times n}$ be a matrix. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m\times n}$ i.e. each element is multiplied by -1 and is denoted by $-A$.

Addition of matrices

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be.

$$A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}.$$
$$=[c_{ij}]_{m\times n} \text{ where } c_{ij}=a_{ij}+b_{ij}\ \forall \text{ i \& j.}$$

e.g.: $A=\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$,

$$A+B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$$

Subtraction of Matrices

Let A & B be two matrices of same order. Then $A - B$ is defined as $A + (-B)$ where $-B$ is $(-1)B$.

Properties of Addition & Scalar Multiplication

1. $A+B=B+A$
2. $A+(B+C)=(A+B)+C$
3. $k(A+B)=kA+kB$ here k is any scalar.
4. $A+O=O+A=A$, here O {null matrix} will be additive identity.
5. If A be a given matrix then the matrix $-A$ is the additive inverse of A for $A+(-A)=$ null matrix O.
6. If A, B and C be three matrices of the same type
   then $A+B=A+C \Rightarrow B=C$ (Left Cancellation Law)
   and $B+A=C+A \Rightarrow B=C$ (Right Cancellation Law)

Multiplication of Matrices

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e.,

$A=[a_{ij}]_{m\times p}$ & $B=[b_{ij}]_{p\times n}$.

Then $AB=[c_{ij}]_{m\times n}$ where $c_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$, which is the dot product of ith row vector of A and j^{th} column vector of B.

e.g.: $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$,

$$AB=\begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

Note:

1. The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\not\Rightarrow$ BA is defined.
2. In general $AB \neq BA$, even when both the products are defined.
3. $A(BC)=(AB)C$, whenever it is defined.

Properties of Matrix Multiplication

Consider all square matrices of order 'n'. Let $M_n(F)$ denote the set of all square matrices of order n. (where F is Q, R or C). Then

(*a*) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(*b*) In general $AB \neq BA$

(*c*) $(AB)C=A(BC)$

(*d*) I_n, the identity matrix of order n, is the multiplicative identity.

$AI_n=A=I_nA \qquad \forall A \in M_n(F)$

(e) For every non singular matrix A (i.e., $|A| \neq 0$) of $M_n(F)$ there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A)B = \lambda(AB) = A(\lambda B)$.

(g) $A(B+C) = AB + AC \; \forall \; A, B, C \in M_n(F)$

(h) $(A+B)C = AC + BC \; \forall \; A, B, C \in M_n(F)$.

Note:

1. Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.
2. For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

SOME IMPORTANT POINTS

1. **Condition for product AB to exist or to be defined:**
   If A and B be two matrices then their product is defined or in other words A is *conformable* to B for multiplication if the number of columns of A is the same as the number of rows in B. i.e. If A be $m \times p$ and B be $p \times n$, the matrix AB will be of the type $m \times n$.
2. **Pre-multiplication and post multiplication:**
   When we say multiply A by B then it could mean both AB or BA where A and B are any numbers. But when A and B are matrices then as seen above AB and BA do not necessarily mean the same thing. If AB is defined for matrix multiplication BA may not be defined. To avoid this when we say product AB it would mean the matrix A post-multiplied by B and when we say product BA it would mean matrix A pre-multiplied by B. In AB, A is called *prefactor* and B *post factor*.
3. In the case when both A and B are square matrices of the same type then also both AB and BA are defined and the product matrix is also a matrix of the same type but still $AB \neq BA$.
4. Again we know that when $ab = 0$ it means that either a or b (or both) is zero. But $AB = O$ i.e. a null matrix does not necessarily imply that either A or $B = O$ as shown above because neither A nor B is null matrix whereas AB is a null matrix.

TRAIN YOUR BRAIN

Example 3: If $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ compute AB and BA.

Sol. Here A is 3×3 and B is 3×3. Hence both AB and BA are defined and each will be 3×3 matrix.

Let $AB = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

where C_{ij} means that take the product of ith row of A with jth column of B.

e.g. C_{23} = product of 2nd row of A with 3rd column of B.

i.e. $[-3 \;\; 2 \;\; -1]\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = -3 \cdot 3 + 2 \cdot 6 - 1 \cdot 3 = 0$

Similarly we can find other elements of C.

We can also say that by the product of first row of A with the three columns of B ; we shall get the three elements of first row of C.

i.e. R_1C_1, R_1C_2, R_1C_3

and similarly take the second row of A and multiply with all the columns of B and you will get the three elements of 2nd row of C i.e. R_2C_1, R_2C_2, R_2C_3 and elements of 3rd row of C will be R_3C_1, R_3C_2, R_3C_3.

$$\therefore \; AB = \begin{bmatrix} 1\cdot1 - 1\cdot2 + 1\cdot1 & 1\cdot2 - 1\cdot4 + 1\cdot2 & 1\cdot3 - 1\cdot6 + 1\cdot3 \\ -3\cdot1 + 2\cdot2 - 1\cdot1 & -3\cdot2 + 2\cdot4 - 1\cdot2 & -3\cdot3 + 2\cdot6 - 1\cdot3 \\ 2\cdot1 + 1\cdot2 + 0\cdot1 & -2\cdot2 + 1\cdot4 + 0\cdot2 & -2\cdot3 + 1\cdot6 + 0\cdot3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ (i.e., null matrix)}$$

Similarly BA can also be computed.

Example 4: $f(x)$ is a quadratic expression such that

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(-1) \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 2b+1 \\ 2c+1 \end{bmatrix}$$ for three unequal numbers a, b, c. Find $f(x)$.

Sol. The given matrix equation implies

$$\begin{bmatrix} a^2 f(0) + af(1) + f(-1) \\ b^2 f(0) + bf(1) + f(-1) \\ c^2 f(0) + cf(1) + f(-1) \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 2b+1 \\ 2c+1 \end{bmatrix}$$

$\Rightarrow$ $x^2 f(0) + xf(1) + f(-1) = 2x + 1$ for three unequal numbers a, b, c ...(i)

$\Rightarrow$ (i) is an identity $\Rightarrow$ $f(0) = 0, f(1) = 2$ & $f(-1) = 1$

$\therefore$ $f(x) = x(ax + b)$

$2 = a + b$ & $-1 = -a + b$.

$\Rightarrow$ $b = \frac{1}{2}$ & $a = \frac{3}{2}$

$\Rightarrow$ $f(x) = \frac{3}{2}x^2 + \frac{1}{2}x$.

Positive Integral Powers of Matrix

Let A be any square matrix of order n.

Then $A^2 = A.A$

$A^3 = A.A.A$

and $A^m = A.A.A...$ m times

All are square matrix of order n.

(i) $A^m . A^n = (A.A.A...\ m \text{ times})\ (A.A.A...n \text{ times})$

$= A.A.A...(m+n)$ times

$= A^{m+n}$

Similarly, (ii) $(A^m)^n = A^{mn}$

Also, we define $A^0 = I$

Note: Matrix multiplication of matrix is not commutative in general *i.e.* $AB \neq BA$.

Example 5: Let $f(x) = x^2 - 5x + 6$. Then the value of $f(A)$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ is:

(a) $\begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & -10 \\ 5 & 4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & -3 \\ 1 & 2 & 10 \\ 6 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 4 & -5 & 4 \end{bmatrix}$

Sol. $f(x) = x^2 - 5x + 6 = x^2 - 5x + 6 \cdot x^0$ $[\because\ x^0 = 1]$

$\therefore\ f(A) = A^2 - 5A + 6A^0$

$= A^2 - 5A + 6I$ ...(i)

Now $A^2 = A.A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A' (or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}\ \ \forall\ i \& j$.

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, $A' = \begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Results :

(*i*) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(*ii*) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(*iii*) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(*iv*) $(A_1 \pm A_2 \pm ..... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_i are comparable.

(*v*) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(*vi*) $(A_1 A_2 .....A_n)' = A_n' . A_{n-1}' ..... A_2' . A_1'$, provided the product is defined.

ORTHOGONAL MATRICES

A square matrix is said to be orthogonal matrix, if $AA^T = I = A^T A$

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then $A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \text{ and}$$

$$\sum_{i=1}^{3} a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$$

Note:

(*i*) All the 3 rows & 3 colums of an orthogonal matrix are pairwise orthogonal triad of 3 unit vectors.

(*ii*) $A^T = A^{-1}$. (Desirable)

SYMMETRIC & SKEW-SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}\ \forall\ i \& j$.

A square matrix A is said to be skew-symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$.

A is skew-symmetric iff $a_{ij} = -a_{ji}\ \forall\ i \& j$.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Notes

1. The pair of conjugate elements are additive invese of each other.
2. Max. number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

$$(1+2+3+.....+n)=\frac{n(n+1)}{2}$$

3. In a skew-symmetric matrix all the diagonal elements are zero but not converse.

$(\because\ a_{ij}=-a_{ji} \Rightarrow a_{ii}=0)$

4. For a skew symmetric matrix A of order $(2p-1)$, $|A|=0$.

Since A is skew symmetric hence $A^T=-A$, then $|A^T|=|-A|$ $=(-1)^{2p-1}|A|=-|A|$ but $|A^T|=|A|$, hence $|A|=-|A| \Rightarrow |A|=0$

5. For any square matrix A, $A+A'$ is symmetric & $A-A'$ is skew-symmetric.
6. Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric.

$$A=B+C, \text{ where } B=\frac{1}{2}(A+A') \& C=\frac{1}{2}(A-A').$$

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Example 6: Show that BAB' is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Sol. Case - I A is symmetric $\Rightarrow A'=A$

$(BAB')'=(B')'A'B'=BAB' \Rightarrow BAB'$ is symmetric.

Case - II A is skew-symmetric $\Rightarrow A'=-A$

$(BAB')'=(B')'A'B'$

$=B(-A)B'$

$=-(BAB')$

$\Rightarrow$ BAB' is skew-symmetric

DETERMINANT OF ANY MATRIX

If matrix $A=[a_{ij}]$ is a square matrix of order 'n' then determinant of A

$$=\left(\sum_{k=1}^{n}a_{1k}C_{1k}\right)=\left(\sum_{k=1}^{n}a_{2k}C_{2k}\right)=.....=....$$

$$=\left(\sum_{k=1}^{n}a_{k1}C_{k1}\right)=\left(\sum_{k=1}^{n}a_{k2}C_{k2}\right)=.....=.....$$

here C_{ik} represents cofactor of the element of ith row and k^{th} column of matrix A.

for 3×3 order matrix A;

$\det A$ (or $|A|$) $=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$

$=a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}$

$=a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}$

$=a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}$

$=a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}$

$=a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33}$

Singular & non singular matrix : A square matrix A is said to be singular or non-singular according as $|A|$ is zero or non-zero respectively.

Cofactor matrix & adjoint matrix : Let $A=[a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A=[a_{ij}]_n$

then cofactor $A=[c_{ij}]_n$ when c_{ij} is the cofactor of $a_{ij}\ \forall\ i\ \&\ j$.

adj $A=[d_{ij}]_n$ where $d_{ij}=c_{ji}\ \forall\ i\ \&\ j$.

Properties of Cofactor A and adj A

(a) A . adj $A=|A|\,I_n=(\text{adj}\,A)\,A$ where $A=[a_{ij}]_n$.

(b) $|\text{adj}\,A|=|A|^{n-1}$, where n is order of A.

In particular, for 3×3 matrix, $|\text{adj}\,A|=|A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

(e) adj $(KA)=K^{n-1}$adj A, where K is any scalar and n is order, then adj $AB=$ (adj B)(adj A)

(f) If A and B are non singular square matrix of the same oder, then adj $AB=$ (adj B)(adj A)

(g) If A is an invertible square matrix, then adj $A^T=(\text{adj}\,A)^T$

(h) If A is a are non singular square matrix, then

(a) adj (adj A) $=|A|^{n-2}A$

(b) $|\text{adj (adj}\,A)|=|A|^{(n-1)^2}$

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Example 7: For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Sol. $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$, cof $A=\begin{bmatrix}c^2 & -bc & ca\\ -bc & b^2 & -ab\\ ca & -ab & a^2\end{bmatrix}$

$$\text{adj}\,A=(\text{cof}\,A)'=\begin{bmatrix}c^2 & -bc & ca\\ -bc & b^2 & -ab\\ ca & -ab & a^2\end{bmatrix}$$

which is symmetric.

INVERSE OF A MATRIX

A square matrix A said to be invertible (non singular) if there exists a matrix B such that, $AB = I = BA$.

B is called the inverse of A and is denoted by A^{-1}. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

Let A be a non-singular matrix.

Then the matrix $\frac{1}{|A|}$ adj A is the inverse of A and is denoted by A^{-1}.

We have A (adj A) $= |A| I_n = (\text{adj } A) A$

$\Rightarrow A\left(\frac{1}{|A|} adj\ A\right) = I_n = \left(\frac{1}{|A|} adj\ A\right) A$, for A is non-singular

$\Rightarrow A^{-1} = \frac{1}{|A|}$ adj A.

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non-singular.
2. A^{-1} is always non-singular.
3. If A = dia $(a_{11}, a_{22}, ..., a_{nn})$ where $a_{ii} \neq 0\ \forall\ i$, then $A^{-1} = \text{diag}\ (a_{11}^{-1}, a_{22}^{-1}, ..., a_{nn}^{-1})$.
4. $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj $(A') = (\text{adj } A)'$.
5. $(A^{-1})^{-1} = A$, if A is non-singular.
6. Let k be a non-zero scalar & A be a non-singular matrix.

   Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.
7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.
8. Let A be a non-singular matrix.

   Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.
9. A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.
10. $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.
11. In general $AB = \mathbf{0}$ does not imply $A = \mathbf{0}$ or $B = \mathbf{0}$. But if A is non-singular and $AB = \mathbf{0}$, then $B = \mathbf{0}$. Similarly B is non-singular and $AB = \mathbf{0} \Rightarrow A = \mathbf{0}$. Therefore,

    $AB = \mathbf{0} \Rightarrow$ either both are singular or one of them is $\mathbf{0}$.

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Example 8: For two non-singular matrices A & B, show that adj $(AB) = (\text{adj } B)(\text{adj } A)$

Sol. We have $(AB)(\text{adj } (AB)) = |AB| I_n$

$= |A||B| I_n$

$A^{-1}(AB)(\text{adj } (AB)) = |A||B| A^{-1}$

$\Rightarrow B\ \text{adj}\ (AB) = |B|\ \text{adj}\ A \quad (\because A^{-1} = \frac{1}{|A|}\ \text{adj}\ A)$

$\Rightarrow B^{-1} B\ \text{adj}\ (AB) = |B| B^{-1}\ \text{adj}\ A$

$\Rightarrow \text{adj}\ (AB) = (\text{adj } B)(\text{adj } A)$

SYSTEM OF LINEAR EQUATIONS & MATRICES

Consider the system

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

.....

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & ..... & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ ... \\ b_n \end{bmatrix}$.

Then the above system can be expressed in the matrix form as $AX = B$.

The system is said to be consistent if it has atleast one solution.

System of Linear equations and Matrix Inverse

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix.

Results :

(1) If A is non-singular, solution is given by $X = A^{-1}B$.

(2) If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.

(3) If A is singular and (adj A) $B \neq 0$, then the system has no solution (we say it is inconsistent).

Homogeneous System and Matrix Inverse

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is $AX = O$.

($\because$ in this case $b_1 = b_2 = ... b_n = 0$), where A is a square matrix.

Results :

(1) If A is non-singular, the system has only the trivial solution (zero solution) $X = 0$

(2) If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non-trivial solutions.

ELEMENTARY ROW/COLUMN TRANSFORMATION OF MATRIX

The following operations on a matrix are called as elementary row transformations.

(*a*) Interchanging two rows.

(*b*) Multiplications of all the elements of row by a nonzero scalar.

(*c*) Addition of constant multiple of a row to another row.

Note: Similar to above we have elementary column transformations also.

Remarks: Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

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Example 9: Find the inverse of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 1 & -2 & 3\\ 1 & 2 & -3\end{bmatrix}$ and hence solve the equations :

$x+2y+3z=11, x-2y+3z=3, x+2y-3z=-1.$

Sol. We have $|A|=1(6-6)-2(-3-3)+3(2+2)=24$

Now as $|A|\neq 0$ then it is a non-singular matrix hence A^{-1} exist and has unique solution:

$$\text{Now adj } A=\begin{pmatrix}0 & 12 & 12\\ 6 & -6 & 0\\ 4 & 0 & -4\end{pmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{|A|}\cdot \text{adj } A=\begin{pmatrix}0 & 1/2 & 1/2\\ 1/4 & -1/4 & 0\\ 1/6 & 0 & -1/6\end{pmatrix}$$

The solution of the given equations is $X=A^{-1}B$

$$\text{i.e.} \quad \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0 & 1/2 & 1/2\\ 1/4 & -1/4 & 0\\ 1/6 & 0 & -1/6\end{pmatrix}\begin{pmatrix}11\\ 3\\ -1\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$$

Hence $x=1, y=2$ and $z=2$ is the required solution.

Example 10: Solve the system $\begin{matrix}x+y+z=6\\ x-y+z=2\\ 2x+y-z=1\end{matrix}$ using matrix inverse.

Sol. Let $A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 1 & -1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ & $B=\begin{bmatrix}6\\ 2\\ 1\end{bmatrix}$.

Then the system is $AX=B$.

$|A|=6$. Hence A is non singular.

$$\text{Cofactor } A=\begin{bmatrix}0 & 3 & 3\\ 2 & -3 & 1\\ 2 & 0 & -2\end{bmatrix} \quad \text{adj } A=\begin{bmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\text{ adj } A=\frac{1}{6}\begin{bmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1/3 & 1/3\\ 1/2 & -1/2 & 0\\ 1/2 & 1/6 & -1/3\end{bmatrix}$$

$$X=A^{-1}B=\begin{bmatrix}0 & 1/3 & 1/3\\ 1/2 & -1/2 & 0\\ 1/2 & 1/6 & -1/3\end{bmatrix}\begin{bmatrix}6\\ 2\\ 1\end{bmatrix} \text{ i.e. } \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$\Rightarrow \quad x=1, y=2, z=3.$

CHARACTERISTIC POLYNOMIAL & CHARACTERISTIC EQUATION

Let A be a square matrix. Then the polynomial $|A-xI|$ is called as characteristic polynomial of A & the equation $|A-xI|=0$ is called as characteristic equation of A.

Cayley - Hamilton theorem : Every square matrix A satisfies its characteristic equation

i.e. $a_0x^n+a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n=0$ is the characteristic equation of A, then $a_0A^n+a_1A^{n-1}+\ldots\ldots+a_{n-1}A+a_nI=0$

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Example 11: If $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$, show that $5A^{-1}=A^2+A-5I$.

Sol. We have the characteristic equation of A.

$|A-xI|=0$

$$\text{i.e.} \quad \begin{bmatrix}1-x & 2 & 0\\ 2 & -1-x & 0\\ 0 & 0 & -1-x\end{bmatrix}=0$$

i.e. $x^3+x^2-5x-5=0.$

Using Cayley - Hamilton theorem.

$A^3+A^2-5A-5I=0$

$\Rightarrow \quad 5I=A^3+A^2-5A$

Multiplying by A^{-1}, we get

$5A^{-1}=A^2+A-5I$

Nilpotent Matrix

A square matrix said to be nilpotent matrix of index P.

$p\in N$ if $A^p=O, A^{p-1}\neq O$ i.e. if p is the least positive integer for which $A^p=O$, then A is said to be nilpotent of index p.

Idempotent Matrix

A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

Involutory Matrix

A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix. An involutory matrix is it own inverse.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

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Example 12: Show that a square matrix A is involutory, iff $(I-A)(I+A)=0$

Sol. Let A be involutory

Then $A^2 = I$

$$(I-A)(I+A) = I.I. + IA - AI - A^2$$
$$= I + A - A - A^2$$
$$= I - A^2$$
$$= 0$$

Conversly, let $(I-A)(I+A)=0$

$\Rightarrow$ $I^2 + IA - AI - A^2 = 0$

$\Rightarrow$ $I + A - A - A^2 = 0$

$\Rightarrow$ $I - A^2 = 0$

$\Rightarrow$ A is involutory

Topicwise Questions

Types of Matrices, Addition, Subtraction

1. If $A+B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $A-2B=\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$, then $A=$

(a) $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$ (b) $\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}$

(c) $\begin{bmatrix}1/3 & 1/3\\ 2/3 & 1/3\end{bmatrix}$ (d) $\begin{bmatrix}1/3 & 2/3\\ 2/3 & 1/3\end{bmatrix}$

Multiplication of Matrices and Identity Matrix

2. If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$ and $B=\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$, then the correct relation is

(a) $A^2=B^2$ (b) $A+B=B-A$

(c) $AB=BA$ (d) $AB=0$

3. If $A=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$ and I is the identity matrix of order 2, then $(A-2I)(A-3I)=$

(a) I (b) O

(c) $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

4. If $A=\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix}$, then

(a) $A^3+3A^2+A-9I_3=O$

(b) $A^3-3A^2+A+9I_3=O$

(c) $A^3+3A^2-A+9I_3=O$

(d) $A^3-3A^2-A+9I_3=O$

5. If matrix $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, then $A^{16}=$

(a) $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

6. If A and B are 3×3 matrices such that $AB=A$ and $BA=B$, then

(a) $A^2=A$ and $B^2\neq B$ (b) $A^2\neq A$ and $B^2=B$

(c) $A^2=A$ and $B^2=B$ (d) $A^2\neq A$ and $B^2\neq B$

7. $\begin{bmatrix}7 & 1 & 2\\ 9 & 2 & 1\end{bmatrix}\begin{bmatrix}3\\ 4\\ 5\end{bmatrix}+2\begin{bmatrix}4\\ 2\end{bmatrix}$ is equal to

(a) $\begin{bmatrix}43\\ 44\end{bmatrix}$ (b) $\begin{bmatrix}43\\ 45\end{bmatrix}$

(c) $\begin{bmatrix}45\\ 44\end{bmatrix}$ (d) $\begin{bmatrix}44\\ 45\end{bmatrix}$

Transpose of Matrices and other properties

8. If $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 5\\ 2 & -5 & 0\end{bmatrix}$, then

(a) $A'=A$ (b) $A'=-A$

(c) $A'=2A$ (d) $A'=-2A$

9. In order that the matrix $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 3 & \lambda & 5\end{bmatrix}$ be non-singular, λ should not be equal to

(a) 1 (b) 2

(c) 3 (d) 4

10. If $A=\begin{bmatrix}1 & -2 & 1\\ 2 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 3 & 2\\ 1 & 1\end{bmatrix}$, then $(AB)^T=$

(a) $\begin{bmatrix}-3 & -2\\ 10 & 7\end{bmatrix}$ (b) $\begin{bmatrix}-3 & 10\\ -2 & 7\end{bmatrix}$

(c) $\begin{bmatrix}-3 & 10\\ 7 & -2\end{bmatrix}$ (d) $\begin{bmatrix}3 & 10\\ 2 & 7\end{bmatrix}$

Symmetric & Skew symmetric Matrices & determinant of matrices and singular Matrix

11. If A is a square matrix $A+A^T$ is symmetric matrix, then $A-A^T=$

(a) Unit matrix (b) Symmetric matrix

(c) Skew symmetric matrix (d) Zero matrix

12. If A is a symmetric matrix and $n\in N$, then A^n is

(a) Symmetric (b) Skew symmetric

(c) A Diagonal matrix (d) Zero matrix

13. Out of the following a skew- symmetric matrix is

(a) $\begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -6 \\ -5 & 6 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 5 \\ -4 & 1 & -6 \\ -5 & 6 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 5 \\ -4 & 2 & -6 \\ -5 & 6 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} i+1 & 4 & 5 \\ -4 & i & -6 \\ -5 & 6 & i \end{bmatrix}$

Adjoint and Inverse of Matrix

14. The element in the first row and third column of the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is

(a) -2 (b) 0

(c) 1 (d) 7

15. If a matrix A is such that $3A^3 + 2A^2 + 5A + I = 0$, then its inverse is

(a) $-(3A^2 + 2A + 5I)$ (b) $3A^2 + 2A + 5I$

(c) $3A^2 - 2A - 5I$ (d) $3A^2 + 2A - 5I$

16. The inverse of matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(a) A (b) A^T

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

17. The inverse matrix of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, is

(a) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ 1 & -6 & 3 \\ 1 & 2 & -1 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 3 \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

18. For any 2×2 matrix A, if $A(\text{adj}\,A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ then $|A|$ is equal

(a) 0 (b) 10

(c) 20 (d) 100

19. If $X = \begin{bmatrix} -x & -y \\ z & t \end{bmatrix}$ then transpose of adj X is

(a) $\begin{bmatrix} t & z \\ -y & -x \end{bmatrix}$ (b) $\begin{bmatrix} t & y \\ -z & -x \end{bmatrix}$

(c) $\begin{bmatrix} t & -z \\ y & -x \end{bmatrix}$ (d) $\begin{bmatrix} -t & z \\ -y & x \end{bmatrix}$

20. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = (\text{adj}\,A)$,, and $C = 5A$, then

$\frac{|\text{adj}\,B|}{|C|} =$

(a) 5 (b) 25

(c) -1 (d) 1

21. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}$, then the value of $|\text{adj}\,A|$ is

(a) 36 (b) 72

(c) 144 (d) 6

22. If A is a matrix of order 3 and $|A| = 8$, then $|\text{adj}\,A| =$

(a) 1 (b) 2

(c) 2^3 (d) 2^6

Special Case of Matrices and System of Equation

23. The matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent of index

(a) 2 (b) 3

(c) 4 (d) 6

24. The matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is

(a) Orthogonal (b) Involutory

(c) Idempotent (d) Nilpotent

Learning Plus

1. The number of different possible orders of matrices having 12 identical elements is

(*a*) 3 (*b*) 1

(*c*) 6 (*d*) 2

2. $\begin{bmatrix} x^2+x & x \\ 3 & 2 \end{bmatrix}+\begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$ then x is equal to -

(*a*) -1 (*b*) 2

(*c*) 1 (*d*) No value of x

3. If $A=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B=\begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

(*a*) $AB=\begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ (*b*) $AB=[-2-14]$

(*c*) $AB=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (*d*) AB does not exist

4. If $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $B=$

(*a*) $I\cos\theta+J\sin\theta$ (*b*) $I\cos\theta-J\sin\theta$

(*c*) $I\sin\theta+J\cos\theta$ (*d*) $-I\cos\theta+J\sin\theta$

5. Let $A=\begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B=[a\ b\ c]$ and C

$$=\begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+1)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$$

Where a, b, c and $x \in R$, Given that tr (AB) = tr(C), then the value of $(a+b+c)$.

(*a*) 7 (*b*) 2

(*c*) 1 (*d*) 4

6. If A = diag (2, –1, 3), B = diag (–1, 3, 2), then $A^2B=$

(*a*) diag (5, 4, 11) (*b*) diag (– 4, 3, 18)

(*c*) diag (3, 1, 8) (*d*) B

7. $A=\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $(A^8+A^6+A^4+A^2+I)\,V=\begin{bmatrix} 31 \\ 62 \end{bmatrix}$.

(Where I is the (2×2) identity matrix), then the product of all elements of matrix V is

(*a*) 2 (*b*) 1

(*c*) 3 (*d*) –2

8. The equation $[1\ x\ y]\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}=[0]$ has

(*i*) for y = 0 (*p*) rational roots

(*ii*) for y = –1 (*q*) irrational roots

(*r*) integral roots

Then

	(*i*)	(*ii*)
(*a*)	(*p*)	(*r*)
(*b*)	(*q*)	(*p*)
(*c*)	(*p*)	(*q*)
(*d*)	(*r*)	(*p*)

9. If $A=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is n^{th} root of I_2, then choose the correct statement :

(*i*) if n is odd, $a=1,\ b=0$

(*ii*) if n is odd, $a=-1, b=0$

(*iii*) if n is even, $a=1,\ b=0$

(*iv*) if n is even, $a=-1, b=0$

(*a*) (*i*), (*ii*), (*iii*) (*b*) (*ii*), (*iii*), (*iv*)

(*c*) (*i*), (*ii*), (*iii*), (*iv*) (*d*) (*i*), (*iii*), (*iv*)

10. If A is a square matrix such that $A^2 = A$, then $(I+A)^3-7A$ is

(*a*) $3I$ (*b*) O

(*c*) I (*d*) $2I$

11. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two-rowed unit matrix, then α, β and γ should staisfy the relation

(*a*) $1-\alpha^2+\beta\gamma=0$ (*b*) $\alpha^2+\beta\gamma-1=0$

(*c*) $1+\alpha^2+\beta\gamma=0$ (*d*) $1-\alpha^2-\beta\gamma=0$

12. Find number of all possible ordered sets of two $(n \times n)$ matrices A and B for which $AB-BA=I$

(*a*) infinite (*b*) n^2

(*c*) $n!$ (*d*) zero

13. If B, C are square matrices of order n and if $A=B+C$, $BC=CB$, $C^2=0$, then which of following is true for any positive integer N.

(*a*) $A^{N+1}=B^N(B+(N+1)C)$

(*b*) $A^N=B^N(B+(N+1)C)$

(*c*) $A^{N+1}=B(B+(N+1)C)$

(*d*) $A^{N+1}=B^N(B+(N+2)C)$

14. Matrix A is such that $A^2=2A-I$, where I is the identity matrix. Then for $n\geq 2$, $A^n=$

(*a*) $nA-(n-1)I$ (*b*) $nA-I$

(*c*) $2^{n-1}A-(n-1)I$ (*d*) $2^{n-1}A-I$

15. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2 =$

(a) $2AB$ (b) $2BA$

(c) $A + B$ (d) AB

16. Let A, B be two matrices such that they commute, then for any positive integer n,

(i) $AB^n = B^nA$

(ii) $(AB)^n = A^nB^n$

(a) only (i) is correct

(b) Both (i) and (ii) are correct

(c) only (ii) is correct

(d) none of (i) and (ii) is correct

17. In an upper triangular matrix $A = [a_{ij}]_{n \times n}$ the elements $a_{ij} = 0$ for

(a) $i < j$ (b) $i = j$

(c) $i > j$ (d) $i \leq j$

18. If A is a skew- symmetric matrix, then trace of A is

(a) 1 (b) -1

(c) 0 (d) 2

19. Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : a_{ij} \in \{-1, 0, 1\} \right\}$ then the number of symmetric matrices with trace equals zero, is

(a) 729 (b) 189

(c) 162 (d) 27

20. If A is a skew - symmetric matrix and n is an even positive integer, then A^n is

(a) a symmetric matrix (b) a skew-symmetric matrix

(c) a diagonal matrix (d) zero matrix

21. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix ?

(a) 1 (b) 2

(c) 3 (d) 4

22. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $kB = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$ where k, $\alpha \in N$. If B is the inverse of the matrix A, then the value of $k + \alpha$ equals

(a) 10 (b) 15

(c) 5 (d) 20

23. A skew - symmetric matrix A satisfies the relation $A^2 + I = 0$, where I is a unit matrix. Then A is

(a) Idempotent matrix (b) Orthogonal matrix

(c) Nilpotent matrix (d) Periodic matrix

24. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a orthogonal matrix, then the value of $(a + b)$ is equal to

(a) 0 (b) ± 1

(c) 3 (d) -3

25. If A is idempotent matrix and I is identity matrix such that $(I + A)^n = I + (2^n + k)A$. Then the value of k is

(a) 0 (b) 1

(c) -1 (d) 2

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/ NUMERICAL

1. If $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, then

(a) $|A| = 2$

(b) A is non-singular

(c) Adj. $A = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$

(d) A is skew symmetric matrix

2. Which of the following statement(s) is/are **CORRECT?**

(a) Every skew-symmetric matrix is non-invertible.

(b) If A and B are two 3×3 matrices such that $AB = \mathbf{O}$ then alteast one of A and B must be null matrix.

(c) If the minimum number of cyphers in an upper triangular matrix of order n is 5050, then the order of matrix is 101.

(d) If A and B are two square matrices of order 3 such that det. $A = 5$ and det. $B = 2$, then det. $(10AB)$ equals 10^4

3. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equations $x^2 + k = 0$, then

(a) $a + d = 0$ (b) $k = -|A|$

(c) $k = |A|$ (d) $a + d \neq 0$

4. Which of the following is true for matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(a) $A + 4I$ is a symmetric matrix

(b) $A^2 - 4A + 5I_2 = 0$

(c) $A - B$ is a diagonal matrix for any value of α if $B = \begin{bmatrix} \alpha & -1 \\ 2 & 5 \end{bmatrix}$

(d) $A - 4I$ is a skew symmetric matrix

5. Which of the following is(are) correct?

(a) If A and B are two square matrices of order 3 and A is a non-singular matrix such that $AB = O$, then B must be a null matrix.

(b) If A, B, C are three square matrices of order 2 and det. $(A) = 2$, det.$(B) = 3$, det. $(C) = 4$, then the value of det. $(3ABC)$ is 216.

(c) If A is a square matrix of order 3 and det. $(A) = \frac{1}{2}$, then det. (adj. A^{-1}) is 8.

(d) Every skew symmetric matrix is singular.

6. Which of the following statement(s) is/are true

(a) The system of equations $\begin{matrix} 4x - 5y - 2z = 2 \\ 5x - 4y + 2z = 3 \\ 2x + 2y + 8z = 1 \end{matrix}$ is Inconsistent.

(b) A matrix 'A' has 6 elements. The number of possible orders of A is 6.

(c) For any 2×2 matrix A, if $A(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A| = 10$.

(d) If A is skew symmetric, then $B'AB$ is also skew symmetric.

7. Let A and B be two 2×2 matrix with real entries. If $AB = O$ and $\text{tr}(A) = \text{tr}(B) = 0$ then

(a) A and B are comutative w.r.t. operation of multiplication.

(b) A and B are not commutative w.r.t. operation of multi plication.

(c) A and B are both null matrices.

(d) $BA = 0$

8. Matrix $\begin{bmatrix} a & b & (a\alpha - b) \\ b & c & (b\alpha - c) \\ 2 & 1 & 0 \end{bmatrix}$ is non invertible if

(a) $\alpha = 1/2$ (b) a, b, c are in A.P.

(c) a, b, c are in G.P. (d) a, b, c are in H.P.

9. Let M be a 3×3 non-singular matrix with det $(M) = 4$. If M^{-1} adj(adj M) $= k^2 I$, then the value of 'k' may be:

(a) +2 (b) 4

(c) –2 (d) –4

10. If A and B are square matrices of order 3, then the true statement is/are (where I is unit matrix).

(a) det $(-A) = -$ det A

(b) If AB is singular then atleast one of A or B is singular

(c) det $(A + I) = 1 +$ det A

(d) det $(2A) = 2^3$ det A

Comprehension – 1 (Q. No. 11 to 13)

Let A be the set of all 3×3 symmetric matrices whose entries are $1,1,1,0,0,0,-1,-1,-1$. B is one of the matrix in set A and

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad U = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

11. Number of such matrices B in set A is λ, then λ lies in the interval

(a) (30,40) (b) (38,40)

(c) (34,38) (d) (25,35)

12. Number of matrices B such that equation $BX = U$ has infinite solutions

(a) is at least 6 (b) is not more than 10

(c) lie between 8 to 16 (c) is zero.

13. The equation $BX = V$

(a) is inconsistent for atleast 3 matrices B.

(b) is inconsistent for all matrices B.

(c) is inconsistent for at most 12 matrices B.

(d) has infinite number of solutions for at least 3 matrices B.

Comprehension – 2 (Q. No. 14 to 16)

Some special square matrices are defined as follows :

Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$. A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$.

Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A' A = I = AA'$.

14. If A and B are two square matrices such that $AB = A$ & $BA = B$, then A & B are

(a) Idempotent matrices (b) Involutory matrices

(c) Orthogonal matrices (d) Nilpotent matrices

15. If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then

(a) $\alpha = \pm \frac{1}{\sqrt{2}}$ (b) $\beta = \pm \frac{1}{\sqrt{6}}$

(c) $\gamma = \pm \frac{1}{\sqrt{3}}$ (c) all of these

16. The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

(a) idempotent matrix (b) involutory matrix

(c) nilpotent matrix (c) none of these

17. Match the column:

Column – I	Column – II
(a) If A and B are square matrices of order 3×3, where $\|A\| = 2$ and $\|B\| = 1$, then $\|(A^{-1}) \cdot \text{adj}\,(B^{-1}) \cdot \text{adj}\,(2A^{-1})\| =$	(p) 7
(b) If A is a square matrix such that $A^2 = A$ and $(I + A)^3 = I + kA$, then k is equal to	(q) 8
(c) If A and B are two invertible matrices such that $AB = C$ and $\|A\| = 2, \|C\| = -2$, then $\det(B)$ is	(r) 0
(d) If $A = [a_{ij}]_{3\times3}$ is a scalar matrix with $a_{11} = a_{22} = a_{33} = 2$ and $A(\text{adj}\,A) = kI_3$ then k is	(s) – 1

18. Match the column:

Column – I	Column – II
(a) $[1 \times 1]\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$ then x =	(p) 2
(b) If square matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $A^2 - 2xA + 5I_2 = 0$, then find x	(q) – 2
(c) If $A = \begin{bmatrix} 2 & \mu \\ \mu^2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \gamma & 7 \\ 49 & \delta \end{bmatrix}$ here $(A - B)$ is upper triangular matrix then number of possible values of μ are	(r) 1
(d) If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,abc\,(a+b+c)^3$ then the value of k is	(s) $-\frac{9}{8}$

NUMERICAL VALUE BASED

19. Let $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ such that $A^TA = 1$. Find the value of $x^2 + y^2 + z^2$

20. If $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, then absolute value of the sum of element of X is equal to

21. If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ then the sum of the elements of the matrix $3X - 4Y$ is equal to

22. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then absolute value of k so that $A^2 = 8A + kI$.

23. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and if $A^6 = KA - 205I$, then K equals

24. Let A is a 3×3 matrix and $A = [a_{ij}]_{3\times3}$. If for every column matrix X, if X^TAX and $a_{23} = -2009$ then $a_{32} =$ .....

25. If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, then $x =$

26. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & k \\ 1 & -2 & 3 \end{bmatrix}$ If B is the inverse of A, then $k =$

27. If A square matrix of order 2×2 and $|A| = 5$, then $|A(\text{adj}\,A)| =$

28. If $\begin{bmatrix} \sin\frac{\pi}{2} & \cos\frac{\pi}{3} \\ 2\tan\frac{\pi}{4} & 2k \end{bmatrix}$ is a singular matrix then $k =$

JEE-MAIN PREVIOUS YEARS

1. If $A=\begin{bmatrix}5a & -b\\ 3 & 2\end{bmatrix}$ and A adj $A = AA^T$, then $5a+b$ is equal to: [JEE Main-2016]

(a) -1 (b) 5

(c) 4 (d) 13

2. If $A=\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$, then adj $(3A^2+12A)$ is equal to : [JEE Main-2017]

(a) $\begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$ (b) $\begin{bmatrix}72 & -84\\ -63 & 51\end{bmatrix}$

(c) $\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$ (d) $\begin{bmatrix}51 & 84\\ 63 & 72\end{bmatrix}$

3. If $\begin{vmatrix}x-4 & 2x & 2x\\ 2x & x-4 & 2x\\ 2x & 2x & x-4\end{vmatrix} = (A+Bx)(x-A)^2$, then the ordered pair (A, B) is equal to : [JEE Main-2018]

(a) $(-4, 3)$ (b) $(-4, 5)$

(c) $(4, 5)$ (d) $(-4, -5)$

4. If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then the matrix A^{-50} when $\theta=\frac{\pi}{12}$, is equal to [JEE Main-2019 (January)]

(a) $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ (b) $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

(c) $\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$ (d) $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

5. Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T)=8$ and $\det(AB^{-1})=8$, then $\det(BA^{-1}B^T)$ is equal to: [JEE Main-2019 (January)]

(a) $\frac{1}{4}$ (b) 1

(c) $\frac{1}{16}$ (d) 16

6. Let $P=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 9 & 3 & 1\end{bmatrix}$ and $Q=[q_{ij}]$ be two 3×3 matrices such that $Q-P^5=I_3$. Then $\frac{q_{21}+q_{31}}{q_{32}}$ is equal to : [JEE Main - 2019 (January)]

(a) 10 (b) 135

(c) 15 (d) 9

7. Let $A=\begin{pmatrix}0 & 2q & r\\ p & q & -r\\ p & -q & r\end{pmatrix}$. If $AA^T=I_3$, then $|p|$ is :- [JEE Main - 2019 (January)]

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{6}}$

8. Let $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$, $(a\in R)$ such that $A^{32}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then a value of α is. [JEE Main -2019 (April)]

(a) $\frac{\pi}{16}$ (b) 0

(c) $\frac{\pi}{32}$ (d) $\frac{\pi}{64}$

9. If $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\cdots\cdots\begin{bmatrix}1 & n-1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 78\\ 0 & 1\end{bmatrix}$, then the inverse of $\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ is [JEE Main -2019 (April)]

(a) $\begin{bmatrix}1 & -13\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ 12 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & -12\\ 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\ 13 & 1\end{bmatrix}$

10. The total number of matrices $A=\begin{pmatrix}0 & 2y & 1\\ 2x & y & -1\\ 2x & -y & 1\end{pmatrix}$, $(x, y\in R, x\neq y)$ for which $A^TA=3I_3$ is :- [JEE Main -2019 (April)]

(a) 6 (b) 2

(c) 3 (d) 4

11. If a is A symmetric matrix and B is a skew-symmetric matrix such that $A+B=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}$, then AB is equal to :

[JEE Main -2019 (April)]

(a) $\begin{bmatrix}-4 & 2\\ 1 & 4\end{bmatrix}$ (b) $\begin{bmatrix}-4 & -2\\ -1 & 4\end{bmatrix}$

(c) $\begin{bmatrix}4 & -2\\ -1 & -4\end{bmatrix}$ (d) $\begin{bmatrix}4 & -2\\ 1 & -4\end{bmatrix}$

12. Let α be a root of equation $x^2+x+1=0$ and the matrix $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha^4\end{bmatrix}$, then the matrix A^{31} is equal to

[JEE Main-2020 (January)]

(a) A^3 (b) A^2

(c) I_3 (d) A

13. The number of all 3×3 matrices A, with entries from the set $\{-1, 0, 1\}$ such that the sum of the diagonal elements of AA^T is 3, is _____.

[JEE Main-2020 (January)]

14. If $A=\begin{pmatrix}2 & 2\\ 9 & 4\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, then $10A^{-1}$ is equal to:

[JEE Main-2020 (January)]

(a) $A-4I$ (b) $A-6I$

(c) $4I-A$ (d) $6I-A$

15. If the matrices $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 3 & 4\\ 1 & -1 & 3\end{bmatrix}$, $B=\text{adj}\,A$ and $C=3A$, then $\frac{|\text{adj}(B)|}{|C|}$ is equal to

[JEE Main-2020 (January)]

(a) 72 (b) 8

(c) 16 (d) 2

16. Let $a, b, c \in R$ be all non- zero and satisfy $a^3+b^3+c^3=2$. If the matrix

$$A=\begin{pmatrix}a & b & c\\ b & c & a\\ c & a & b\end{pmatrix}$$

satisfies $A^TA=I$, then a value of abc can be :

[JEE Main-2020 (September)]

(a) 3 (b) $\frac{1}{3}$

(c) $-\frac{1}{3}$ (d) $\frac{2}{3}$

17. Let A be a 2×2 real matrix with entries from $\{0, 1\}$ and $|A|\neq 0$. Consider the following two statements :

(P) If $A\neq I_2$, then $|A|=-1$

(Q) If $|A|=1$, then $\text{tr}(A)=2$,

where I_2 denotes 2×2 identity matrix and $\text{tr}(A)$ denotes the sum of the diagonal entries of A.

Then : [JEE Main-2020 (September)]

(a) (P) is true and (Q) is false

(b) Both (P) and (Q) are false

(c) Both (P) and (Q) are true

(d) (P) is false and (Q) is true

18. Let A be a 3×3 matrix such that

$$\text{adj}\,A=\begin{bmatrix}2 & -1 & 1\\ -1 & 0 & 2\\ 1 & -2 & -1\end{bmatrix} \text{ and } B=\text{adj}(\text{adj}\,A)$$

If $|A|=\lambda$ and $|(B^{-1})^T|=\mu$, then the ordered pair, $(|\lambda|, \mu)$ is equal to [JEE Main-2020 (September)]

(a) (3,81) (b) $\left(9,\frac{1}{9}\right)$

(c) $\left(3,\frac{1}{81}\right)$ (d) $\left(9,\frac{1}{81}\right)$

19. Let $A=\begin{bmatrix}x & 1\\ 1 & 0\end{bmatrix}$, $x\in R$ and $A^4=[a_{ij}]$. if $a_{11}=109$, then a_{22} is equal to _____. [JEE Main-2020 (September)]

20. Suppose the vectors x_1, x_2 and x_3 are the solutions of the system of linear equations, $Ax=b$ when the vector b on the right side is equal to b_1, b_2 and b_3 respectively. If

$x_1=\begin{bmatrix}1\\1\\1\end{bmatrix}, x_2=\begin{bmatrix}0\\2\\1\end{bmatrix}, x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}, b_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, b_2=\begin{bmatrix}0\\2\\0\end{bmatrix}$ and $b_3=\begin{bmatrix}0\\0\\2\end{bmatrix}$, then the determinant of A is equal to :

[JEE Main-2020 (September)]

(a) 4 (b) $\frac{1}{2}$

(c) 2 (d) $\frac{3}{2}$

21. If $A=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix}$, $\left(\theta=\frac{\pi}{24}\right)$ and $A^5=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ where $i=\sqrt{-1}$, then which one of the following is not true? [JEE Main-2020 (September)]

(a) $a^2-b^2=\sqrt{-1}$ (b) $a^2-c^2=1$

(c) $a^2-d^2=0$ (d) $0\le a^2+b^2\le 1$

22. Let $\theta = \frac{\pi}{5}$ and $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. If $B = A + A^4$, then det (B) : [JEE Main-2020 (September)]

(a) lies in (2, 3). (b) is zero.

(c) is one. (d) lies in (1, 2)

23. If $A = \begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{pmatrix}$ and $\det\left(A^2 - \frac{1}{2}I\right) = 0$, then a possible value of α is : [JEE Main-2021 (March)]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

24. The system of equations $k\alpha + y + z = 1, x + ky + z = k$ and $x + y + zk = k^2$ has no solution if k is equal to : [JEE Main-2021 (March)]

(a) 0 (b) –1

(c) –2 (d) 1

25. Consider the following system of equations:

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

where a, b and c are real constants. Then the system of equations: [JEE Main-2021 (February)]

(a) has no solution for all a, b and c

(b) has a unique solution when $5a = 2b + c$

(c) has infinite number of solutions when $5a = 2b + c$

(d) has a unique solution for all a, b and c

26. Consider the three planes

$P_1 : 3x + 15y + 21z = 9$

$P_2 : x - 3y - z = 5$, and

$P_3 : 2x + 10y + 14z = 5$

Then, which one of the following is true? [JEE Main-2021 (February)]

(a) P_1 and P_2 are parallel.

(b) P_1, P_2 and P_3 all are parallel.

(c) P_1 and P_3 are parallel.

(d) P_2 and P_3 are parallel.

27. The value of $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$ is : [JEE Main-2021 (February)]

(a) –2 (b) 0

(c) $(a+2)(a+3)(a+4)$ (d) $(a+1)(a+2)(a+3)$

JEE-ADVANCED PREVIOUS YEARS

1. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is [IIT JEE-2012]

(a) 2^{10} (b) 2^{11}

(c) 2^{12} (d) 2^{13}

2. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that [IIT JEE-2012]

(a) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $PX = X$

(c) $PX = 2X$ (d) $PX = -X$

3. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) [IIT JEE-2012]

(a) –2 (b) –1

(c) 1 (d) 2

4. For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ? [JEE Advanced-2013]

(a) $N^T M N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(b) $MN - NM$ is skew symmetric for all symmetric matrices M and N

(c) MN is symetric for all symmetric matrices M and N

(d) (adj M) (adj N) = adj(MN) for all invertible matrices M and N

5. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if [JEE Advanced-2014]

(a) the first column of M is the transpose of the second row of M

(b) the second row of M is the transpose of first column of M

(c) M is a diagonal matrix with nonzero entries in the main diagonal

(d) the product of entries in the main diagonal of M is not the square of an integer

6. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ?

[*JEE Advanced-2015*]

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$

(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

7. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then [*JEE Advanced-2016*]

(a) $\alpha = 0, k = 8$ (b) $4\alpha - k + 8 = 0$

(c) $\det(P\,\text{adj}(Q)) = 2^9$ (d) $\det(Q\,\text{adj}(P)) = 2^{13}$

8. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals [*JEE Advanced-2016*]

(a) 52 (b) 103

(c) 201 (d) 205

9. Which of the following is (are) NOT the square of a 3×3 matrix with real entries? [*JEE Advanced-2017*]

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, the $1 + \alpha + \alpha^2 =$ [*JEE Advanced-2017*]

11. How many 3×3 matrices M with entries from $\{0, 1, 2\}$ are there, for which the sum of the diagonal entries of $M^T M$ is 5 ? [*JEE Advanced-2017*]

(a) 198 (b) 162

(c) 126 (d) 135

12. Let S be the of all column metrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then which of the folowing system (s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? [*JEE Advanced-2018*]

(a) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(b) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(c) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(d) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

13. Let $M = \begin{bmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is [*JEE Advanced-2019*]

(a) $-\frac{37}{16}$ (b) $-\frac{29}{16}$

(c) $-\frac{31}{16}$ (d) $-\frac{17}{16}$

14. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is are correct ? [*JEE Advanced -2019*]

(a) $a + b = 3$

(b) $\det(\text{adj } M^2) = 81$

(c) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$

(d) if $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

15. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$

[JEE Advanced -2019]

where P_k^T denotes the transpose of the matrix P_k. Then which of the following options is/are correct ?

(*a*) $X - 30I$ is an invertible matrix

(*b*) The sum of diagonal entries of X is 18

(*c*) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$

(*d*) X is a symmetric matrix

16. Let $x \in R$ and lat $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following.options is/are correct ?

[JEE Advanced - 2019]

(*a*) For $x = 1$, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(*b*) There exists a real number x such that $PQ = QP$

(*c*) $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in R$

(*d*) For $x = 0$, if $R \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then $a + b = 5$

17. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statement is/are ALWAYS TRUE ?

[JEE(Advanced) - 2020]

(*a*) $M = I$ (*b*) $\det M = I$

(*c*) $M^2 = I$ (*d*) $(\text{adj } M)^2 = I$

18. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18, then the value of the determinant of A is _____ **[JEE(Advanced) - 2020]**

ANSWER KEY

Topicwise Questions

1. (*c*)	**2.** (*c*)	**3.** (*b*)	**4.** (*d*)	**5.** (*d*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*d*)	**10.** (*b*)
11. (*c*)	**12.** (*a*)	**13.** (*a*)	**14.** (*d*)	**15.** (*a*)	**16.** (*a*)	**17.** (*a*)	**18.** (*b*)	**19.** (*c*)	**20.** (*d*)
21. (*c*)	**22.** (*d*)	**23.** (*a*)	**24.** (*a*)						

Learning Plus

1. (*c*)	**2.** (*a*)	**3.** (*d*)	**4.** (*a*)	**5.** (*a*)	**6.** (*b*)	**7.** (*a*)	**8.** (*c*)	**9.** (*d*)	**10.** (*c*)
11. (*b*)	**12.** (*d*)	**13.** (*a*)	**14.** (*a*)	**15.** (*c*)	**16.** (*b*)	**17.** (*c*)	**18.** (*c*)	**19.** (*b*)	**20.** (*a*)
21. (*b*)	**22.** (*b*)	**23.** (*b*)	**24.** (*d*)	**25.** (*c*)					

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*b,c*)	**2.** (*c,d*)	**3.** (*a,c*)	**4.** (*b,c*)	**5.** (*a,b*)	**6.** (*a,c,d*)	**7.** (*a,d*)	**8.** (*a,c*)	**9.** (*a,c*)	**10.** (*a,b,d*)
11. (*a,c*)	**12.** (*a,c*)	**13.** (*a,c*)	**14.** (*a*)	**15.** (*d*)	**16.** (*c*)	**17.** (*a*) → (q); (*b*) → (p); (*c*) → (s); (*d*) → (q)			
18. (*a*) → (s); (*b*) → (p); (*c*) → (p); (*d*) → (p)				**19.** [0001]	**20.** [005]	**21.** [0014]	**22.** [0007]	**23.** [0044]	**24.** [2009]
25. [–4]	**26.** [5]	**27.** [25]	**28.** [0.5]						

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*b*)	**2.** (*c*)	**3.** (*b*)	**4.** (*c*)	**5.** (*c*)	**6.** (*a*)	**7.** (*c*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*c*)	**12.** (*a*)	**13.** [672]	**14.** (*b*)	**15.** (*b*)	**16.** (*b*)	**17.** (*d*)	**18.** (*c*)	**19.** [10]	**20.** (*c*)
21. (*a*)	**22.** (*d*)	**23.** (*a*)	**24.** (*c*)	**25.** (*c*)	**26.** (*c*)	**27.** (*a*)			

JEE-ADVANCED PREVIOUS YEARS

1. (*d*)	**2.** (*d*)	**3.** (*a,d*)	**4.** (*c,d*)	**5.** (*c,d*)	**6.** (*c,d*)	**7.** (*b,c*)	**8.** (*b*)	**9.** (*a,c*)	**10.** (*a*)
11. (*a*)	**12.** (*a,c,d*)	**13.** (*b*)	**14.** (*a,c,d*)	**15.** (*b,c,d*)	**16.** (*c,d*)	**17.** (*b,c,d*)	**18.** (5)		

PRAYAS 1.0 t.me/realprayasians

PRAYAS 2.0 t.me/realprayasians2

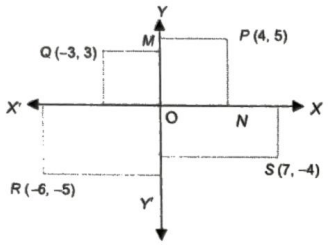
13 STRAIGHT LINE

PRAYAS 1.0 time/realprayasion

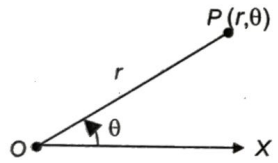
PRAYAS 2.0 time/realprayasion2

RECTANGULAR CARTESIAN CO-ORDINATE SYSTEMS

Let P be a point in a plane; $X'OX, Y'OY$ are known as the coordinate axes. the x-coordinate, called the abscissa, and NP is the y-coordinate called the ordinate of P.



POLAR COORDINATES



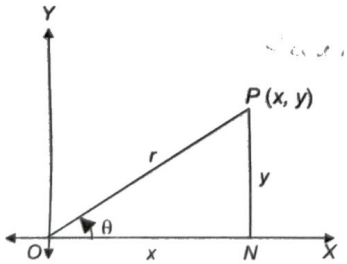
Relation between Cartesian and Polar Coordinates

Let P be the point (x, y) with reference to rectangular axes OX, OY and the point (r, θ) with reference to the pole O and initial line OX. i.e., the X-axis.

$\therefore$ we have $x = r\cos\theta$; $y = r\sin\theta$

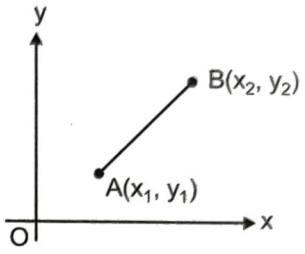
Hence $\tan\theta = \frac{y}{x}$ and $x^2 + y^2 = r^2$

These relations enable us to change from one system of coordinates to the other.



DISTANCE FORMULA

(*a*) Let A and B be two given points, whose coordinates are given by



$A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

Then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(*b*) Distance of (x_1, y_1) from origin : $\sqrt{x_1^2 + y_1^2}$

Notes:

(*i*) If two vertex $A(x_1, y_1)$, $B(x_2, y_2)$ are given then third vertex of equilateral triangle C is

$$\left[\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right]$$

(*ii*) If the three points P, Q, R are collinear, then $|PQ| \pm |QR| = |PR|$.

(*iii*) When 3 points are given then

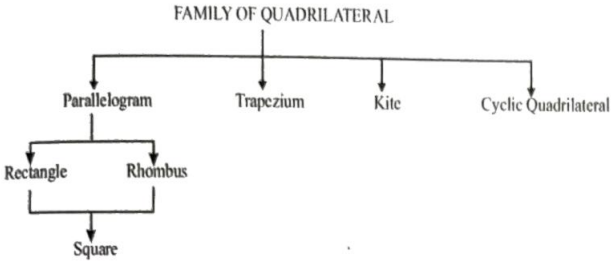
(*a*) an isosceles triangle, when any two sides are equal.

(*b*) an equilateral triangle, when all sides are equal

(*c*) a scalene triangle, when all sides are unequal.

(*d*) a right-angled triangle, say ΔABC, when $AB^2 + BC^2 = AC^2$

TYPES OF QUADRILATERAL (EUCLIDIAN FIGURES):



- **Parallelogram**

Definition: If opposite sides of quadrilateral are parallel and equal, then quadrilateral is called parallelorgam.

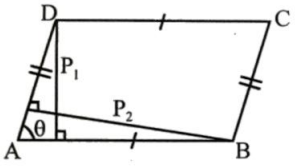
Four ways to prove that a quadrilateral is parallogram.

(*i*) Opposite sides are parallel
i.e. $AB \parallel DC$ and $AD \parallel BC$

(*ii*) Opposite sides are equal
i.e. $AB = DC$ and $AD = BC$

(*iii*) One pair of opposite sides are equal and parallel.

(*iv*) Diagonals bisect each other

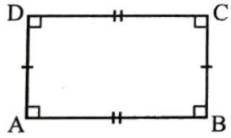


Note: Area of parallelogram $= \frac{P_1P_2}{\sin\theta}$

- **Rectangle**

Definition: If all angles of parallelogram are equal then it is called rectangle.

(*i*) Diagonal are equal and bisect each other.

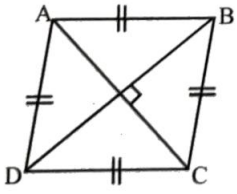


(*ii*) Each diagonal divides the rectangle into two triangles of equal area.

- **Rhombus**

Definition: If all sides of a parallelogram are equal then it is called Rhombus

(*i*) Diagonals are perpendicular



(*ii*) Area $= \frac{1}{2} d_1d_2$ where d_1 and d_2 are diagonal.

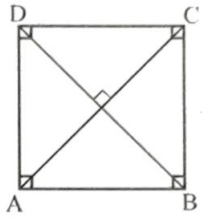
Note: If distance between pair of parallel sides are equal then it is a rhombus.

- **Square**

Definition: If all the sides and all the interior angles of a parallelogram are equal then it is called a square.

(*i*) All sides are equal
$AB = BC = CD = DA$

(*ii*) Diagonals are equal and bisect each other at 90°.



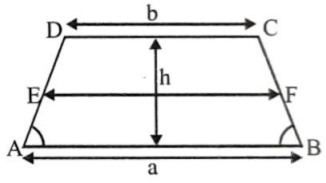
(*iii*) Area $= \frac{d^2}{2}$ (d = diagonal)

Note: Every square is a rectangle but not the converse.

- **Trapezium**

Definition: Trapezium is a quadrilateral which has exactly one pair of opposite sides parallel.

(*i*) Area of trapezium $= \frac{1}{2}$ (sum of parallel sides) × distance between sides



(*ii*) $\angle DAB$ and $\angle CBA$ = Base angle

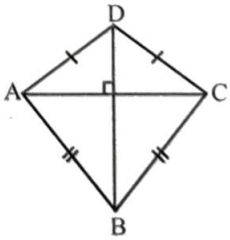
(*iii*) Median $(EF) = \frac{1}{2}(a + b)$

(*iv*) For equilateral /isosceles trapezium, non parallel sides are equal i.e. $AD = BC$

- **Kite**

Definition: It is a quadrilateral in which two pairs of adjacent sides are equal.

(*i*) $AD = DC$ and $AB = BC$



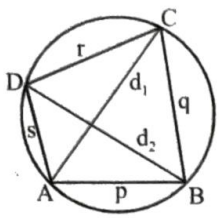
(*ii*) Diagonals are perpendicular but not bisect

(*iii*) Only one diagonal divide the figure into two congruent triangles.

- **Cyclic Quadrilateral**

Definition: If all vertices of quadrilateral lies on the circumference of a circle, then it is called cyclic quadrilateral.

(*i*) Opposite angles are supplementary.



(*ii*) Sum of product of opposite sides are equal to product of diagonals $pr + qs = d_1 d_2$

TRAIN YOUR BRAIN

Example 1: If $A(0, -1)$, $B(6, 7)$, $C(-2, 3)$ and $D(\lambda, 3)$ forms a rectangle then find the value of λ.

Sol. $AB = CD$ and $AC = BD$.

$AB = \sqrt{6^2 + 8^2} = 10$

$CD = \sqrt{(\lambda+2)^2} = |\lambda + 2|$

$\lambda + 2 = 10$ or $\lambda + 2 = -10$

$\lambda = 8$ or $\lambda = -12$ ...(1)

Now $AC = BD$

$4 + 16 = (6 - \lambda)^2 + 4^2$

$4 = 36 + \lambda^2 - 12\lambda$

$\Rightarrow \quad \lambda^2 - 12\lambda + 32 = 0$

$\lambda = 4, 8$

Hence from (1) and (2)

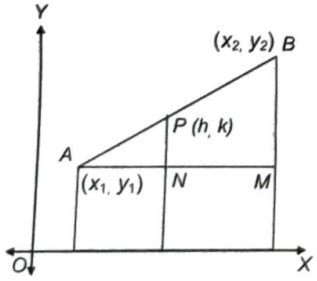
$\lambda = 8$

SECTION FORMULA

Let A, B are the points (x_1, y_1), (x_2, y_2), here we will find the coordinates of the point P on AB such that $AP : PB = l : m$.

Let the coordinates of P be (h, k)

In the Figure, triangles ANP and AMB are similar.



$\therefore \quad k = \dfrac{ly_2 + my_1}{l + m}$

similarly, $h = \dfrac{lx_2 + mx_1}{l + m}$

Note: Putting $l = m$, we find that the mid-point of the line joining the points (x_1, y_1), (x_2, y_2).

i.e. $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Note: When P lies outside AB i.e., external to AB such that $AP : BP = l : m$.

We have $h = \dfrac{lx_2 - mx_1}{l - m}, k = \dfrac{ly_2 - my_1}{l - m}$

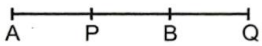
Notes:

(*i*) If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio $\lambda_1 : \lambda_2$), we take the ratio $k : 1$. If the value of k turns out to be positive, it is an internal division otherwise it is an external division.

(*ii*) The coordinates of the mid-point of the line-segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

HARMONIC CONJUGATE

If P is a point that divides AB internally in the ratio $m_1 : m_2$ and Q is another point which divides AB externally in the same ratio $m_1 : m_2$, then the point P and Q are said to be Harmonic conjugate to each other with respect to A and B.



i.e. AP, AB and AQ forms a $HP \Rightarrow \dfrac{1}{AP} + \dfrac{1}{AQ} = \dfrac{2}{AB}$

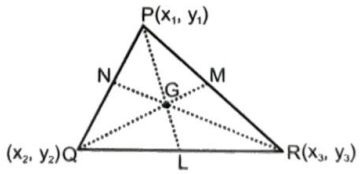
Note: Internal and External angle bisector of an angle divides the base harmonically.

COORDINATES OF SPECIAL POINTS WITH RESPECT TO A TRIANGLE

(*a*) Centroid (G) :

Definition: The point of concurrence of the medians of a triangle is called the centroid of the triangle.

(*i*) ***G*** divides median into 2 : 1.



(*ii*) ***G*** always lies inside the triangle.

(*iii*) Co-ordinates of ***G*** is

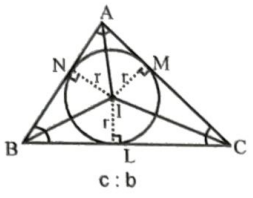
$\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ or $\left(\dfrac{\Sigma x_1}{3}, \dfrac{\Sigma y_1}{3}\right)$

(b) Incentre (I)

Definition: The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle.

(i) I always lies inside the triangle.

(ii) Internal angle bisector divides the base in the ratio of adjacent sides.



c : b

(iii) Co-ordinates of **I** is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the sides of the Δ

(c) Ex-centres (I_1, I_2, I_3):

Definition: The centre of the escribed circle which is opposite to vertices.

To get I_1 (or I_2 or I_3) replace a by $-a$ (b by $-b$ or c by $-c$) in formula of coordinate of **I**

$$I_1=\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$I_2=\left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3=\left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

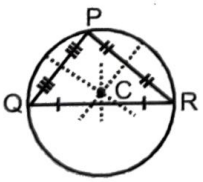
(d) Circumcentre (C):

Definition: The point of concurrency of the perpendicular bisectors of the sides of a triangle is called circumcentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

(ii) For obtuse angle $\Delta \Rightarrow$ lies outside

(iii) For right angle $\Delta \Rightarrow$ Mid point of hypotenuse



(iv) Co-ordinates of circumcentre is

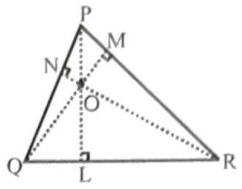
$$\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$$

(e) Orthocentre (O):

Definition: The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

(ii) For obtuse angle $\Delta \Rightarrow$ lies outside



(iii) For right angle Δ
$\Rightarrow$ vertex having right angle

(iv) Co-ordinates of orthocentre is

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$

Notes:

(i) In any triangle **O, G, C** are collinear.

(ii) In any triangle **G** divides the line joining **O** & **C** in ratio 2 : 1.

(iii) In an equilateral triangle **O, G, C, I** are coincident.

(iv) In an isosceles triangle **O, G, C, I** are collinear.

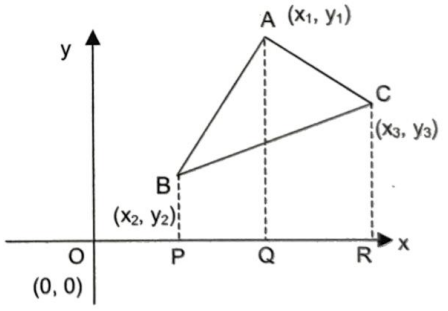
(v) The orthocentre of ΔABC are (a, b), (b, a) and (a, a) is (a, a).

AREA OF A TRIANGLE

Let the vertices of the triangle be

$$A(x_1, y_1),\ B(x_2, y_2),\ C(x_3, y_3)$$

Area of ΔABC



$$= \frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

expressed in the determinant form,

$$\Rightarrow \Delta = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Notes:

(*i*) If three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear,

then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(*ii*) Equation of straight line passing through (x_1, y_1) and (x_2, y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(*iii*) The area of a polygon, with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots (x_n, y_n)$ is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots\ldots + \frac{1}{2}\begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

TRAIN YOUR BRAIN

Example 2: The vertices of quadrilateral in order are $(-1, 4)$, $(5, 6)$, $(2, 9)$ and (x, x^2). The area of the quadrilateral is $\frac{15}{2}$ sq. units, then find the point (x, x^2)

Sol. Area of quadrilateral

$$= \frac{1}{2}\left\| \begin{vmatrix} -1 & 4 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 6 \\ 2 & 9 \end{vmatrix} + \begin{vmatrix} 2 & 9 \\ x & x^2 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ -1 & 4 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\left|-26 + 33 + 2x^2 - 9x + 4x + x^2\right| = \frac{15}{2}$$

$\therefore \quad 3x^2 - 5x + 7 = \pm 15$

$\therefore \quad 3x^2 - 5x - 8 = 0,\ 3x^2 - 5x + 22 = 0 \Rightarrow x = \frac{8}{3}, x = -1$

Hence point is $\left(\frac{8}{3}, \frac{64}{9}\right)$ or $(-1, 1)$. But $(-1, 1)$ will not form a quadrilateral as per given order of the points. Hence the required point is $\left(\frac{8}{3}, \frac{64}{9}\right)$

LOCUS

The locus of a point, as it moves in accordance with a given geometrical condition, is the path traced out by the moving point.

Note: All those points which satisfy the given geometrical condition will definitely lie on the locus. But converse is not true always.

EQUATION OF LOCUS

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

Procedure to find the equation of the locus of a point

(*i*) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(*ii*) Express the given conditions as equations in terms of the known quantities to facilitate calculations. Sometimes we include some unknown quantities known as parameters.

(*iii*) Eliminate the parameters. so that the eliminate contains only h, k and known quantities.

(*iv*) Replace h by x, and k by y, in the eliminate. The resulting would be the equation of the locus of P.

(*v*) If x and y coordinates of the moving point are obtained in terms of a third variable t (called the parameter), eliminate t to obtain the relation in x and y and simplify this relation. This will give the required equation of locus.

TRAIN YOUR BRAIN

Example 3: Find the locus of point P if P is equidistant from points

(*i*) $A(3, 4)$ and $B(5, -2)$

(*ii*) $A(a + b, a - b)$ and $B(a - b, a + b)$

Sol. (*i*) If point $P(h, k)$ is equidistant from $A(3, 4)$ & $B(5, -2)$ then $PA = PB$

$\Rightarrow \sqrt{(h-3)^2 + (k-4)^2} = \sqrt{(h-5)^2 + (k+2)^2}$

$\Rightarrow h^2 - 6h + 9 + k^2 - 8k + 16 = h^2 - 10h + 25 + k^2 + 4k + 4$

$\Rightarrow 4h - 12k = 4$

$\Rightarrow h - 3k = 1$

hence locus of P is $x - 3y = 1$

(*ii*) $PA = PB$

$\Rightarrow [h - (a + b)]^2 + [k - (a - b)]^2 = [h - (a - b)]^2 + [k - (a + b)]^2$

$\Rightarrow -2h(a + b) - 2k(a - b) = -2h(a - b) - 2k(a + b)$

$\Rightarrow 2h(2b) + 2k(2b) = 0$

$\Rightarrow h + k = 0$

Hence locus of P is $x + y = 0$

Example 4: Find the equation to the locus of a point which moves so that

(*i*) Its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

(*ii*) Sum of the squares of its distances from the axes is equal to 3.

(*iii*) Its distance from x-axis is 3 times of its distance from y-axis.

Sol. (*i*) Let the point be $P(h, k)$

Distance of P from axis of $y = |h|$

Distance of P from $(a, 0) = \sqrt{(h-a)^2 + k^2}$

$\Rightarrow \sqrt{(h-a)^2 + k^2} = 4|h|$

$\Rightarrow (h-a)^2 + k^2 = 16h^2$

$\Rightarrow h^2 - 2ah + a^2 + k^2 = 16h^2$

$\Rightarrow 15h^2 - k^2 + 2ah = a^2$

hence locus of P is $15x^2 - y^2 + 2ax = a^2$

(*ii*) Let the point be $P(h, k)$

Distance of P from y-axis $= |h|$

Distance of P from x-axis $= |k|$

$h^2 + k^2 = 3$

hence Locus of P is

$x^2 + y^2 = 3$

(*iii*) Let the point be $P(h, k)$

Distance from x-axis $= |k|$

Distance from y-axis $= |h|$

$|k| = 3|h|$

$3h - k = 0$ or $3h + k = 0$

$3x - y = 0$ or $3x + y = 0$

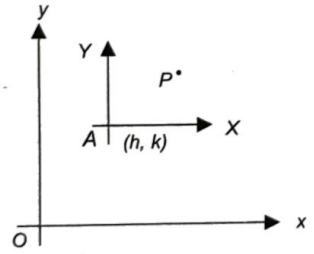
SHIFTING OF AXES

Shifting of origin

When origin is shifted to a new position $A(h, k)$ without changing the directions of the axes, then coordinates of all points in the plane are obtained by the formulae:

$$x = X + h \quad y = Y + k$$

where, $P(x, y)$ referred to ox, oy and $P(X, Y)$ referred to AX, AY

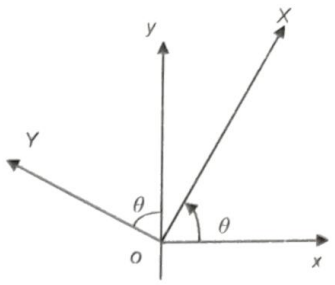


Rotation of Coordinate Axes

When coordinate axes are rotated in anti-clockwise direction through an angle θ, about the origin, then coordinates of a point P in the plane are obtained by the formulae.

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\} \text{ and } \left\{\begin{aligned} X &= x\cos\theta + y\sin\theta \\ Y &= y\cos\theta - x\sin\theta \end{aligned}\right.$$

Where P is (x, y) referred to coordinate system oxy and P is (X, Y) referred to new coordinate system oXY.



Example 5: Find the equation of the curve $x^2 + 2y^2 + 3x - 4y + 7 = 0$, when the origin is transferred to the point $(1, -2)$ without changing the direction of axes.

Sol. If $P(x, y)$ be any point on the curve and (x_1, y_1) be the coordinates of P. w. r. to new axes then $x_1 = x - \alpha = x - 1$ and $y_1 = y - \beta = y + 2$

$\therefore \quad x = x_1 + 1, y = y_1 - 2$

Hence new equation will be

$$(x_1 + 1)^2 + 2(y_1 - 2)^2 + 3(x_1 + 1) - 4(y_1 - 2) + 7 = 0$$

or

$$x_1^2 + 2x_1 + 1 + 2(y_1^2 - 4y_1 + 4) + 3x_1 + 3 - 4y_1 + 8 + 7 = 0$$

or $\quad x_1^2 + 2y_1^2 + 5x_1 - 12y_1 + 27 = 0$

Thus new equation of the curve will be

$$x^2 + 2y^2 + 5x - 12y + 27 = 0$$

Note: New equation of the curve can be directly obtained by putting $(x + 1)$ in place of x and $(y - 2)$ in place of y in the given equation of the curve.

DEFINITION OF STRAIGHT LINE

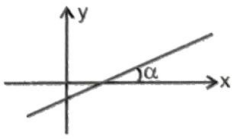
It is a locus of a point $P(h, k)$ which moves in such a way that point $P(h, k)$ is collinear with the two given points.

'or'

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Inclination of a line:

Its a measure of the smallest non-negative angle which the line makes with +ve direction of the x-axis [angle being measured in anti-clockwise direction]. $0 \le \alpha < \pi$



Slope of the line:

If the inclination of line is θ and $\theta \neq \frac{\pi}{2}$ then its slope is defined as tan θ and denoted by 'm'

(*i*) If $\theta = 0$, then $m = 0$ i.e. line parallel to x-axis.

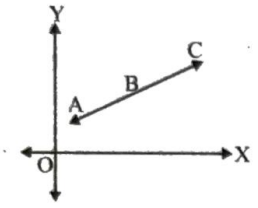
(*ii*) If $\theta = 90°$, then m does not exist i.e. line parallel to y-axis

(*iii*) Slope of line joining two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

(*iv*) If a line equally inclined with co-ordinate axes then slope is ± 1.

(*v*) If three points A, B, C are collinear, then $m(AB) = m(BC) = m(CA)$.



(*vi*) If two lines, having slopes m_1 and m_2 are parallel, then $m_1 = m_2$

If two lines, having slopes m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$

Intercepts:

The point where a line cuts the x-axis (or y-axis) is called its x-intercept (or y-intercept).

(*i*) Intercepts may be +ve, –ve or zero.

(*ii*) A line making an intercept of $-a$ with y-axis means the line passing through $(0, -a)$

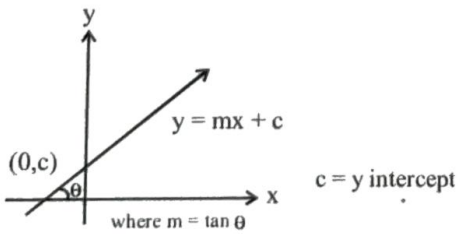
(*iii*) A line makes equal non-zero intercept with both co-ordinate axes then slope is -1.

(*iv*) A line makes non-zero intercept with both co-ordinate axes equal in magnitude then slope is ± 1.

STANDARD EQUATIONS OF STRAIGHT LINES

1. **General Form:** Any first degree equation of the form $Ax + By + C = 0$,
   where A, B, C are constant always represents general equation of a straight line (at least one out of A and B is non zero.)

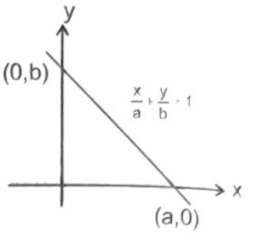
2. **Slope - Intercept Form :**
   $$y = mx + c$$
   where m = slope of the line = $\tan\theta$



c = y intercept

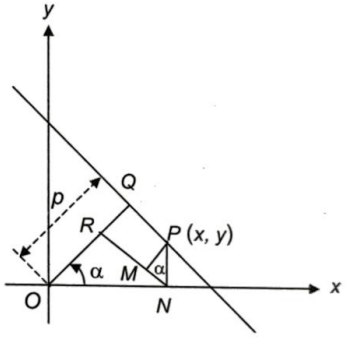
The general equation of any line passing through the origin is $y = mx$.

3. **Intercept Form:**
   $$\frac{x}{a} + \frac{y}{b} = 1$$
   x intercept $= a$
   y intercept $= b$



Notes:

1. The intercepts a and b may be positive or negative.
2. The intercept cut on negative side of x and y axes are taken as negative.
3. Area of the triangle formed by the line
   $$\frac{x}{a} + \frac{y}{b} = 1$$ and coordinate axes is
   $$= \frac{1}{2}|ab| \text{ sq. unit}$$
4. **Normal Form:** If p is the length of the perpendicular from the origin upon a straight line, and that α is the angle the perpendicular makes with the axis of x,
   $p = x\cos\alpha + y\sin\alpha$.
   $\therefore x\cos\alpha + y\sin\alpha = p$ is the required equation.
   This is called the perpendicular form.



$\therefore$ by substitution, the equation becomes

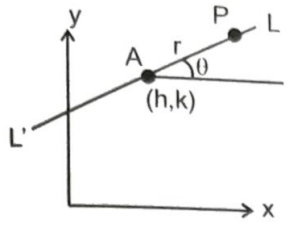
$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1 \text{ or } x\cos\alpha + y\sin\alpha = p$$

This is known as the normal form of the equation to a straight line.

5. **Slope Point Form:** Equation : $y - y_1 = m(x - x_1)$, is the equation of line passing through the point (x_1, y_1) and having the slope 'm'

6. **Two points Form:** Equation: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of line passing through two points (x_1, y_1) and (x_2, y_2)

7. **Parametric Form:** To find the equation of a straight line which passes through a given point $A(h, k)$ and makes a given angle θ with the positive direction of the x-axis. $P(x, y)$ is any point on the line LAL'. Let $AP = r$, $x - h = r\cos\theta$, $y - k = r\sin\theta$



$\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r$ is the equation of the straight line LAL'. Any point on the line will be of the form $(h+r\cos\theta, k+r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k)

Note : If point P is taken relatively upward to A then r is positive otherwise negative. If line is parallel to x-axis then for the point right to A, r is positive and for left to A, r is negative.

TRAIN YOUR BRAIN

Example 6: What is the slope of a line whose inclination with the positive direction of x-axis is:

(*i*) $0°$ (*ii*) $90°$

(*iii*) $120°$ (*iv*) $150°$

Sol. (*i*) Here $\theta=0°$

Slope $=\tan\theta=\tan 0°=0$.

(*ii*) Here $\theta=90°$

$\therefore$ The slope of line is not defined.

(*iii*) Here $\theta=120°$

$\therefore$ Slope $=\tan\theta=\tan 120°=\tan(180°-60°)$

$=-\tan 60°=-\sqrt{3}$.

(*iv*) Here $\theta=150°$

$\therefore$ Slope $=\tan\theta=\tan 150°=\tan(180°-30°)$

$=-\tan 30°=-\frac{1}{\sqrt{3}}$

Example 7: Find the slope of the line passing through the points:

(*i*) $(1, 6)$ and $(-4, 2)$

(*ii*) $(5, 9)$ and $(2, 9)$

Sol. (*i*) Let $A=(1, 6)$ and $B=(-4, 2)$

$\therefore$ Slope of $AB=\frac{2-6}{-4-1}=\frac{-4}{-5}=\frac{4}{5}$

$\left(\text{Using slope } \frac{y_2-y_1}{x_2-x_1}\right)$

(*ii*) Let $A=(5, 9), B=(2, 9)$

$\therefore$ Slope of $AB=\frac{9-9}{2-5}=\frac{0}{-3}=0$

REDUCTION OF GENERAL EQUATION TO DIFFERENT STANDARD FORMS

1. **Slope Form:** To reduce the equation $Ax+By+C=0$ to the form $y=mx+c$

Given equation is $Ax+By+C=0 \Rightarrow m=\frac{-A}{B}$,

$c=-\frac{C}{B}\ (B\neq 0)$

Note: Slope of the line $Ax+By+c=0$ is $-\frac{A}{B}$.

i.e $-\left(\frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$. y intercept the line $=-\frac{C}{B}$

2. **Intercept Form:** To reduce the equation $Ax+By+C=0$ the form $\frac{x}{a}+\frac{y}{b}=1$. This reduction is possible only when $C\neq 0$

Given equation is $Ax+By=-C$

$\Rightarrow \quad \frac{x}{\frac{-C}{A}}+\frac{y}{\frac{-C}{B}}=1$ which is the form $\frac{x}{a}+\frac{y}{b}=1$.

where $a=\frac{-C}{A}, b=-\frac{C}{B}$

3. **Normal form:** To reduce the equation $Ax+By+C=0$ to the form $x\cos\alpha+y\sin\alpha=p$

Given equation is $Ax+By+C=0$ or, $Ax+By=-C$

Case I: When $-C>0$, then normal form is $\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=\frac{-C}{\sqrt{A^2+B^2}}$

where $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\frac{B}{\sqrt{A^2+B^2}}$;

$p=\frac{-C}{\sqrt{A^2+B^2}}$

Case II: When $-C<0$, the write the equation as $-Ax-By=C$

$\frac{-A}{\sqrt{A^2+B^2}}x+\frac{-B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$

where $\cos\alpha$

$=\frac{-A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\frac{-B}{\sqrt{A^2+B^2}}$, $p=\frac{C}{\sqrt{A^2+B^2}}$

Note: In the normal form $x\cos\alpha+y\sin\alpha=p$, p is always taken as positive.

TRAIN YOUR BRAIN

Example 8: Find the equation of a line passing through $(2, -3)$ and inclined at an angle of 135° with the positive direction of x-axis.

Sol. Here, m = slope of the line = $\tan 135° = \tan(90° + 45°) = -\cot 45° = -1$, $(x_1, y_1) = (2, -3)$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or $y + 3 = -x + 2$ or $x + y + 1 = 0$

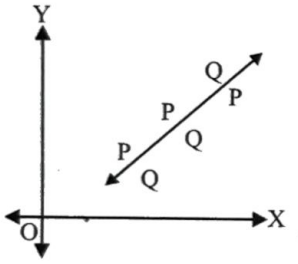
Example 9: Find the equation of a line with slope -1 and cutting off an intercept of 4 units on negative direction of y-axis.

Sol. Here $m = -1$ and $c = -4$. So, the equation of the line is $y = mx + c$ i.e. $y = -x - 4$ or $x + y + 4 = 0$

POSITION OF A POINT WITH RESPECT TO A LINE

$L : Ax + By + C = 0$

(*i*) If the points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lies on the same side of the line $Ax + By + C = 0$ then the expressions $Ax_1 + By_1 + C$ & $Ax_2 + By_2 + C$ have same sign otherwise if P and Q lies on opposite side then $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will have opposite sign.



(*ii*) If only one point is given then position of that point is checked w.r. to origin.

TRAIN YOUR BRAIN

Example 10: Find the range of θ in the interval $(0, \pi)$ such that the points $(3, 5)$ and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y - 1 = 0$

Sol. Here $3 + 5 - 1 = 7 > 0$.

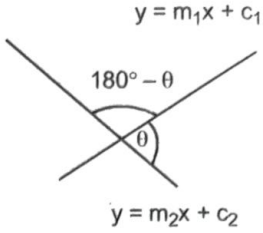
Hence $\sin\theta + \cos\theta - 1 > 0$

$$\Rightarrow \quad \sin\left(\frac{\pi}{4} + \theta\right) > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\pi}{4} < \frac{\pi}{4} + \theta < \frac{3\pi}{4} \Rightarrow 0 < \theta < \frac{\pi}{2}$$

ANGLE BETWEEN TWO STRAIGHT LINES

If θ is the acute angle between two lines, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$



where m_1 and m_2 are the slopes of the two lines and are finite.

Notes:

(*i*) If the two lines are perpendicular to each other then $m_1 m_2 = -1$

(*ii*) Any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$

(*iii*) If the two lines are parallel or coincident, then $m_1 = m_2$

(*iv*) Any line parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$

(*v*) If any of the two lines is perpendicular to x-axis, then the slope of that line is infinite.

Let $m_1 = \infty$,

Then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$

or $\theta = |90° - \alpha|$, where $\tan\alpha = m_2$

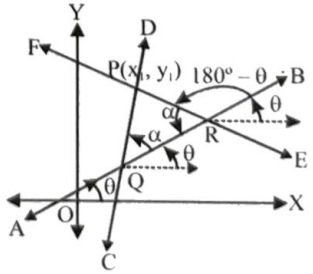
i.e. angle θ is the complimentary to the angle which the oblique line makes with the x-axis.

(*vi*) If lines are equally inclined to the coordinate axis then $m_1 + m_2 = 0$

- **Equation of straight lines passing through a given point and making a given angle with a given line.**

The equation of the straight lines which pass through a given point (x_1, y_1) and makes an angle α with the given straight line $y = mx + c$ are $y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$,

Where $m = \tan\theta$.

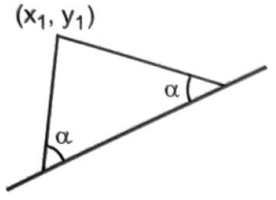


Equation of straight Lines passing through a given point and equally inclined to a given line :

Let the straight passing through the point (x_1, y_1) and make equal angles with the given straight line $y = mx + c$. If m is the slope of the required line and α is the angle which this line makes with the given line

then $\tan\alpha = \pm\dfrac{m_1 - m}{1 + m_1 m}$

The above expression for tan α, given two values of m, say m_A and m_B.



The required equations of the lines through the point (x_1, y_1) and making equal angles α with the given line are $y - y_1 = m_A(x - x_1), y - y_1 = m_B(x - x_1)$

TRAIN YOUR BRAIN

Example 11: Find the equations of the straight lines passing through the point (2, 3) and inclined at $\pi/4$ radians to the line $2x + 3y = 5$.

Sol. Let the line $2x + 3y = 5$ make an angle θ with positive x-axis.

Then $\tan\theta = -\dfrac{2}{3}$

Now $\tan\theta . \tan\dfrac{\pi}{4} = -\dfrac{2}{3}\times 1 = -\dfrac{2}{3} \neq \pm 1$

Slopes of required lines are $\tan\left(\theta + \dfrac{\pi}{4}\right)$ and $\tan\left(\theta - \dfrac{\pi}{4}\right)$

$\therefore$

$$\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\theta\tan\left(\frac{\pi}{4}\right)} = \frac{\left(-\frac{2}{3}\right) + 1}{1 - \left(-\frac{2}{3}\right)(1)} = \frac{1}{5}$$

and $$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\theta\tan\left(\frac{\pi}{4}\right)} = \frac{\left(-\frac{2}{3}\right) - 1}{1 + \left(-\frac{2}{3}\right)(1)} = -5$$

$\therefore$ Equations of required lines are

$$y - 3 = \frac{1}{5}(x - 2) \quad \text{and} \quad y - 3 = -5(x - 2)$$

i.e., $x - 5y + 13 = 0$ and $5x + y - 13 = 0$

DISTANCE BETWEEN POINT & LINE AND TWO PARALLEL LINES

1. **Length of the Perpendicular from a Point on a Line:**

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

The length of the perpendicular from origin on $ax + by + c = 0$ is $\left|\dfrac{c}{\sqrt{a^2 + b^2}}\right|$

TRAIN YOUR BRAIN

Example 12: Find the point on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3

Sol. Let the point on y-axis be $P(0, k)$

Distance of $P(0, k)$ from $4x - 3y - 12 = 0$ is

$$\left|\frac{4(0) - 3(k) - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$$

$\Rightarrow \quad |3k + 12| = 15$

$\Rightarrow \quad |k + 4| = 5$

$\Rightarrow \quad k = 1, -9$

Hence the points are (0, 1) & (0, –9)

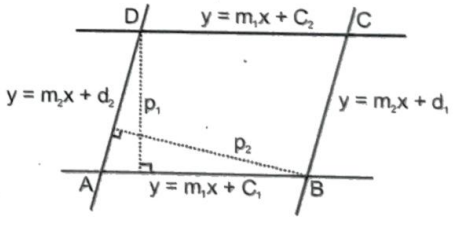
2. **The distance between two parallel lines:**

The distance between two parallel lines :

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 - b^2}}$

3. **Area of parallelogram with given sides :**

$$\boxed{Area = \left|\frac{(C_1 - C_2)(d_1 - d_2)}{m_1 - m_2}\right|}$$

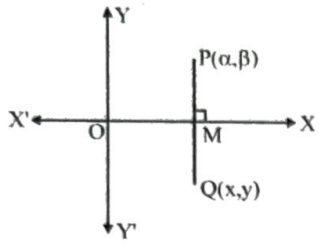


4. **Condition of parallelogram as shown becomes a rhombus:**

$$\boxed{p_1 = p_2 \Rightarrow \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2}}\right|}$$

IMAGE OR REFLECTION OF A POINT IN DIFFERENT CASES

1. **The image or reflection of a point with respect to x-axis:** Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about x-axis, then (M is the mid point of P and Q)

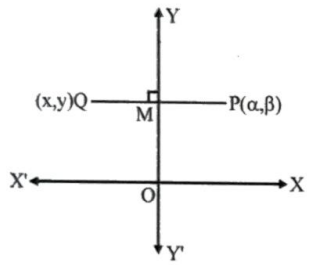


$x = \alpha;\ y = -\beta$

$\therefore\ Q \equiv (\alpha, -\beta)$ i.e., sign change of ordinate.

Note : The image of the line $ax + by + c = 0$ about x-axis is $ax - by + c = 0$

2. **The image or reflection of a point with respect to y-axis:** Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about y-axis, then (M is the mid point of PQ)

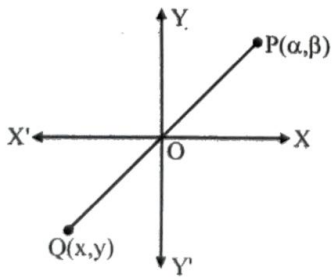


$x = -\alpha$ and $y = \beta$

$\therefore\ Q \equiv (-\alpha, \beta)$ i.e., sign change of abscissae.

Note: The image of the line $ax + by + c = 0$ about y-axis is $-ax + by + c = 0$

3. **The image or reflection of a point with respect to origin:** Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the origin (O is the mid point of PQ), then



$x = -\alpha$ and $y = -\beta$

$\therefore\ Q \equiv (-\alpha, -\beta)$

i.e., sign change of abscissae and ordinate.

4. **Foot of the perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

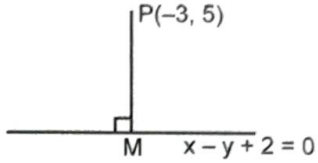
5. **The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

TRAIN YOUR BRAIN

Example 13: Find the foot of perpendicular of the line drawn from $P(-3, 5)$ on the line $x - y + 2 = 0$.

Sol. Slope of $PM = -1$



$\therefore$ Equation of PM is

$$x + y - 2 = 0 \qquad \ldots(i)$$

Solving equation (i) with $x - y + 2 = 0$, we get co-ordinates of $M(0, 2)$

TRAIN YOUR BRAIN

Example 14: Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line $2x - 3y = 7$.

Sol. Given line is $2x - 3y = 7$ ...(1)

$$\Rightarrow \quad 3y = 2x - 7 \quad \Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}$$

$\therefore$ Slope of (1) is 2/3

The required line is parallel to (1), so its slope is also 2/3, y-intercept of required line = 4

$\therefore$ By using $y = mx + c$ form, the equation of the required line is

$$y = \frac{2}{3}x + 4 \quad \text{or} \quad 2x - 3y + 12 = 0$$

BISECTORS OF THE ANGLES BETWEEN TWO GIVEN LINES

The equation of the two bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

If the two given lines are not perpendicular i.e. $a_1 a_2 + b_1 b_2 \neq 0$, then one of these equation is the equation of the bisector of the acute angle and the other that of the obtuse angle.

Note: Whether both lines are perpendicular or not but the angular bisectors of these lines will always be mutually perpendicular.

The equation of the bisector of the angle containing the origin

Write the equations of the two lines so that the constants c_1 and c_2 become positive. Then the equation $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector containing the origin.

Notes:

(*i*) If $a_1a_2+b_1b_2<0$, then the origin will lie in the acute angle and if $a_1a_2+b_1b_2>0$, then origin will lie in the obtuse angle.

(*ii*) The note (*i*) is helpful in finding the equation of bisector of the obtuse angle or acute angle directly.

The equation of the bisector of the angle which contains a given point

The equation of the bisector of the angle between the two lines containing the point (α, β) is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

or $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right).$$

according as $a_1\alpha+b_1\beta+c_1$ and $a_2\alpha+b_2\beta+c_2$ are of the same sings or of opposite signs.

TRAIN YOUR BRAIN

Example 15: For the straight line $4x+3y-6=0$ and $5x+12y+9=0$, find the equation of the

(*i*) bisector of the obtuse angle between them.

(*ii*) bisector of the acute angle between them.

(*iii*) bisector of the angle which contains (1, 2).

Sol. Equations of bisectors of the angles between the given lines are $\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x-7y-41=0$ and $7x+9y-3=0$.

If θ is the acute angle between the line $4x+3y-6=0$ and the bisector $9x-7y-41=0$, then

$$\tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(*i*) The bisector of the obtuse angle is $9x-7y-41=0$

(*ii*) The bisector of the acute angle is $7x+9y-3=0$

(*iii*) The bisector of the angle containing the origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow 7x+9y-3=0$

(*i*) For the point (1, 2), $4x+3y-6 = 4\times1+3\times2-6>0$

$\Rightarrow 5x+12y+9 = 5\times1+12\times2+9>0$

Hence equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13} \Rightarrow 9x-7y-41=0$$

CONDITION OF CONCURRENCY

Three lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Alternatively : If three constants A, B & C (not all zero) can be found such that

$A(a_1x+b_1y+c_1)+B(a_2x+b_2y+c_2)+C(a_3x+b_3y+c_3)\equiv0$, then the three straight lines are concurrent.

Example 16: Prove that the straight lines $4x+7y=9$, $5x-8y+15=0$ and $9x-y+6=0$ are concurrent.

Sol. Given lines are

$4x+7y-9=0$ ...(1)

$5x-8y+15=0$ ...(2)

and $9x-y+6=0$ ...(3)

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix}$$

$= 4(-48+15)-7(30-135)-9(-5+72) = -132+735-603 = 0$

Hence lines (1), (2) and (3) are concurrent.

OPTICS BASED PROBLEM

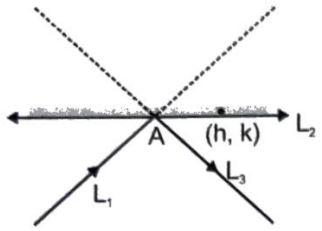
The equation of reflected ray:

Let $L_1 \equiv a_1x+b_1y+c_1=0$ be the incident ray in the line mirror $L_2 \equiv a_2x+b_2y+c_2=0$

Let L_3 be the reflected ray from the line L_2. Clearly L_3 will be one of the bisectors of the angles between L_1 and L_2. Since L_3 passes through A, so $L_3 \equiv L_1+\lambda L_2=0$.

Let (h, k) be a point on L_2. Then,

$$\frac{|a_1h+b_1k+c_1|}{\sqrt{a_1^2+b_1^2}} = \frac{|a_1h+b_1k+c_1+\lambda(a_2h+b_2k+c_2)|}{\sqrt{(a_1+\lambda a_2)^2+(b_1+\lambda b_2)^2}}$$



Since (h, k) lies on L_2, $a_2h + b_2k + c_2 = 0$

$\Rightarrow \quad a_1^2 + a_2^2\lambda^2 + 2a_1a_2\lambda + b_1^2 + b_2^2\lambda^2 + 2b_1b_2\lambda + a_1^2 + b_1^2$

$\Rightarrow \quad \lambda = 0$ or $\lambda = \dfrac{-2(a_1a_2+b_1b_2)}{a_2^2+b_2^2}$

But $\lambda = 0$ gives $L_3 = L_1$.

Hence $L_3 = L_1 - \dfrac{2(a_1a_2+b_1b_2)}{a_2^2+b_2^2} L_2 = 0$.

Note: Some times the reflected ray L_3 is also called the mirror image of L_1 in L_2.

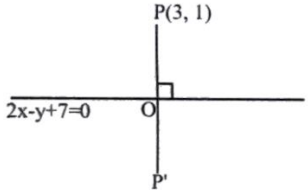
TRAIN YOUR BRAIN

Example 17: Find the image of (3, 1) across the line $y = 2x + 7$.

Sol. Let the point $P(3, 1)$ has image P' across the line $2x - y + 7 = 0$

Now PP' is perpendicular to $2x - y + 7 = 0$

Slope of $PP' = -\dfrac{1}{2}$



Equation of PP' is $y - 1 = -\dfrac{1}{2}(x-3)$

$$2y - 2 = -x + 3 \quad \Rightarrow \quad x + 2y = 5$$

Point of intersection of lines $2x - y + 7 = 0$ and $x + 2y = 5$ is

$$O\left(\frac{-9}{5}, \frac{17}{5}\right)$$

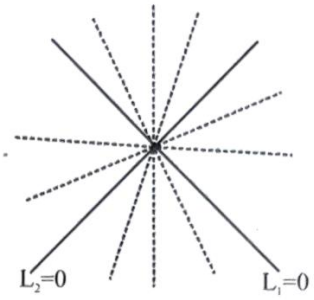
O is the mid point of PP'

Let $P'(h, k)$, $\dfrac{h+3}{2} = \dfrac{-9}{5}$, $\dfrac{k+1}{2} = \dfrac{17}{5}$

$$h = \frac{-33}{5}, k = \frac{29}{5} \Rightarrow \text{Image} \left(-\frac{33}{5}, \frac{29}{5}\right)$$

FAMILY OF LINES

The general equation of the family of lines through the point of intersection of two given lines is $L + \lambda L' = 0$, where $L = 0$ and $L' = 0$ are the two given lines, and λ is a parameter.



Conversely, any line of the form $L_1 + \lambda L_2 = 0$ passes through a fixed point which is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.

The family of lines perpendicular to a given line $ax + by + c = 0$ is given by, $bx - ay + k = 0$, where k is a parameter. The family of lines parallel to a given line $ax + by + c = 0$ is given by $ax + by + k = 0$, where k is a parameter.

TRAIN YOUR BRAIN

Example 18: Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin pass through a fixed point. Find that the point.

Sol. Let the equation of chord be $lx + my = 1$. So equation of pair of straight line joining origin to the points of intersection of chord and curve.

$3x^2 - y^2 - 2x(lx + my) + 4y(lx + my) = 0$, which subtends right angle at origin.

$\Rightarrow \quad (3 - 2l + 4m - 1) = 0$

$\Rightarrow \quad l = 2m + 1.$

Hence chord becomes $(2m + 1)x + my = 1$

$$x - 1 + m(2x + y) = 0$$

$$L_1 \qquad L_2$$

Which will pass through point of intersection of $L_1 = 0$ and $L_2 = 0$.

$\Rightarrow \quad x = 1, y = -2$. Hence fixed point is $(1, -2)$.

One Parameter Family of Straight Lines

If a linear expression L_1 contains an unknown coefficient, then the line $L_1 = 0$ can not be a fixed line. Rather it represents a family of straight lines known as one parameter family of straight lines. e.g. family of lines parallel to the x-axis i.e. $y = c$ and family of straight lines passing through the origin i.e. $y = mx$.

Each member of the family passes a fixed point. We have two methods to find the fixed point.

Method (1) : Let the family of straight lines of the form $ax + by + c = 0$ where a, b, c are variable parameters satisfying the condition $al + bm + cn = 0$, where l, m, n, are given and $n \neq 0$. Rewriting the condition as $a\left(\frac{l}{n}\right) + b\left(\frac{m}{n}\right) + c = 0$ and comparing with the given family of straight lines, we find that each member of it passes through the fixed point $\left(\frac{l}{n}, \frac{m}{n}\right)$

Example 19: If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Sol. Let n given points be (x_1, y_1) where $i = 1, 2 \ldots n$ and the variable line is $ax + by + c = 0$, Given that $\sum_{i=1}^{n}\left(\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right) = 0$.

$$\Rightarrow \quad a\Sigma x_1 + a\Sigma y_1 + cn = 0$$

$$\Rightarrow \quad a\frac{\sum x_i}{n} + b\frac{\sum y_i}{n} + c = 0.$$

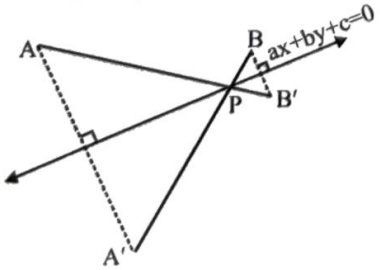
Hence the variable straight line always passes through the fixed point $\left(\frac{\sum x_i}{n}, \frac{\sum y_i}{n}\right)$.

Method (2): If a family of straight lines can be written as $L_1 + \lambda L_2 = 0$ where L_1, L_2 are two fixed lines and λ is a parameter, then each member of it will pass through a fixed point given by point of intersection of $L_1 = 0$ and $L_2 = 0$.

Note: If $L_1 = 0$ an $L_2 = 0$ are parallel lines, they will meet at infinity.

OPTIMIZATION (MINIMIZATION OR MAXIMIZATION)

1. **Minimization:** Let A and B are two given points on the same side of $ax + by + c = 0$ Suppose we want to determine a point P on $ax + by + c = 0$ such that $PA + PB$ is minimum. Then find the image of A or B about the line $ax + by + c = 0$ (say A' or B') then join B' with A or A' with B wherever it intersects $ax + by + c = 0$ is the required point.



$\therefore \quad PA + PB = PA + PB'$

or $\quad PA + PB = PA' + PB$

Note: By triangle in equality

Sum of two sides of a triangle > Third side

i.e., $|PA + PB| = |PA + PB'| = |AB'|$ (minimum value).

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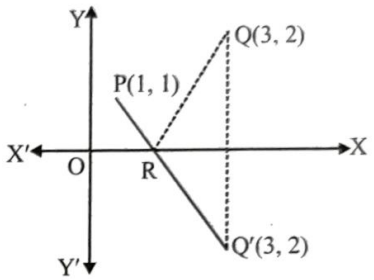
Example 20: Find a point R on the x-axis such that $PR + RQ$ is the minimum when $P = (1, 1)$ and $Q = (3, 2)$.

Sol.



Since P and Q lie on the same side of x-axis.

The image of $Q(3, 2)$ about x-axis is $Q'(3, -2)$ then the equation of line PQ' is

$$y - 1 = \frac{-2-1}{3-1}(x-1) \quad \Rightarrow \quad 3x + 2y - 5 = 0$$

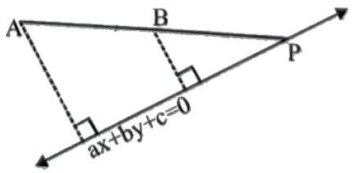
This line meets x-axis at $R\left(\frac{5}{3}, 0\right)$ which is the required point.

***(ii)* Maximization :** Let A and B are two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $|PA - PB|$ is maximum, then find the equation of line AB wherever it intersects $ax + by + c = 0$ is the required point.

Note: By triangle inequality

Difference of two sides of a triangle < Third side

i.e., $|PA - PB| = |AB|$ (maximum value)



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Example 21: Find a point P on the line $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum where A is $(4, 2)$ and B is $(2, 4)$.

Sol. Let $L(x, y) = 3x + 2y + 10$

$\therefore$ $L(4, 2) = 12 + 4 + 10 = 26$

and $L(2, 4) = 6 + 8 + 10 = 24$

$\therefore$ A and B lie on the same side of the line

$$3x + 2y + 10 = 0 \quad ...(1)$$

Equation of line AB is $y - 2 = \frac{4-2}{2-4}(x-4)$

or $x + y - 6 = 0$ ...(2)

This line (2) meets (1) at $P = (-22, 28)$ which is the required point.

PAIR OF STRAIGHT LINES

The combined equation of pair of straight lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ i.e. $L_1L_2 = 0$. Opening the brackets and comparing the terms with the terms of general equation of 2nd degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we can get all the following results for a pair of straight lines.

The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 \geq ab$.

$\Rightarrow$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$.

The homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin if $h^2 \geq ab$.

If the lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$, then $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$ and $y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$ are identical, so that

$$m_1 + m_2 = -\frac{2h}{b}, \; m_1m_2 = \frac{a}{b}$$

Angle between pair of straight lines: If θ be the angle between two lines, through the origin, then

$$\tan\theta = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

Lines are perpendicular if $a + b = 0$ coincident if $h^2 = ab$.

Note: Point of intersection of the two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is obtained by solving the equations $\frac{\partial f}{\partial x} = ax + hy + g = 0$ and $\frac{\partial f}{\partial y} = hx + by + f = 0$ where $\frac{\partial f}{\partial x}$ denotes the derivative of f with respect to y, keeping x constant. The fact can be used in splitting $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ into equations of two straight lines. With the above method, the point of intersection can be found. Now only the slopes need to be determined.

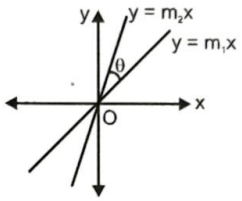
It should be noted that the line $ax + hy + g = 0$ and $hx + by + f = 0$ are not the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. These are the lines concurrent with the lines represented by given equation.

A Pair of straight lines through origin:

(*i*) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(*a*) $h^2 > ab \Rightarrow$ lines are real & distinct.

(*b*) $h^2 = ab \Rightarrow$ lines are coincident.



(*c*) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

This equation is obtained by multiplying the two equations of lines $(m_1x - y)(m_2x - y) = 0$

$\Rightarrow$ $m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

(*ii*) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \;\&\; m_1m_2 = \frac{a}{b}.$$

(*iii*) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(*iv*) The condition that these lines are :

(*a*) at right angles to each other is $a + b = 0$.

i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(*b*) coincident is $h^2 = ab$.

(*c*) equally inclined to the axis of x is $h = 0$.i.e. coeff. of $xy = 0$.

(*v*) The equation to the pair of straight lines bisecting the angles between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

Notes:

- If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, then the homogeneous equation $ax^2 + 2hxy + by^2 = 0$ also represents pair of lines passing through origin and parallel to the lines $S = 0$.
- If $h^2 = ab$, then $ax^2 + 2hxy + by^2 = 0$ represents pair of coincidental lines so that in this case, $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines.
- Procedure to Find the lines Represented by the Second-Degree General Equation

  $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

  Step 1: Factorise the homogeneous part $ax^2 + 2hxy + by^2$ and suppose $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

  Step 2: $S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

  Step 3: Equate the corresponding coefficients of x and y and also the constant terms on both sides and solve for n_1 and n_2.

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Example 22: Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 3y^2 = 0$.

Sol. We have $2x^2 - 7xy + 3y^2 = 0$.

$\Rightarrow \quad 2x^2 - 6xy - xy + 3y^2 = 0$

$\Rightarrow \quad 2x(x - 3y) - y(x - 3y) = 0$

$\Rightarrow \quad (x - 3y)(2x - y) = 0$

$\Rightarrow \quad x - 3y = 0$ or $2x - y = 0$

Thus the given equation represents the lines $x - 3y = 0$ and $2x - y = 0$. The equations of the lines passing through the origin and perpendicular to the given lines are $y - 0 = -3(x - 0)$ and $y - 0 = -\frac{1}{2}(x - 0)$ [$\because$ (Slope of $x - 3y = 0$) is 1/3 and (Slope of $2x - y = 0$) is 2]

$\Rightarrow \quad y + 3x = 0$ and $2y + x = 0$

Example 23: Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$

Here a = coeff. of $x^2 = 4$, b = coeff. of $y^2 = 11$

and $2h$ = coeff. of $xy = 24$

$\therefore \quad h = 12$

Now $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{144 - 44}}{4 + 11}\right| = \frac{4}{3}$

Where θ is the acute angle between the lines.

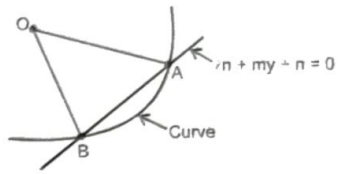
$\therefore$ acute angle between the lines is $\tan^{-1}\left(\frac{4}{3}\right)$ and obtuse angle between them is $\pi - \tan^{-1}\left(\frac{4}{3}\right)$

HOMOGENIZATION

This method is used to write the joint equation of two lines connecting origin to the points of intersection of a given line and a given second degree curve.

The equation of a pair of straight lines joining origin to the points of intersection of the line $L \equiv \ell x + my + n = 0$ and a second degree curve $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0.$

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



Note: (i) Here we have written 1 as $\frac{\ell x + my}{-n}$ and converted all terms of the curve to second degree expressions

(ii) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$, where λ & μ are parameters.

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Example 24: Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$.

Sol. Equation of the given curve is

$$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0 \qquad ...(1)$$

and equation of the given straight line is $y - 3x = 2$;

$$\therefore \quad \frac{y - 3x}{2} = 1 \qquad ...(2)$$

Making equation (1) homogeneous equation of the second degree in x any y with the help of (1), we have

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y - 3x}{2}\right) + 8y\left(\frac{y - 3x}{2}\right) - 11\left(\frac{y - 3x}{2}\right)^2 = 0$$

or $\quad x^2 + 2xy + 3y^2 + \frac{1}{2}(4xy + 8y^2 - 12x^2 - 24xy) - \frac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $\quad 4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$

or $\quad -119x^2 + 34xy + 17y^2 = 0$

or $\quad 119x^2 - 34xy - 17y^2 = 0$

or $\quad 7x^2 - 2xy - y^2 = 0 \qquad ...(3)$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$

We have $a = 7$, $b = -1$ and $2h = -2$ i.e. $h = -1$

If θ be the acute angle between pair of lines (3), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$$\therefore \theta = \tan^{-1}\frac{2\sqrt{2}}{3}.$$

BISECTORS OF THE ANGLE BETWEEN THE LINES GIVEN BY A HOMOGENEOUS EQUATION OR COMBINED EQUATION OF ANGLE BISECTOR

Theorem: The joint equation of the bisectors of the angles between the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0 \text{ is } \frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

Note: The joint equation of the bisectors is

$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

or $hx^2 - (a-b)xy - hx^2 = 0$

i.e., coeffi. of x^2 + coeffi. of $y^2 = 0$

Hence the bisectors of the angle between the lines are always perpendicular to each other.

Corollary :

1. If $a = b$, the bisectors are $x^2 - y^2 = 0$

   i.e., $x - y = 0,\ x + y = 0$

2. If $h = 0$, the bisectors are $xy = 0$

   i.e., $x = 0, y = 0$

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Example 25: Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$.

Sol. Given equation is $3x^2 - 5xy + 4y^2 = 0$

Comparing it with the equation

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots(2)$$

then $a = 3, h = -\frac{5}{2}, b = 4$

Hence equation of bisectors of the angle between the pair of the lines (1) is

$$\frac{x^2 - y^2}{3-4} = \frac{xy}{-5/2} \Rightarrow \frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$$

$$\Rightarrow 5x^2 - 2xy - 5y^2 = 0$$

Topicwise Questions

Coordinate system, Distance formula and its Application

1. The triangle formed by the lines $x+y-4=0$, $3x+y=4$, $x+3y=4$ is

(a) Isosceles (b) Equilateral

(c) Right–angled (d) Right angled Isosceles

Section formula and its and collinearity

2. The equation of the line parallel to the line $2x-3y=1$ and passing through the middle point of the line segment joining the points $(1, 3)$ and $(1, -7)$, is

(a) $2x-3y+8=0$ (b) $2x-3y=8$

(c) $2x-3y+4=0$ (d) $2x-3y=4$

3. A line passes through the point of intersection of $2x+y=5$ and $x+3y+8=0$ and parallel to the line $3x+4y=7$ is

(a) $3x+4y+3=0$ (b) $3x+4y=0$

(c) $4x-3y+3=0$ (d) $4x-3y=3$

Locus

4. A point moves so that square of its distance from the point $(3, -2)$ is numerically equal to its distance from the line $5x-12y=13$. The equation of the locus of the point is

(a) $13x^2+13y^2-83x+64y+182=0$

(b) $x^2+y^2-11x+16y+26=0$

(c) $x^2+y^2-11x+16y=0$

(d) $x^2+y^2+11x+16y+26=0$

5. Locus of the points which are at equal distance from $3x+4y-11=0$ and $12x+5y+2=0$ and which is near the origin is

(a) $21x-77y+153=0$ (b) $99x+77y-133=0$

(c) $7x-11y=19$ (d) $99x+77y+133=0$

6. A point moves such that the area of the triangle formed by it with the points (1, 5) and (3, –7) is 21*sq. unit*. The locus of the point is

(a) $6x+y-32=0$ (b) $6x-y+32=0$

(c) $x+6y-32=0$ (d) $6x-y-32=0$

Different forms of a straight line, Parametire form

7. The equations of the lines which cuts off an intercept – 1 from y-axis are equally inclined to the axes are

(a) $x-y+1=0$, $x+y+1=0$

(b) $x-y-1=0$, $x+y-1=0$

(c) $x-y-1=0$, $x+y+1=0$

(d) $x+y+1=0, x+2y+1=0$

8. A line L is perpendicular to the line $5x-y=1$ and the area of the triangle formed by the line L and coordinate axes is 5. The equation of the line L is

(a) $x+5y=5$ (b) $x+5y=\pm 5\sqrt{2}$

(c) $x-5y=5$ (d) $x-5y=5\sqrt{2}$

9. The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y=x$ is

(a) $x-y=5$ (b) $x+y=5$

(c) $x+y=1$ (d) $x-y=1$

10. The equation of the lines which passes through the point $(3, -2)$ and are inclined at 60° to the line $\sqrt{3}x+y=1$

(a) $y+2=0$, $\sqrt{3}x-y-2-3\sqrt{3}=0$

(b) $x-2=0$, $\sqrt{3}x-y+2+3\sqrt{3}=0$

(c) $\sqrt{3}x-y-2-3\sqrt{3}=0$

(d) $\sqrt{3}x+y+2+3\sqrt{3}=0$

11. The intercept cut off from y–axis is twice that from x–axis by the line and line is passes through (1, 2) then its equation is

(a) $2x+y=4$ (b) $2x+y+4=0$

(c) $2x-y=4$ (d) $2x-y+4=0$

12. Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the x–axis in counterclockwise sense, is

(a) $x\sqrt{3}+y+8=0$ (b) $x\sqrt{3}-y=8$

(c) $x\sqrt{3}-y=8$ (d) $x-\sqrt{3}y+8=0$

13. The equation to the straight line passing through the point of intersection of the lines $5x-6y-1=0$ and $3x+2y+5=0$ and perpendicular to the line $3x-5y+11=0$ is

(a) $5x+3y+8=0$ (b) $3x-5y+8=0$

(c) $5x+3y+11=0$ (d) $3x-5y+11=0$

14. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

(a) 1/3 (b) 2/3

(c) 1 (d) 4/3

15. A straight line moves so that the sum of the reciprocals of its intercepts on two perpendicular lines is constant, then the line passes through

(a) A fixed point (b) A variable point

(c) Origin (d) Two variable point

Angle between lines

16. If the coordinates of the vertices A, B, C of the triangle ABC be $(-4, 2)$, $(12, -2)$ and $(8, 6)$ respectively, then $\angle B =$

(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{6}{7}\right)$

(c) $\tan^{-1}\left(\frac{5}{6}\right)$ (d) $\tan^{-1}\left(\frac{7}{6}\right)$

17. If $\frac{1}{ab'} + \frac{1}{ba'} = 0$, then lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b'} + \frac{y}{a'} = 1$ are

(a) Parallel

(b) Inclined at 60° to each other

(c) Perpendicular to each other

(d) Inclined at 30° to each other

18. The line passing through the points (3, – 4) and (–2, 6) and a line passing through (–3,6) and (9, –18) are

(a) Perpendicular

(b) Parallel

(c) Makes an angle 60° with each other

(d) Make an angle 30° with each other

19. If vertices of a parallelogram are respectively (0, 0), (1, 0), (2, 2) and (1, 2), then angle between diagonals is

(a) $\pi/3$ (b) $\pi/2$

(c) $3\pi/2$ (d) $\pi/4$

Points and a line, Position and distance of a point form the line

20. The point on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$, are

(a) (3,1),(–7,11) (b) (3,1),(7,11)

(c) (–3,1),(–7,11) (d) (1,3),(–7,11)

21. The perpendicular distance of the straight line $12x + 5y = 7$ from the origin is given by

(a) $\frac{7}{13}$ (b) $\frac{12}{13}$

(c) $\frac{5}{13}$ (d) $\frac{1}{13}$

22. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. The length of the side of the triangle is

(a) $\sqrt{3/2}$ (b) $\sqrt{2}$

(c) $\sqrt{2/3}$ (d) $\sqrt{1/3}$

23. Choose the correct statement which describe the position of the point (– 6, 2) relative to straight lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$

(a) Below both the lines (b) Above both the lines

(c) In between the lines (d) Both (a) & (b)

24. The image of a point $A(3, 8)$ in the line $x + 3y - 7 = 0$, is

(a) (–1, – 4) (b) (–3, –8)

(c) (1, – 4) (d) (3, 8)

Distance between lines of concurrenay of lines

25. The distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$ is

(a) 3/2 (b) 3/10

(c) 6 (d) 12

26. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ be concurrent, then

(a) $a^3 + b^3 + c^3 + 3abc = 0$

(b) $a^3 + b^3 + c^3 - abc = 0$

(c) $a^3 + b^3 + c^3 - 3abc = 0$

(d) $a^3 + b^3 + c^3 + 2abc = 0$

27. The straight lines $4ax + 3by + c = 0$ where $a + b + c = 0$, will be concurrent at point

(a) (4, 3) (b) (1/4, 1/3)

(c) (1/2, 1/3) (d) (1/3, 1/2)

Family of lines

28. The equation of straight line passing through point of intersection of the straight lines $3x - y + 2 = 0$ and $5x - 2y + 7 = 0$ and having infinite slope is

(a) $x = 2$ (b) $x + y = 3$

(c) $x = 3$ (d) $x = 4$

Angle Bisectors

29. The equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$, is

(a) $(4 - \sqrt{5})x - (3 - 2\sqrt{5})y + (2 - 4\sqrt{5}) = 0$

(b) $(4 + \sqrt{5})x - (3 + 2\sqrt{5})y + (2 + 4\sqrt{5}) = 0$

(c) $(4 + \sqrt{5})x + (3 + 2\sqrt{5})y + (2 + 4\sqrt{5}) = 0$

(d) $(4 - \sqrt{5})x + (3 - 2\sqrt{5})y - (2 + 4\sqrt{5}) = 0$

Optics Problem

30. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The coordinates of the point A are

(a) (13/5, 0) (b) (5/13, 0)

(c) (– 7, 0) (d) (–5/ 13, 0)

Learning Plus

1. The points $A(-4,-1)$, $B(-2,-4)$, $C(4,0)$ and $D(2,3)$ are the vertices of
   (a) square (b) rectangle
   (c) rhombus (d) none of these

2. The ratio in which the line joining the points $(3,-4)$ and $(-5,6)$ is divided by x-axis
   (a) 2:3 (b) 6:4
   (c) 3:2 (d) none of these

3. If the points $(k, 2-2k)$, $(1-k, 2k)$ and $(-k-4, 6-2k)$ be collinear, the possible values of k are
   (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$
   (c) 1 (d) None of these

4. The points with the co-ordinates $(2a, 3a)$, $(3b, 2b)$ & (c, c) are collinear
   (a) for no value of a, b, c (b) for all values of a, b, c
   (c) If $a, \frac{c}{5}, b$ are in H.P. (d) if $a, \frac{2}{5}c, b$ are in H.P.

5. The orthocentre of the triangle ABC is 'B' and the circumcentre is 'S' (a, b). If A is the origin, then the co–ordinates of C are :
   (a) $(2a, 2b)$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$
   (c) $\left(\sqrt{a^2+b^2}, 0\right)$ (d) none

6. Find the area of the triangle formed by the mid points of sides of the triangle whose vertices are $(2, 1)$, $(-2, 3)$, $(4, -3)$
   (a) 1.5 sq. units (b) 3 sq. units
   (c) 6 sq. units (d) 12 sq. units

7. The circumcentre of the triangle with vertices $(0, 0)$, $(3, 0)$ and $(0, 4)$ is
   (a) $(1, 1)$ (b) $(2, 3/2)$
   (c) $(3/2, 2)$ (d) none of these

8. The mid points of the sides of a triangle are $(5, 0)$, $(5, 12)$ and $(0, 12)$, then orthocentre of the triangle is
   (a) $(0,0)$ (b) $(0,24)$
   (c) $(10,0)$ (d) $\left(\frac{13}{3}, 8\right)$

9. Area of a triangle whose vertices are $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$ and $(-a\cos\theta, -b\sin\theta)$ is
   (a) $|ab\sin\theta\cos\theta|$ (b) $|a\cos\theta\sin\theta|$
   (c) $\left|\frac{1}{2}ab\right|$ (d) $|ab|$

10. The point A divides the join of the points $(-5, 1)$ and $(3, 5)$ in the ratio $k : 1$ and coordinates of points B and C are $(1, 5)$ and $(7, -2)$ respectively. If the area of ΔABC be 2 units, then k equals
    (a) 7, 9 (b) 6, 7
    (c) 7, 31/9 (d) 9, 31/9

11. Given the points $A(0, 4)$ and $B(0, -4)$, the equation of the locus of the point $P(x, y)$ such that $|AP - BP| = 6$ is:
    (a) $9x^2 - 7y^2 + 63 = 0$ (b) $9x^2 - 7y^2 - 63 = 0$
    (c) $7x^2 - 9y^2 + 63 = 0$ (d) $7x^2 - 9y^2 - 63 = 0$

12. The locus of the mid-point of the intercept between the axes of the variable line $x\cos\alpha + y\sin\alpha = p$, where p is constant, is
    (a) $x^2 + y^2 = 4p^2$ (b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$
    (c) $x^2 + y^2 = \frac{4}{p^2}$ (d) $\frac{1}{x^2} - \frac{1}{y^2} = \frac{2}{p^2}$

13. A variable straight line passes through a fixed point (a, b) intersecting the co–ordinates axes at A & B. If 'O' is the origin, then the locus of the centroid of the triangle OAB is :
    (a) $bx + ay - 3xy = 0$ (b) $bx + ay - 2xy = 0$
    (c) $ax + by - 3xy = 0$ (d) $ax + by - 2xy = 0$

14. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, Then locus of its centroid is
    (a) $x^2 + y^2 - 2x - 4y + 3 = 0$
    (b) $x^2 + y^2 - 2x - 4y + 1 = 0$
    (c) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
    (d) None of these

15. A stick of length 10 units rests against the floor along x-axis and a wall of a room along y-axis. If the stick begins to slide on the floor then the locus of its middle point is
    (a) $x^2 + y^2 = 2.5$ (b) $x^2 + y^2 = 25$
    (c) $x^2 + y^2 = 100$ (d) None

16. If $P(1, 0)$; $Q(-1, 0)$ & $R(2, 0)$ are three given points, then the locus of the points S satisfying the relation, $SQ^2 + SR^2 = 2\,SP^2$ is

(*a*) A straight line parallel to x-axis

(*b*) A circle passing through the origin

(*c*) A circle with the centre at the origin

(*d*) A straight line parallel to y-axis

17. Slope of line joining points $(5, 3)$ and $(k^2, k+1)$ is $1/2$, then k is

(*a*) 1 (*b*) $1+\sqrt{2}$

(*c*) $\sqrt{2}-1$ (*d*) $-1-\sqrt{2}$

18. In a ΔABC, side AB has the equation $2x + 3y = 29$ and the side AC has the equation $x + 2y = 16$. If the mid point of BC is $(5, 6)$, then the equation of BC is

(*a*) $2x+y=7$ (*b*) $x+y=11$

(*c*) $2x-y=17$ (*d*) None of these

19. In a triangle ABC, co-ordinates of A are $(1, 2)$ and the equations to the medians through B and C are $x + y = 5$ and $x = 4$ respectively. Then the co-ordinates of B and C will be

(*a*) $(-2, 7), (4, 3)$ (*b*) $(7, -2), (4, 3)$

(*c*) $(2, 7), (-4, 3)$ (*d*) $(2, -7), (3, -4)$

20. A straight line through $P(1, 2)$ is such that its intercept between the axes is bisected at P. Its equation is :

(*a*) $x+2y=5$ (*b*) $x-y+1=0$

(*c*) $x+y-3=0$ (*d*) $2x+y-4=0$

21. The line joining two points $A\,(2, 0)$ and $B\,(3, 1)$ is rotated about A in the anticlock wise direction through an angle of 15°. The equation of the line in the new position is :

(*a*) $x-\sqrt{3}y-2=0$ (*b*) $x-2y-2=0$

(*c*) $\sqrt{3}x-y-2\sqrt{3}=0$ (*d*) None

22. The equation of the line cutting an intercept of 3 on negative y-axis and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the x-axis is

(*a*) $5y-3x+15=0$ (*b*) $5y-3x=15$

(*c*) $3y-5x+15=0$ (*d*) None of these

23. The equation of a straight line which passes through the point $(-3, 5)$ such that the portion of it between the axes is divided by the point in the ratio 5 : 3 (reckoning from x-axis) will be

(*a*) $x+y-2=0$ (*b*) $2x+y+1=0$

(*c*) $x+2y-7=0$ (*d*) $x-y+8=0$

24. The equation of perpendicular bisector of the line segment joining the points $(1, 2)$ and $(-2, 0)$ is

(*a*) $5x+2y=1$ (*b*) $4x+6y=1$

(*c*) $6x+4y=1$ (*d*) None of these

25. A line is perpendicular to $3x + y = 3$ and passes through a point $(2, 2)$. Its y intercept is

(*a*) 2/3 (*b*) 1/3

(*c*) 1 (*d*) 4/3

26. The equation of the line passing through the point (c, d) and parallel to the line $ax + by + c = 0$ is

(*a*) $a(x+c)+b(y+d)=0$ (*b*) $a(x+c)-b(y+d)=0$

(*c*) $a(x-c)+b(y-d)=0$ (*d*) None of these

27. The area of triangle formed by the lines $x + y - 3 = 0$, $x-3y+9=0$ and $3x-2y+1=0$

(*a*) $\frac{16}{7}$ sq. units (*b*) $\frac{10}{7}$ sq. units

(*c*) 4 sq. units (*d*) 9 sq. units

28. The angle between the lines $y - x + 5 = 0$ and $\sqrt{3}x-y+7=0$ is

(*a*) 15° (*b*) 60°

(*c*) 45° (*d*) 75°

29. The distance of the point $(2, 3)$ from the line $2x-3y+9=0$ measured along a line $x-y+1=0$ is :

(*a*) $5\sqrt{3}$ (*b*) $4\sqrt{2}$

(*c*) $3\sqrt{2}$ (*d*) $2\sqrt{2}$

30. The position of the point $(8, -9)$ with respect to the lines $2x+3y-4=0$ and $6x+9y+8=0$ is

(*a*) point lies on the same side of the lines

(*b*) point lies on one of the lines

(*c*) point lies on the different sides of the line

(*d*) none of these

31. The lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$, are concurrent at the point :

(*a*) $\left(\frac{1}{2}, \frac{3}{4}\right)$ (*b*) $(1, 3)$

(*c*) $(3, 1)$ (*d*) $\left(\frac{3}{4}, \frac{1}{2}\right)$

32. The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through a fixed point whose co-ordinates are

(*a*) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (*b*) $\left(\frac{2}{5}, \frac{2}{5}\right)$

(*c*) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (*d*) $\left(\frac{2}{5}, \frac{3}{5}\right)$

33. Given the family of lines, $a(3x + 4y + 6) + b(x + y + 2) = 0$. The line of the family situated at the greatest distance from the point $P(2, 3)$ has equation

(*a*) $4x+3y+8=0$ (*b*) $5x+3y+10=0$

(*c*) $15x+8y+30=0$ (*d*) None

34. The base BC of a triangle ABC is bisected at the point (p, q) and the equation to the side AB & AC are $px + qy = 1$ & $qx + py = 1$. The equation of the median through A is

(a) $(p-2q)x + (q-2p)y + 1 = 0$

(b) $(p+q)x + y - 2 = 0$

(c) $(2pq-1)(px+qy-1) = (p^2+q^2-1)(qx+py-1)$

(d) None

35. The equation of the internal bisector of $\angle BAC$ of ΔABC with vertices $A(5, 2)$, $B(2, 3)$ and $C(6, 5)$ is

(a) $2x + y + 12 = 0$ (b) $x + 2y - 12 = 0$

(c) $2x + y - 12 = 0$ (d) None of these

36. The co-ordinates of foot of the perpendicular drawn on line $3x - 4y - 5 = 0$ from the point $(0, 5)$ is

(a) $(1, 3)$ (b) $(2, 3)$

(c) $(3, 2)$ (d) $(3, 1)$

37. The co-ordinates of the point of reflection of the origin $(0, 0)$ in the line $4x - 2y - 5 = 0$ is

(a) $(1, -2)$ (b) $(2, -1)$

(c) $\left(\frac{4}{5}, \frac{2}{5}\right)$ (d) $(2, 5)$

38. If the slope of one line of the pair of lines represented by $ax^2 + 10xy + y^2 = 0$ is four times the slope of the other line, then $a =$

(a) 1 (b) 2

(c) 4 (d) 16

39. The combined equation of the bisectors of the angle between the lines represented by $(x^2 + y^2)\sqrt{3} = 4xy$ is

(a) $y^2 - x^2 = 0$ (b) $xy = 0$

(c) $x^2 + y^2 = 2xy$ (d) $\frac{x^2 - y^2}{\sqrt{3}} = \frac{xy}{2}$

40. The straight lines joining the origin to the points of intersection of the line $2x + y = 1$ and curve $3x^2 + 4xy - 4x + 1 = 0$ include an angle:

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. The points which trisect the line segment joining the points (0, 0) and (9, 12) are

(a) (3, 4) (b) (8, 6)

(c) (6, 8) (d) (4, 0)

2. One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its vertices are (–3, 1) and (1, 1). Then the equations of other sides are :

(a) $7x - 4y + 25 = 0$ (b) $7x + 4y + 25 = 0$

(c) $7x - 4y - 3 = 0$ (d) $4x + 7y = 11$

3. The equation of a straight line which passes through the point (2, 1) and makes an angle of $\pi/4$ with the straight line $2x + 3y + 4 = 0$ is

(a) $x - 5y + 3 = 0$ (b) $x - 5y + 1 = 0$

(c) $5x + y - 11 = 0$ (d) $5x - y + 1 = 0$

4. The area of a triangle is 5. Two of its vertices are (2, 1) & (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex.

(a) $\left(\frac{7}{2}, -\frac{13}{2}\right)$ (b) $\left(\frac{-3}{2}, \frac{3}{2}\right)$

(c) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (d) $\left(\frac{3}{2}, \frac{3}{2}\right)$

5. The sides of a triangle are the straight line $x + y = 1$, $7y = x$ and $\sqrt{3}y + x = 0$. Then which of the following is an interior points of triangle ?

(a) circumcentre (b) centroid

(c) incentre (d) orthocentre

6. The straight lines $x + y = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ form a triangle which is

(a) isosceles (b) right angled

(c) obtuse angled (d) equilateral

7. In the xy plane, the line 'ℓ_1' passes through the point (1, 1) and the line 'ℓ_2' passes through the point (–1, 1). If the difference of the slopes of the lines is 2. Find the locus of the point of intersection of the lines ℓ_1 and ℓ_2.

(a) $y = -x^2$ (b) $y = 2 + x^2$

(c) $y = 2 - x^2$ (d) $y = x^2$

8. The points (1, 3) & (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$.

(a) Vertices are (2, 0) & (4, 4)

(b) Value of c is 4

(c) Vertices are (0, 2) & (4, 4)

(d) Value of c is –4

9. Equation of a straight line passing through the point (4, 5) and equally inclined to the lines $3x = 4y + 7$ and $5y = 12x + 6$ is

(a) $9x - 7y = 1$ (b) $9x + 7y = 71$

(c) $7x + 9y = 73$ (d) $7x - 9y + 17 = 0$

10. Straight lines $2x + y = 5$ and $x - 2y = 3$ intersect at the point A. Points B and C are chosen on these two lines such that $AB = AC$. Then the equation of a line BC passing through the point (2, 3) is

(a) $3x - y - 3 = 0$ (b) $x + 3y - 11 = 0$

(c) $3x + y - 9 = 0$ (d) $x - 3y + 7 = 0$

Comprehenssion – 1 (Q. No. 11 to 13)

Let ABC be an acute angled triangle and AD, BE and CF are its medians, where E and F are the points (3, 4) and (1, 2) respectively and centroid of ΔABC is $G(3, 2)$, then answer the following questions:

11. The equation of side AB is

(a) $2x + y = 4$ (b) $x + y - 3 = 0$

(c) $4x - 2y = 0$ (d) None of these

12. Co-ordinates of D are

(a) $(7, -4)$ (b) $(5, 0)$

(c) $(7, 4)$ (d) $(-3, 0)$

13. Height of altitude drawn from point A is (in units)

(a) $4\sqrt{2}$ (b) $3\sqrt{2}$

(c) $6\sqrt{2}$ (d) $2\sqrt{3}$

Comprehension – 2 (Q. No. 14 to 16)

Given two straight lines AB and AC whose equations are $3x + 4y = 5$ and $4x - 3y = 15$ respectively. Then the possible equation of line BC through (1, 2), such that ΔABC is isosceles, is $L_1 = 0$ or $L_2 = 0$, then answer the following questions

14. If $L_1 \equiv ax + by + c = 0$ & $L_2 \equiv dx + ey + f = 0$ where $a, b, c, d, e, f \in I$, and $a, d > 0$, then $c + f =$

(a) 1 (b) 2

(c) 3 (d) 4

15. A straight line through $P(2, c + f - 1)$, inclined at an angle of 60° with positive Y-axis in clockwise direction. The co-ordinates of one of the points on it at a distance $(c + f)$ units from point P is $(c, f$ obtained from previous question)

(a) $(2 + 2\sqrt{3}, 5)$ (b) $(3 + 2\sqrt{3}, 3)$

(c) $(2 + 3\sqrt{2}, 4)$ (d) $(2 + 3\sqrt{2}, 3)$

16. If (a, b) is the co-ordinates of the point obtained in previous question, then the equation of line which is at the distance $|b - 2a - 1|$ units from origin and make equal intercept on co-ordinate axes in first quadrant, is

(a) $x + y + 4\sqrt{6} = 0$

(b) $x + y + 2\sqrt{6} = 0$

(c) $x + y - 4\sqrt{6} = 0$

(d) $x + y - 2\sqrt{6} = 0$

17. **Match the column**

Column –I	**Column – II**
(a) Slope of line bisecting the angle between co-ordinate axes, is	(p) 3
(b) Area of Δ formed by line $3x + 4y + 12 = 0$ with co-ordinate axis is	(q) 1
(c) If the equation $2x^2 - 2xy - y^2 - 6x + 6y + c = 0$ represents a pair of lines, then 'c' is	(r) 6
(d) If distance between the pair of parallel lines $x^2 + 2xy + y^2 - 8ax - 8ay - 9a^2 = 0$ is $25\sqrt{2}$, then '$a/5$' is equal to	(s) – 1

18. Let ABC be a triangle such that the coordinates of A are (–3, 1). Equation of the median through B is $2x + y - 3 = 0$ and equation of the angular bisector of C is $7x - 4y - 1 = 0$. Then match the entries of column-I with their corresponding correct entries of column-II.

Column–I	**Column–II**
(a) Equation of the line AB is	(P) $2x + y - 3 = 0$
(b) Equation of the line BC is	(Q) $2x - 3y + 9 = 0$
(c) Equation of the line CA is	(R) $4x + 7y + 5 = 0$
	(S) $18x - y - 49 = 0$

NUMERICAL BASED QUESTIONS

19. The ordinate of a point P on the line $6x + y = 9$, which is closest to the point (–3, 1) can be expressed in the form a/b. Where $a, b \in N$ and are in lowest form, then find the value $(a + b)$.

20. If the lines $2x + ky = 3$, $5x - 2y = 3$ and $3x - y = 2$ are concurrent then the value of k is

21. Area of the triangle formed by the line $x + y = 3$ and angle bisector of pair of straight lines $x^2 - y^2 + 2y = 1$

22. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x - \lambda$, then λ is

23. A line which is parallel to $y = x$ is rotated about the point (2, 0) through angle 15° anticlockwise direction, then y intercept of the line passing through the point of intersection of new line with $y = x$ and at right angle to new line, is $a + b\sqrt{3}$, then find value of $a + b$ equals

24. The area of the triangle formed by points (a, a^2), (b, b^2), (c, c^2) is (a, b, c are three consecutive odd integers).

25. If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the side of the triangle is $\sqrt{b/a}$, then find value of $b + a$

26. If $ax^2 - y^2 + 4x - y = 0$ represents a pair of lines, then a is equal to

27. Find the area of the quadrilateral formed by the lines $|x| + |y| = 2$.

28. If the straight line drawn through the point $P(\sqrt{3}, 2)$ and inclined at an angle $\pi/6$ with the x-axis, meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q. Find the length PQ.

29. If the points of intersection of curves $C_1 = 4y^2 - \lambda x^2 - 2xy - 9x + 3$ and $C_2 = 2x^2 + 3y^2 - 4xy + 3x - 1$ subtends a right angle at origin, then find the the value of λ.

30. The portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angle at the origin and the condition in a and b is $\lambda a + b + b^2 = 0$, then find value of λ.

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. Two sides of a rhombous are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombous? [*JEE Main-2016*]

(a) $(-3, -9)$ (b) $(-3, -8)$

(c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$

2. Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point : [*JEE Main-2017*]

(a) $\left(2, \frac{1}{2}\right)$ (b) $\left(2, -\frac{1}{2}\right)$

(c) $\left(1, \frac{3}{4}\right)$ (d) $\left(1, -\frac{3}{4}\right)$

3. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is : [*JEE Main-2018*]

(a) $2x + 3y = xy$ (b) $3x + 2y = xy$

(c) $3x + 2y = 6xy$ (d) $3x + 2y = 6$

4. Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true ? [*JEE Main-2019 (January)*]

(a) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$

(b) Each line passes through the origin

(c) The lines are all parallel

(d) The lines are not concurrent

5. Let the equation of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at (1, 1), then the equation of its third side is: [*JEE Main-2019 (January)*]

(a) $122y - 26x - 1675 = 0$ (b) $26y - 61x + 1675 = 0$

(c) $122y + 26x + 1675 = 0$ (d) $26x - 122y - 1675 = 0$

6. A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line: [*JEE Main-2019 (January)*]

(a) With slope 3/2 (b) Parallel to x-axis

(c) With slope 2/3 (d) Parallel to y-axis

7. If the line $3x + 4y - 24 = 0$ intersects the x-axis at the point A and the y-axis at the point B, then the incentre of the triangle OAB, where O is the origin, is: [*JEE Main-2019 (January)*]

(a) (3, 4) (b) (2, 2)

(c) (4, 3) (d) (4, 4)

8. Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at (2, 4), then one of its vertex is: [*JEE Main-2019 (January)*]

(a) (3, 5) (b) (2, 1)

(c) (2, 6) (d) (3, 6)

9. Two vertices of a triangle are (0, 2) and (4, 3). If its orthocenter is at the origin, then its third vertex lies in which quadrant? [*JEE Main-2019 (January)*]

(a) third (b) second

(c) first (d) fourth

10. If in a parallelogram $ABCD$, the coordinates of A, B and C are respectively $(1, 2)$, $(3, 4)$ and $(2, 5)$, then the equation of the diagonal AD is: [JEE Main-2019 (January)]

(a) $5x-3y+1=0$ (b) $5x+3y-11=0$

(c) $3x-5y+7=0$ (d) $3x+5y-13=0$

11. If the straight line, $2x-3y+17=0$ is perpendicular to the line passing through the points $(7, 17)$ and $(15, \beta)$, then β equals: [JEE Main-2019 (January)]

(a) $\frac{35}{3}$ (b) -5

(c) $-\frac{35}{3}$ (d) 5

12. If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is: [JEE Main-2019 (January)]

(a) $3x-4y+25=0$

(b) $4x-3y+24=0$

(c) $x-y+7=0$

(d) $4x+3y=0$

13. A point on the straight line, $3x+5y=15$ which is equidistant from the coordiante axes will lie only in: [JEE Main-2019 (April)]

(a) 1st and 2nd quadrants (b) 4th quadrant

(c) 1st, 2nd and 4th quadrant (d) 1st quadrant

14. Let $O(0, 0)$ and $A(0, 1)$ be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4, is : [JEE Main-2019 (April)]

(a) $8x^2-9y^2+9y=18$

(b) $9x^2+8y^2-8y=16$

(c) $8x^2+9y^2-9y=18$

(d) $9x^2-8y^2+8y=16$

15. Suppose that the points (h, k), $(1, 2)$ and $(-3, 4)$ lie on the line L_1. If a line L_2 passing through the points (h, k) and $(4, 3)$ is perpendicular to L_1, then $\frac{k}{h}$ equals : [JEE Main-2019 (April)]

(a) 3 (b) $-\frac{1}{7}$

(c) $\frac{1}{3}$ (d) 0

16. Slope of a line passing through $P(2, 3)$ and intersecting the line, $x+y=7$ at a distance of 4 units from P, is [JEE Main-2019 (April)]

(a) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (b) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$

(c) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (d) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

17. If the two lines $x+(a-1)y=1$ and $2x+a^2y=1$ $(a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is: [JEE Main-2019 (April)]

(a) $\frac{2}{5}$ (b) $\frac{2}{\sqrt{5}}$

(c) $\frac{\sqrt{2}}{5}$ (d) $\sqrt{\frac{2}{5}}$

18. A rectangle is inscribed in a circle with a diameter lying along the line $3y=x+7$. If the two adjacent vertices of the rectangle are $(-8, 5)$ and $(6, 5)$, then the area of the rectangle (in sq. units) is: [JEE Main-2019 (April)]

(a) 72 (b) 84

(c) 98 (d) 56

19. Lines are drawn parallel to the line $4x-3y+2=0$, at a distance 3/5 from the origin. Then which one of the following points lies on any of these lines ? [JEE Main-2019 (April)]

(a) $\left(-\frac{1}{4}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{3}\right)$

(c) $\left(-\frac{1}{4}, -\frac{2}{3}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{3}\right)$

20. The region represented by $|x-y| \le 2$ and $|x+y| \le 2$ is bounded by a; [JEE Main-2019 (April)]

(a) square of side length $2\sqrt{2}$ units

(b) rhombus of side length 2 units

(c) square of area 16 sq. units

(d) rhombus of area $8\sqrt{2}$ sq. units

21. A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of $60°$ with the line $x+y=0$. Then an equation of the line L is: [JEE Main-2019 (April)]

(a) $(\sqrt{3}+1)x+(\sqrt{3}-1)y=8\sqrt{2}$

(b) $(\sqrt{3}-1)x+(\sqrt{3}+1)y=8\sqrt{2}$

(c) $\sqrt{3}x+y=8$

(d) $x+\sqrt{3}y=8$

22. A triangle has a vertex at $(1, 2)$ and the mid points of the two sides through it are $(-1, 1)$ and $(2, 3)$. Then the centroid of this triangle is: [JEE Main-2019 (April)]

(a) $\left(\frac{1}{3}, 1\right)$ (b) $\left(\frac{1}{3}, 2\right)$

(c) $\left(1, \frac{7}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{5}{3}\right)$

23. Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of a triangle ABC. If P is a point inside the triangle ABC such that the triangles APC, APB and BPC have equal areas, then the length of the line segment PQ, where Q is the point $\left(\frac{-7}{6}, \frac{-1}{3}\right)$, is _____.

[JEE Main-2020 (January)]

24. The locus of the mid-point of the perpendiculars drawn from points on the line, $x = 2y$ to the line $x = y$ is:

[JEE Main-2020 (January)]

(a) $3x - 2y = 0$ (b) $3x - 3y = 0$

(c) $5x - 7y = 0$ (d) $7x - 5y = 0$

25. Let two points be $A(1, -1)$ and $B(0, 2)$. If a point $P(x', y')$ be such that the area of $\Delta PAB = 5$ square units and it lies on the line, $3x + y - 4\lambda = 0$, then a value of λ is

[JEE Main-2020 (January)]

(a) 3 (b) -3

(c) 1 (d) 4

26. Let C be the centroid of the triangle with vertices $(3, -1)$, $(1, 3)$ and $(2, 4)$. Let P be the point of intersection of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$. Then the line passing through the points C and P also passes through the point [JEE Main-2020 (January)]

(a) $(7, 6)$ (b) $(-9, -7)$

(c) $(9, 7)$ (d) $(-9, -6)$

27. The set of all possible values of θ in the interval $(0, \pi)$ for which the points $(1, 2)$ and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y = 1$ is: [JEE Main-2020 (September)]

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(c) $\left(0, \frac{\pi}{4}\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

28. If a ΔABC has vertices $A(-1, 7)$, $B(-7, 1)$ and $C(5, -5)$, then its orthocentre has coordinates.

[JEE Main-2020 (September)]

(a) $(-3, 3)$ (b) $\left(-\frac{3}{5}, \frac{3}{5}\right)$

(c) $(3, -3)$ (d) $\left(\frac{3}{5}, -\frac{3}{5}\right)$

29. If the perpendicular bisector of the line segment joining the points $P(1, 4)$ and $Q(k, 3)$ has y-intercept equal to -4, then a value of k is: [JEE Main-2020 (September)]

(a) $\sqrt{14}$ (b) $\sqrt{15}$

(c) -4 (d) -2

30. A triangle ABC lying in the first quadrant has two vertices as $A(1, 2)$ and $B(3, 1)$. If $\angle BAC = 90°$, and $\text{ar}(\Delta ABC) = 5\sqrt{5}$ sq. units, then the abscissa of the vertex C is :

[JEE Main-2020 (September)]

(a) $1 + \sqrt{5}$ (b) $1 + 2\sqrt{5}$

(c) $2\sqrt{5} - 1$ (d) $2 + \sqrt{5}$

31. If the line, $2x - y + 3 = 0$ is at a distance $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ from the lines $4x - 2y + \alpha = 0$ and $6x - 3y + \beta = 0$, respectively, then the sum of all possible values of α and β is _____.

[JEE Main-2020 (September)]

32. A ray of light coming from the point $(2, 2\sqrt{3})$ is incident at an angle $30°$ on the line $x = 1$ at the point A. The ray gets reflected on the line $x = 1$ and meets x-axis at the point B. Then, the line AB passes through the point :

[JEE Main-2020 (September)]

(a) $\left(3, -\frac{1}{\sqrt{3}}\right)$ (b) $(3, -\sqrt{3})$

(c) $(4, -\sqrt{3})$ (d) $\left(4, -\frac{\sqrt{3}}{2}\right)$

33. A man starts walking from the point $P(-3, 4)$, touches the x-axis at R, and then turns to reach at the point $Q(0, 2)$. The man is walking at a constant speed. If the man reaches the point Q in the minimum time, then $50((PR)^2 + (RQ)^2)$ is equal to [JEE Main-2021 (September)]

34. Consider a triangle having vertices $A(-2, 3)$, $B(1, 9)$ and $C(3, 8)$. If a line L passing through the circumcentre of triangle ABC, bisects line BC, and intersects y-axis at point $\left(0, \frac{\alpha}{2}\right)$, then the value of real number α is

[JEE Main-2021 (July)]

35. Let $\tan\alpha, \tan\beta$ and $\tan\gamma, \alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}, n \in \mathbb{N}$ be the slopes of three line segments OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha \cos\beta \cos\gamma}\right)^2$ is equal to

[JEE Main-2021 (March)]

JEE-ADVANCED PREVIOUS YEARS

1. For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then

[JEE Advanced-2013]

(a) $a + b - c > 0$ (b) $a - b + c < 0$

(c) $a - b + c > 0$ (d) $a + b - c < 0$

2. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distance of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is

[*JEE Advanced-2014*]

3. Let $a, \lambda, \mu \in \mathbb{R}$. Consider the system of linear equations

[*JEE Advanced-2016*]

$ax + 2y = \lambda$

$3x + 2y = \mu$

Which of the following statement(s) is(are) correct?

(*a*) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.

(*b*) If $a \neq -3$, then the system has a unique solutions for all values of λ and μ.

(*c*) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(*d*) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

ANSWER KEY

Topicwise Questions

1. (*a*)	**2.** (*b*)	**3.** (*a*)	**4.** (*a*)	**5.** (*b*)	**6.** (*a*)	**7.** (*c*)	**8.** (*b*)	**9.** (*b*)	**10.** (*a*)	**11.** (*a*)	**12.** (*a*)
13. (*a*)	**14.** (*d*)	**15.** (*a*)	**16.** (*d*)	**17.** (*c*)	**18.** (*b*)	**19.** (*d*)	**20.** (*a*)	**21.** (*a*)	**22.** (*c*)	**23.** (*a*)	**24.** (*a*)
25. (*b*)	**26.** (*c*)	**27.** (*b*)	**28.** (*c*)	**29.** (*a*)	**30.** (*a*)						

Learning Plus

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*d*)	**5.** (*a*)	**6.** (*a*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*)	**11.** (*a*)	**12.** (*b*)
13. (*a*)	**14.** (*c*)	**15.** (*b*)	**16.** (*d*)	**17.** (*b*)	**18.** (*b*)	**19.** (*b*)	**20.** (*d*)	**21.** (*c*)	**22.** (*a*)	**23.** (*d*)	**24.** (*c*)
25. (*d*)	**26.** (*c*)	**27.** (*b*)	**28.** (*a*)	**29.** (*b*)	**30.** (*a*)	**31.** (*d*)	**32.** (*d*)	**33.** (*a*)	**34.** (*c*)	**35.** (*c*)	**36.** (*d*)
37. (*b*)	**38.** (*d*)	**39.** (*a*)	**40.** (*a*)								

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *c*)	**2.** (*a*, *c*, *d*)	**3.** (*a*, *c*)	**4.** (*c*, *d*)	**5.** (*b*, *c*)	**6.** (*a*, *c*)	**7.** (*c*, *d*)	**8.** (*ad*)
9. (*a*, *c*)	**10.** (*a*, *b*)	**11.** (*a*)	**12.** (*b*)	**13.** (*c*)	**14.** (*d*)	**15.** (*a*)	**16.** (*c*)

17. (*a*) → (q, s), (*b*) → (r), (*c*) → (p), (*d*) → (q, s)

18. (*a*) → (R), (*b*) → (S), (*c*) → (Q)

19. (0100)	**20.** (0001)	**21.** (0002)	**22.** (0004)	**23.** (0006)	**24.** (0008)
25. (0023)	**26.** (16)	**27.** (8)	**28.** (6)	**29.** (19)	**30.** (2)

JEE Mains & Advanced Past Years Questions

JEE-MAIN

PREVIOUS YEARS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*a*)	**5.** (*d*)	**6.** (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*b*)	**10.** (*a*)	**11.** (*d*)	**12.** (*b*)
13. (*a*)	**14.** (*b*)	**15.** (*c*)	**16.** (*c*)	**17.** (*d*)	**18.** (*b*)	**19.** (*a*)	**20.** (*a*)	**21.** (*a*)	**22.** (*b*)	**23.** [5]	**24.** (*c*)
25. (*a*)	**26.** (*d*)	**27.** (*a*)	**28.** (*a*)	**29.** (*c*)	**30.** (*b*)	**31.** (30)	**32.** (*b*)	**33.** (1250)	**34.** (9)	**35.** (144)	

JEE-ADVANCED

PREVIOUS YEARS

1. (*a*) or (*c*) or Bonus **2.** (6) **3.** (*b*, *c*, *d*)

PRAYAS 2.0 time / real prayas 2

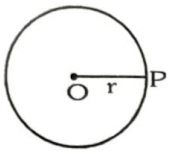
14

CIRCLE

Pravas 1 t.me/realprayasiam

DEFINITION

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre and the constant distance is called the radius of the circle. The given figure consists of a circle with centre O and radius equal to r units.

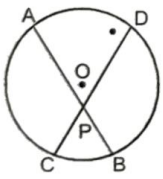


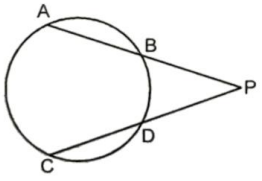
TANGENTS TO A CIRCLE

Theorem 1: (Intersecting chord Theorem.)

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the segments of one chord is equal in area to the rectangle formed by the two segments of the other chord

$$PA \times PB = PC \times PD$$

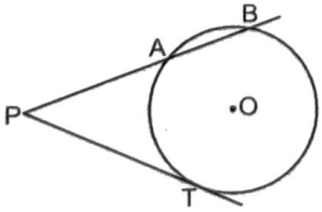




Theorem 2: (Tangent Secant Theorem.)

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, the $PA \times PB = PT^2$

OR

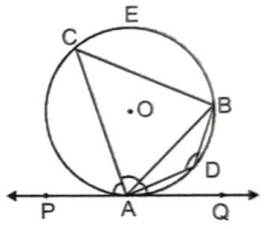


Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 3: (Alternate Segment Theorem.)

If a chord is drawn through the point of contact of tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$$\angle BAQ = \angle ACB \text{ and } \angle BAP = \angle ADB$$



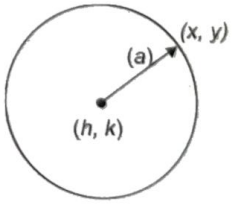
Converse: If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

EQUATION OF CIRCLE IN VARIOUS FORMS

We present below the equation of circle in various forms:

(*i*) **Circle with Centre at the point (h, k) and radius a:** The equation of the circle is $(x-h)^2 + (y-k)^2 = a^2$.

Particular Case: The equation of a circle with centre at the origin and radius a is $x^2 + y^2 = a^2$.



(ii) General equation of a circle:

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(9)$$

where g, f and c are constants.

To find the centre and radius.

Equation (*iii*) can be written as

$$(x+g)^2 + (y+f)^2 = [\sqrt{(g^2 + f^2 - c)}]^2$$

Comparing with the equation of the circle,

$h = -g, k = -f$ and $a = \sqrt{(g^2 + f^2 - c)}$.

$\therefore$ Coordinates of the centre are $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

(iii) Important remarks

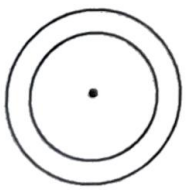
(*i*) If $g^2 + f^2 - c > 0$, equation represents real circle with centre $(-g, -f)$.

(*ii*) If $g^2 + f^2 - c = 0$, the equation represents a circle whose centre is $(-g, -f)$ and radius is zero i.e., the circle coincides with the centre and so it represents a point $(-g, -f)$. It is, therefore, called a point circle.

(*iii*) If $g^2 + f^2 < c$, the radius of the circle is imaginary. In this case the equation does not represent any real geo-metrical locus. It is better not to say that the circle does not exist, but to say that it is a circle with a real centre and an imaginary radius.

Notes:

1. Circle may exist in any quadrant hence for general cases use ± sign before h and k.
2. A general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in x, y represent a circle if
   (*i*) coefficient of x^2 = coefficient of y^2, i.e. $a = b$.
   (*ii*) coefficient of xy is zero, i.e. $h = 0$
   (*iii*) $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$
   However for a point circle (Whose radius is zero), $\Delta = 0$
3. (*i*) Two circles are said to be concentric if they have the same centre.



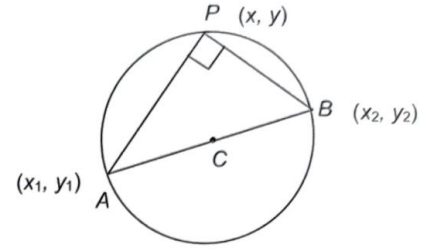
(*ii*) Equations of two concentric circles differ by a constant only.

(*iii*) The equation of the circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is of the form $x^2 + y^2 + 2gx + 2fy + k = 0$, k is an unknown constant.

(iv) Diameter form i.e., circle with the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter: Let $P(x, y)$ be any point on the circle with AB as diameter then $\angle APB = \pi/2$ (angle in semicircle). i.e., AP is $\perp$ to BP.

$$\therefore \quad \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, is the required equation.



Note:

(*i*) Equation of circle with least radius passess through (x_1, y_1) and (x_2, y_2) is diametric form with those points as extermeties of diameter.

(v) Parametric Equation of a Circle

In the right angles ΔPOA

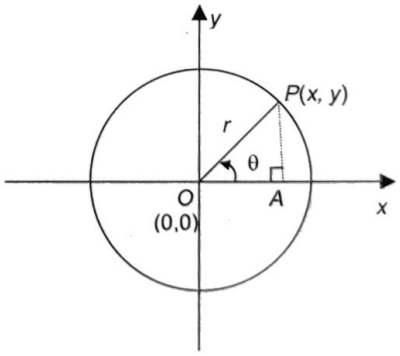
$$x = r\cos\theta, \quad y = r\sin\theta$$

where, $0 \leq \theta < 2\pi$

The point $P(r\cos\theta, r\sin\theta)$ satisfies the equation of the circle.

$\therefore \quad x = r\cos\theta$

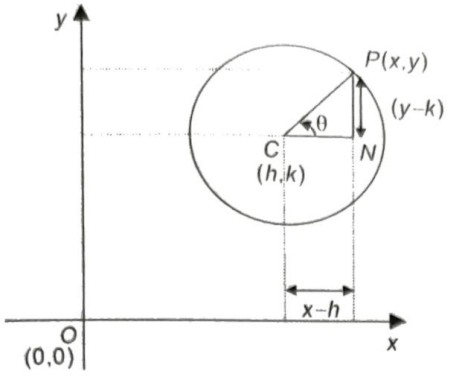
$y = r\sin\theta$ are said to be the parametric equations of the circle.



Now lets consider in the figure below. The equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \qquad ...(i)$$

where, r is the radius of the circle.



From, ΔCPN,

$$(x-h) = r\cos\theta$$

$$(y-k) = r\sin\theta$$

$$\Rightarrow \quad x = h + r\cos\theta \qquad \ldots(ii)$$

$$y = k + r\sin\theta \qquad \ldots(iii)$$

where, $0 \le \theta < 2\pi$

$P(h + r\cos\theta, k + r\sin\theta)$ satisfies equation (i)

$\therefore$ (*ii*) and (*iii*) represent the parametric equations of the circle.

Note: Parametric equation of circle is very useful when we need to write the equation for part of the circle only

Eg: If we have to find the equation of quarter of the circle then we have to restrict the range of θ from $[\alpha, \alpha + \pi/2]$ where value of α is according to the problem $(0 \le \alpha \le 3\pi/2)$.

(*vi*) Circle Through Three Points

If three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are non-collinear, an unique circle passes through A, B and C. To find the equation of this circle, we can proceed in the following way:

First method : Find the circum-centre and radius as in the lesson on 'Points and Straight lines'. We recall that:

If (α, β) be the circum-centre, then $(x_1-\alpha)^2 + (y_1-\beta)^2 = (x_2-\alpha)^2 + (y_2-\beta)^2 = (x_3-\alpha)^2 + (y_2-\beta)^2$.

Solving these, we find α, β. Also radius

$$= \sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2} = r \text{ (say)}.$$

Then equation of the circle is $(x-\alpha)^2 + (y-\beta)^2 = r^2$.

Second method: Let required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \ldots(ii)$$

Since this circle passes through $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \ldots(iii)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \ldots(iv)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad \ldots(v)$$

Solving these three, we can find g, f, c and hence the equation of the circle.

Students acquainted with determinants, can put the equation of this circle in determinant form as follows:

Eliminating g, f and c from (*ii*) and (*v*), we have

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad \ldots(vi)$$

(*vi*) is an equation of the circle passing through the points A, B and C.

INTERCEPTS CUT BY THE CIRCLE ON AXES

The intercepts cut by the circle

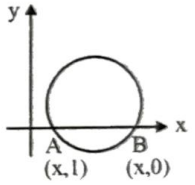
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ on :}$$

(*i*) x-axis $= \left|2\sqrt{g^2 - c}\right|$

(*ii*) y-axis $= \left|2\sqrt{f^2 - c}\right|$

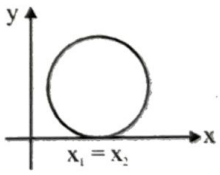
Note:

(*i*) If the circle cuts the x-axis at two distinct point then $g^2 - c > 0$



$\Rightarrow$ circle cuts the x-axis at 2 distinct points.

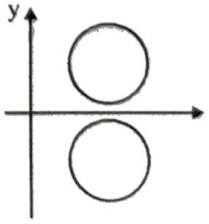
(*ii*) If circle touches x-axis then $g^2 = c$.



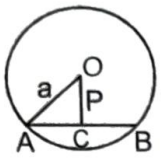
$\Rightarrow$ circle touches the x-axis

(*iii*) If circle touches y-axis then $f^2 = c$.

(*iv*) Circle lies completely above or below the x-axis then $g^2 < c$.



(v) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.



CIRCLE IN DIFFERENT CASES

(i) Circle with centre at the point (h, k) and which touches the axis of x.

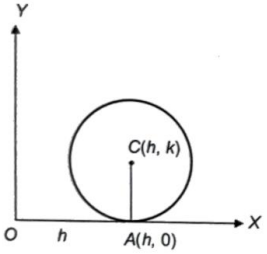
If the circle touches the axis of x, then its radius is equal to the numerical value of y-coordinate of the centre.

$\therefore$ Equation of the circle is

$$(x - h)^2 + (y - k)^2 = |k|^2 = k^2$$

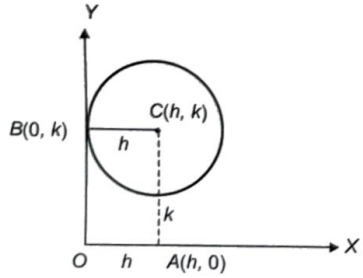
or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$ ...(i)

This circle meets the x-axis, at the points where $y = 0$.



(ii) Circle with centre at the point (h, k) and which touches the axis of y.

If the circle touches the axis of y, then its radius is equal to the numerical value of x-coordinate of the centre.



$\therefore$ Equation of the circle is given by

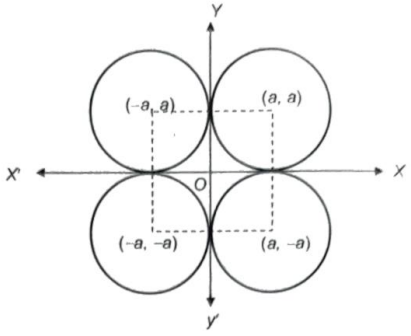
$$(x - h)^2 + (y - k)^2 = h^2$$

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$. ...(i)

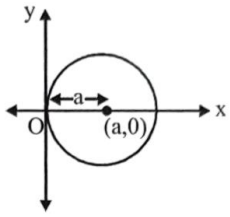
(iii) Circle which touches both the coordinate axes.

Let a be the radius of the circle. Since the centre of the circle may be in any of the four quadrants, therefore it will be any one of the four points $(\pm a, \pm a)$. Thus there are four circles of radius a touching both the coordinate axes, and their equations are

$(x \pm a)^2 + (y \pm a)^2 = a^2$ or $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$



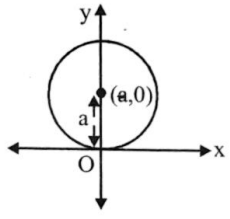
(iv) Circle passing through origin and centre lies on x-axis (Circle touching y-axis at origin)



As centre lies on x-axis so let co-ordinates of centre by $c(a, 0)$ then radius of circle is a units.

Equation of circle is $(x - a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0$

(v) Circle touching x-axis at origin



As centre of the circle lies on Y-axis, so co-ordinates of centre are $(0, a)$, and radius of circle is a units.

$\therefore$ Equation of circle is $x^2 + (y - a)^2 = a^2$

$\Rightarrow$ $x^2 + y^2 - 2ay = 0$

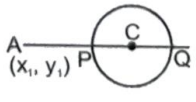
POSITION OF A POINT W.R.T. CIRCLE

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$

or $S_1 >, =, < 0$

Where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$



(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

TRAIN YOUR BRAIN

Example 1: If $P(2, 8)$ is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is

Sol. For internal point

$P(2, 8), 4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$ ...(1)

and x intercept $= 2\sqrt{1+p}$ therefore

$1 + p < 0 \Rightarrow \quad p < -1$ ...(2)

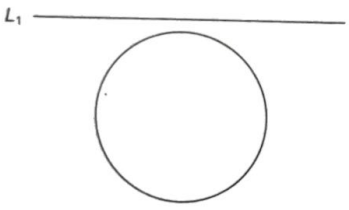
and y intercept $= 2\sqrt{4+p} \quad \Rightarrow \quad p < -4$ ...(3)

by intersection of (1), (2) and (3), $p \in \phi$

CIRCLE AND A LINE

We consider 3 cases:

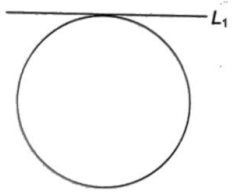
Case I:



Line L_1 does not intersect the circle

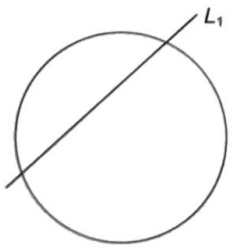
Case II:

Line L_1 touches the circle



Case III:

Line L_1 intersects the circle in two points.



Now we proceed to derive the conditions of the three cases.

For the ease of calculations we take the circle as,

$x^2 + y^2 = a^2$ ...(i)

and the equation of the line as

$y = mx + c$ ...(ii)

substituting the value of y in equation (i),

$x^2 + (mx + c)^2 = a^2$

$\Rightarrow \quad x^2(1 + m^2) + 2mcx + (c^2 - a^2) = 0$ ...(iii)

Equation (iii) represents a quadratic equation in x. The discriminant of the equation is

$\Delta = 4m^2c^2 - 4(1 + m^2)(c^2 - a^2)$

Case I:

There is no real point of intersection.

i.e., $\Delta < 0$

$\Rightarrow \quad a^2 < \dfrac{c^2}{(1+m^2)} \Rightarrow a < \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

Case II:

There are two coincident points of intersection.

i.e., $\Delta = 0$

$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0 \Rightarrow a^2(1 + m^2) = c^2$

$\Rightarrow \quad a = \left|\dfrac{c}{\sqrt{1+m^2}}\right| \quad \Rightarrow \quad c = \pm a\sqrt{1+m^2}$

This is the required condition of tangency.

$\therefore$ the equation of the tangents are

$y = mx + a\sqrt{1+m^2}$ and $y = mx - a\sqrt{1+m^2}$

Case III:

There are two different real points of intersection.

i.e., $\Delta > 0$

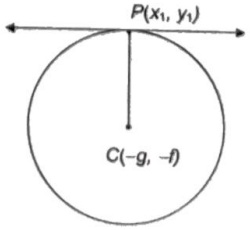
$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0.$

$\therefore \quad a^2(1 + m^2) > c^2$ or $a > \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

TANGENT LINE OF CIRCLE

For the Circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Consider a point $P(x_1, y_1)$ on the circle, where the tangent touches the circle



Equation of the tangent at P is,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 .$$

$\therefore$ We conclude that in order to write the equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we put,

- xx_1 in place of x^2
- yy_1 in place of y^2
- $\frac{(x + x_1)}{2}$ in place of x
- $\frac{(y + y_1)}{2}$ in place of y

Note : Let $P(x_1, y_1)$ be any point. If we replace x^2 by xx_1, y^2 by yy_1, x by $\frac{1}{2}(x + x_1)$ and y by $\frac{1}{2}(y + y_1)$ in the left hand side of the equation of circle, we get $T = 0$.

Point form

(*i*) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(*ii*) Parametric form

1. The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is $x \cos t + y \sin t = a$
2. The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

(*iii*) Slope form

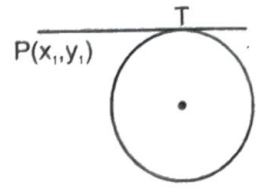
If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and point of contacts are $\left(\mp\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\frac{a^2m}{c}, \pm\frac{a^2}{c}\right)$ and equation of tangent is

$$y = mx \pm a\sqrt{1+m^2}$$

(*iv*) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is

$$(y-k) = m(x-h) \pm a\sqrt{1+m^2}$$

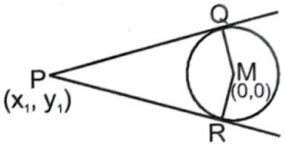
(*c*) **Length of tangent ($\sqrt{S_1}$) :** The length of tangent draw from point (x_1, y_1) out side the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,



$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Note: When we use this formula the coefficient of x^2 and y^2 must be 1.

(*d*) **Equation of Pair to tangents ($SS_1 = T^2$) :** Let the equation of circle $X \equiv x^2 + y^2 = a^2$ and $P(x_1, y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS_1 = T^2$



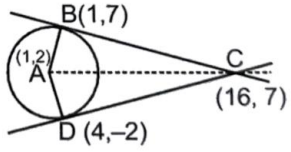
TRAIN YOUR BRAIN

Example 2: Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and $B(1, 7)$ and $D(4, -2)$ are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral $ABCD$ is

Sol. $x.1 + y.7 - 1(x+1) - 2(y+7) - 20 = 0$ or $y = 7$ ...(*i*)

Tangent at $D(4, -2)$ is $3x - 4y - 20 = 0$ ...(*ii*)

Solving (*i*) and (*ii*), C is (16, 7)



Area $ABCD = AB \times BC = 5 \times 15 = 75$ units.

Example 3: Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1).

Sol. Given circle is $S \equiv x^2 + y^2 - 2x + 4y = 0$ ...(*i*)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5 > 0$

Point P lies outside the circle

and $T \equiv x.0 + y.1 - (x + 0) + 2(y + 1)$

i.e., $T = -x + 3y + 2$

Now equation of the pair of tangents from $P(0, 1)$ to circle (i) is

$$SS_1 = T^2$$

or $5(x^2+y^2-2x+4y) = (-x+3y+2)^2$

or $5x^2+5y^2-10x+20y = x^2+9y^2+4-6xy-4x+12y$

or $4x^2-4y^2-6x+8y+6xy-4=0$

or $2x^2-2y^2+3xy-3x+4y-2=0$ ...(ii)

From (ii), $2x^2+3(y-1)x-(2y^2-4y+2)=0$

$$\therefore \quad x = \frac{3(y-1)\pm\sqrt{9(y-1)^2+8(2y^2-4y+2)}}{4}$$

or $4x-3y+3 = \pm\sqrt{25y^2-50y+25} = \pm 5(y-1)$

$\therefore$ Separate equations of tangents are

$4x-8y+8=0$ i.e., $x-2y+2=0$

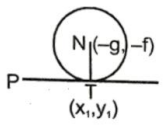
and $4x+2y-2=0$ or $2x+y-1=0$.

NORMAL OF CIRCLE

Normal at a point of the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

(*a*) Equation of normal at point (x_1, y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is

$$y-y_1 = \left(\frac{y_1+f}{x_1+g}\right)(x-x_1)$$



(*b*) The equation of normal on any point (x_1, y_1) of circle $x^2+y^2=a^2$ is $\left(\frac{y}{x}=\frac{y_1}{x_1}\right)$

(*c*) If $x^2+y^2=a^2$ is the equation of the circle then at any point 't' of this circle ($a\cos t$, $a\sin t$), the equation of normal is $x\sin t - y\cos t = 0$

TRAIN YOUR BRAIN

Example 4: Find the equation of the normal to the circle $x^2+y^2-5x+2y-48=0$ at the point $(5, 6)$.

Sol. Since normal to the circle always passes through the centre so equation of the normal will be the line passing through $(5, 6)$ & $\left(\frac{5}{2}, -1\right)$ i.e. $y+1 = \frac{7}{5/2}\left(x-\frac{5}{2}\right)$

$\Rightarrow \quad 5y+5 = 14x-35 \quad \Rightarrow 14x-5y-40=0$

Example 5: If the straight line $ax+by=2$; $a, b \neq 0$ touches the circle $x^2+y^2-2x=3$ and is normal to the circle $x^2+y^2-4y=6$, then the values of a and b are respectively

Sol. Given $x^2+y^2-2x=3$

$\therefore$ centre is $(1, 0)$ and radius is 2 and for circle $x^2+y^2-4y=6$, centre is $(0, 2)$ and radius is $\sqrt{10}$. Since line $ax+by=2$ touches the first circle

$$\therefore \quad \frac{a(1)+b(0)-2}{\sqrt{a^2+b^2}} = 2 \text{ or } (a-2) = [2\sqrt{a^2+b^2}] \qquad ...(i)$$

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore \quad a(0)+b(B)=2$ or $2b=2$ or $b=1$

Putting this value in equation (i) we get $a-2 = 2\sqrt{a^2+1^2}$ or $(a-2)^2 = 4(a^2+1)$

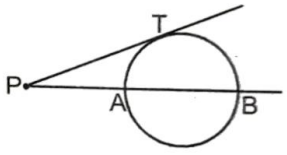
or $\quad a^2+4-4a = 4a^2+4$ or $3a^2+4a=0$

or $\quad a(3a+4)=0$ or $a=0, -\frac{4}{3}$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$ respectively according to the given choices.

POWER OF THE POINT

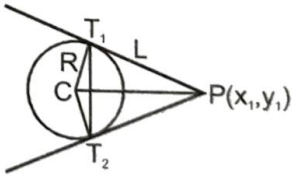
Square of the length of the tangent from the point P is defined as power of the point 'P' with respect to given circle $\Rightarrow PT^2 = S_1$



Note: Power of a point remains constant with respect to a circle $PA.PB = PT^2$

CHORD OF CONTACT

A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.



If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2+y^2+2gx+2fy+c=0$, then the equation of the chord of contact T_1T_2 is : $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$ (i.e. $T=0$ same as equation of tangent).

Note:

- For the existence of chord of contact points should be outside the circle.
- The above result is valid for all second degree conic.

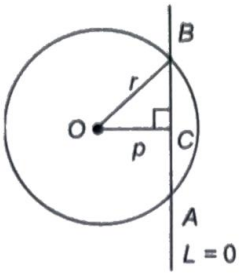
Length of the Chord

Let O be the centre of the circle $S = 0$ and the line of the chord AB be $L = 0$.

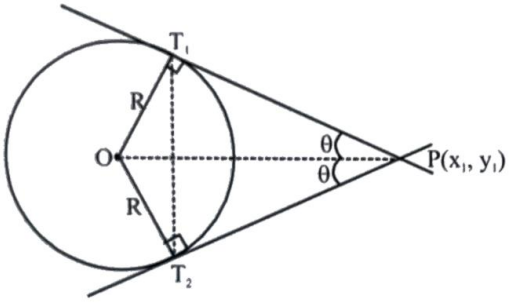
Let p = length of $\perp$ from centre to the line $L = 0$

Then AB = length of the chord $= 2\sqrt{(r^2 - p^2)}$.

(by Pythagoras theorem)



Remember :



(*a*) Angle between the pair of tangents from $P(x_1, y_1)$ is equal to 2θ

where, $\tan\theta = \frac{R}{L}$ so, $2\theta = \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$

(*b*) Length of chord of contact, $T_1T_2 = 2L\sin\theta$

(where, $\tan\theta = \frac{R}{L}$)

so, $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$.

(*c*) Area of the triangle formed by the pair of the tangents & its chord of contact

$$\text{area} = \frac{1}{2} \times T_1T_2 \times L\cos\theta$$

so, $\text{area} = \frac{RL^3}{R^2 + L^2}$

(*d*) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(since P, T_1, O, T_2 are concyclic)

(*e*) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$;

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

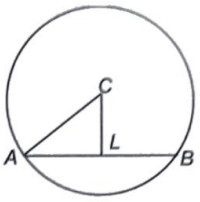
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

TRAIN YOUR BRAIN

Example 6: Find the length of the chord $4x - 3y = 5$ of the circle $x^2 + y^2 + 3x - y - 10 = 0$.

Sol. Given line is $4x - 3y - 5 = 0$ ...(*i*)

and given circle is $x^2 + y^2 + 3x - y - 10 = 0$ ...(*ii*)



Let C be the centre and a the radius of circle (*ii*), then

$C \equiv \left(-\frac{3}{2}, \frac{1}{2}\right)$ and $a^2 = \frac{9}{4} + \frac{1}{4} + 10 = \frac{25}{2}$

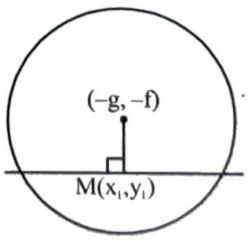
CL = length of perpendicular from C to line (*i*)

$$= \frac{\left|4\left(-\frac{3}{2}\right) - 3\left(\frac{1}{2}\right) - 5\right|}{\sqrt{4^2 + (-3)^2}} = \frac{5}{2}.$$

Let line (*i*) cut the circle (*ii*) at A and B, then,

$$AB = 2AL = 2\sqrt{a^2 - CL^2} = 2\sqrt{\frac{25}{2} - \frac{25}{4}} = 5 \text{ units.}$$

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT (T = S_1)



The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

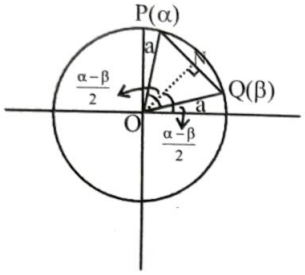
$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Note:

1. The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.
2. Chord bisected at point (x_1, y_1) is the farthest from centre among all the chords passing through the point (x_1, y_1). Also for such chord, the length of the chord is minimum.

EQUATION OF CHORD WHEN END POINTS OF CHORD GIVEN IN PARAMETRIC FORM

The equation to the straight line joining two points, α and β, on the circle $x^2 + y^2 = a^2$.



$$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos\left(\frac{\alpha-\beta}{2}\right).$$

TRAIN YOUR BRAIN

Example 7: Let a circle be given by $2x(x-a)+y(2y-b)=0$ $(a\neq 0, b\neq 0)$ Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to t he circle from $(a, b/2)$.

Sol. The given circle is $2x(x-a)+y(2y-b)=0$

or $x^2+y^2-ax-by/2=0$

Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be $M(h, 0)$.

and $S\equiv x^2+y^2-ax-by/2=0$

$\therefore$ Equation of chord AB is $T=S_1$

$$hx+0-\frac{a}{2}(x+h)-\frac{b}{4}(y+0)=h^2+0-ah-0$$

Since its passes through $(a, b/2)$ we have $ah-\frac{a}{2}(a+h)-\frac{b^2}{8}=h^2-ah$

$$\Rightarrow \quad h^2-\frac{3ah}{2}+\frac{a^2}{2}+\frac{b^2}{8}=0$$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

$\therefore \quad B^2-4AC>0$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2-4.1.\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0$$

$$\Rightarrow \quad a^2>2b^2.$$

DIRECTOR CIRCLE

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. The circle $x^2+y^2=a^2$. Then the equation of the pair of tangents is $SS_1=T^2$.

$$h^2+k^2=2a^2$$

$\therefore$ locus of (h, k) is $x^2+y^2=2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note: The director circle of $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+2c-g^2-f^2=0$

TRAIN YOUR BRAIN

Example 8: Let P be any moving point on the circle $x^2+y^2-2x=1$, from this point chord of contact is drawn w.r.t the circle $x^2+y^2-2x=0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle.

Sol. The two circles are

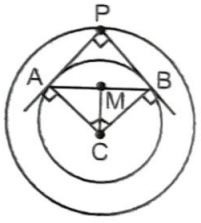
$$(x-1)^2+y^2=1 \qquad ...(i)$$

$$(x-1)^2+y^2=2 \qquad ...(ii)$$

So the second circle is the director circle of the first. So $\angle APB=\pi/2$. Also $\angle ACB=\pi/2$

Now circumcentre of the right angled triangle CAB would lie on the mid point of AB

So let the point be $M\equiv(h, k)$



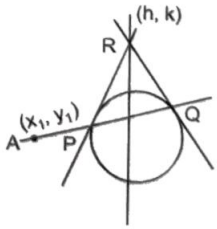
Now, $CM=CB\sin 45^\circ=\frac{1}{\sqrt{2}}$

So, $(h-1)^2+k^2=\left(\frac{1}{\sqrt{2}}\right)^2$

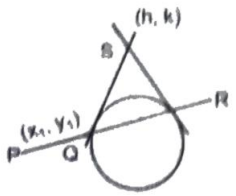
So, locus of M is $(x-1)^2+y^2=\frac{1}{2}$.

POLE AND POLAR

Let any straight line through the given point $A(x_1, y_1)$ intersect the given circle $S=0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



(a) The equation of the polar of point (x_1, y_1) w.r.t.circle $x^2 + y^2 = a^2$.



$\therefore$ locus of point $S(h, k)$ is $xx_1 + yy_1 = a^2$ which is the equation of the polar.

Note:

(*i*) The equation of the polar is, $T = 0$, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

(*ii*) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(*iii*) If point is inside the circle then chord of contact does not exist but polar exists.

(*iv*) If point lies on the circle then polar, chord of contact and tangent on that point are same.

(*v*) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P and Q are conjugate points of each other w.r.t. the given circle.

(*vi*) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

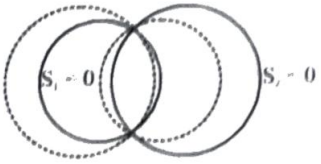
(b) Pole of a given line with respect to a circle.

The pole of $\ell x + my + n = 0$,

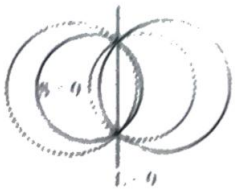
w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$.

FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is ; $S_1 + KS_2 = 0$ $(K \neq 1)$.



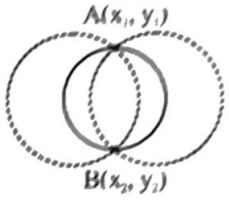
(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ and line $L = 0$ is given by $S + KL = 0$.



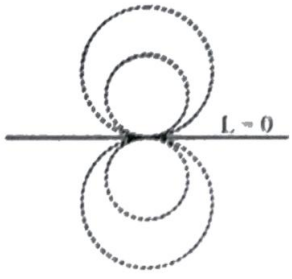
(c) The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

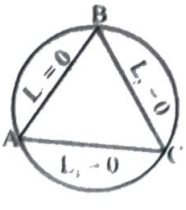
where K is a parameter.



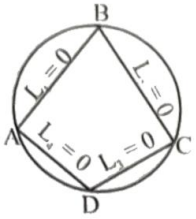
(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.



(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2



(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the line $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



(g) Equation of circle (if exist) passing through point of intersection of two straight lines $L_1 = 0$ and $L_2 = 0$ with co-ordinate axes in $L_1L_2 + \lambda xy = 0$

(h) Equation of circle passing through four points of intersection of two second degree conics

$S = 0$ and $S' = 0$ is $S + \lambda Ss' = 0$

TRAIN YOUR BRAIN

Example 9: The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is

Sol. Family of circle is $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$$(1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + (1-\lambda) = 0$$

or $$x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

Centre is $\left[\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right]$ and

radius $= \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{1+\lambda}$

Since it touches the line $x + 2y = 0$, hence,

Radius = Perpendicular distance from centre to the line.

i.e., $$\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{1+\lambda} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2}$$

$\Rightarrow \lambda = \pm 1$

$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$. Thus, we get the equation of circle

DIRECT AND TRANSVERSE COMMON TANGENTS

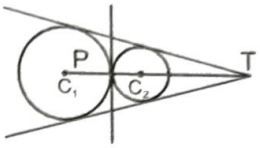
Let two circles having centre c_1 and c_2 and radii, r_1 and r_2 and c_1c_2 is the distance between their centres then :

Direct common tangent : The common tangent having centre of both circle on same side.

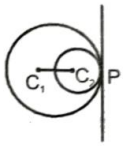
Transverse common tangents: The common tangent having centre of both circle on opposite side.

(a) Both circles will touch:

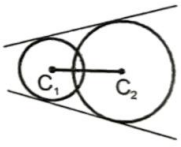
(i) Externally: If $c_1c_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally). In this case there will be **three common tangents.**



(ii) Internally: If $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be **only one common tangent.**



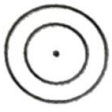
(b) The circles will intersect: When $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there will be **two common tangents.**



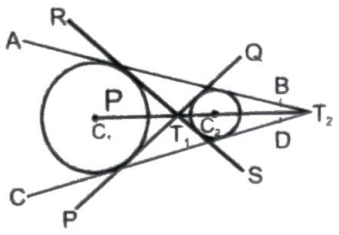
(c) The circles will not intersect:

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$

In this case there will be **no common tangent.**



(ii) When circle are apart from each other the $C_1C_2 > r_1 + r_2$ and in this case there will be four common tangents. Lines PQ and RS are called **transverse or indirect or internal common tangents** and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct or external common tangents.** These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note: Length of direct common tangent

$$= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

Length of transverse common tangent

$$= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$$

TRAIN YOUR BRAIN

Example 10: Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Sol. Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

and $x^2 + y^2 + 2by + c^2 = 0$ ...(*ii*)

Let C_1 and C_2 be the centre of circle (i) and (ii), respectively and r_1 and r_2 be their radii, then

$$C_1 = (-a, 0),\ C_2 = (0, -b),$$

$$r_1 = \sqrt{a^2 - c^2},\ r_2 = \sqrt{b^2 - c^2}$$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2 + b^2)} = |\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}|$

On squaring

$$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$$

or $$c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$$

Again squaring, $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

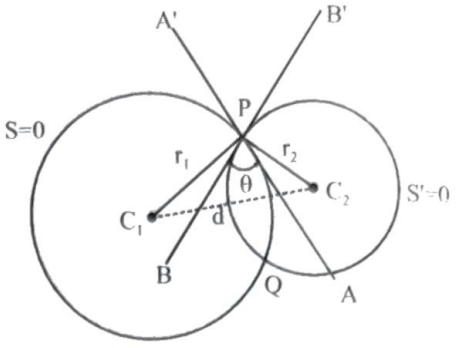
or $$c^2(a^2 + b^2) = a^2b^2 \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

ANGLE OF INTERSECTION OF TWO CIRCLES

Let the two circles $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

and $S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

intersect each other at the points P and Q. The angle θ between two circles $S = 0$ and $S' = 0$ is defined as the angle between the tangents to the two circles at the points of intersection.



C_1, C_2 and r_1, r_2 are centres and radius of circles $S = 0$ and $S' = 0$ then

$C_1 \equiv (-g_1, -f_1)$ and $C_2 \equiv (-g_2, -f_2)$

and $r_1 = \sqrt{(g_1^2 + f_1^2 - c_1)}$

and $r_2 = \sqrt{(g_2^2 + f_2^2 - c_2)}$

Let $d = |C_1C_2|$ = Distance between their centres

$$= \sqrt{(-g_1 + g_2)^2 + (-f_1 + f_2)^2}$$

$$= \sqrt{(g_1^2 + f_1^2 + g_2^2 + f_2^2 - 2g_1g_2 - 2f_1f_2)}$$

Now in ΔC_1PC_2, $\cos\alpha = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$ ...(i)

($\because \alpha + \theta + 90° + 90° = 360°$)

or $\cos(180° - \theta) = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$

($\therefore \alpha = 180° - \theta$)

If the angle between the circles is 90°, i.e., $\theta = 90°$, then the circles are said to be orthogonal circles or we say that the circles cut each other ORTHOGONALLY.

Then from (i)

$$0 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \text{or} \quad r_1^2 + r_2^2 - d^2 = 0$$

or $r_1^2 + r_2^2 = d^2$

$\Rightarrow g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = g_1^2 + f_1^2 + g_2^2 + f_2^2 - 2g_1g_2 - 2f_1f_2$

or **Condition of orthogonality**

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

COMMON CHORD OF TWO CIRCLES

The common chord joining the point of intersection of two given circles is called their common chrod.

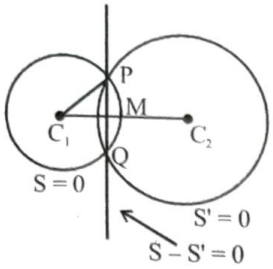
If $S = 0$ and $S' = 0$ be two intersecting circles, the equation of their common chord is

$$S - S' = 0$$

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 10$

and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 10$

be two circles intersecting at P and Q.



Then PQ is their common chord.

$\therefore \quad S - S' = 0$

$\Rightarrow \quad 2(g - g')x + 2(f - f')y + c - c' = 0$

is the common chord of two circles $S = 0$ and $S' = 0$.

Length of the Common Chord

$$PQ = 2(PM) = 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where C_1P = radius of the circle $S = 0$ and C_1M is the length of perpendicular from C_1 on common Chord PQ.

Notes:

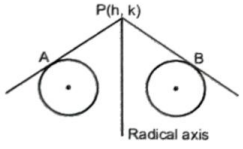
1. The length of common chord PQ of two cirlces is maximum when it is a diameter of the smaller circle.
2. IF circle is described on the common chord as a diameter then centre of the circle passing through P and Q lie on the common chord of two circles i.e., $S - S' = 0$.
3. If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

RADICAL AXIS AND RADICAL CENTRE

(*a*) **Radical axis:** The locus of a point, which moves in such a way that the whose power with respect to the circles are equal, is called the radical axis. If two circles are:

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$



Let $P(h, k)$ be the point, then from definition :

$$h^2 + k^2 + 2g_1h + 2f_1k + c_1 = h^2 + k^2 + 2g_2h + 2f_2k + c_2$$

or $\quad 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$

$\therefore$ locus of (h, k)

$\Rightarrow \quad 2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$

$\Rightarrow \quad S_1 - S_2 = 0$

which is the equation of radical axis.

Note:

(*i*) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$

(*ii*) If circles touch each other then Radical axis is the common tangent to both the circles.

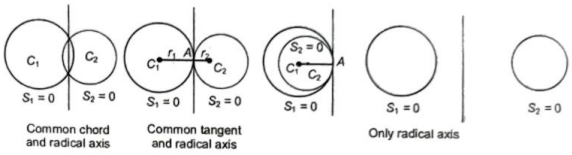
(*iii*) When the two circles intersect on real points then common chord is the Radical axis of the two circles.

(*iv*) The Radical axis of the two circles is perpendicular to the line joining the centre of two circle of two circles but not always pass through mid point of it.

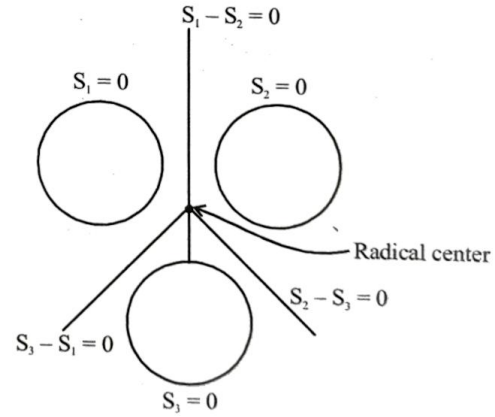
(*v*) If circles are concentric then the Radical axis does not always exist but if circles are not concentric then Radical axis always exists.

(*vi*) If two circles are orthogonal to the third circle then Radical axis of both circles passes through the centre of the third circle.

(*vii*) A system of circle, every pair of which have the same radical axis, is called a coaxial system of circles.



(*b*) **Radical centre:** The Radical axis of three circles (Taking two at a time) meet on a point. This point of intersection of radical axes is knonw as radical centre.



Radical Axis of S_1 & S_2 :

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

Radical Axis of S_2 & S_3:

$$2(g_2 - g_3)x + 2(f_2 - f_3)y + c_2 - c_3 = 0$$

Radical Axis of S_3 & S_1:

$$2(g_3 - g_1)x + 2(f_3 - f_1)y + c_3 - c_1 = 0$$

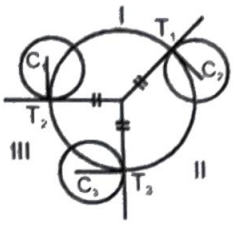
$$D = \begin{vmatrix} g_1 - g_2 & f_1 - f_2 & c_1 - c_2 \\ g_2 - g_3 & f_2 - f_3 & c_2 - c_3 \\ g_3 - g_1 & f_3 - f_1 & c_3 - c_1 \end{vmatrix} = 0 \ ;$$

Use $R_1 \to R_1 + R_2 + R_3 \Rightarrow D = 0$

(solve any 2 radical axes to get radical centre which is a point from which tangent to the three circles are equal.)

Note:

(*i*) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(*ii*) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocentre will be its radical centre.

(*iii*) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

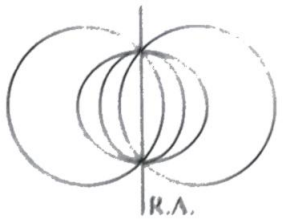
(*iv*) If two circles are orthogonal, then the polar of a point '*P*' on first circle w.r.t. the second circle passes through the point *Q* which is the other end of the diameter through *P*. Hence locus of a point which moves such that its polars w.r.t. the circles, $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

AN IMPORTANT RESULT ABOUT RADICAL AXES

Let $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ be three circles. Then their radical axes taken in pairs are either parallel or concurrent. Radical axes are parallel if the centres of the circles are collinear.

COAXIAL SYSTEM OF CIRCLES

Definition : A system of circles, every 2 of which have the same radical axis, is called Coaxial system of circles.



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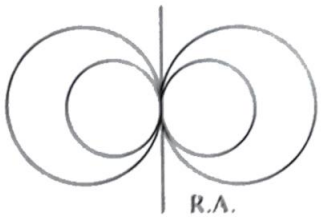


TRAIN YOUR BRAIN

Example 11: Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.

Sol. Given circles are

$$S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0 \quad ...(i)$$

and $$S_2 \equiv x^2 + y^2 - 8x - 6y + 23 = 0 \quad ...(ii)$$

Equation of common chord of circles (*i*) and (*ii*) which is also the radical axis of circles (*i*) and (*ii*) is

$$S_1 - S_2 = 0$$

or $$2x + 2y - 14 = 0$$

or $$x + y - 7 = 0 \quad ...(iii)$$

Centre of circle (*ii*) is (4, 3). Clearly line (*iii*) passes through the point (4, 3) and hence line (*iii*) is the diameter of circle (*ii*). Hence circle (*i*) bisects circumference of circle (*ii*).

Example 12: Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Sol. The equation of the circle through the intersection of the given circles is

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0 \quad ...(i)$$

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

and $$x^2 + y^2 + 6x + 4y - 12 = 0$$

Equation (*i*) can be rearranged as

$$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow \quad 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4$

$\Rightarrow \quad \lambda = -2$

Hence the required circle is

$$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

Example 13: Find the radical centre of circle

$$x^2 + y^2 + 3x + 2y + 1 = 0,$$

$$x^2 + y^2 - x + 6y + 5 = 0$$

and $\quad x^2 + y^2 + 5x - 8y + 15 = 0.$

Also find the equation of the circle cutting them orthogonally.

Sol. Equation of two radical axes are

$$S_1 - S_2 \equiv 4x - 4y - 4 = 0 \text{ or } x - y - 1 = 0$$

and $\quad S_2 - S_3 \equiv -6x + 14y - 10 = 0$

or $\quad 3x - 7y + 5 = 0$

Solving them the radical centre is (3, 2) also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, S_1, we have

$$r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore \quad$ Its equation is $(x-3)^2 + (y-2)^2 = r^2 = 27$

$\Rightarrow \quad x^2 + y^2 - 6x - 4y - 14 = 0$

Example 14: Find the equation of a circle coaxial with $S_1 : x^2 + y^2 + 4x + 2y + 1 = 0$ and $S_2 : 2x^2 + 2y^2 - 2x - 4y - 3 = 0$ and the centre of the circle to be determined lies on the radical axes of these 2 circle.

Sol. Let equation of circle coaxial with S_1 & S_2 be

$$S_1 + \lambda S_2 = 0$$

So, radical axis will be same also centre lies on the radical axis, so intercept of radical axis cut by S_1 or S_2 is act as diameter of the new circle equation of radical axis, is

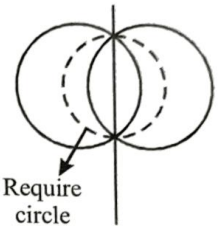
$$S_1 - S_2 \equiv 10x + 8y + 5 = 0$$

it cut the circle $\quad x^2 + y^2 + 4x + 2y + 1 = 0$

so $\quad x^2 + \left(\frac{10x+5}{-8}\right)^2 + 4x + 2\left(\frac{10x+5}{-8}\right) + 1 = 0$

$$41x^2 + 49x + 36 = 0$$

Similarly putting $x = \frac{8y+5}{-10}$



$$\Rightarrow \quad 41\left(\frac{8y+5}{-10}\right)^2 + 49\left(\frac{8y+5}{-10}\right) + 36 = 0$$

then equation of circle in diameter form will be

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$$

Topicwise Questions

Different forms of a circle

1. The centres of the circles $x^2 + y^2 - 6x - 8y - 7 = 0$ and $x^2 + y^2 - 4x - 10y - 3 = 0$ are the ends of the diameter of the circle
   (a) $x^2 + y^2 - 5x - 9y + 26 = 0$
   (b) $x^2 + y^2 + 5x - 9y + 14 = 0$
   (c) $x^2 + y^2 + 5x - y - 14 = 0$
   (d) $x^2 + y^2 + 5x + y + 14 = 0$

2. The intercepts made by the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the x-axis and y-axis are respectively
   (a) 9, 13 (b) 5, 13
   (c) 9, 15 (d) None

3. Equation of line passing through mid point of intercepts made by circle $x^2 + y^2 - 4x - 6y = 0$ on co-ordinate axis is
   (a) $3x + 2y - 12 = 0$ (b) $3x + y - 6 = 0$
   (c) $3x + 4y - 12 = 0$ (d) $3x + 2y - 6 = 0$

4. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. The equation of the circle is
   (a) $x^2 + y^2 - 2x - 2y = 47$
   (b) $x^2 + y^2 - 2x - 2y = 62$
   (c) $x^2 + y^2 - 2x + 2y = 47$
   (d) $x^2 + y^2 - 2x + 2y = 62$

5. If a be the radius of a circle which touches x-axis at the origin, then its equation is
   (a) $x^2 + y^2 + ax = 0$ (b) $x^2 + y^2 \pm 2ya = 0$
   (c) $x^2 + y^2 \pm 2xa = 0$ (d) $x^2 + y^2 + ya = 0$

6. The equation of the circle passing through (3, 6) and whose centre is (2, –1) is
   (a) $x^2 + y^2 - 4x + 2y = 45$ (b) $x^2 + y^2 - 4x - 2y + 45 = 0$
   (c) $x^2 + y^2 + 4x - 2y = 45$ (d) $x^2 + y^2 - 4x + 2y + 45 = 0$

7. The equation of a circle which passes through the three points (3, 0) (1, –6), (4, –1) is
   (a) $2x^2 + 2y^2 + 5x - 11y + 3 = 0$
   (b) $x^2 + y^2 - 5x + 11y - 3 = 0$
   (c) $x^2 + y^2 + 5x - 11y + 3 = 0$
   (d) $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

8. B and C are fixed point having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation
   (a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 2$
   (c) $9(x^2 + y^2) = 1$ (d) $9(x^2 + y^2) = 4$

9. The length of intercept on y-axis, by a circle whose diameter is the line joining the points (– 4, 3) and (12, 1) is
   (a) $3\sqrt{2}$ (b) $\sqrt{13}$
   (c) $4\sqrt{13}$ (d) None of these

10. The equation of the circle having the lines $y^2 - 2y + 4x - 2xy = 0$ as its normals & passing through the point (2, 1) is
    (a) $x^2 + y^2 - 2x - 4y + 3 = 0$
    (b) $x^2 + y^2 - 2x + 4y - 5 = 0$
    (c) $x^2 + y^2 + 2x + 4y - 13 = 0$
    (d) None

11. A circle is drawn touching the x-axis and centre at the point which is the reflection of (a, b) in the line $y - x = 0$. The equation of the circle is
    (a) $x^2 + y^2 - 2bx - 2ay + a^2 = 0$
    (b) $x^2 + y^2 - 2bx - 2ay + b^2 = 0$
    (c) $x^2 + y^2 - 2ax - 2by + b^2 = 0$
    (d) $x^2 + y^2 - 2ax - 2by + a^2 = 0$

12. (6, 0), (0, 6) and (7, 7) are the vertices of a triangle. The circle inscribed in the triangle has the equation
    (a) $x^2 + y^2 - 9x + 9y + 36 = 0$
    (b) $x^2 + y^2 - 9x - 9y + 36 = 0$
    (c) $x^2 + y^2 + 9x - 9y + 36 = 0$
    (d) $x^2 + y^2 - 9x - 9y - 36 = 0$

POSITION OF A POINT

13. Find the co-ordinates of a point p on line $x + y = -13$, nearest to the circle $x^2 + y^2 + 4x + 6y - 5 = 0$
    (a) (– 6, – 7) (b) (– 15, 2)
    (c) (– 5, – 6) (d) (– 7, – 6)

14. The greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is
    (a) 5 (b) 15
    (c) 10 (d) None of these

15. The parametric coordinates of any point on the circle $x^2 + y^2 - 4x - 4y = 0$ are
    (a) $(-2 + 2\cos\alpha, -2 + 2\sin\alpha)$
    (b) $(2 + 2\cos\alpha, 2 + 2\sin\alpha)$
    (c) $(2 + 2\sqrt{2}\cos\alpha, 2 + 2\sqrt{2}\sin\alpha)$
    (d) None of these

Line and Circle

16. Two lines through (2, 3) from which the circle $x^2 + y^2 = 25$ intercepts chords of length 8 units have equations

(a) $2x + 3y = 13, x + 5y = 17$

(b) $y = 3, 12x + 5y = 39$

(c) $x = 2, 9x - 11y = 51$

(d) None of these

17. The line $3x + 5y + 9 = 0$ w.r.t. the circle $x^2 + y^2 - 4x + 6y + 5 = 0$ is

(a) chord (b) diameter

(c) tangent (d) None

18. The co-ordinates of the middle point of the chord cut off on $2x - 5y + 18 = 0$ by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ are

(a) (1, 4) (b) (2, 4)

(c) (4, 1) (d) (1, 1)

Tangent and Normal

19. $y = \sqrt{3}x + c_1$ & $y = \sqrt{3}x + c_2$ are two parallel tangents of a circle of radius 2 units, then $|c_1 - c_2|$ is equal to

(a) 8 (b) 4

(c) 2 (d) 1

20. The number of tangents that can be drawn from the point (8, 6) to the circle $x^2 + y^2 - 100 = 0$ is

(a) 0 (b) 1

(c) 2 (d) None

21. The gradient of the tangent line at the point ($a \cos \theta$, $a \sin \theta$) to the circle $x^2 + y^2 = a^2$, is

(a) $\tan(\pi - \theta)$ (b) $\tan \theta$

(c) $\cot \theta$ (d) $-\cot \theta$

22. The tangent lines to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the line $4x + 3y + 5 = 0$ are given by:

(a) $4x + 3y - 7 = 0, 4x + 3y + 15 = 0$

(b) $4x + 3y - 31 = 0, 4x + 3y + 19 = 0$

(c) $4x + 3y - 17 = 0, 4x + 3y + 13 = 0$

(d) None of these

23. The tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at

(a) (–2, 1) (b) (–3, 0)

(c) (–1, –1) (d) (3, –1)

24. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ equals

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) None

25. If $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at (1, 1), then the value of c is

(a) 1 (b) 2

(c) –1 (d) –2

26. Line $3x + 4y = 25$ touches the circle $x^2 + y^2 = 25$ at the point

(a) (4, 3) (b) (3, 4)

(c) (–3, –4) (d) None

27. The equations of the tangents drawn from the point (0, 1) to the circle $x^2 + y^2 - 2x + 4y = 0$ are

(a) $2x - y + 1 = 0, x + 2y - 2 = 0$

(b) $2x - y - 1 = 0, x + 2y - 2 = 0$

(c) $2x - y + 1 = 0, x + 2y + 2 = 0$

(d) $2x - y - 1 = 0, x + 2y + 2 = 0$

28. The equation of the normal to the circle $x^2 + y^2 = 9$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

(a) $x - y = \frac{\sqrt{2}}{3}$ (b) $x + y = 0$

(c) $x - y = 0$ (d) None

29. The equation of the diameter of the circle $(x - 2)^2 + (y + 1)^2 = 16$ which bisects the chord cut off by the circle on the line $x - 2y - 3 = 0$ is

(a) $x + 2y = 0$ (b) $2x + y - 3 = 0$

(c) $3x + 2y - 4 = 0$ (d) None

30. The equation of normal to the circle $x^2 + y^2 - 4x + 4y - 17 = 0$ which passes through (1, 1) is

(a) $3x + y - 4 = 0$ (b) $x - y = 0$

(c) $x + y = 0$ (d) None

LENGTH OF TANGENT AND POWER OF A POINT

31. A line segment through a point P cuts a given circle in 2 points A & B, such that $PA = 16$ & $PB = 9$, find the length of tangent from points to the circle

(a) 7 (b) 25

(c) 12 (d) None

32. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + p = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + q = 0$ is: $(q > p)$

(a) $\sqrt{q - p}$ (b) $\sqrt{p - q}$

(c) $\sqrt{q + p}$ (d) None

33. The length of the tangent drawn from the point (2, 3) to the circles $2(x^2 + y^2) - 7x + 9y - 11 = 0$.

(a) 18 (b) 14

(c) $\sqrt{14}$ (d) $\sqrt{28}$

Director Circle & Chord of Contact

34. The distance between the chords of contact of tangents to the circle $x^2+y^2+2gx+2fy+c=0$ from the origin and from the point (g, f) is

(a) $\sqrt{g^2+f^2}$ (b) $\frac{\sqrt{g^2+f^2-c}}{2}$

(c) $\frac{g^2+f^2-c}{2\sqrt{g^2+f^2}}$ (d) $\frac{\sqrt{g^2+f^2+c}}{2\sqrt{g^2+f^2}}$

Chord with a given Mid Point

35. The locus of the mid-points of the chords of the circle $x^2+y^2-2x-4y-11=0$ which subtend 60° at the centre is

(a) $x^2+y^2-4x-2y-7=0$

(b) $x^2+y^2+4x+2y-7=0$

(c) $x^2+y^2-2x-4y-7=0$

(d) $x^2+y^2+2x+4y+7=0$

36. The locus of the centres of the circles such that the point (2, 3) is the mid point of the chord $5x+2y=16$ is

(a) $2x-5y+11=0$ (b) $2x+5y-11=0$

(c) $2x+5y+11=0$ (d) None

Pair of Tangents

37. Number of common tangents of the circles $(x+2)^2+(y-2)^2=49$ and $(x-2)^2+(y+1)^2=4$ is:

(a) 0 (b) 1

(c) 2 (d) 3

38. The equation of the common tangent to the circle $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+18y+26=0$ at their point of contact is

(a) $12x+5y+19=0$ (b) $5x+12y+19=0$

(c) $5x-12y+19=0$ (d) $12x-5y+19=0$

Position of two circle and Angle of intersection of two circles

39. The number of common tangents of the circles $x^2+y^2-2x-1=0$ and $x^2+y^2-2y-7=0$

(a) 1 (b) 3

(c) 2 (d) 4

40. If the circle $x^2+y^2=9$ touches the circle $x^2+y^2+6y+c=0$, then c is equal to

(a) –27 (b) 36

(c) –36 (d) 27

Radical Axis and Radical Centre

41. The locus of the centre of the circle which bisects the circumferences of the circles $x^2+y^2=4$ & $x^2+y^2-2x+6y+1=0$ is:

(a) a straight line (b) a circle

(c) a parabola (d) None of these

42. The point from which the tangents to the circles

$x^2+y^2-8x+40=0$

$5x^2+5y^2-25x+80=0$

$x^2+y^2-8x+16y+160=0$

are equal in length is

(a) $\left(8, \frac{15}{2}\right)$ (b) $\left(-8, \frac{15}{2}\right)$

(c) $\left(8, -\frac{15}{2}\right)$ (d) None

Family of Circles and Coaxial System

43. Tangents are drawn to the circle $x^2+y^2=1$ at the points where it is met by the circles. $x^2+y^2-(\lambda+6)x+(8-2\lambda)y-3=0$, λ being the variable. The locus of the point of intersection of these tangents is

(a) $2x-y+10=0$ (b) $x+2y-10=0$

(c) $x-2y+10=0$ (d) $2x+y-10=0$

Pole and Polar and Orthogonality

44. Equation of the circle cutting orthogonally the three circles $x^2+y^2-2x+3y-7=0$, $x^2+y^2+5x-5y+9=0$ and $x^2+y^2+7x-9y+29=0$ is

(a) $x^2+y^2-16x-18y-4=0$

(b) $x^2+y^2-7x+11y+6=0$

(c) $x^2+y^2+2x-8y+9=0$

(d) None of these

45. The locus of the centers of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-5x+4y-2=0$ orthogonally is

(a) $9x+10y-7=0$

(b) $x-y+2=0$

(c) $9x-10y+11=0$

(d) $9x+10y+7=0$

46. The equation of the circle which touches both the axes and whose radius is a, can be

(a) $x^2+y^2-2ax-2ay+a^2=0$

(b) $x^2+y^2+ax+ay-a^2=0$

(c) $x^2+y^2+2ax+2ay-a^2=0$

(d) $x^2+y^2-ax-ay+a^2=0$

47. The centre and radius of the circle $2x^2+2y^2-x=0$ are

(a) $\left(\frac{1}{4}, 0\right)$ and $\frac{1}{4}$ (b) $\left(-\frac{1}{2}, 0\right)$ and $\frac{1}{2}$

(c) $\left(\frac{1}{2}, 0\right)$ and (d) $\left(0, -\frac{1}{4}\right)$ and $\frac{1}{4}$

Learning Plus

1. The equation of the circle whose radius is 5 and which touches the circle $x^2+y^2-2x-4y-20=0$ externally at the point (5, 5), is

(a) $x^2+y^2-18x-16y-120=0$

(b) $x^2+y^2-18x-16y+120=0$

(c) $x^2+y^2+18x+16y-120=0$

(d) $x^2+y^2+18x-16y+120=0$

2. The locus of the centre of a circle which touches externally the circle $x^2+y^2-6x-6y+14=0$ and also touches the y-axis, is given by the equation

(a) $x^2-6x-10y+14=0$

(b) $x^2-10x-6y+14=0$

(c) $y^2-6x-10y+14=0$

(d) $y^2-10x-6y+14=0$

3. If one end of the diameter is (1, 1) and other end lies on the line $x+y=3$, then locus of centre of circle is

(a) $x+y=1$ (b) $2(x-y)=5$

(c) $2x+2y=5$ (d) $2x+2y=1$

4. If the line $x+2by+7=0$ is a diameter of the circle $x^2+y^2-6x+2y=0$, then $b=$

(a) 3 (b) -5

(c) -1 (d) 5

5. If the line $3x-4y=\lambda$ touches the circle $x^2+y^2-4x-8y-5=0$, then λ is equal to

(a) $-35, -15$ (b) $-35, 15$

(c) $35, 15$ (d) $35, -15$

6. The angle between the two tangents from the origin to the circle $(x-7)^2+(y+1)^2=25$ is

(a) 0 (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

7. A pair of tangents are drawn from the origin to the circle $x^2+y^2+20(x+y)+20=0$. The equation of the pair of tangents is

(a) $x^2+y^2+10xy=0$ (b) $x^2+y^2+5xy=0$

(c) $2x^2+2y^2+5xy=0$ (d) $2x^2+2y^2-5xy=0$

8. The two circles $x^2+y^2-2x+6y+6=0$ and $x^2+y^2-5x+6y+15=0$ touch each other. The equation of their common tangent is

(a) $x=3$ (b) $y=6$

(c) $7x-12y-21=0$ (d) $7x+12y+21=0$

9. The tangents are drawn from the point (4, 5) to the circle $x^2+y^2-4x-2y-11=0$. The area of quadrilateral formed by these tangents is

(a) 15 *sq. units* (b) 75 *sq. units*

(c) 8 *sq. units* (d) 4 *sq. units*

10. The co-ordinates of the point from where the tangents are drawn to the circles $x^2+y^2=1$, $x^2+y^2+8x+15=0$ and $x^2+y^2+10y+24=0$ are of same length, are

(a) $\left(2,\frac{5}{2}\right)$ (b) $\left(-2,-\frac{5}{2}\right)$

(c) $\left(-2,\frac{5}{2}\right)$ (d) $\left(2,-\frac{5}{2}\right)$

11. Length of the tangent drawn from any point on the circle $x^2+y^2+2gx+2fy+c_1=0$ to the circle $x^2+y^2+2gx+2fy+c=0$ is $(c>c_1)$

(a) $\sqrt{c_1-c}$ (b) $\sqrt{c-c_1}$

(c) $\sqrt{c_1+c}$ (d) $\sqrt{2c-c_1}$

12. The equation of pair of tangents to the circle $x^2+y^2-2x+4y+3=0$ from $(6,-5)$, is

(a) $7x^2+23y^2+30xy+66x+50y-73=0$

(b) $7x^2+23y^2+30xy-66x-50y-73=0$

(c) $7x^2+23y^2-30xy-66x-50y+73=0$

(d) $7x^2+23y^2+30xy-66x+50y+73=0$

13. The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then the equation of the circle is

(a) $x^2+y^2+2x-2y-13=0$

(b) $x^2+y^2-2x-2y-11=0$

(c) $x^2+y^2-2x+2y+12=0$

(d) $x^2+y^2-2x-2y+14=0$

14. The common chord of the circle $x^2+y^2+4x+1=0$ and $x^2+y^2+6x+2y+3=0$ is

(a) $x+y+1=0$ (b) $5x+y+2=0$

(c) $2x+2y+5=0$ (d) $3x+y+3=0$

15. If the line $x-2y=k$ cuts off a chord of length 2 from the circle $x^2+y^2=3$, then $k=$

(a) 0 (b) ± 1

(c) $\pm\sqrt{10}$ (d) $\pm\sqrt{5}$

16. The equation of the common chord of the circles $(x-a)^2+(y-b)^2=c^2$ and $(x-b)^2+(y-a)^2=c^2$ is

(a) $x-y=0$ (b) $x+y=0$

(c) $x+y=a^2+b^2$ (d) $x-y=a^2-b^2$

17. The length of common chord of the circles $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$ is

(a) $2\sqrt{a^2+b^2}$ (b) $\dfrac{ab}{\sqrt{a^2+b^2}}$

(c) $\dfrac{|2ab|}{\sqrt{a^2+b^2}}$ (d) $\dfrac{ab}{\sqrt{a^2-b^2}}$

18. Length of the common chord of the circles $x^2+y^2+5x+7y+9=0$ and $x^2+y^2+7x+5y+9=0$ is

(a) 9 (b) 8

(c) 7 (d) 6

19. Middle point of the chord of the circle $x^2+y^2=25$ intercepted on the line $x-2y=2$ is

(a) $\left(\frac{3}{5},\frac{4}{5}\right)$ (b) $(-2,-2)$

(c) $\left(\frac{2}{5},-\frac{4}{5}\right)$ (d) $\left(\frac{8}{3},\frac{1}{3}\right)$

20. The locus of the middle points of chords of the circle $x^2+y^2-2x-6y-10=0$ which passes through the origin, is

(a) $x^2+y^2+x+3y=0$ (b) $x^2+y^2-x+3y=0$

(c) $x^2+y^2+x-3y=0$ (d) $x^2+y^2-x-3y=0$

21. Circles $x^2+y^2-2x-4y=0$ and $x^2+y^2-8y-4=0$

(a) Touch each other internally

(b) Touch each other externally

(c) Cuts each other at two points

(d) Cuts each other more than two points

22. The equation of radical axis of the circles $2x^2+2y^2-7x=0$ and $x^2+y^2-4y-7=0$ is

(a) $7x+8y+14=0$ (b) $7x-8y+14=0$

(c) $7x-8y-14=0$ (d) $-7x-8y+14=0$

23. The radical centre of the circles $x^2+y^2-16x-60=0$, $x^2+y^2-12x+27=0$, $x^2+y^2-12y+8=0$ is

(a) (13, 33/4) (b) (33/4, 13)

(c) (33/4, 13) (d) (33/4, 20/3)

24. The equation of a circle passing through points of intersection of the circles $x^2+y^2+13x-3y=0$ and $2x^2+2y^2+4x-7y-25=0$ and point (1, 1) is

(a) $4x^2+4y^2-30x-10y-25=0$

(b) $4x^2+4y^2+30x-13y-25=0$

(c) $4x^2+4y^2-17x-10y+25=0$

(d) $4x^2+4y^2+30x+10y-25=0$

25. The locus of centre of a circle passing through (a, b) and cuts orthogonally to circle $x^2+y^2=p^2$, is

(a) $2ax+2by-(a^2+b^2+p^2)=0$

(b) $2ax+2by-(a^2-b^2+p^2)=0$

(c) $x^2+y^2-3ax-4by+(a^2+b^2-p^2)=0$

(d) $x^2+y^2-2ax-3by+(a^2-b^2-p^2)=0$

26. The two circles $x^2+y^2-2x+22y+5=0$ and $x^2+y^2+14x+6y+k=0$ intersect orthogonally provided k is equal to

(a) 47 (b) -47

(c) 49 (d) -49

27. The equation of the circle having the lines $x^2+2xy+3x+6y=0$ as its normals and having size just sufficient to contain the circle $x(x-4)+y(y-3)=0$ is

(a) $x^2+y^2+3x-6y-40=0$

(b) $x^2+y^2+6x-3y-45=0$

(c) $x^2+y^2+8x+4y-20=0$

(d) $x^2+y^2+4x+8y+20=0$

28. The polar of the point $(5,-1/2)$ *w.r.t* circle $(x-2)^2+y=4$ is

(a) $5x-10y+2=0$ (b) $6x-y-20=0$

(c) $10x-y-10=0$ (d) $x-10y-2=0$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. The equation of the circle which touches both the axes and the line $\frac{x}{3}+\frac{y}{4}=1$ and lies in the first quadrant is $(x-c)^2+(y-c)^2=c^2$ where c is

(a) 1 (b) 2

(c) 4 (d) 6

2. Equations of circles which pass through the points $(1,-2)$ and $(3,-4)$ and touch the x-axis is

(a) $x^2+y^2+6x+2y+9=0$

(b) $x^2+y^2+10x+20y+25=0$

(c) $x^2+y^2-6x+4y+9=0$

(d) None

3. The equation of circles passing through $(3,-6)$ touching both the axes is

(a) $x^2+y^2-6x+6y+9=0$

(b) $x^2+y^2+6x-6y+9=0$

(c) $x^2+y^2+30x-30y+225=0$

(d) $x^2+y^2-30x+30y+225=0$

4. The circle $x^2+y^2-2x-3ky-2=0$ passes through two fixed points, (k is the parameter)

(a) $(1+\sqrt{3},0)$ (b) $(-1+\sqrt{3},0)$

(c) $(-\sqrt{3}-1,0)$ (d) $(1-\sqrt{3},0)$

5. Slope of tangent to the circle $(x-r)^2+y^2=r^2$ at the point (x,y) lying on the circle is

(a) $\frac{x}{y-r}$ (b) $\frac{r-x}{y}$

(c) $\frac{y^2-x^2}{2xy}$ (d) $\frac{y^2+x^2}{2xy}$

6. $x^2+y^2+6x=0$ and $x^2+y^2-2x=0$ are two circles, then

(a) They touch each other externally

(b) They touch each other internally

(c) Area of triangle formed by their common tangents is $3\sqrt{3}$ sq. units.

(d) Their common tangents do not form any triangle.

7. The centre(s) of the circle(s) passing through the points $(0,0)$, $(1,0)$ and touching the circle $x^2+y^2=9$ is/are

(a) $\left(\frac{3}{2},\frac{1}{2}\right)$ (b) $\left(\frac{1}{2},\frac{3}{2}\right)$

(c) $\left(\frac{1}{2},2^{1/2}\right)$ (d) $\left(\frac{1}{2},-2^{1/2}\right)$

8. For the circles $S_1\equiv x^2+y^2-4x-6y-12=0$ and $S_2\equiv x^2+y^2+6x+4y-12=0$ and the line $L\equiv x+y=0$

(a) L is common tangent of S_1 and S_2

(b) L is common chord of S_1 and S_2

(c) L is radical axis of S_1 & S_2

(d) L is Perpendicular to the line joining the centre of S_1 & S_2

9. The centre of a circle C lies on the line $2x-2y+9=0$ and the circle C cuts orthogonally the circle $x^2+y^2=4$. The circle C passes through fixed points

(a) $(-3,3)$ (b) $\left(\frac{-1}{2},\frac{1}{2}\right)$

(c) $(-4,4)$ (d) $(-2,2)$

PARAGRAPH FOR QUESTION NOS. 10 TO 12

Consider the relation $4l^2-5m^2+6l+1=0$, where $l, m\in R$, and the line $lx+my+1=0$ touches a fixed circle S.

10. Centre and radius of circle S are

(a) $(2,0),3$ (b) $(-2,0),\sqrt{5}$

(c) $(3,0),\sqrt{5}$ (d) $(-3,0),3$

11. Tangents PA and PB are drawn to the circle S from the points P on the line $x+y=1$. The chord of contact AB passes through the fixed point

(a) $\left(\frac{-1}{2},\frac{3}{2}\right)$ (b) $\left(\frac{1}{2},\frac{-5}{2}\right)$

(c) $\left(\frac{1}{2},\frac{5}{2}\right)$ (d) $\left(\frac{1}{2},\frac{-3}{2}\right)$

12. If the line $x-2y+c=0$ intersects the fixed circle S orthogonally, then c equals

(a) -3 (b) -2

(c) 2 (d) 3

PARAGRAPH FOR QUESTION NOS. 13 AND 14

Let $P(x)$ be a quadratic polynomial with real coefficients such that $2P(x) = P(x-1) + P(x+1)$ is true for all $x \in R$. Given $P(0) = 8$ and $P(2) = 32$.

13. Sum of coefficients of $P(x)$ is

(a) 20 (b) 19

(c) 17 (d) 15

14. The abscissa of 2 points A and B are the roots of the equation $P(x) - 2 = 0$ and the ordinates are the roots of $P(y) - 2 = 0$, then x-intercept of circle with AB as diameter is

(a) 3 (b) 4

(c) 5 (d) 6

15. Match the column

Column – I	Column – II
(a) Number of values of a for which the common chord of the circles $x^2 + y^2 = 8$ and $(x-a)^2 + y^2 = 8$ subtends a right angle at the origin is	(p) 4
(b) A chord of the circle $(x-1)^2 + y^2 = 4$ lies along the line $y = 2\sqrt{3}(x-1)$. The length of the chord is equal to	(q) 2
(c) The number of circles touching all the three lines $3x + 7y = 2$, $21x + 49y = 5$ and $9x + 21y = 0$ are	(r) 0
(d) If radii of the smallest and largest circle passing through the point $(\sqrt{3}, \sqrt{2})$ and touching the circle $x^2 + y^2 - 2\sqrt{2}y - 2 = 0$ are r_1 and r_2 respectively, then the mean of r_1 and r_2 is	(s) 1

16. Match the column

Column – I	Column – II
(a) Number of common tangents of the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x + 6y + 2 = 0$ is	(p) 1
(b) Number of transverse common tangents of the circles $x^2 + y^2 - 4x - 10y + 4 = 0$ and $x^2 + y^2 + 6x + 12y - 55 = 0$ is	(q) 2
(c) Number of common tangents of the circles $x^2 + y^2 - 2x - 4y = 0$ & $x^2 + y^2 - 8y + 4 = 0$ is	(r) 3
(d) Number of direct common tangents of the circles $x^2 + y^2 + 2x - 8y + 13 = 0$ and $x^2 + y^2 - 6x - 2y + 6 = 0$ is	(s) 0

NUMERICAL BASED QUESTION

17. The number of such points $(a-1, \sqrt{3}a)$, where a is any integer, lying inside the region bounded by the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 2x - 15 = 0$, is

18. The number of tangents which can be drawn from the point $(1, 2)$ to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ are

19. The radius of the circle passing through the points $(1, -2)$, $(3, -4)$ and $(5, -6)$ is

20. The greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

21. The radius of the circle passing through the point $(6, 2)$ two of whose diameters are $x + y = 6$ and $x - 2y = 4$ is $\sqrt{a}$, then find value of a

22. The number of common tangents that can be drawn to the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is

23. Two diameters of the circle $3x^2 + 3y^2 - 6x - 18y - 7 = 0$ are along the lines $3x + y = c_1$ and $x - 3y = c_2$. Then, the value of c_1c_2 is

24. The line segment joining the points $(4, 7)$ and $(-2, -1)$ is a diameter of a circle. If the circle intersects the x–axis at A and B, then AB is equal to

25. How many common tangents can be drawn to the following circles $x^2 + y^2 = 6$ and $x^2 + y^2 + 6x + 2y + 1 = 0$

26. The axes are translated so that the new equation of the circle $x^2 + y^2 - 5x + 2y - 5 = 0$ has no first degree terms and the new equation $x^2 + y^2 = \frac{\lambda}{4}$, then find the value of λ.

27. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is $\frac{\lambda}{\sqrt{5}}$, then find λ.

28. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C. Find the area of the quadrilateral $ABCD$.

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The centers of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on: [JEE Main-2016]

(a) a circle

(b) an ellipse which is not a circle

(c) a hyperbola

(d) a parabola

2. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$, then the radius of S is: [JEE Main-2016]

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$

(c) 5 (d) 10

3. Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$ respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is: [JEE Main-2018]

(a) $2\sqrt{10}$ (b) $3\sqrt{\frac{5}{2}}$

(c) $\frac{3\sqrt{5}}{2}$ (d) $\sqrt{10}$

4. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is: [JEE Main-2018]

(a) 185 (b) 85

(c) 95 (d) 195

5. Three circles of radii a, b, c $(a < b < c)$ touch each other externally. If they have x–axis as a common tangent, then: [JEE Main-2019 (January)]

(a) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$ (b) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

(c) a, b, c are in A.P. (d) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A.P

6. If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then [JEE Main-2019 (January)]

(a) $0 < r < 1$ (b) $1 < r < 11$

(c) $r > 11$ (d) $r = 11$

7. If a circle C passing through the point $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point $(1, -1)$, then the radius of C is: [JEE Main-2019 (January)]

(a) $2\sqrt{5}$ (b) 4

(c) 5 (d) $\sqrt{57}$

8. If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units then c is equal to: [JEE Main-2019 (January)]

(a) 13 (b) 20

(c) –25 (d) 25

9. A circle cuts a chord of length $4a$ on the x-axis and passes through a point on the y – axis, distance $2b$ form the origin. Then the locus of the centre of this circle, is: [JEE Main-2019 (January)]

(a) a hyperbola (b) an ellipse

(c) a straight line (d) a parabola

10. A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the coordinate axes. Then the distance of the vertex of the square which is nearest to the origin is : [JEE Main-2019 (January)]

(a) 6 (b) $\sqrt{137}$

(c) $\sqrt{41}$ (d) 13

11. Two circles with equal radii intersecting at the points $(0, 1)$ and $(0, -1)$. The tangent at the point $(0, 1)$ to one of the circles passes through the centre of the other circle. Then the distance between the centres of these circles is: [JEE Main-2019 (January)]

(a) 1 (b) 2

(c) $2\sqrt{2}$ (d) $\sqrt{2}$

12. The straight line $x + 2y = 1$ meets the coordinate axes at A and B. A circle is drawn through A, B and the origin. Then the sum of perpendicular distances from A and B on the tangent to the circle at the origin is: [JEE Main-2019 (January)]

(a) $\frac{\sqrt{5}}{2}$ (b) $2\sqrt{5}$

(c) $\frac{\sqrt{5}}{4}$ (d) $4\sqrt{5}$

13. Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$ respectively. If P and Q are the points of intersection of these circles, then the area (in sq. units) of the quadrilateral PC_1QC_2 is:

[JEE Main-2019 (January)]

(a) 8 (b) 6

(c) 9 (d) 4

14. If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval:

[JEE Main-2019 (January)]

(a) (2, 17) (b) [13, 23]

(c) [12, 21] (d) (23, 31)

15. Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then the minimum value of $|z_1 - z_2|$ is:

[JEE Main-2019 (January)]

(a) 0 (b) $\sqrt{2}$

(c) 1 (d) 2

16. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is :

[JEE Main-2019 (January)]

(a) $(x^2 + y^2)^2 = 4R^2x^2y^2$ (b) $(x^2 + y^2)^3 = 4R^2x^2y^2$

(c) $(x^2 + y^2)^2 = 4Rx^2y^2$ (d) $(x^2 + y^2)(x + y) = R^2xy$

17. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is

[JEE Main-2019 (April)]

(a) 320 (b) 160

(c) 105 (d) 210

18. The tangent and the normal lines at the point $(\sqrt{3}, 1)$ to the circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The area of this triangle (in square units) is :

[JEE Main-2019 (April)]

(a) $\frac{1}{3}$ (b) $\frac{4}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

19. If a tangent to the circle $x^2 + y^2 = 1$ intersects the corrdinate axes at distinct points P and Q, then the locus of the mid-point of PQ is

[JEE Main-2019 (April)]

(a) $x^2 + y^2 - 2xy = 0$ (b) $x^2 + y^2 - 16x^2y^2 = 0$

(c) $x^2 + y^2 - 4x^2y^2 = 0$ (d) $x^2 + y^2 - 2x^2y^2 = 0$

20. The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point:

[JEE Main-2019 (April)]

(a) $(-4, 6)$ (b) $(6, -2)$

(c) $(-6, 4)$ (d) $(4, -2)$

21. If the circles $x^2 + y^2 + 5Kx + 2y + K = 0$ and $2(x^2 + y^2) + 2Kx + 3y - 1 = 0$, $(K \in R)$, intersect at the points P and Q, then the line $4x + 5y - K = 0$ passes through P and Q for :

[JEE Main-2019 (April)]

(a) exactly two values of K

(b) exactly one value of K

(c) no value of K.

(d) infinitely many values of K

22. The locus of the centres of the circles, which touch the circle, $x^2 + y^2 = 1$ externally, also touch the y-axis and lie in the first quadrant, is :

[JEE Main-2019 (April)]

(a) $y = \sqrt{1 + 4x}$, $x \geq 0$ (b) $x = \sqrt{1 + 4y}$, $y \geq 0$

(c) $x = \sqrt{1 + 2y}$, $y \geq 0$ (d) $y = \sqrt{1 + 2x}$, $x \geq 0$

23. The line $x = y$ touches a circle at the point $(1, 1)$. If the circle also passes through the point $(1, -3)$, then its radius is;

[JEE Main-2019 (April)]

(a) $3\sqrt{2}$ (b) 3

(c) $2\sqrt{2}$ (d) 2

24. If the angle of intersection at a point where the two circles with radii 5 cm and 12 cm intersect is 90°, then the length (in cm) of their common chord is :

[JEE Main-2019 (April)]

(a) $\frac{60}{13}$ (b) $\frac{120}{13}$

(c) $\frac{13}{2}$ (d) $\frac{13}{5}$

25. A circle touching the x-axis at $(3, 0)$ and making an intercept of length 8 on the y-axis passes through the point

[JEE Main-2019 (April)]

(a) $(3, 10)$ (b) $(2, 3)$

(c) $(1, 5)$ (d) $(3, 5)$

26. If a line, $y = mx + c$ is a tangent to the circle, $(x - 3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tagnent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; then :

[JEE Main-2020 (January)]

(a) $c^2 + 7c + 6 = 0$ (b) $c^2 + 6c + 7 = 0$

(c) $c^2 - 6c + 7 = 0$ (d) $c^2 - 7c + 6 = 0$

27. A circle touches the y-axis at the point $(0, 4)$ and passes through the point $(2, 0)$. Which of the following lines is not a tangent to this circle ?

[JEE Main-2020 (January)]

(a) $4x - 3y + 17 = 0$ (b) $3x - 4y - 24 = 0$

(c) $3x + 4y - 6 = 0$ (d) $4x + 3y - 8 = 0$

28. If the curves, $x^2 - 6x + y^2 + 8 = 0$ and $x^2 - 8y + y^2 + 16 - k = 0$, $(k > 0)$ touch each other at a point, then the largest value of k is_____. [JEE Main-2020 (January)]

29. The number of integral values of k for which the line, $3x + 4y = k$ intersects the circle, $x^2 + y^2 - 2x - 4y + 4 = 0$ at two distinct points is _____.

[JEE Main-2020 (September)]

30. Let the latus rectum of the parabola $y^2 = 4x$ be the common chord to the circles C_1 and C_2 each of them having radius $2\sqrt{5}$. Then, the distance between the centres of the circles C_1 and C_2 is : [JEE Main-2020 (September)]

(a) 8 (b) 12

(c) $8\sqrt{5}$ (d) $4\sqrt{5}$

31. The diameter of the circle, whose centre lies on the line $x + y = 2$ in the first quadrant and which touches both the lines $x = 3$ and $y = 2$, is _____.

[JEE Main-2020 (September)]

32. Let PQ be a diameter of the circle $x^2 + y^2 = 9$. If α and β are the lengths of the perpendiculars from P and Q on the straight line, $x + y = 2$ respectively, then the maximum value of $\alpha\beta$ is_____. [JEE Main-2020 (September)]

33. The circle passing through the intersection of the circles, $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 - 4y = 0$, having its centre on the line, $2x - 3y + 12 = 0$, also passes through the point:

[JEE Main-2020 (September)]

(a) $(-3, 6)$ (b) $(-1, 3)$

(c) $(-3, 1)$ (d) $(1, -3)$

34. If the length of the chord of the circle, $x^2 + y^2 = r^2$ $(r > 0)$ along the line, $y - 2x = 3$ is r, then $\frac{5}{4}r^2$ is equal to:

[JEE Main-2020 (September)]

35. Let B be the centre of the circle $x^2 + y - 2x + 2y + 1 = 0$. Let the tangents at two points P and Q on the circle intersect at the point $A(3,1)$. Then $8.\left(\frac{\text{area } \Delta APQ}{\text{area } \Delta BPQ}\right)$ is equal to

[JEE Main-2021 (August)]

36. If the variable line $3x + 4y = \alpha$ lies between the two circles $(x-1)^2 + (y-1)^2 = 1$ and $(x-9)^2 + (y-1)^2 = 4$, without intercepting a chord on either circle, then the sum of all the integral values of α is.....

37. Two circles each of radius 5 units touch each other at the point (1, 2). If the equation of their common tangent is $3x + 4y = 10$, and $C_1(\alpha, \beta)$ and $C_2(\gamma, \delta)$, $C_1 \neq C_2$ are their centres, then $|(\alpha + \beta)(\gamma + \delta)|$ equal To

[JEE Main-2021 (August)]

38. Let the equation $x^2 + y^2 + px + (1 - p)y + 5 = 0$ represent circles of varying radius $r \in (0, 5]$. Then the number of elements in the set $S = \{q : q = p^2$ and q is an integer $\}$ is [JEE Main-2021 (August)]

39. The locus of a point, which moves such that the sum of squares of its distances from the points (0, 0), (1, 0) (0, 1), (1, 1) is 18 units, is a circle of diameter d. Then d^2 is equal to [JEE Main-2021 (August)]

JEE-ADVANCED PREVIOUS YEARS

1. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is [IIT JEE-2012]

(a) $20(x^2 + y^2) - 36x + 45y = 0$

(b) $20(x^2 + y^2) + 36x - 45y = 0$

(c) $36(x^2 + y^2) - 20x + 45y = 0$

(d) $36(x^2 + y^2) + 20x - 45y = 0$

Comprehension – 1 (No. 2 & 3)

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2 + y^2 = 1$. [IIT JEE-2012]

2. A common tangent of the two circles is

(a) $x = 4$ (b) $y = 2$

(c) $x + \sqrt{3}y = 4$ (d) $x + 2\sqrt{2}y = 6$

3. A possible equation of L is

(a) $x - \sqrt{3}y = 1$ (b) $x + \sqrt{3}y = 1$

(c) $x - \sqrt{3}y = -1$ (d) $x + \sqrt{3}y = 5$

4. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are)

[JEE Advanced-2013]

(a) $x^2 + y^2 - 6x + 8y + 9 = 0$

(b) $x^2 + y^2 - 6x + 7y + 9 = 0$

(c) $x^2 + y^2 - 6x - 8y + 9 = 0$

(d) $x^2 + y^2 - 6x - 7y + 9 = 0$

5. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then [JEE Advanced-2014]

(a) radius of S is 8

(b) radius of S is 7

(c) centre of S is $(-7, 1)$

(d) centre of S is $(-8, 1)$

6. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s)

[JEE Advanced-2016]

(a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

7. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

[JEE Advanced-2017]

Comprehension–2 (No. 8 & 9)

Let S be the circle in the xy-plane defiend by the equation $x^2 + y^2 = 4$.

8. let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop -1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the point E_3, F_3 and G_3 lie on the curve [JEE Advanced-2018]

(a) $x + y = 4$ (b) $(x-4)^2 + (y-4)^2 = 16$

(c) $(x-4)(y-4) = 4$ (d) $xy = 4$

9. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve -

(a) $(x+y)^2 = 3xy$ (b) $x^{2/3} + y^{2/3} = 2^{4/3}$

(c) $x^2 + y^2 = 2xy$ (d) $x^2 + y^2 = x^2y^2$

10. Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1, 1)$ be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statemebt(s) is (are) TRUE ?

[JEE Advanced-2018]

(a) The point $(-2,7)$ lies in E_1

(b) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does NOT lie in E_2

(c) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2

(d) The point $\left(0, \frac{3}{2}\right)$ does NOT lie in E_1

11. A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ? [JEE Advanced-2019]

(a) $6 \le m < 8$ (b) $2 \le m < 4$

(c) $4 \le m < 6$ (d) $-3 \le m < -1$

12. Let the point B be the reflection of the point $A(2, 3)$ with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is ____

[JEE Advanced-2019]

Comprehension–3 (No. 13 & 14)

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y.

[JEE Advanced-2019]

Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions:

(*i*) centre of C_3 is collinear with the centres of C_1 and C_2

(*ii*) C_1 and C_2 both lie inside C_3, and

(*iii*) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below :

List-I	List-II
(I) $2h + k$	(P) 6
(II) $\frac{\text{Length of } ZW}{\text{Length of } XY}$	(Q) $\sqrt{6}$
(III) $\frac{\text{Area of triangle } MZN}{\text{Area of triangle } ZMW}$	(R) $\frac{5}{4}$
(IV) α	(S) $\frac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\frac{10}{3}$

13. Which of the following is the only INCORRECT combination ?

Options

(a) (IV), (S) (b) (IV), (U)

(c) (III), (R) (d) (I), (P)

14. Which of the following is the only INCORRECT combination?

Options [JEE Advanced-2019]

(a) (II), (T) (b) (I), (S)

(c) (I), (U) (d) (II), (Q)

ANSWER KEY

Topicwise Questions

1. (*a*)	**2.** (*c*)	**3.** (*d*)	**4.** (*c*)	**5.** (*b*)	**6.** (*a*)	**7.** (*d*)	**8.** (*a*)	**9.** (*c*)	**10.** (*a*)	**11.** (*b*)	**12.** (*b*)
13. (*a*)	**14.** (*b*)	**15.** (*c*)	**16.** (*b*)	**17.** (*b*)	**18.** (*a*)	**19.** (*a*)	**20.** (*b*)	**21.** (*d*)	**22.** (*b*)	**23.** (*d*)	**24.** (*c*)
25. (*a*)	**26.** (*b*)	**27.** (*a*)	**28.** (*c*)	**29.** (*b*)	**30.** (*a*)	**31.** (*c*)	**32.** (*a*)	**33.** (*c*)	**34.** (*c*)	**35.** (*c*)	**36.** (*a*)
37. (*b*)	**38.** (*b*)	**39.** (*a*)	**40.** (*a*)	**41.** (*a*)	**42.** (*c*)	**43.** (*a*)	**44.** (*a*)	**45.** (*c*)	**46.** (*a*)	**47.** (*a*)	

Learning Plus

1. (*b*)	**2.** (*d*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)	**6.** (*d*)	**7.** (*c*)	**8.** (*a*)	**9.** (*c*)	**10.** (*b*)	**11.** (*b*)	**12.** (*a*)
13. (*b*)	**14.** (*a*)	**15.** (*c*)	**16.** (*a*)	**17.** (*c*)	**18.** (*d*)	**19.** (*c*)	**20.** (*d*)	**21.** (*a*)	**22.** (*c*)	**23.** (*d*)	**24.** (*b*)
25. (*a*)	**26.** (*a*)	**27.** (*b*)	**28.** (*b*)								

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a,d*)	**2.** (*b,c*)	**3.** (*a,d*)	**4.** (*a,d*)	**5.** (*b,c*)	**6.** (*a,c*)	**7.** (*c,d*)	**8.** (*b,c,d*)	**9.** (*b,c*)	**10.** (*c*)	**11.** (*b*)	**12.** (*a*)

13. (*b*) **14.** (*d*) **15.** (*a*) → (q), (*b*) → (*p*), (*c*) → (r), (*d*) → (s) **16.** (*a*) → (r), (*b*) → (s), (*c*) → (*p*), (*d*) → (q)

17. (0000)	**18.** (0000)	**19.** (0010)	**20.** (0015)	**21.** (20)	**22.** (0003)	**23.** [–48]	**24.** [8]	**25.** [4]	**26.** [49]	**27.** [16]	**28.** [75]

JEE Mains & Advanced Past Years Questions

JEE-MAIN

PREVIOUS YEARS

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*b*)	**7.** (*c*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)	**11.** (*b*)	**12.** (*a*)
13. (*d*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*d*)	**18.** (*d*)	**19.** (*c*)	**20.** (*b*)	**21.** (*c*)	**22.** (*d*)	**23.** (*c*)	**24.** (*b*)
25. (*a*)	**26.** (*b*)	**27.** (*d*)	**28.** [36]	**29.** [9]	**30.** (*a*)	**31.** [3]	**32.** [7]	**33.** (*a*)	**34.** [3]	**35.** (18)	**36.** (165)
37. (40)	**38.** (61)	**39.** (16)									

JEE-ADVANCED
PREVIOUS YEARS

1. (*a*)	**2.** (*d*)	**3.** (*a*)	**4.** (*a,c*)	**5.** (*b,c*)	**6.** (*a,c*)	**7.** [2]	**8.** (*a*)	**9.** (*d*)	**10.** (*a,d*)	**11.** (*b*)	**12.** [10]
13. (*a*)	**14.** (*d*)										

PRAYAS 2.0 t.me/realprayasians2

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ELLIPSE

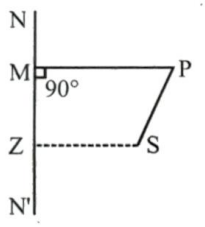
PRAYAS 1.0 time/real prayasam

DEFINITION

An ellipse is the locus of the point which moves in a plane such that the ratio of its distance from a fixed point (focus) to fixed straight line (directrix) is always constant (called eccentricity) and is less than 1.

In the given figure, S is the focus and NN' is the directrix. Let P be a point on the ellipse, then

$$\frac{PS}{PM} = e, \quad e < 1 \quad \text{(for ellipse)}$$



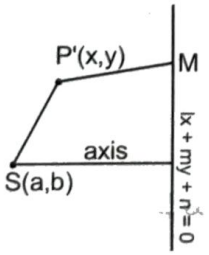
GENERAL EQUATION OF AN ELLIPSE

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let $P(x, y)$ be any point on the ellipse. Then by definition.

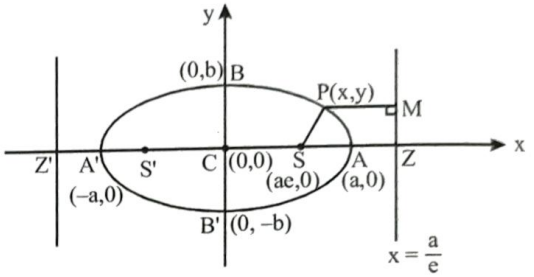
$\Rightarrow$ $SP = e\,PM$ (e is the eccentricity)

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = e^2 \frac{(lx+my+n)^2}{(l^2+m^2)}$$



$\Rightarrow$ $(l^2 + m^2)\{(x - a)^2 + (y - b)^2\} = e^2\{lx + my + n\}^2$ reduces to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, in which $\Delta \neq 0$, $h^2 < ab$, if $(h = 0, a \neq b)$

STANDARD EQUATION



By defintion of ellipse.

$$PS = ePM$$

$$\Rightarrow \quad (PS)^2 = e^2 (PM)^2$$

$$\Rightarrow \quad (x-ae)^2 + (y-0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

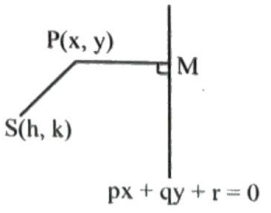
or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2)$

FACTS ABOUT AN ELLIPSE

1. If focus of the ellipse is taken as $(ae, 0)$, then corresponding directrix is $x = \frac{a}{e}$ and if focus is $(-ae, 0)$, then corresponding directrix is $x = \frac{-a}{e}$.
2. If equation of directrix is $px + qy + r = 0$ & focus is (h, k) then its equation will be $PS^2 = e^2\,PM^2$

$$(x-h)^2 + (y-k)^2 = e^2 \cdot \left(\frac{px+qy+r}{\sqrt{p^2+q^2}}\right)^2$$



3. Distance between foci $SS' = 2ae$ & distance between directrix $ZZ' = 2\frac{a}{e}$.
4. Degree of flatness of an ellipse is also called on eccentricity & written as

$$e = \frac{CS}{CA} = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

If $e \to 0 \Rightarrow b \to a \Rightarrow$ foci becomes closer & move towards centre and ellipse becomes circle.

If $e \to 1 \Rightarrow b \to 0 \Rightarrow$ ellipse get thinner & thinner

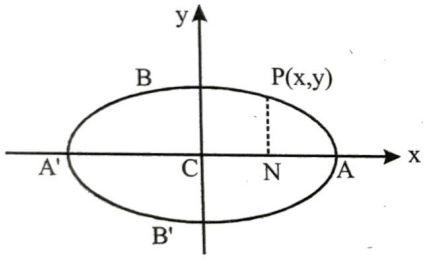
5. Distance of focus from the extremity of minor axis is equal to 'a' because $a^2e^2 + b^2 = a^2$
6. Let $P(x, y)$ be any point on the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{(a-x)(a+x)}{a^2}$$

$$\Rightarrow \quad \frac{PN^2}{b^2} = \frac{AN \cdot A'N}{a^2}$$



$$\Rightarrow \quad \frac{PN^2}{AN \cdot A'N} = \frac{b^2}{a^2}$$

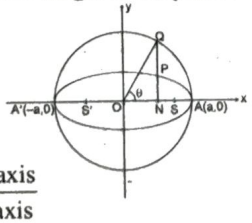
AUXILIARY CIRCLE / ECCENTRIC ANGLE

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x–axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \le \pi)$.

$Q \equiv (a\cos\theta, a\sin\theta)$

$P \equiv (a\cos\theta, b\sin\theta)$



Note that:

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

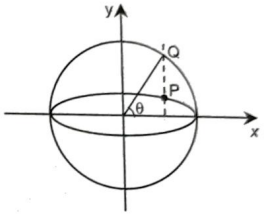
Note:

If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

PARAMETRIC EQUATION OF ELLIPSE

If ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, then auxiliary circle will be $x^2 + y^2 = a^2$ and point Q will be $(a\cos\theta, a\sin\theta)$, hence point P will be $(a\cos\theta, b\sin\theta)$.

Here $x = a\cos\theta, y = b\sin\theta$ is called as parametric equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

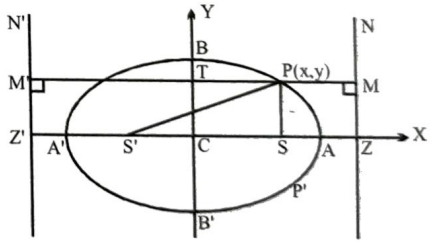


FOCAL DISTANCE OF A POINT

Let $P(x, y)$ be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$$

Then by definition of ellipse,



$$SP = ePM = e(MT - PT) = e\left(\frac{a}{e} - x\right) = a - ex$$

& $\quad S'P = ePM' = e(M'T + PT) = e\left(\frac{a}{e} + x\right) = a + ex$

Hence $SP + S'P = 2a$

TWO STANDARD FORMS OF ELLIPSE

There are two standard forms of ellipse with centre at the origin and axes along coordinate axes. The foci of the ellipse are either on the x-axis or on the y-axis.

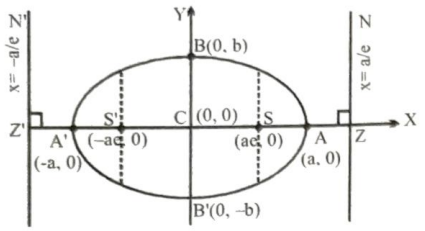
1. **Major axis along x–axis:** The equation of this type of ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$ and $b = a\sqrt{1 - e^2}$.

For this ellipse:

(*i*) Major axis is $2a$

(*ii*) Minor axis is $2b$.

(*iii*) Centre is $(0, 0)$



(*iv*) Vertices are $(\pm a, 0)$

(*v*) Foci are $(\pm ae, 0)$

(*vi*) Equation of directrices are $x = \pm\frac{a}{e}$

(*vii*) Equation of major axis is $y = 0$

(*viii*) Equation of minor axis is $x = 0$

(*ix*) Length of latus rectum $= \frac{2b^2}{a}$

(*x*) Extremity of latus rectum is $\left(\pm ae, \pm\frac{b^2}{a}\right)$

2. Major axis along *y*–axis: The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where $0 < a < b$ and $a = b\sqrt{1-e^2}$.

For this ellipse:

(*i*) Major axis is $2b$

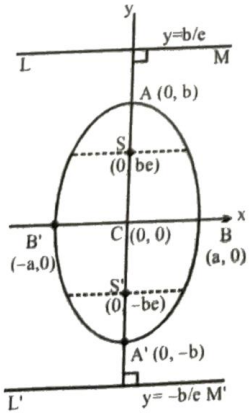
(*ii*) Minor axis is $2a$.

(*iii*) Centre is $(0, 0)$

(*iv*) Vertices are $(0, \pm b)$

(v) Foci are $(0, \pm be)$

(*vi*) Equation of directrices are $y = \pm\frac{b}{e}$



(*vii*) Equation of major axis is $x = 0$

(*viii*) Equation of minor axis is $y = 0$

(*ix*) Length of latus rectum $= \frac{2a^2}{b}$

(*x*) Extremity of latus rectum is $\left(\pm\frac{a^2}{b}, \pm be\right)$

TRAIN YOUR BRAIN

Example 1: If latus rectum of an ellipse is half of its minor axis, then its eccentricity is

Sol. As given $\frac{2b^2}{a} = b \Rightarrow 2b = a \Rightarrow 4b^2 = a^2$

$\Rightarrow 4a^2(1-e^2) = a^2 \Rightarrow 1 - e^2 = 1/4 \ \therefore\ e = \sqrt{3}/2$

Example 2: Find the equation of the ellipse whose foci are $(2, 3)$, $(-2, 3)$ and whose semi minor axis is of length $\sqrt{5}$.

Sol. Here S is $(2, 3)$ & S' is $(-2, 3)$ and $b = \sqrt{5}$

$\Rightarrow SS' = 4 = 2ae \Rightarrow ae = 2$

but $b^2 = a^2(1 - e^2) \Rightarrow 5 = a^2 - 4 \Rightarrow a = 3$. Hence the equation to major axis is $y = 3$

Centre of ellipse is midpoint of SS' i.e. $(0, 3)$

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2}+\frac{(y-3)^2}{b^2}=1$

or $\frac{x^2}{9}+\frac{(y-3)^2}{5}=1$

Example 3: Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus-rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25}+\frac{(y-2)^2}{16}=1$.

Sol. Let $x - 3 = X, y - 2 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2}+\frac{Y^2}{4^2}=1$.

Equation of major axis is $Y = 0 \Rightarrow y = 2$.

equation of minor axis is $X = 0 \Rightarrow x = 3$.

Centre $(X = 0, Y = 0) \Rightarrow x = 3, y = 2$ $C \equiv (3, 2)$

Length of semi-major axis $a = 5$

Length of major axis $2a = 10$

Length of semi-minor axis $b = 4$

Length of minor axis $= 2b = 8$.

Let 'e' be eccentricity

$\therefore\ b^2 = a^2(1 - e^2)$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}.$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Coordinates foci are $X = \pm ae$, $Y = 0$

$S = (X = 3, Y = 0)$ & $S' = (X = 3, Y = 0)$

$S = (6, 2)$ & $S' = (0, 2)$

Extremities of major axis

$A = (X = a, Y = 0)$ & $A' = (X = a, Y = 0)$

$A = (x = 8, y = 2)$ & $A' = (x = 2, 2)$

$A = (8, 2)$ & $A' = (2, 2)$

Extremities of minor axis $B = (X = 0, Y = b)$

& $B' = (X = 0, Y = b)$

$B = (x = 3, y = 6)$ & $B' = (x = 3, y = -2)$

$B = (3, 6)$ & $B' = (3, -2)$

Equation of directrix $X = \pm \dfrac{a}{e}$

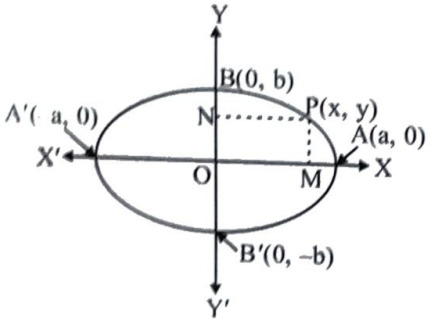
$$x - 3 = \pm \frac{25}{3} \Rightarrow x = \frac{34}{3} \text{ \& } x = -\frac{16}{3}$$

OBLIQUE ELLIPSES

Equation of Ellipse Referred to Two Perpendicular Line

Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as shown in the figure.

Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.



$$\Rightarrow \quad \frac{PN^2}{a^2} + \frac{PM^2}{b^2} = 1$$

It follows from this, that if perpendicular distance p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight line $L_1 = a_1x + b_1y + c_1 = 0$, $L_2 = b_1x - a_1y + c_2 = 0$, respectivley, satisfy the equation

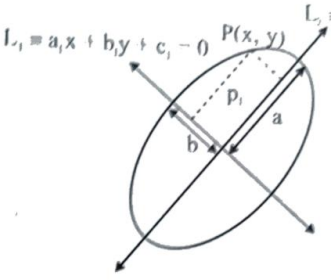
$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$$

i.e.
$$\frac{\left(\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} + \frac{\left(\dfrac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1$$

then locus of the point P is an ellipse in the plane of the given lines such that

(i) The centre of the ellipse is the point of the intersection of the lines $L_1 = 0$ and $L_2 = 0$

(ii) The major axis lies along $L_2 = 0$ and the minor axis lies along $L_1 = 0$, if $a > b$



Example 4: Find the major axis, minor axis, centre and eccentricity of the ellipse

$$4(x-2y+1)^2 + 9(2x+y+2)^2 = 180$$

Sol. We have,

$$4(x-2y+1)^2 + 9(2x+y+2)^2 = 1s80$$

$$\Rightarrow \quad \frac{(x-2y+1)^2}{45} + \frac{(2x+y+2)^2}{20} = 1$$

$$\Rightarrow \quad \frac{\left(\dfrac{x-2y+1}{\sqrt{1+4}}\right)^2}{3^2} + \frac{\left(\dfrac{2x+y+2}{\sqrt{4+1}}\right)^2}{2^2} = 1$$

$$\Rightarrow \quad \frac{X^2}{3^2} + \frac{Y^2}{2^2} = 1,$$

We have

$$X = \frac{x-2y+1}{\sqrt{5}} \text{ and } Y = \frac{2x+y+2}{\sqrt{5}}$$

It follows from this that :

Length of the major axis $= 2 \times 3 = 6$

Length of the minor axis $= 2 \times 2 = 4$

Equation of major axis is $Y = 0$ i.e., $2x + y + 2 = 0$

Equation of the minor axis is $X = 0$ i.e., $x - 2y + 1 = 0$

Center of the ellipse is the point of intersection of the lines $2x + y + 2 = 0$ and $x - 2y + 1 = 0$ i.e., $(-1, 0)$.

Let e be the ecentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}.$$

EQUATION OF CHORD OF AN ELLIPSE

Equation of a chord of an ellipse joining two points $P(\alpha)$ and $Q(\beta)$ on it is equal to

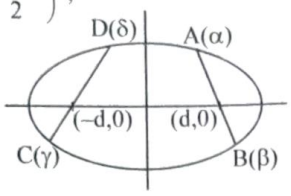
$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

(use formula of line joining points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$)

If this particular chord passes through $(d, 0)$ then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right);$$

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{d}$$



Using componendo and dividendo rule

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a-d}{a+d}$$

or $$-\frac{2\sin\alpha/2\,\sin\beta/2}{2\cos\alpha/2\,\cos\beta/2}=\frac{a-d}{a+d}$$

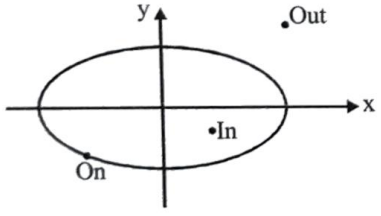
i.e. $$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}$$

if $d=ae$ i.e. PQ is a focal chord then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$

POSITION OF A POINT W.R.T AN ELLIPSE

Let $S(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ be the given ellipse and $P(x_1,y_1)$ is the given point.

(*i*) If $S(x_1,y_1)>0$ then $P(x_1,y_1)$ lie outside the ellipse.



(*ii*) If $S(x_1,y_1)<0$ then $P(x_1,y_1)$ lie inside the ellipse.

(*iii*) If $S(x_1,y_1)=0$ then $P(x_1,y_1)$ lie on the ellipse. This result holds true for circle and parabola also.

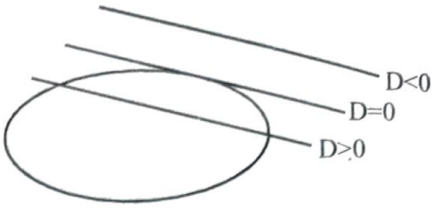
LINE AND AN ELLIPSE

Let the equations of the line is $y=mx+c$ ...(1)

and equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(2)

The points of intersection of the line and the ellipse can be obtained by solving the two equations simultaneously.

Hence by eliminating y from (1) & (2), we get



$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1.$$

i.e. $(b^2+a^2m^2)x^2+2a^2cmx+a^2(c^2-b^2)=0$ .....(3)

Note: $y=mx+c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$c^2=a^2m^2+b^2$.

TRAIN YOUR BRAIN

Example 5: For what value of λ does the line $y=x+\lambda$ touches the ellipse $9x^2+16y^2=144$.

Sol. $\because$ Equation of ellipse is $9x^2+16y^2=144$ or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then we get $a^2=16$ and $b^2=9$ and comparing the line $y=x+\lambda$ with $y=mx+c$

$\therefore\quad m=1$ and $c=\lambda$

If the line $y=x+\lambda$ touches the ellipse $9x^2+16y^2=144$, then $c^2=a^2m^2+b^2$

$\Rightarrow\ \lambda^2=16\times1^2+9\Rightarrow\lambda^2=25\ \ \therefore\ \lambda=\pm5$

Example 6: Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Sol. Equation of chord is

$$\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}+\frac{y}{4}\cdot\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}=\cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$$

$$\frac{x}{5}\cdot\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}\cdot\sin\left(\frac{3\pi}{4}\right)=0$$

$$-\frac{x}{5}+\frac{y}{4}=0\ \Rightarrow\ 4x=5y$$

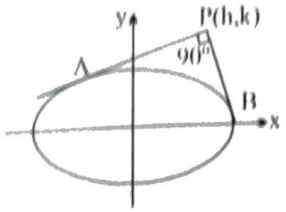
DIRECTOR CIRCLE

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle.

Let equation of any tangent is $y = mx + \sqrt{a^2m^2 + b^2}$

If it passes through (h, k) then

$k = mh + \sqrt{a^2m^2 + b^2}$



i.e. $k^2 - h^2 = a^2 - h^2$;

i.e. $x^2 + y^2 = a^2 + b^2$

which is the director circle of the ellipse. Hence director circle of an ellipse is a circle whose centre is the centre of ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

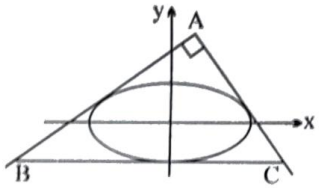
Equation (3) can be used to determine the locus of the point of intersection of two tangents enclosing.

If from any point $P(h, k)$ pair of tangents are drawn to the ellipse which include an angle α, then

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

By putting value of $m_1 + m_2$ and m_1m_2 in above equation we will get the angle between pair of tangents.

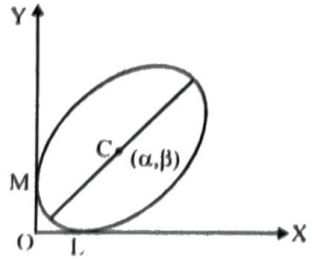
Note: If a right triangle, right angled at A circumscribes an ellipse then locus of the point A is the director circle of the ellipse.



TRAIN YOUR BRAIN

Example 7: An ellipse slides between two lines at right angles to another. Show that the locus of its centre is a circle.

Sol. Let the two given perpendicular lines be taken as the x and y axes respectivley.



Let $C(\alpha, \beta)$ be the centre of the ellipse in any position. Here the position of centre C changes as the ellipse slides.

Let a and b be the semimajor and minor axes of the elipse.

Equation of the director circle of the ellipse is

$(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2$ ...(1)

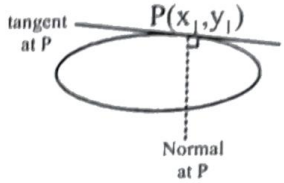
Since OX and OY are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, $O(0, 0)$ will lie on its director circle (1)

$\therefore \quad \alpha^2 + \beta^2 = a^2 + b^2$

Hence locus of $C(a, b)$ is $x^2 + y^2 = a^2 + b^2$ ...(2)

TANGENT TO THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

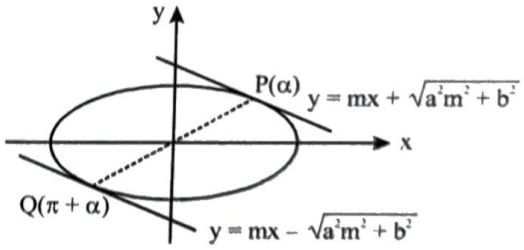
(a) Point form: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse at (x_1, y_1).



Since point (x_1, y_1) lie on the curve therefore we can use standard substitution to obtain the equation of tangent.

(b) Slope form: Let the given line is $y = mx + c$ and given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$c^2 = b^2 + a^2m^2$ or $c = \pm\sqrt{a^2m^2 + b^2}$



which is the required **condition of tangency**.

Substituting this value of c in $y = mx + c$, we have

$y = mx + \sqrt{a^2m^2 + b^2}$ or $y = mx - \sqrt{a^2m^2 + b^2}$, which are tangents to the ellipse for all values of m.

Here $\pm$ sign represents two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

The equation of any tangent to the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

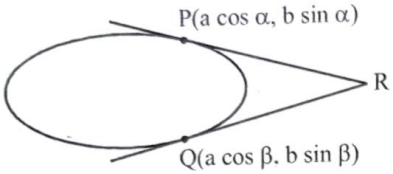
$(y - k) = m(x - h) \pm \sqrt{a^2m^2 + b^2}$

(c) Parametric form: $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

Note:

(i) Point of intersection of thse tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $P(\alpha)$ and $Q(\beta)$ with *C.O.C.* of the pair of tangents from $R(x_1, y_1)$ on the ellipse, where

$$x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \; ; \; y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$



(ii) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is π then the tangents at these points are parallel.

TRAIN YOUR BRAIN

Example 8: A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Sol. Given ellipse are $\frac{x^2}{4}+\frac{y^2}{1}=1$ ...(i)

and, $\frac{x^2}{6}+\frac{y^2}{3}=1$ ...(ii)

any tangent to (i) is $\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1$ ...(iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\frac{hx}{6}+\frac{ky}{3}=1$ ...(iv)

comparing (iii) and (iv), we get $\frac{\cos\theta}{h/3}=\frac{\sin\theta}{k/3}=1$

$\Rightarrow \cos\theta=\frac{h}{3}$ and $\sin\theta=\frac{k}{3}$ $\Rightarrow h^2+k^2=9$ locus of the point (h, k) is $x^2+y^2=9$

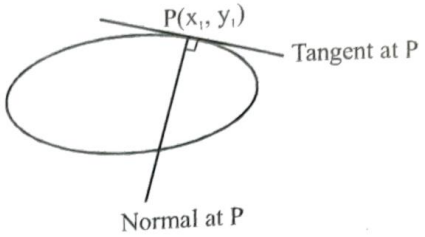
$\Rightarrow x^2+y^2=6+3=a^2+b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) **Point form:** Equation of the tangent to the ellipse at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

The slope of the tangent at $(x_1, y_1) = \frac{-x_1}{a^2}\times\frac{b^2}{y_1}$



$\therefore$ Slope of the normal at $(x_1, y_1) = \frac{a^2}{x_1}\times\frac{y_1}{b^2}=\frac{a^2y_1}{b^2x_1}$

Hence the equation of the normal at (x_1, y_1) is $y-y_1$

$=\frac{a^2y_1}{b^2x_1}(x-x_1)$

or $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$

(b) **Slope form:** Equation of a normal to the given ellipse whose slope is 'm' is $y=mx\mp\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

Note: Maximum four normals can be drawn to an ellipse from a given point.

(c) **Parametric form:** Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax\cdot\sec\theta - by\cdot\operatorname{cosec}\theta = (a^2-b^2)$.

TRAIN YOUR BRAIN

Example 9: If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Sol. The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \text{ or } \frac{ax}{e}-ay=a^2-b^2$$

It passes through one extremity of the minor axis whose co-ordinates are $(0, -b)$

$\therefore \quad 0 + ab = a^2 - b^2 \quad \Rightarrow \quad (a^2b^2) = (a^2 - b^2)^2$

$\Rightarrow \quad a^2.a^2(1 - e^2) = (a^2\, e^2)^2 \quad \Rightarrow \quad 1 - e^2 + e^4$

$\Rightarrow \quad e^4 + e^2 - 1 = 0 \quad \Rightarrow \quad (e^2)^2 + e^2 - 1 = 0$

$\therefore \quad e^2 = \dfrac{-1 \pm \sqrt{1+4}}{2} \quad \Rightarrow \quad e = \sqrt{\dfrac{\sqrt{5}-1}{2}}$

(taking positive sign)

REMARK

Conormal Points Lie on a Fixed Curve

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be the conormal points. Let normals at P, Q, R, S meet in $T(h, k)$.

Then equation of normal at $P(x_1, y_1)$ is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$

The point $T(h, k)$ lies on it

$$(a^2 - b^2)x_1y_1 + b^2kx_1 - a^2hy_1 = 0 \qquad ...(i)$$

Similarly, for points Q, R and S in equation (i), x_1, y_1 are just replaced by (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively.

$\therefore$ P, Q, R, S all lie on the curve $(a^2 - b^2)xy + b^2kx - a^2hy = 0$, which is called Apollonius a rectangular hyperbola.

Note:

(i) Condition of conormal points: If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ and $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha + \beta + \gamma + \delta = (2n+1)\pi$.

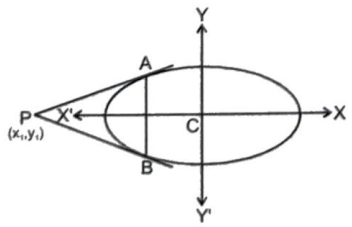
(ii) Condition of concyclic points: If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ and $D(\delta)$ are four concyclic point them sum of their eccentric angles is even multiple of π. i.e. $\alpha + \beta + \gamma + \delta = 2n\pi$.

PAIR OF TANGENTS

If $P(x_1, y_1)$ be any point lies outside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and a pair of tangents PA, PB can be drawn to it from P.

Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$, $T = \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$



i.e. $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right)\left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1\right) = \left(\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1\right)^2$

EQUATION OF CHORD WITH MID POINT (X_1, Y_1)

The equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, whose mid-point be (x_1, y_1) is $T = S_1$

where $T = \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$, $S_1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$

i.e. $\left(\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1\right) = \left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1\right)$

TRAIN YOUR BRAIN

Example 10: Find the locus of the mid - point of focal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Sol. Let $P = (h, k)$ be the mid-point

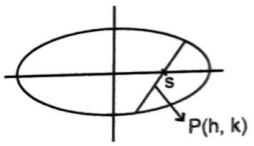
$\therefore$ Equation of chord whose mid-point is given

$$\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$

Since it is a focal chord, therefore it passes through focus, either $(ae, 0)$ or $(-ae, 0)$

If it passes through $(ae, 0)$

$\therefore$ locus is $\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



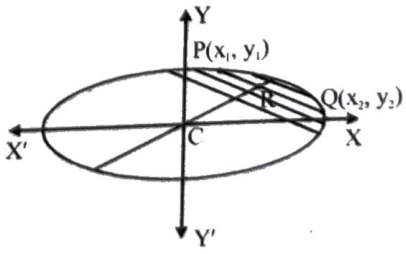
If it passes through $(-ae, 0)$

$\therefore$ locus is $-\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ **Ans.**

DIAMETER

The locus of the middle point of a system of parallel chords of an ellipse is called a diameter and the point where the diameter intersect the ellipse is called the vertex of the diameter.

Let $y = mx + c$ be system of parallel chords to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for different chords c varies, m remains constant.

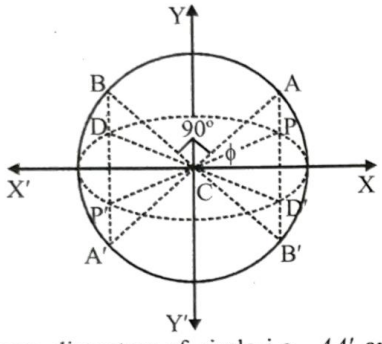


Hence locus of $R(h, k)$ is $y = -\frac{b^2 x}{a^2 m}$ which is diameter of the ellipse passing through (0, 0).

CONJUGATE DIAMETERS

Two diameters are said to be conjugate when each bisects all chords parallel to the other. If $y = mx$ and $y = m_1 x$ be two conjugate diameters of an ellipse then

$$mm_1 = \frac{b^2}{a^2}$$



Conjugate diameters of circle i.e., AA' and BB' are perpendicular to each other. Hence conjugate diameters of ellipse are PP' and DD'

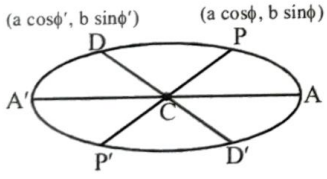
Hence angle between conjugate diameters of ellipse $\neq 90°$

Now the co-ordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$, $P'(-a\cos\phi - b\sin\phi)$,
$D(-a\sin\phi, b\cos\phi)$, $D'(a\sin\phi, -b\cos\phi)$

Properties of Conjugate Diameters

Prop. 1: The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.



$$\Rightarrow \quad \phi = \frac{\pi}{2} + \phi' \Rightarrow \qquad \phi - \phi' = \frac{\pi}{2}$$

Prop. 2: The sum of the squares of any two conjugate semi diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$

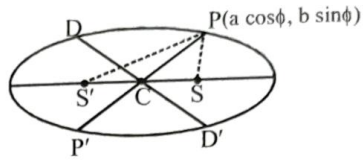
Let CP and CD be two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let eccentric angle of P is ϕ. The eccentric angle of D is $\frac{\pi}{2} + \phi$. So the co-ordinates of P and D are :

$(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$

i.e., $(-a\sin\phi, b\cos\phi)$ respectively

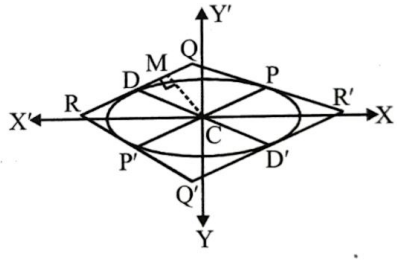
$$\therefore\ CP^2 + CD^2 = (a^2\cos^2\phi + b^2\sin^2\phi) + (a^2\sin^2\phi + b^2\cos^2\phi) = a^2 + b^2$$

Prop. 3: The product of the focal distances of a point on an ellipse is equal to the square of the semi diameter which is conjugate to the diameter through the point.



$$\begin{aligned} SP \cdot S'P &= (a - ae\cos\phi) \times (a + ae + \cos\phi) \\ &= a^2 - a^2 e^2 \cos^2\phi \\ &= a^2 - (a^2 - b^2)\cos^2\phi \qquad \{\because b^2 = a^2(1 - e^2)\} \\ &= a^2\sin^2\phi + b^2\cos^2\phi = CD^2 \qquad \{\because a^2 - b^2 = a^2e^2\} \end{aligned}$$

Prop. 4 : The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes.



Area of parallelogram $QRQ'R'$

$$= 4 \times \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)} \times \frac{ab}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

$$= 4ab\ (= \text{constant}) = (2a)(2b)$$

Prop. 5: The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is (h, k).

Let (h, k) be the point on the diameter $y = m_1 x$

$\therefore \quad m_1 = k/h$

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \text{ i.e., } T = S_1$$

which is equation of chord of the ellipse if mid point (h, k).

EQUI-CONJUGATE DIAMETERS

Two conjugate diameters are called equi-conjugate if their lengths are equal. In such cases therefore.

$(CP)^2 = (CD)^2$

$\Rightarrow (a^2 - b^2)\cos^2\phi = 0$

$\because (a^2 - b^2) \neq 0$

$\therefore \cos^2\phi = 0$

$\therefore \phi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

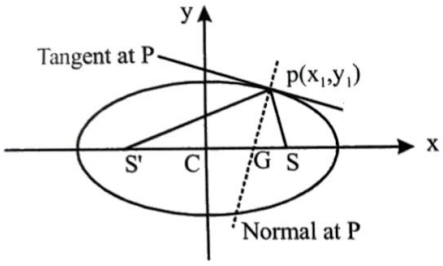
$\therefore (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}}$

IMPORTANT HIGHLIGHTS

1. Let the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \ldots(i)$$

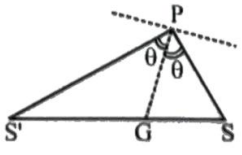
& Normal at $P(x_1, y_1)$ is



$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} \qquad \ldots(ii)$$

The normal meet x-axis at $G \Rightarrow$ Put $y = 0$ in equation (ii) we get $CG = x = \left(\frac{a^2 - b^2}{a^2}\right) x_1 = e^2 x_1$

$\therefore SG = CS - CG = ae - e^2x_1 = e(a - ex_1) = eSP$



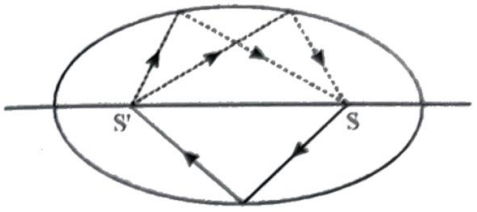
Similarly $S'G = eS'P$

$\therefore \frac{SG}{S'G} = \frac{eSP}{eS'P} = \frac{SP}{S'P}$

$\Rightarrow$ PG is bisector of angle $\angle P$ in ΔSPS.

Tangent & normal at any point P bisect the external & internal angles between the focal distances of SP & SP.

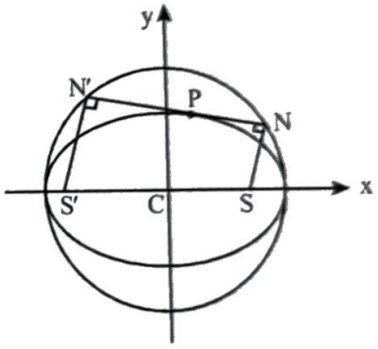
This lead to reflection property of ellipse.



If incoming light ray passes through focus S'(or S), strike the concave side of ellipse the after reflection it will pass through other focus.

2. Let ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

its tangent is $y - mx = \sqrt{a^2m^2 + b^2}$ ...(ii)



Equation of perpendicular to above the passes through focus $(ae, 0)$ is

$$my + x = ae \qquad \ldots(iii)$$

Eliminate m from (ii) & (iii) we will get focus of intersection point.

For that square & add (ii) & (iii) we will get an answer

$\therefore y^2 + m^2x^2 - 2mxy + x^2 + my^2 + 2mxy = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow x^2(1 + m^2) + y^2(1 + m^2) = a^2m^2 + a^2$

$\Rightarrow x^2 + y^2 = a^2$ which is the auxilairy circle

The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.

3. From previous equation of any tangent is $mx - y + \sqrt{a^2m^2 + b^2} = 0$

$$SN = \left|\frac{\sqrt{a^2m^2 + b^2} + ame}{\sqrt{1 + m^2}}\right|$$

& $S'N' = \left|\dfrac{\sqrt{a^2m^2+b^2}-ame}{\sqrt{1+m^2}}\right|$

$$SN \cdot S'N' = \frac{(a^2m^2+b^2)-a^2m^2e^2}{(1+m^2)}$$

$$= \frac{a^2m^2+b^2-(a^2-b^2)m^2}{1+m^2} = b^2$$

$\because \quad a^2e^2 = a^2 - b^2$

The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.

4. Equation of tangent at $L\left(ae, \dfrac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2} + \frac{y\left(\frac{b^2}{a}\right)}{b^2} = 1 \Rightarrow xe + y = a \qquad \ldots(i)$$

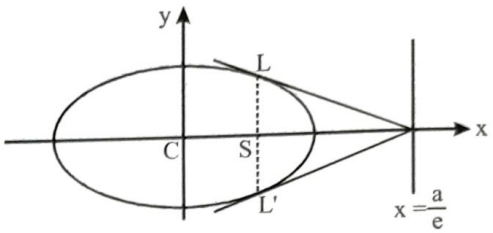
& equation of tangent at $L'\left(ae, -\dfrac{b^2}{a}\right)$ is

$$ex - y = a \qquad \ldots(ii)$$

Solve (i) & (ii) we get $x = \dfrac{a}{e}$ & $y = 0$

i.e. at the directrix of ellipse.

Tangents at the extremities of latus-rectum of an ellipse intersect on the corresponding directrix.



5. If P be any point on the ellipse with S & S' as its foci then $\ell(SP) + \ell(S'P) = 2a$.

6. If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively & if CF be perpendicular upon this normal then

   (i) $PF \cdot PG = b^2$ (ii) $PF \cdot Pg = a^2$

7. The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

Topicwise Questions

EQUATION OF ELLIPSE

1. The equation to the ellipse (referred to its axes as the axes of x and y respectively) whose foci are $(\pm 2, 0)$ and eccentricity 1/2, is

(a) $\frac{x^2}{12}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}+\frac{y^2}{12}=1$

(c) $\frac{x^2}{16}+\frac{y^2}{8}=1$ (d) None of these

2. The eccentricity of the ellipse $9x^2+5y^2-30y=0$ is-

(a) 1/3 (b) 2/3

(c) 3/4 (d) None of these

3. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is

(a) 8 (b) 12

(c) 18 (d) 24

4. The distance between the foci of the ellipse $3x^2+4y^2=48$ is

(a) 2 (b) 4

(c) 6 (d) 8

5. The equation of the ellipse whose one of the vertices is (0,7), centre at (0, 0) and the corresponding directrix is $y=12$, is

(a) $95x^2+144y^2=4655$ (b) $144x^2+95y^2=4655$

(c) $95x^2+144y^2=13680$ (d) $144x^2+95y^2=13680$

6. The foci of $16x^2+25y^2=400$ are

(a) $(\pm 3, 0)$ (b) $(0, \pm 3)$

(c) $(3, -3)$ (d) $(-3, 3)$

7. The locus of a variable point whose distance from $(-2, 0)$ is $\frac{2}{3}$ times its distance from the line $x=-\frac{9}{2}$, is

(a) Ellipse (b) Parabola

(c) Hyperbola (d) Circle

8. If the eccentricity of the two ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of $|a/b|$ (where $|a|>|b|$) is

(a) 5/13 (b) 6/13

(c) 13/5 (d) 13/6

9. The centre of the ellipse $4x^2+9y^2-16x-54y+61=0$ is

(a) (1,3) (b) (2, 3)

(c) (3,2) (d) (3, 1)

10. Latus rectum of ellipse $4x^2+9y^2-8x-36y+4=0$ is

(a) 8/3 (b) 4/3

(c) $\frac{\sqrt{5}}{3}$ (d) 16/3

11. The equation of the ellipse whose centre is (2, –3), one of the foci is (3, –3) and the corresponding vertex is (4, –3) is

(a) $\frac{(x-2)^2}{3}+\frac{(y+3)^2}{4}=1$

(b) $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$

(c) $\frac{x^2}{3}+\frac{y^2}{4}=1$

(d) $\frac{(x-3)^2}{4}+\frac{(y-2)^2}{3}=1$

12. The equation $14x^2-4xy+11y^2-44x-58y+71=0$ represents

(a) A circle (b) An ellipse

(c) A hyperbola (d) A rectangular hyperbola

13. The foci of the ellipse $25(x+1)^2+9(y+2)^2=225$ are at

(a) (–1, 2) and (–1, –6) (b) (–1, 2) and (6, 1)

(c) (1, –2) and (1, –6) (d) (–1, –2) and (1, 6)

14. The co-ordinates of the foci of the ellipse $3x^2+4y^2-12x-8y+4=0$ are

(a) (1, 2), (3, 4) (b) (1, 4), (3, 1)

(c) (1, 1), (3, 1) (d) (2, 3), (5, 4)

15. For the ellipse $25x^2+9y^2-150x-90y+225=0$ the eccentricity $e=$

(a) 2/5 (b) 3/5

(c) 4/5 (d) 1/5

16. The eccentric angles of the extremities of latus recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $(|a|>|b|)$

(a) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$ (b) $\tan^{-1}\left(\pm\frac{be}{a}\right)$

(c) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$ (d) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

17. If the foci of an ellipse are $(\pm\sqrt{5}, 0)$ and its eccentricity is $\frac{\sqrt{5}}{3}$, then the equation of the ellipse is

(a) $9x^2 + 4y^2 = 36$ (b) $4x^2 + 9y^2 = 36$

(c) $36x^2 + 9y^2 = 4$ (d) $9x^2 + 36y^2 = 4$

PARAMATRIC EQUATION

18. The parametric representation of a point on the ellipse whose foci are $(-1, 0)$ and $(7, 0)$ and eccentricity 1/2 is-

(a) $(3 + 8\cos\theta, 4\sqrt{3}\sin\theta)$

(b) $(8\cos\theta, 4\sqrt{3}\sin\theta)$

(c) $(3 + 4\sqrt{3}\cos\theta, 8\sin\theta)$

(d) None of these

AUXILLARY CIRCLE, POINT AND ELLIPSE, CHORD JOINING TWO POINTS

19. The position of the point $(4, -3)$ with respect to the ellipse $2x^2 + 5y^2 = 20$ is

(a) Outside the ellipse (b) On the ellipse

(c) On the major axis (d) Inside the ellipse

TANGENT AND NORMAL

20. The equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$ are

(a) $y = 3x \pm \sqrt{\frac{155}{3}}$ (b) $y = 3x \pm \sqrt{\frac{155}{12}}$

(c) $y = 3x \pm \sqrt{\frac{95}{12}}$ (d) None of these

21. Find the equations of tangents to the ellipse $9x^2 + 16y^2 = 144$ which pass through the point (2,3).

(a) $y = 3$ and $y = -x + 5$ (b) $y = 5$ and $y = -x + 3$

(c) $y = 3$ and $y = x - 5$ (d) None of these

22. The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is

(a) 27/4 sq. unit (b) 9 sq. unit

(c) 27/2 sq. unit (d) 27 sq. unit

23. The line $lx + my - n = 0$ will be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

(a) $a^2l^2 + b^2m^2 = n^2$ (b) $al^2 + bm^2 = n^2$

(c) $a^2l + b^2m = n$ (d) $al^2 + bm^2 = n$

24. The angle between the tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2), is

(a) $\tan^{-1}\left(\frac{12}{5}\right)$ (b) $\tan^{-1}(6\sqrt{5})$

(c) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (d) $\tan^{-1}(12\sqrt{5})$

25. The locus of the point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is

(a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = 4$

(c) $x^2 + y^2 = 13$ (d) $x^2 + y^2 = 5$

26. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$, are

(a) $y = \pm 3$ (b) $x = \pm\sqrt{5}$

(c) $y = 0, y = 6$ (d) $x = \pm\sqrt{6}$

27. The equation of normal at the point (0, 3) of the ellipse $9x^2 + 5y^2 = 45$ is

(a) $y - 3 = 0$ (b) $y + 3 = 0$

(c) x-axis (d) y-axis

28. The equation of tangent and normal at point (3, –2) of ellipse $4x^2 + 9y^2 = 36$ are

(a) $\frac{x}{3} - \frac{y}{2} = 1, \frac{x}{2} + \frac{y}{3} = \frac{5}{6}$

(b) $\frac{x}{3} + \frac{y}{2} = 1, \frac{x}{2} - \frac{y}{3} = \frac{5}{6}$

(c) $\frac{x}{2} + \frac{y}{3} = 1, \frac{x}{3} - \frac{y}{2} = \frac{5}{6}$

(d) $\frac{x}{2} + \frac{y}{3} = 1, \frac{x}{3} + \frac{y}{2} = \frac{5}{6}$

29. The value of λ, for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{3}{2}$ (d) $\frac{3}{8}$

Learning Plus

1. The equation of the ellipse whose focus is $(1, -1)$, directrix is the line $x - y - 3 = 0$ and the eccentricity is $\frac{1}{2}$, is

   (a) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

   (b) $7x^2 + 2xy + 7y^2 + 7 = 0$

   (c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

   (d) None of these

2. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is

   (a) $\frac{5}{6}$ (b) $\frac{3}{5}$

   (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{5}}{3}$

3. If distance between the directrices be thrice the distance between the focii, then eccentricity of ellipse is

   (a) $\frac{1}{2}$ (b) $\frac{2}{3}$

   (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{4}{5}$

4. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if

   (a) $r > 2$ (b) $2 < r < 5$

   (c) $r > 5$ (d) $r \in \{2, 5\}$

5. If the eccentricity of an ellipse be 5/8 and the distance between its foci be 10, then its latus rectum is

   (a) $\frac{39}{4}$ (b) 12

   (c) 15 (d) $\frac{37}{2}$

6. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$, is

   (a) ellipse (b) parabola

   (c) hyperbola (d) circle

7. An ellipse having foci at (3, 3) and (–4, 4) and passing through the origin has eccentricity equal to

   (a) $\frac{3}{7}$ (b) $\frac{2}{7}$

   (c) $\frac{5}{7}$ (d) $\frac{3}{5}$

8. Let P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ with focii at S and S'. If A be the area of triangle PSS', then the maximum value of A is

   (a) 24 sq. units (b) 12 sq. units

   (c) 36 sq. units (d) none of these

9. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

   (a) outside the ellispe (b) on the ellipse

   (c) on the major axis (d) on the minor axis

10. If the distance of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ from the centre is 2, then the eccentric angle is

    (a) $\pi/3$ (b) $\pi/4$

    (c) $\pi/6$ (d) $\pi/2$

11. The eccentricity of the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is decreasing at the rate of 0.1/second due to change in semi minor axis only. The time at which ellipse become auxiliary circle is

    (a) 2 seconds (b) 3 seconds

    (c) 4 seconds (d) 5 seconds

12. If $\tan\theta_1 \cdot \tan\theta_2 = -\frac{a^2}{b^2}$ then the chord joining two points θ_1 & θ_2 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will subtend a right angle at

    (a) focus (b) centre

    (c) end of the major axis (d) end of the minor axis

13. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point P, then eccentric angle of P is

    (a) 0 (b) 45°

    (c) 60° (d) 90°

14. If the line $y = 2x + c$ be a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, then c is equal to

    (a) ± 4 (b) ± 6

    (c) ± 1 (d) ± 8

15. If the line $3x + 4y = -\sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$ then, the point of contact is

(a) $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ (b) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

(c) $\left(\frac{1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$ (d) $\left(\frac{-1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$

16. If the line $x\cos\alpha + y\sin\alpha = p$ be normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

(a) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = a^2 - b^2$

(b) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = (a^2 - b^2)^2$

(c) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = a^2 - b^2$

(d) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = (a^2 - b^2)^2$

17. The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ included between the co-ordinate axes is the curve

(a) $9x^2 + 16y^2 = 4x^2y^2$

(b) $16x^2 + 9y^2 = 4x^2y^2$

(c) $3x^2 + 4y^2 = 4x^2y^2$

(d) $9x^2 + 16y^2 = x^2y^2$

18. Which of the following is the common tangent to the ellipses $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ & $\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1$?

(a) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$

(b) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(c) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(d) $by = ax - \sqrt{a^4 - a^2b^2 + b^4}$

19. The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive end of latus rectum is

(a) $x + ey + e^2a = 0$ (b) $x - ey - e^3a = 0$

(c) $x - ey - e^2a = 0$ (d) None of these

20. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its corresponding point Q on the auxiliary circle, lies on the line :

(a) $x = a/e$ (b) $x = 0$

(c) $y = 0$ (d) None

21. The angle between the tangents drawn from the point $(2, 2)$ to the ellipse, $3x^2 + 5y^2 = 15$ is:

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

22. The eccentric angle of the point where the line, $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\tan^{-1} 2$

23. PQ is a double ordinate of the ellipse $x^2 + 9y^2 = 9$, the normal at P meets the diameter through Q at R, then the locus of the mid point of PR is

(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

24. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point $(2, 1)$ is

(a) $4x + 5y + 13 = 0$ (b) $4x + 5y = 13$

(c) $5x + 4y + 13 = 0$ (d) $5x + 4y = 13$

25. Q is a point on the auxiliary circle of an ellipse. P is the corresponding point on ellipse. N is the foot of perpendicular from focus S, to the tangent of auxiliary circle at Q. Then

(a) $SP = SN$ (b) $SP = PQ$

(c) $PN = SP$ (d) $NQ = SP$

26. Q is a point on the auxiliary circle corresponding to the point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If T is the foot of the perpendicular dropped from the focus S onto the tangent to the auxiliary circle at Q then the ΔSPT is

(a) isosceles (b) equilateral

(c) right angled (d) right isosceles

27. If F_1 & F_2 are the feet of the perpendiculars from the focii S_1 & S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to :

(a) 2 (b) 3

(c) 4 (d) 5

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. A latus rectum of an ellipse is a line
   (*a*) passing through a focus
   (*b*) through the centre
   (*c*) perpendicular to the major axis
   (*d*) parallel to the minor axis

2. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = C$.
   (*a*) cannot represent a real pair of straight lines for any value of C
   (*b*) represents an ellipse, if $C > 0$
   (*c*) no locus, if $C < 0$
   (*d*) a point, if $C = 0$

3. If the distance between the focii of an ellipse is equal to the length of its latus rectum, the eccentricity of the ellipse is:
   (*a*) $\frac{\sqrt{5}+1}{2}$ (*b*) $\frac{\sqrt{5}-1}{2}$
   (*c*) $\frac{\sqrt{5}-2}{2}$ (*d*) $\frac{2}{\sqrt{5}+1}$

4. If P is a point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose focii are S and S'. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then
   (*a*) $PS + PS' = 2a$, if $a > b$
   (*b*) $PS + PS' = 2b$, if $a < b$
   (*c*) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$
   (*d*) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{a^2-b^2}}{b^2}\left[a - \sqrt{a^2-b^2}\right]$ when $a > b$

5. If the chord through the points whose eccentric angles are θ & ϕ on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the focus, then the value of $\tan(\theta/2)\tan(\phi/2)$ is
   (*a*) $\frac{e+1}{e-1}$ (*b*) $\frac{e-1}{e+1}$
   (*c*) $\frac{1+e}{1-e}$ (*d*) $\frac{1-e}{1+e}$

6. Point/points, from which tangents to the ellipse $5x^2 + 4y^2 = 20$ are perpendicular, is/are:
   (*a*) $(1, 2\sqrt{2})$ (*b*) $(2\sqrt{2}, 1)$
   (*c*) $(2, \sqrt{5})$ (*d*) $(\sqrt{5}, 2)$

7. The tangent at any point 'P'
   (*a*) $(AV)(A'V') = b^2$
   (*b*) $(AV)(A'V') = a^2$
   (*c*) $\angle V'SV = 90^\circ$
   (*d*) $V'S'SV$ is a cyclic quadrilateral

8. Identify the statements which are True.
   (*a*) the equation of the director circle of the ellipse, $5x^2 + 9y^2 = 45$ is $x^2 + y^2 = 14$.
   (*b*) the sum of the focal distances of the point $(0, 6)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{36} = 1$ is 10.
   (*c*) the point of intersection of any tangent to a parabola & the perpendicular to it from the focus lies on the tangent at the vertex.
   (*d*) the line through focus and $(at^2_1, 2at_1)$ on $y^2 = 4ax$, meets it again in the point $(at^2_2, 2at_2)$ if $t_1t_2 = -1$.

9. A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is less than
   (*a*) 12 (*b*) 13
   (*c*) 14 (*d*) 15

10. Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If $OF = 6$ and the diameter of the inscribed circle of triangle OCF is 2, then the product $(AB)(CD)$ is less than
    (*a*) 65 (*b*) 52
    (*c*) 78 (*d*) 87

Comprehension – 1 (Q. No. 11 to 13)

$\frac{(3x-4y+10)^2}{2} + \frac{(4x+3y-15)^2}{3} = 1$ is an ellipse

11. Major and minor axes of the ellipse are
    (*a*) 6 and 4 (*b*) 150 and 100
    (*c*) $10\sqrt{3}$ and $10\sqrt{2}$ (*d*) $\frac{2\sqrt{3}}{5}$ and $\frac{2\sqrt{2}}{5}$

12. Eccentricity of the ellipse is
    (*a*) $\frac{1}{\sqrt{3}}$ (*b*) $\frac{\sqrt{3}}{5}$
    (*c*) $\frac{\sqrt{2}}{5}$ (*d*) None of these

13. Centre of the ellipse is

(a) $(0, 0)$ (b) $\left(\frac{6}{5}, \frac{17}{5}\right)$

(c) $\left(\frac{\sqrt{2}}{5}, \frac{\sqrt{3}}{5}\right)$ (d) None of these

Comprehension – 2 (Q. No. 14 to 16)

Two tangents PA and PB are drawn from a point $P(h, k)$ to the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$. Angle of the tangents with the positive x-axis are θ_1 and θ_2. Normals at A and B are intersecting at Q point.

On the basis of above information answer the following questions.

14. Locus of P, if $\tan\theta_1 . \tan\theta_2 = 4$, is

(a) $\frac{y-b}{x-a} = 2(x+a)$ (b) $y^2 - b^2 = 2(x^2 + a^2)$

(c) $\frac{y+b}{x+a} = \frac{4(x-a)}{y-b}$ (d) $\frac{y+b}{x-b} = \frac{x+a}{y-b}$

15. Circumcentre of ΔQAB is

(a) mid point of AB (b) mid point of PQ

(c) orthocentre of ΔPAB (d) can't say

16. Locus of P, if $\cot\theta_1 + \cot\theta_2 = \lambda$, is

(a) $2xy = \lambda(y^2 - b^2)$

(b) $2xy - \lambda(b^2 - y^2) = 0$

(c) $xy = \lambda$

(d) $x^2 + xy = \lambda$

17. A tangent having slope $-\frac{4}{3}$ touches the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$ at point P and intersects the major and minor axes at A & B respectively, O is the centre of the ellipse

Column - I	**Column - II**
(a) Distance between the parallel tangents having slopes $-\frac{4}{3}$, is	(p) 24
(b) Area of ΔAOB is	(q) 7/24
(c) If the tangent in first quadrant touches the ellipse at (h, k) then value of hk is	(r) 48/5
(d) If equation of the tangent intersecting positive axes is $\ell x + my = 1$, then $\ell + m$ is equal to	(s) 12

18. **Match the column:**

Column – I	**Column – II**
(a) If the mid point of a chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is $(0, 3)$, then length of the chord is $\frac{4k}{5}$, then k is	(p) 6
(b) Eccentric angle of one of the points on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is $\frac{k\pi}{4}$, then k is	(q) 8
(c) If the distance between a focus and corresponding directix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$, then length of the minor axis is $\frac{k}{\sqrt{3}}$, then k is	(r) 3
(d) Sum of distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ from the focii	(s) 16

NUMERICAL TYPE

19. The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of coordinates is $\frac{2\sqrt{6}}{\lambda}$. Find λ

20. The length of major axis of the ellipse $(5x-10)^2+(5y+15)^2 = \frac{(3x-4y+7)^2}{4}$ is

21. The distance between the directrices of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ is $\frac{a}{\sqrt{b}}$, then a/b equals

22. An ellipse has foci at $F_1(9, 20)$ and $F_2(49, 55)$ in the xy-plane and is tangent to the x-axis. Find the length of its major axis.

23. Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If $OF = 6$ & the diameter of the inscribed circle of triangle OCF is 2. Find the product $(AB)(CD)$.

24. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32} = 1$ intersects the major & minor axes in points A & B respectively. If C is the centre of the ellipse find the area (sq. units) of the triangle ABC.

25. An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Find the radius of the circle.

26. Given the equation of the ellipse $\frac{(x-3)^2}{16}+\frac{(y+4)^2}{49}=1$, a parabola is such that its vertex is the lowest point of the ellipse and it passes through the ends of the minor axis of the ellipse. The equation of the parabola is in the form $16y$ $16y = a(x-h)^2 - k$. Determine the value of $(a+h+k)$.

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directices is $x=-4$, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is: [*JEE Main-2017*]

   (*a*) $x+2y=4$ (*b*) $2y-x=2$
   (*c*) $4x-2y=1$ (*d*) $4x+2y=7$

2. If the curves $y^2=6x$, $9x^2+by^2=16$ intersect each other at right angles, then the value of b is: [*JEE Main-2018*]

   (*a*) $\frac{7}{9}$ (*b*) 4
   (*c*) $\frac{9}{2}$ (*d*) 6

3. Let $S=\left\{(x,y)\in R^2 : \frac{y^2}{1+r}-\frac{x^2}{1-r}=1\right\}$, where $r \neq \pm 1$. Then S represents. [*JEE Main-2019 (January)*]

   (*a*) a hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0<r<1$
   (*b*) an ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r>1$.
   (*c*) a hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0<r<1$.
   (*d*) an ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, when $r>1$.

4. Let the length of the latus rectum of an ellipse with its major axis long $x-$axis and center at the origin, be 8. If the distance the foci of this ellipse is equal to the length of the length of its minor axis, then which one of hte following points lies on it? [*JEE Main-2019 (January)*]

   (*a*) $(4\sqrt{2},2\sqrt{2})$ (*b*) $(4\sqrt{3},2\sqrt{2})$
   (*c*) $(4\sqrt{3},2\sqrt{3})$ (*d*) $(4\sqrt{2},2\sqrt{3})$

5. If tangents are drawn to the ellipse $x^2+2y^2=2$ at all points on the ellipse other than its four vertices than the mid points of the tangents intercepted between the coordinate axes lie on the curve: [*JEE Main-2019 (January)*]

   (*a*) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$ (*b*) $\frac{x^2}{4}+\frac{y^2}{2}=1$
   (*c*) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (*d*) $\frac{x^2}{2}+\frac{y^2}{4}=1$

6. Let S and S' be the foci of an ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and area $(\Delta S'BS) = 8$ *sq.* units, then the length of a latus rectum of the ellipse is : [*JEE Main-2019 (January)*]

   (*a*) 4 (*b*) $2\sqrt{2}$
   (*c*) $4\sqrt{2}$ (*d*) 2

7. If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points (1, 2) and (a, b) are perpendicular to each other, then a^2 is equal to: [JEE Main-2019 (April)]

(a) $\frac{64}{17}$ (b) $\frac{2}{17}$

(c) $\frac{128}{17}$ (d) $\frac{4}{17}$

8. In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latus rectum is: [JEE Main-2019 (April)]

(a) 10 (b) 8

(c) 5 (d) 6

9. The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meet the x-axis at Q and R, respectively. Then the area (in sq. units) of the triangle PQR is: [JEE Main-2019(April)]

(a) $\frac{14}{3}$ (b) $\frac{16}{3}$

(c) $\frac{68}{15}$ (d) $\frac{34}{15}$

10. If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left(3, \frac{-9}{2}\right)$, then the length of the latus rectum of the ellipse is : [JEE Main-2019(April)]

(a) 9 (b) $8\sqrt{3}$

(c) $12\sqrt{2}$ (d) 5

11. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent of the ellipse at P passes through $Q(4,4)$ then PQ is equal to : [JEE Main-2019 (April)]

(a) $\frac{\sqrt{221}}{2}$ (b) $\frac{\sqrt{157}}{2}$

(c) $\frac{\sqrt{61}}{2}$ (d) $\frac{5\sqrt{5}}{2}$

12. An ellipse, with foci at (0, 2) and (0, –2) and minor axis of length 4, passes through which of the following points? [JEE Main-2019 (April)]

(a) $(1, 2\sqrt{2})$ (b) $(2, \sqrt{2})$

(c) $(2, 2\sqrt{2})$ (d) $(\sqrt{2}, 2)$

13. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is [JEE Main-2020 (January)]

(a) $\sqrt{3}$ (b) $3\sqrt{2}$

(c) $\frac{3}{\sqrt{2}}$ (d) $2\sqrt{3}$

14. If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in R$ then the distance between the foci of the ellipse is: [JEE Main-2020 (January)]

(a) $2\sqrt{2}$ (b) $2\sqrt{7}$

(c) 4 (d) $2\sqrt{5}$

15. Let the line $y = mx$ and the ellipse $2x^2 + y^2 = 1$ intersect at a point P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(\frac{-1}{3\sqrt{2}}, 0\right)$ and $(0, \beta)$, then β is equal to [JEE Main-2020(January)]

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{2}}{3}$

(c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

16. The length of the minor axis (along y-axis) of an ellipse in the standard form is $\frac{4}{\sqrt{3}}$. If this ellipse touches the line, $x + 6y = 8$; then its eccentricity is: [JEE Main-2020(January)]

(a) $\frac{1}{3}\sqrt{\frac{11}{3}}$ (b) $\sqrt{\frac{5}{6}}$

(c) $\frac{1}{2}\sqrt{\frac{11}{3}}$ (d) $\frac{1}{2}\sqrt{\frac{5}{3}}$

17. Let $x = 4$ be a directrix to an ellipse whose centre is at the origin and its eccentricity is $\frac{1}{2}$. If $P(1, \beta), \beta > 0$ is a point on this ellipse, then the equation of the normal to it at P is [JEE Main-2020(September)]

(a) $7x - 4y = 1$ (b) $4x - 2y = 1$

(c) $4x - 3y = 2$ (d) $8x - 2y = 5$

18. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ be a given ellipse, length of whose latus rectum is 10. If its eccentricity is the maximum value of the function, $\phi(t) = \frac{5}{12} + t - t^2$, then $a^2 + b^2$ is equal to: [JEE Main-2020 (September)]

(a) 135 (b) 116

(c) 126 (d) 145

19. If the point P on the curve, $4x^2 + 5y^2 = 20$ is farthest from the point $Q(0, -4)$, then PQ^2 is equal to : [JEE Main-2020(September)]

(a) 29 (b) 48

(c) 21 (d) 36

20. If the co-ordinates of two points A and B are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ respectively and P is any point on the conic, $9x^2 + 16y^2 = 144$, then $PA + PB$ is equal to:
[JEE Main-2020(September)]
(a) 9 (b) 16
(c) 6 (d) 8

21. If the normal at an end of a latus rectum of an ellipse passes through an extremity of the minor axis, then the eccentricity e of the ellipse satisfies:
[JEE Main-2020(September)]
(a) $e^2 + 2e - 1 = 0$ (b) $e^2 + e - 1 = 0$
(c) $e^4 + 2e^2 - 1 = 0$ (d) $e^4 + e^2 - 1 = 0$

22. Which of the following points lies on the locus of the foot of perpendicular drawn upon any tangent to the ellipse, $\frac{x^2}{4} + \frac{y^2}{2} = 1$ from any of its foci?
[JEE Main-2020(September)]
(a) $(1, 2)$ (b) $(-2, \sqrt{3})$
(c) $(-1, \sqrt{3})$ (d) $(-1, \sqrt{2})$

23. If the curve $x^2 + 2y^2 = 2$ intersects the line $x + y = 1$ at two points P and Q, then the angle subtended by the line segment PQ at the origin is:
[JEE Main-2021 (Fabruary)]
(a) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{4}\right)$ (b) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{4}\right)$
(c) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{3}\right)$ (d) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)$

24. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is
[JEE Main-2021 (Fabruary)]

25. If the point of intersections of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = 4b$, $b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to:
[JEE Main-2021 (March)]
(a) 12 (b) 5
(c) 6 (d) 10

26. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at $(6, 2)$. If L is also a tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$, then the value of b is equal to;
[JEE Main-2021 (March)]
(a) 11 (b) 14
(c) 16 (d) 20

JEE-ADVANCED PREVIOUS YEARS

1. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is
[IIT JEE-2012]
(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

2. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin $O(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this elipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE?
[JEE Advanced-2018]
(a) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectrum is 1
(b) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$
(c) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$
(d) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

3. Define the collections $\{E_1, E_2, E_3, \ldots\}$ of ellipes and $\{R_1, R_2, R_3, \ldots\}$ of rectangles as follows:
$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$;
R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1;
[JEE Advanced-2019]
E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;
R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.
Then which of the following options is/are correct?
(a) The eccentricities of E_{18} and E_{19} are NOT equal
(b) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$
(c) The length of latus rectum of E_9 is $\frac{1}{6}$
(d) $\sum_{n=1}^{N}$ (area of R_n) < 24, for each positive integer N

4. Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let $M(P, Q)$ be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is _____
[JEE Advance-2021]

ANSWER KEY

Topicwise Questions

1. (*b*) **2.** (*b*) **3.** (*c*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*a*) **8.** (*c*) **9.** (*b*) **10.** (*a*) **11.** (*b*) **12.** (*b*)
13. (*a*) **14.** (*c*) **15.** (*c*) **16.** (*c*) **17.** (*b*) **18.** (*a*) **19.** (*a*) **20.** (*b*) **21.** (*a*) **22.** (*d*) **23.** (*a*) **24.** (*c*)
25. (*c*) **26.** (*c*) **27.** (*d*) **28.** (*a*) **29.** (*a*)

Learning Plus

1. (*a*) **2.** (*d*) **3.** (*c*) **4.** (*b*) **5.** (*a*) **6.** (*a*) **7.** (*c*) **8.** (*b*) **9.** (*c*) **10.** (*b*) **11.** (*d*) **12.** (*b*)
13. (*b*) **14.** (*b*) **15.** (*d*) **16.** (*d*) **17.** (*a*) **18.** (*b*) **19.** (*b*) **20.** (*c*) **21.** (*d*) **22.** (*b*) **23.** (*c*) **24.** (*b*)
25. (*a*) **26.** (*a*) **27.** (*b*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *c*, *d*) **2.** (*a*, *b*, *c*, *d*) **3.** (*b*, *d*) **4.** (*a*, *b*, *c*) **5.** (*a*, *b*) **6.** (*a*, *b*, *c*, *d*) **7.** (*a*, *c*, *d*)
8. (*a*, *c*, *d*) **9.** (*c*) **10.** (*a*) **11.** (*d*) **12.** (*a*) **13.** (*b*) **14.** (*c*) **15.** (*b*) **16.** (*a*)
17. (*a*) → (*r*), (*b*) → (*p*), (*c*) → (*s*), (*d*) → (*q*) **18.** (*a*) → (*q*), (*b*) → (*r*), (*c*) → (*s*), (*d*) → (*q*)
19. [0007] **20.** [6.67] **21.** [3.6] **22.** [85] **23.** [65] **24.** [24] **25.** [2] **26.** [186]

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*c*) **2.** (*c*) **3.** (*b*) **4.** (*b*) **5.** (*c*) **6.** (*a*) **7.** (*b*) **8.** (*c*) **9.** (*c*) **10.** (*a*) **11.** (*d*) **12.** (*d*)
13. (*b*) **14.** (*b*) **15.** (*b*) **16.** (*c*) **17.** (*b*) **18.** (*c*) **19.** (*d*) **20.** (*d*) **21.** (*d*) **22.** (*c*) **23.** (*a*) **24.** (3)
25. (*a*) **26.** (*b*)

PRAYAS 2.0 t.me/realprayasians2

JEE-ADVANCED PREVIOUS YEARS

1. (*c*) **2.** (*a*, *c*) **3.** (*c*, *d*) **4.** (4)

15 PARABOLA

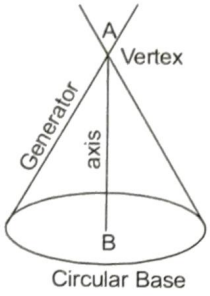
PRAYAS 1.0 time/real projections

CONIC SECTION

Conic sections are the curve generated by intersection of a plane with one or two right circular cone.

Section of right circular cone by different planes

A right circular cone is as shown in the figure–1



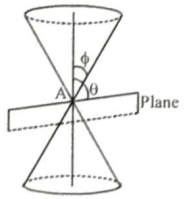
Case-1: When plane is passing through *A*

(*i*) When $\theta > \phi$: Two imaginary lines

(*ii*) When $\theta = \phi$: Two real and coincident lines

(*iii*) When $\theta < \phi$: Two real and distinct lines

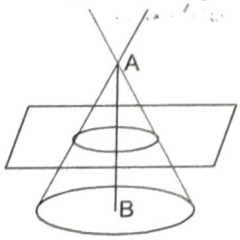
Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure



Case-2 : When plane is not passing through *A*

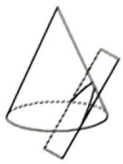
(*i*) When $\theta = 90°$: Circle

Section of a right circular cone by a plane parallel to its base is a circle as shown in figure.

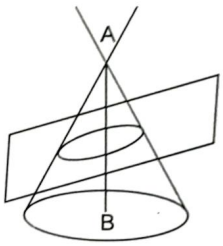


(*ii*) When $\theta = \phi$: Parabola

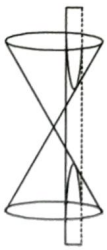
Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in figure.



(*iii*) When $\theta < \phi < 90°$: Ellipse



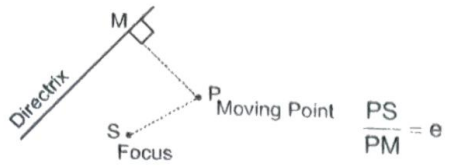
(*iv*) When $0 \le \theta < \phi$: Hyperbola



Mathematical definition

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.



- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

If S is (p, q) & directrix is $\ell x + my + n = 0$

then $PS = \sqrt{(x-p)^2 + (y-q)^2}$

& $PM = \dfrac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$$\frac{PS}{PM} = e \Rightarrow (\ell^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(\ell x + my + n)^2$$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is

$$(l^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

DISTINGUISHING BETWEEN THE CONIC

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (*i*) When the focus lies on the directrix

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will be coincident.

$e < 1$ the lines will be imaginary.

Case (*ii*) When the focus does not lie on the directrix

The conic represents :

A parabola	An ellipse	A hyperbola	A rectangular hyperbola
$e = 1 ; D \neq 0$	$0 < e < 1 ; D \neq 0$	$D \neq 0 ; e > 1 ;$	$e > 1 ; D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab ; a + b = 0$

Note:

(*i*) For pair of straight lines $e \to \infty$

(*ii*) All second degree terms in parabola form a perfect square

DEFINITION OF VARIOUS TERMS RELATED TO A CONIC

1. **Focus:** The fixed point is called a focus of the conic.
2. **Directrix:** The fixed line is called a directrix of the conic.
3. **Axis:** The line passing through the focus and perpendicular to the directrix is called the axis of the conic.
4. **Vertex:** The points of intersection of the conic and the axis are called vertices of the conic.
5. **Centre:** The point which bisects every chord of the conic passing through it, is called the centre of the conic.
6. **Latus-rectum:** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.
7. **Double ordinate:** A chord which is perpendicular to the axis of parabola or parallel to its directrix.

TRAIN YOUR BRAIN

Example 1: What conic does $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent ?

Sol. Compare the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore$ $a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$

then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162$

$= -1600 \neq 0$

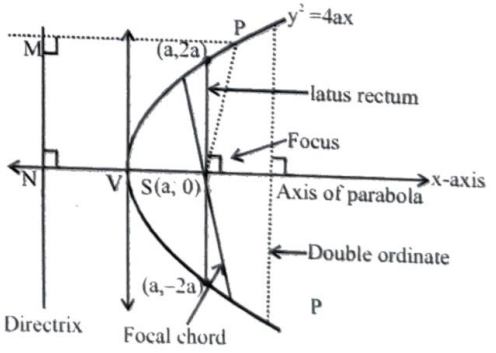
and also $h^2 = (-9)^2 = 81$ and $ab = 13 \times 37 = 481$

Here $h^2 - ab < 0$

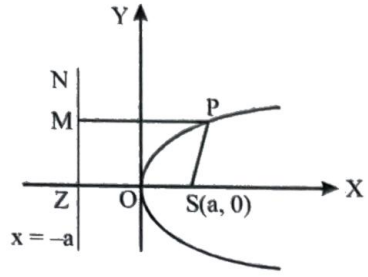
So we have $h^2 - ab < 0$ and $\Delta \neq 0$. Hence the given equation represents an ellipse.

PARABOLA

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).



STANDARD EQUATION OF A PARABOLA



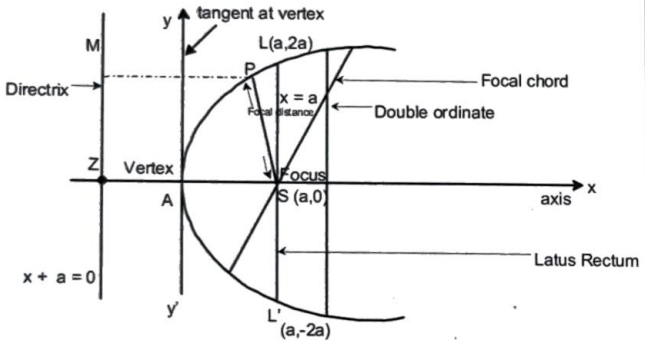
Now, $S \equiv (a, 0)$ and the equation of ZN is $x = -a$ or $x + a = 0$.

Let $P(x, y)$ be any point on the parabola.

$\therefore PS = PM$ (by definition of parabola).

$y^2 = 4ax$ which is the required equation.

TERMS RELATED TO PARABOLA



Note:

(*i*) The length of the latus rectum = 2 × perpendicular distance of focus from the directrix.

(*ii*) If $y^2 = lx$ then length of the latus rectum = l.

(*iii*) Two parabolas are said to be equal if they have same latus rectum.

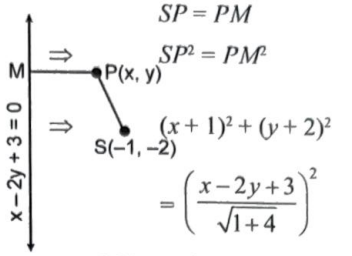
(*iv*) The ends of a double ordinate of a parabola can be taken as $(at^2, 2at)$ and $(at^2, -2at)$.

(*v*) Parabola has no centre, but circle, ellipse, hyperbola have centre.

TRAIN YOUR BRAIN

Example 2: Find the equation of the parabola whose focus is at $(-1, -2)$ and the directrix is $x - 2y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,



$$SP = PM$$

$$\Rightarrow SP^2 = PM^2$$

$$\Rightarrow (x+1)^2 + (y+2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$$\Rightarrow 5[(x+1)^2 + (y+2)^2] = (x-2y+3)^2$$

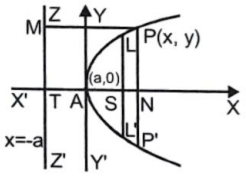
$$\Rightarrow 5(x^2+y^2+2x+4y+5) = (x^2+4y^2+9-4xy+6x-12y)$$

$$\Rightarrow 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$

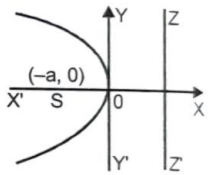
This is the equation of the required parabola.

TYPE OF PARABOLA

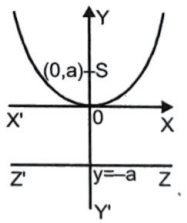
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



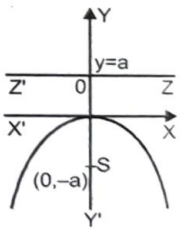
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends points of Latus rectum	Parametric co-ordinates
$y^2 = 4ax$	(0, 0)	(a, 0)	y = 0	x = –a	4a	(a, ±2a)	$(at^2, 2at)$
$y^2 = -4ax$	(0, 0)	(–a, 0)	y = 0	x = a	4a	(–a, ±2a)	$(-at^2, 2at)$
$x^2 = +4ay$	(0, 0)	(0, a)	x = 0	y = –a	4a	(± 2a, a)	$(2at, at^2)$
$x^2 = -4ay$	(0, 0)	(0, –a)	x = 0	y = a	4a	(± 2a, –a)	$(2at, -at^2)$
$(y-k)^2 = 4a(x-h)$	(h, k)	(h + a, k)	y = k	x + a – h = 0	4a	(h + a, k ± 2a)	$(h + at^2, k + 2at)$
$(x-p)^2 = 4b(y-q)$	(p, q)	(p, b + q)	x = p	y + b – q = 0	4b	(p ± 2a, q + a)	$(p + 2at, q + at^2)$

REDUCTION TO GENERALIZED EQUATION OF PARABOLA

(A) When the equation of parabola is :

$(y-k)^2 = 4a(x-h)$ ...(*i*)

Equation (*i*) is of the form $Y^2 = 4aX$

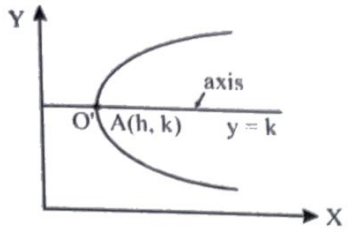
where $Y = y-k$ and $X = x-h$

1. Axis of parabola is $Y = 0$, i.e., $y-k=0 \Rightarrow y=k$
2. Coordinates of vertex of parabola are given by

$X=0$ and $Y=0$

i.e. $x-h=0$ and $y-k=0$

$\therefore$ Vertex is (h, k)



(B) When the equation of parabola is:

$(x-h)^2 = 4a(y-k)$ ...(*i*)

Equation (*i*) is of the form $X^2 = 4aY$

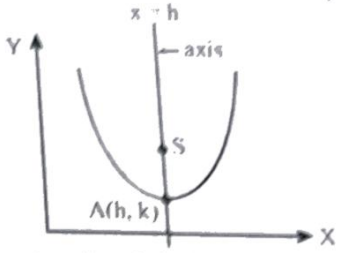
where $X = x-h$ and $Y = y-k$

1. Axis of parabola is $X = 0$, i.e., $x-h=0$
2. Coordinates of vertex of parabola is given by

$X=0$ and $Y=0$

i.e., by $x-h=0$ and $y-k=0$

$\therefore$ $x=h$ and $y=k$



Hence vertex of parabola is (h, k)

Equation of Parabola	Vertex	Axis	Focus	Directrix	Parametric equation
$(y-k)^2 = 4a(x-h)$	(h, k)	$y=k$	$(h+a, k)$	$x+a-h=0$	$x = h+at^2, y = k+2at$
$(x-h)^2 = 4a(y-k)$	(h, k)	$x=h$	$(h, a+k)$	$y+a-k=0$	$x = h+2at, y = k+at^2$

Note:

(*i*) For the parabola $y = Ax^2 + Bx + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to y-axis. If A is positive then it is concave up parabola, if A is negative then it is concave down parabola.

(*ii*) For the parabola $x = Ay^2 + By + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to x-axis. If A is positive then it is opening right and if A is negative then it is opening left parabola.

TRAIN YOUR BRAIN

Example 3: Find the parametric equation of the parabola $(x-1)^2 = -12(y-2)$

Sol. $\because 4a = -12 \Rightarrow a = -3$,

parametric equation is $y-2 = -3t^2$

$x-1 = -6t \Rightarrow x = 1-6t,\ y = 2-3t^2$

Example 4: Find the equation of parabola, whose axis is parallel to y-axis and which passes through points $(0, 2), (-1, 0)$ and $(1, 6)$.

Sol. General equation of such parabola is

$y = Ax^2 + Bx + C$ ...(*i*)

Since it passes throught $(0, 2)$ $(-1, 0)$ and $(1, 6)$, then we have

$C = 2$ ...(*ii*)

$A - B + C = 0$ ...(*iii*)

$A + B + C = 6$ ...(*iv*)

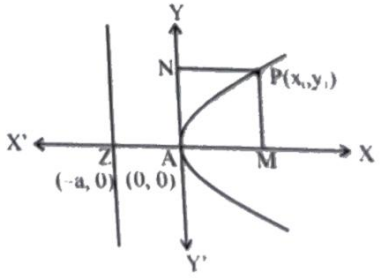
from (*ii*), (*iii*) and (*iv*),

$A = 1, B = 3, C = 2$

$\therefore$ equation of parabola is $y = x^2 + 3x + 2$. **Ans.**

EQUATION OF PARABOLA WITH RESPECT TO TWO PERPENDICULAR LINES

Let $P(x_1, y_1)$ is any point on the parabola then equation of parabola $y^2 = 4ax$ is consider as



$y_1^2 = 4ax_1$

$(PM)^2 = 4a(PN)$

$$\left\{\begin{matrix}\text{Perpendicular distance}\\ \text{from a point P to axis}\end{matrix}\right\}^2 =$$

$$\left\{\begin{matrix}\text{length of the}\\ \text{latusrectum}\end{matrix}\right\} \cdot \left\{\begin{matrix}\text{Perpendicular distance from a}\\ \text{point P to tangent at vertex}\end{matrix}\right\}$$

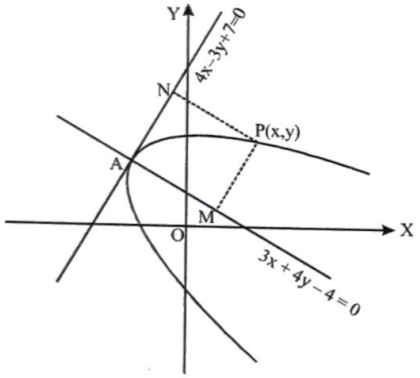
TRAIN YOUR BRAIN

Example 5: Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively then

$$(PM)^2 = (\text{latus re-ctum})\,(PN)$$

$$\Rightarrow \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

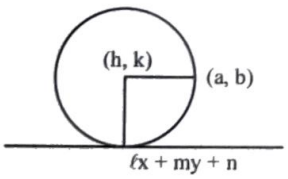


$$\Rightarrow (3x+4y-4)^2 = 20\,(4x-3y+7)$$

which is required parabola.

SPECIAL CASE OF PARABOLA

The locus of centres of a circle which passes through a fixed point and tangential to a line, is a parabola whose focus is fixed point, directrix is tangent of a circle.

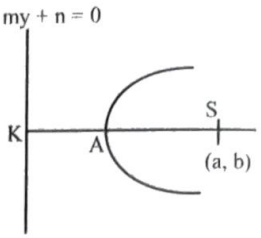


$$\sqrt{(h-a)^2+(k-b)^2} = \left|\frac{\ell h+mk+n}{\sqrt{\ell^2+m^2}}\right|$$

Locus of centre of circle

$$\sqrt{(x-a)^2+(y-b)^2} = \left|\frac{\ell x+my+n}{\sqrt{\ell^2+m^2}}\right|$$

LENGTH OF THE LATUSRECTUM



$$= 2(KS) = 2\left|\frac{\ell a+mb+n}{\sqrt{\ell^2+m^2}}\right|$$

POSITION OF A POINT WITH RESPECT TO A PARABOLA $y^2 = 4ax$:

Equation of parabola $S(x, y) = y^2 - 4ax$.

(*i*) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(*ii*) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(*iii*) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

TRAIN YOUR BRAIN

Example 6: Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Sol. $\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

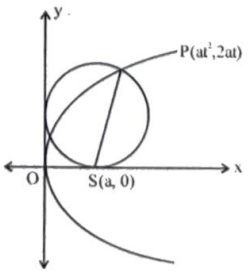
$$\therefore \quad y_1^2 - 4x_1 < 0 \Rightarrow \alpha^2 - 4(\alpha - 1) < 0$$

$$\Rightarrow \quad \alpha^2 - 4\alpha + 4 < 0 \Rightarrow (\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$$

LENGTH OF FOCAL CHORD

If point P is $(at^2, 2at)$, then length of focal chord PQ is $a\left(t + \frac{1}{t}\right)^2$

- The length of the focal chord which makes an angle θ with positive direction of x-axis is **$4a\,\text{cosec}^2\theta$.**
- If ℓ_1 and ℓ_2 are the lengths of the segments of a focal of a parabola, then its latusrectum is $\frac{4\ell_1\ell_2}{\ell_1+\ell_2}$
- **Circle described on the focal as diameter touches the tangent at vertex**



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Example 7: Circles are drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line, find its equation.

Sol. $y^2 - 4x - y - 4 = 0$

$$\Rightarrow y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$$

$$\Rightarrow \left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$$

Circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus, circle will touch the line $x + \frac{17}{16} = -1$,

i.e., $16x + 33 = 0$.

Example 8: Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Sol. The given parabola is $y^2 = 4x$ ...(*i*)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$.

Slope of $OP = \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of $OQ = \frac{2}{t_2}$

Since $OP \perp OQ$, $\frac{4}{t_1 t_2} = -1$ or $t_1 t_2 = -4$ ...(*ii*)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1 t_2)$

$$\Rightarrow y\left(t_1 - \frac{4}{t_1}\right) = 2(x - 4) \text{ [from (ii)]}$$

$$\Rightarrow 2(x-4) - y\left(t_1 - \frac{4}{t_1}\right) = 0$$

$\Rightarrow L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$ i.e. (4, 0)

LINE & A PARABOLA

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm \Rightarrow$ condition of tangency is, $c = a/m$.

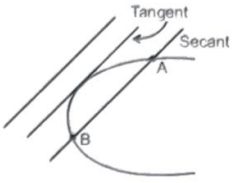
Note:

(*a*) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm \Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

(*b*) Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = am^2$

(*c*) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is

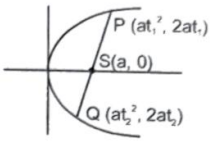
$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}.$$



TRAIN YOUR BRAIN

Example 9: If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Sol. Let parabola is $y^2 = 4ax$



since P, S & Q are collinear

$\therefore \quad m_{PQ} = m_{PS}$

$$\Rightarrow \frac{2}{t_1 + t_2} = \frac{2t_1}{t_1^2 - 1} \Rightarrow t_1^2 - 1 = t_1^2 + t_1 t_2$$

$$\Rightarrow t_1 t_2 = -1$$

EQUATION OF TANGENT

(*a*) **Point form :** Equation of parabola is

$y^2 = 4ax$ ...(1)

Let $P \equiv (x_1, y_1)$

$yy_1 = 2ax + 2ax_1$

which is the required equation of tangent at (x_1, y_1).

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + yx_1}{2}$ and without changing the constant (if any) in the equation of curve.

Equation of tangent of standard parabola:

Equation of Parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

(b) Slope form

$y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, where m is the slope of the tangent and the co-ordinates of the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Thus $y = mx + c$ is the tangent of $y^2 = 4ax$ for all values of m if only if $c = \frac{a}{m}$ and $(y-k) = m(x-h) + \frac{a}{m}$ is tangent to the parabola $(y-k)^2 = 4a(x-h)$.

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of standard parabolas are shown below in the table.

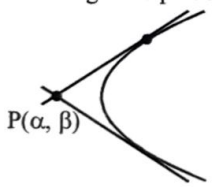
Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangnecy
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$

Two tangents can be drawn from a point $P(a, b)$ to a parabola if P lies outside the parabola :

Let the parabola be $y^2 = 4ax$ ...(1)

Let $P(\alpha, \beta)$ be the given point

The equation of a tangent to parabola (1) is



$$y = mx + \frac{a}{m} \quad ...(2)$$

If line (2) passes through $P(\alpha, \beta)$, then

$$\beta = m\alpha + \frac{a}{m} \text{ or } m^2\alpha - \beta m + a = 0 \quad ...(3)$$

There will be two tangents to parabola (1) from $P(\alpha, \beta)$ if roots of equation (3) are real and distinct i.e., $D > 0$ i.e. if $\beta^2 - 4\alpha a > 0 \Rightarrow P(\alpha, \beta)$ lies outside parabola (1).

We can also find the angle between two tangents from point $P(\alpha, \beta)$ using the formula $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(c) Parametric form : We have to find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'

Since the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad ...(1)$$

replacing x_1 by at^2 and y_1 by $2at$, then (1) becomes

$$y(2at) = 2a(x + at^2) \Rightarrow ty = x + at^2$$

Point of intersection of tangents at any two points on the parabola:

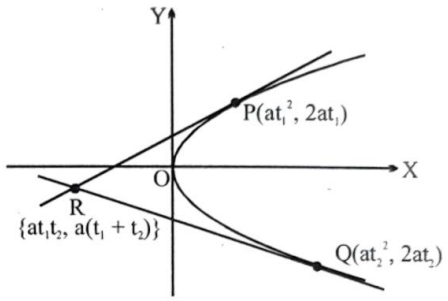
Let the given parabola be $y^2 = 4ax$ and two points on the parbaola are

$$P = (at_1^2, 2at_1) \text{ and } Q = (at_2^2, 2at_2)$$

Equation of tangents at

$$P(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2)$$

are $t_1 y = x + at_1^2$ ...(1)



and $t_2 y = x + at_2^2$ ...(2)

Solving these equations we get

$$x = at_1t_2, y = a(t_1 + t_2)$$

Thus, the co-ordinates of the point of intersection of tangents at

$P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $R(at_1t_2, a(t_1 + t_2))$.

Note :

(*i*) The Arithmetic mean of the y-co-ordinates of P and Q

$$\left(i.e., \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$$

is the y - co-ordinate of the point of intersection of tangents at P and Q on the parabola.

(*ii*) The Geometric mean of the x-co-ordinates of P and Q $\left(i.e., \sqrt{at_1^2 \times at_2^2} = at_1t_2\right)$ is the x co-ordinate of the point of intersection of tangents at P and Q on the parabola.

TRAIN YOUR BRAIN

Example 10: If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, then find the value of k.

Sol. Line $y = \frac{2}{3}x - \frac{k}{3}$ touch is $y^2 = 6x$ then use the condition of tangency $c = \frac{a}{m}$

$$\Rightarrow -\frac{k}{3} = \frac{9}{4} \Rightarrow k = -\frac{27}{4}$$

DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to the parabola is called the DIRECTOR CIRCLE. In case of parabola its own directrix representation director circle.

For standard parabola, $y^2 = 4ax$ equation of directrix is $x + a = 0$.

TRAIN YOUR BRAIN

Example 11: The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is

Sol. The given point $(-a, 2a)$ lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle.

Example 12: The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line

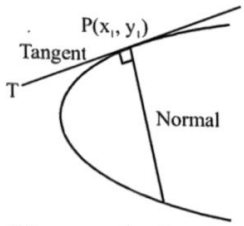
Sol. Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. $(5, 0)$. Thus the drawn circle will always touch the directrix of the parabola i.e.. the line $x + 5 = 0$.

NORMAL TO THE PARABOLA: $y^2 = 4ax$

(a) Point form: Since the equation of the tangent to the parabola $y^2 = 4ax$ *at* (x_1, y_1) is

$yy = 2a(x + x_1)$ ...(1)

The slope of the tangent at $(x_1, y_1) = 2a/y_1$.



$\therefore$ Slope of the normal at $(x_1, y_1) = -y_1/2a$

Hence the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form: The equation of normal to the parabola $y^2 = 4ax$ *at* (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ...(1)$$

Since m is the slope of the normal

then $m = -\frac{y_1}{2a}$ or $y_1 = -2am$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1$ or $4a^2m^2 = 4ax_1$

$\therefore$ $x_1 = am^2$

Substituting the values of x_1 and y_1 in (1) we get

$y + 2am = m(x - am^2)$ ...(2)

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal. The co-ordinates of the point of contact are $(am^2, -2am)$.

Hence $y = mx + c$ will be normal to parabola. If and only if $c = -2am - am^3$

Note :

Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$x^2 = 4ay$	$\left(\frac{-2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = -2a + \frac{a}{m^2}$

(c) Parametric form: Equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ...(1)$$

Replacing x_1 by at^2 and y_1 by $2at$ then (1) becomes

$y - 2at = -t(x - at)^2$

or $y = -tx + 2at + at^3$

Three Supplementry Results:

(a) Point of intersection of normals at any two points on the parabola :

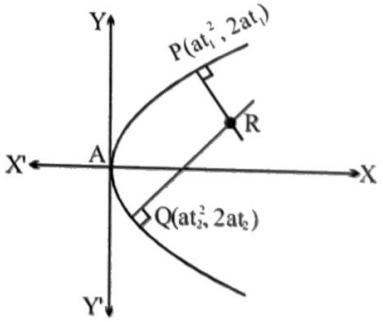
Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$

The equations of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$y = -t_1x + 2at_1 + at_1^3$ ...(1)

and $y = -t_2x + 2at_2 + at_2^3$ ...(2)

Hence point of intersection of above normals will be obtained by solving (1) and (2), we get



$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$

$y = -at_1t_2(t_1 + t_2)$

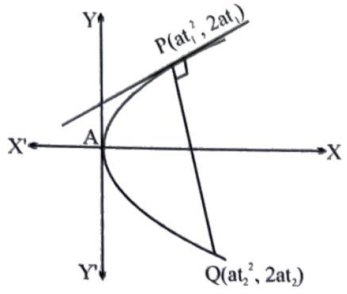
If R is the point of intersection then it is

$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$

(b) Relation between t_1 and t_2 if normal at t_1 meets the parabola again at t_2 :

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$y = -t_1x + 2at_1 + at_1^3$ ...(1)



$\therefore \quad 2 + t_1(t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \dfrac{2}{t_1}$

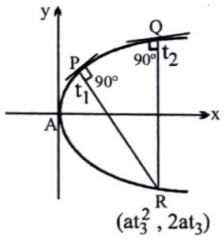
(c) If normal to the parabola $y^2 = 4ax$ drawn at any point $(at^2, 2at)$ meet the parabola at t_3 then

$$t_3 = -t - \frac{2}{t}$$

$\Rightarrow \quad t^2 + t\,t_3 + 2 = 0 \qquad \ldots(i)$

It has two roots t_1 & t_2. Hence there are two such point $P(t_1)$ & $Q(t_2)$ on the parabola from where normals are drawn and which meet parabola at $R(t_3)$

$\Rightarrow \quad t_1 + t_2 = -t_3 \;\; \& \;\; t_1 t_2 = 2$



Thus the line joining $P(t_1)$ & $Q(t_2)$ meet x-axis at $(-2a, 0)$

TRAIN YOUR BRAIN

Example 13: Find the equation of a normal at the parabola $y^2 = 4x$ which passes through (3,0).

Sol. Equation of Normal $y = mx - 2am - am^3$

Here $a = 1$ and it passes through (3,0)

$0 = 3m - 2m - m^3$

$\Rightarrow \quad m^3 - m = 0$

$\Rightarrow \quad m = 0, \pm 1$

for $\quad m = 0 \quad \Rightarrow \quad y = 0$

$\quad m = 1 \quad \Rightarrow \quad y = x - 3$

$\quad m = -1 \quad \Rightarrow \quad y = -x + 3$ **Ans.**

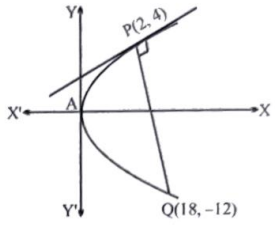
Example 14: Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, – 12). Find also the length of the normal chord.

Sol. Comparing the given parabola with $y^2 = 4ax$

$\therefore \quad 4a = 8$

$\therefore \quad a = 2$

$P(at_1^2, 2at_1) \equiv (2, 4)$



$\Rightarrow \quad t_1 = 1$

$\Rightarrow$ parameter at $Q(t_2) = -t_1 - \dfrac{2}{t_1} = -3$

$\therefore \quad Q(2(-3)^2, 2 \times 2(-3))$

i.e. $Q(18, -12)$

$\therefore$ Length of normal chord PQ = Distance between points P and Q.

$$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}\,.$$

CO-NORMAL POINTS

Maximum three normals can be drawn from a point to a parabola and their feet (points where the normal meet the parabola) are called co-normal points.

Let $P(h, k)$ be any given point and

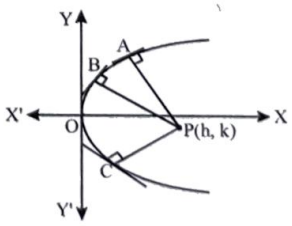
$y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

If it passes through (h, k) then

$k = mh - 2am - am^3$



$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \qquad \ldots(i)$

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0, \qquad \ldots(ii)$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{(2a - h)}{a}, \qquad \ldots(iii)$$

$$m_1 m_2 m_3 = -\frac{k}{a} \qquad \ldots(iv)$$

1. If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) is real then $h > 2a$.

When normals are real, then all the three roots of equation (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \text{ (for any values of } m_1, m_2, m_3)$$

$$\Rightarrow (m_1 + m_2 + m_3)^2 - 2 \quad (m_1m_2 + m_2m_3 + m_3m_1) > 0$$

$$\Rightarrow (0)^2 - \frac{2(2a-h)}{a} > 0 \Rightarrow h - 2a > 0 \quad \text{or} \quad h > 2a$$

2. The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$

$\therefore$ Centroid of ΔABC is $\left(\frac{2h - 4a}{3}, 0\right)$

TRAIN YOUR BRAIN

Example 15: If from point $P(h, k)$ three normals are drawn to the parabola $(y-1)^2 = 8(x-2)$ then find the condition

Sol. Here $a = 2$ and abscissa of point from where three normals are drawn must be greater than $2a$.

i.e. $x - 2 > 2a$ i.e. $x > 6$

Hence $h > 6$ **Ans.**

Example 16: Find the locus of points through which three normals to parabola $y^2 = 4ax$ passes and two of them are perperdicular to each other.

Sol. Let $P(h, k)$ be the point of intersection of three normals to be parabola $y^2 = 4ax$ then it will be

$y = mx - 2am - am^3$ and it passes through $P(h, k)$

$$k = mh - 2am - am^3$$

$$\therefore \quad am^3 + m(2a - h) + k = 0 \qquad ...(i)$$

It has three roots m_1, m_2 and m_3

$$m_1 + m_2 + m_3 = 0 \qquad ...(ii)$$

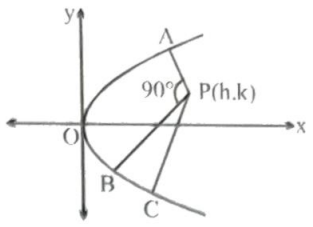
$$m_1m_2m_3 = \frac{-k}{a} \qquad ...(iii)$$

Given condition, $m_1m_2 = -1$ ...(iv)

as given in figure.

from (iii) and (iv)

$$m_3 = \frac{k}{a}$$



$\because$ m_3 is also a root of equation (1) therefore it will satify equation (i)

$$\therefore \quad a\left(\frac{k}{a}\right)^3 + (2a - h)\left(\frac{k}{a}\right) + k = 0$$

$$\Rightarrow \quad k^3 + (2a - h)\,ka + ka^2 = 0$$

$$\Rightarrow \quad k^2 + 3a^2 - ah = 0$$

$\therefore$ locus of $P(h, k)$ is $y^2 + 3a^2 - ax = 0$. **Ans.**

PAIR OF TANGENTS

Let the parabola be $y^2 = 4ax$ ...(1)

Let $P(x_1, y_1)$ be a point outside the parabola.

$$[yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

$$\Rightarrow \quad \boldsymbol{SS_1 = T^2}$$

where S, S_1 and T have usual meanings.

The same result holds true for cicle, ellipse and hyperbola also.

TRAIN YOUR BRAIN

Example 17: Find the locus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(i) $m_1 + m_2 = m_0$ (const) (ii) $\theta_1 + \theta_2 = \theta_0$ (const)

Sol. Equation of tangent to $y^2 = 4ax$, is

$$y = mx + \frac{a}{m}$$

Let it passes through $P(h, k)$

$$\therefore \quad m^2h - mk + a = 0$$

(i) $m_1 + m_2 = m_0 = \frac{k}{h} \Rightarrow y = m_0x$

(ii) $\tan\theta_0 = \frac{m_1 + m_2}{1 - m_1m_2} = \frac{k/h}{1 - a/h}$

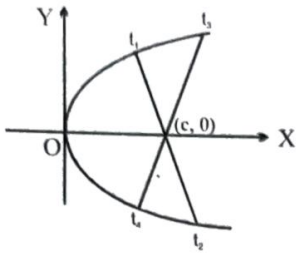
$$\Rightarrow \quad y = (x - a)\tan\theta_0$$

CHORD JOINING TWO POINTS

Line joining any two points on the parabola is called its chords.

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, lie on the parabola then equation of chord is

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$



$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\Rightarrow \quad (t_1 + t_2)\, y = 2x + 2at_1 t_2$$

If this chord meet the x-axis at point $(c, 0)$ then from above equation $c + at_1t_2 = 0$ i.e. $t_1t_2 = -c/a$.

Note:

(*i*) If the chord joining t_1, t_2 & t_3, t_4 pass through a point $(c, 0)$ on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

(*ii*) If PQ is a focal chord then $t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$. which is required relation.

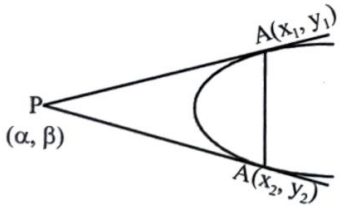
Hence if one extremity of a focal chord is $(at^2, 2at)$ then the other extremity will be $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

CHORD OF CONTACT

Let the parabola be $y^2 = 4ax$ ...(1)

Let $P(\alpha, \beta)$ be a point outside the parabola.

Let PA and PB be the two tangents from $P(\alpha, \beta)$ to parabola (1).



Now we consider the equation

$$y\beta = 2a(x + \alpha) \quad (T = O) \qquad ...(6)$$

TRAIN YOUR BRAIN

Example 18: If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Sol. Let (h, k) be point of intersection of tangents then chord of contact is

$$yk = 4(x + h)$$

$$4x - yk + 4h = 0 \qquad ...(i)$$

But given is $x - y - 1 = 0$

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1}$$

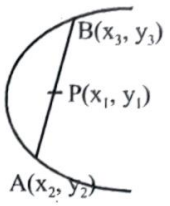
$$\Rightarrow \quad h = -1, k = 4$$

$$\therefore \quad \text{point} \equiv (-1, 4)$$

CHORD WITH A GIVEN MIDDLE POINT

Equation of the parabola is $y^2 = 4ax$ ...(1)

Let AB be a chord of the parabola whose middle point is $P(x_1, y_1)$.



In usual notations, equation (5) can be written as $T = S_1$.

The same result holds true for cicle, ellipse and hyperbola also.

AN IMPORTANT CONCEPT

If a family of straight lines can be represented by an equation $\lambda^2 P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

Example 35: If the equation $m^2(x+1) + m(y-2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Sol. $m^2(x + 1) + m(y - 2) + 1 = 0$. The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

$$\therefore \quad (y-2)^2 - 4(x+1) = 0$$

$$\Rightarrow \quad y^2 - 4y + 4 - 4x - 4 = 0 \quad \Rightarrow \quad y^2 = 4(x + y)$$

POLAR & POLE

Equation of the Polar of the point $P(x_1, y_1)$ *w.r.t.* the parabola $y^2 = 4ax$ is, $yy_1 = 2a(x + x_1)$

The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Properties of polar:

(*i*) The polar of the focus of the parabola is the directrix.

(*ii*) When the point (x_1, y_1) lies on the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(*iii*) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the

other. Similarly two points P and Q are said to be conjugate points if polar of P passes through Q and vice versa.

(iv) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

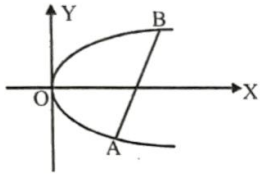
DIAMETER OF A PARABOLA

Diameter of a conic is the locus of middle points of a series of its parallel chords.

Equation of diameter of a parabola:

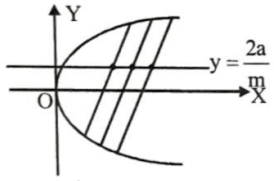
Let the parabola be $y^2 = 4ax$ ...(1)

Let AB be one of the chords of a series of parallel chords having slope m.



Hence locus of $P(\alpha, \beta)$ i.e. equation of diameter

(which is the locus of a series of a parallel chords having slope m) is



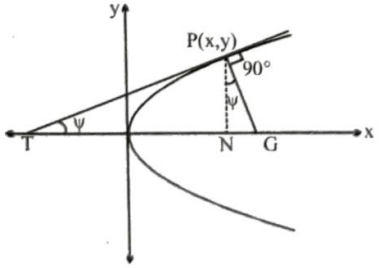
$$y = \frac{2a}{m} \qquad ...(2)$$

Clearly line is parallel to the axis of the parabola. Thus a diameter of a parabola is parallel to its axis.

LENGTH OF TANGENT, SUBTANGNET, NORMAL AND SUB-NORMAL

Let the parabola is $y^2 = 4ax$. Let the tangent at any point $P(x, y)$ meet the axis of parabola at T and G respectively and tangent makes an angle ψ with x-axis.

$$\therefore \qquad \tan\psi = \left(\frac{dy}{dx}\right)_{P(x,y)} \quad \text{and } PN = y$$



$\therefore$ PT = length of tangent = PN cosec ψ = y cosec ψ

PG = length of normal = y sec ψ

TN = length of sub-tangent = PN cot ψ = y cot ψ

NG = length of sub-normal = y tan ψ

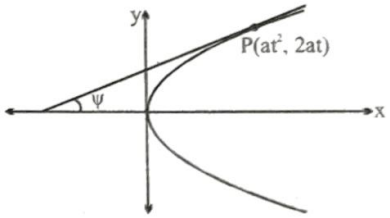
TRAIN YOUR BRAIN

Example 19: Find the length of tangent, sub-tangent, normal and sub-normal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$\therefore \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t} = \tan\psi \Rightarrow \cot\psi = t$$



$\therefore$ l (tangent) $= y$ cosec $\psi = 2at\sqrt{1+\cot^2\psi}$

$$= 2at\sqrt{1+t^2}$$

l (normal) $= y$ sec $\psi = 2at\sqrt{1+\tan^2\psi}$

$$= 2a\sqrt{1+t^2}$$

l (sub-tangent) $= y$ cot $\psi = 2at \cdot t = 2at^2$

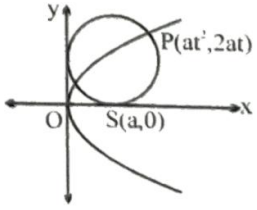
l (sub-normal) $= y$ tan $\psi = 2at \cdot \frac{1}{t} = 2a$

Thus length of sub-normal of parabola is constant and equal to semi-latus rectum.

IMPORTANT HIGHLIGHTS OF PARABOLA

1. Circle described on the focal length (distance) as diameter touches the tangent at the vertex.

   Equation of the circle described on SP as diameter is $(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$



Solve it with y-axis i.e. $x = 0$, we get

$y^2 - 2aty + a^2t^2 = 0 \Rightarrow (y - at)^2 = 0$

circle touches y-axis at $(0, at)$.

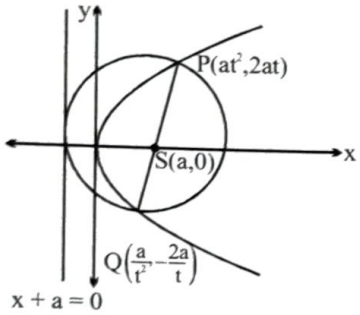
2. Circle described on the focal chord as diameter touches directrix

Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Solving it with $x = -a$

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$



$$\Rightarrow y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0$$

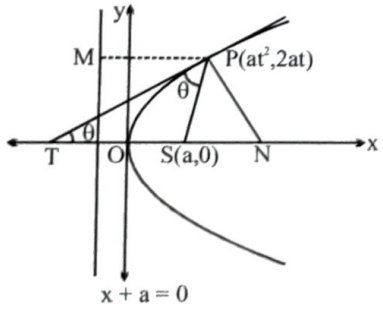
$$\Rightarrow \left[y - a\left(t - \frac{1}{t}\right)\right]^2 = 0$$

$\Rightarrow$ circle tocuhes the directrix.

3. Tangent at P is $yt = x + at^2$, meet x-axis at T, then $T(-at^2, 0)$. Normal at P is $y + xt = 2at + at^3$, meet x-axis at N,

then $N(2a + at^2, 0)$

$\Rightarrow ST = SN = a + at^2 = PM = PS$



$\Rightarrow \angle PTS = \angle TPS = \theta$

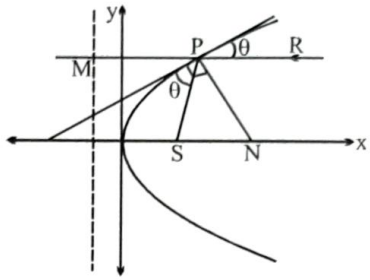
$\because TS = PS = PM \Rightarrow \angle TPM = \theta$

Tangent and Normal at any point P bisect the angle between PS and PM internally and externally. This property leads to the *reflection property of parabola.*

Circle circumscribing the triangle formed by any tangent normal and x-axis, has its centre at focus.

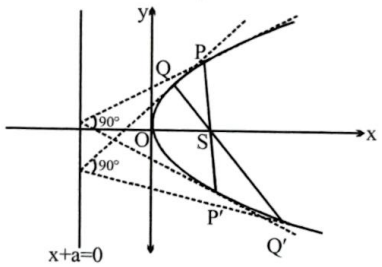
If we extend MP, then from figure

$\angle RPN = \angle SPN = 90 - \theta$



Thus ray parallel axis meet parabola at P and after reflection from P it passes through the focus.

4. The tangents at the extremities of a focal chord intersect at right angles on the directrix.



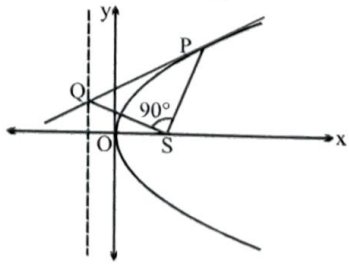
5. The portion of tangent to the parabola intercepted between the directrix and the curve subtends a right angle at the focus.

Tangent at $P(at^2, 2at)$ is $yt = x + at^2$ meet the directrix at

$x = -a \Rightarrow Q\left(-a, \frac{at^2 - a}{t}\right)$ and $S(a, 0)$.

Slope at $SP = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = m_1$

Slope at $SQ = \frac{\frac{at^2 - a}{t} - 0}{-a - 0} = \frac{t^2 - 1}{-2t} = m_2$.



$\Rightarrow m_1 m_2 = -1$

$\Rightarrow SP \perp SQ$

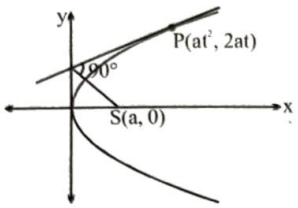
$\Rightarrow \angle PSQ = 90°$

6. Tangent at P is $yt = x + at^2$ ...(*i*)

Line perpendicular to above line is $xt + y = \lambda$

and passes through $(a, 0)$ gives $\lambda = at$

$\therefore$ Perpendicular line will be $xt + y = at$ ...(*ii*)



Solve (i) and (ii), we get $x = 0$

i.e., these two line intersect at y-axis i.e. tangnet at the vertex.

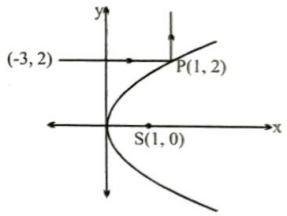
The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at vertex.

7. **Tangents and Normals** at the extremities of the latus rectum of a parabola $y^2 = 4ax$ **constitute a square**, their points of intersection being $(-a, 0)$ & $(3a, 0)$.
8. The **circle circumscribing the triangle** formed by **any three tangents** to a parabola passes through the focus.
9. The **orthocentre** of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $(-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3))$.
10. The area of the triangle formed by three points on a parabola is **twice the area** of the triangle formed by the tangents at these points.

TRAIN YOUR BRAIN

Example 20: If incident ray from point $(-3, 2)$ parallel to the axis of parabola $y^2 = 4x$ strike the parabola, then find the equation of reflected ray.

Sol. Since incident ray strikes parabola at $P(1, 2)$ i.e. extremity of latus rectum and it will pass through the focus of parabola therefore reflected ray will be parallel to y-axis and its equation will be $x = 1$.



Topicwise Questions

DIFFERENT FORMS OF PARABOLA

1. The focus of the parabola $x^2 = -16y$ is
   (a) (4, 0) (b) (0, 4)
   (c) (-4, 0) (d) (0, -4)

2. If the vertex of a parabola be at origin and directrix be $x + 5 = 0$, then its latus rectum is
   (a) 5 (b) 10
   (c) 20 (d) 40

3. The latus rectum of a parabola whose directrix is $x + y - 2 = 0$ and focus is $(3, -4)$, is
   (a) $-3\sqrt{2}$ (b) $3\sqrt{2}$
   (c) $-3/\sqrt{2}$ (d) $3/\sqrt{2}$

4. If (2, 0) is the vertex and y-axis the directrix of a parabola, then its focus is
   (a) (2, 0) (b) (–2, 0)
   (c) (4, 0) (d) (-4, 0)

5. If the parabola $y^2 = -4ax$ passes through (–3, 2), then length of its latus rectum is
   (a) 2/3 (b) 1/3
   (c) 4/3 (d) 4

6. The ends of latus rectum of parabola $x^2 + 8y = 0$ are
   (a) (–4, –2) and (4, 2) (b) (4, –2) and (–4, 2)
   (c) (–4, –2) and (4, –2) (d) (4, 2) and (–4, 2)

7. Focus and directrix of the parabola $x^2 = -8ay$ are
   (a) $(0, -2a)$ and $y = 2a$ (b) $(0, 2a)$ and $y = -2a$
   (c) $(2a, 0)$ and $x = -2a$ (d) $(-2a, 0)$ and $x = 2a$

8. The equation of latus rectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$, then length of the latus rectum is
   (a) $4\sqrt{2}$ (b) $2\sqrt{2}$
   (c) 8 (d) $8\sqrt{2}$

9. The equation of the parabola whose vertex is (–1, –2), axis is vertical and which passes through the point (3, 6), is
   (a) $x^2 + 2x - 2y - 3 = 0$ (b) $2x^2 = 3y$
   (c) $x^2 - 2x - y + 3 = 0$ (d) $x^2 = 3y$

10. Axis of the parabola $x^2 - 4x - 3y + 10 = 0$ is
    (a) $y + 2 = 0$ (b) $x + 2 = 0$
    (c) $y - 2 = 0$ (d) $x - 2 = 0$

11. The equation of the latus rectum of the parabola represented by equation $y^2 + 2Ax + 2By + C = 0$ is
    (a) $x = \frac{B^2 + A^2 - C}{2A}$ (b) $x = \frac{B^2 - A^2 + C}{2A}$
    (c) $x = \frac{B^2 - A^2 - C}{2A}$ (d) $x = \frac{A^2 - B^2 - C}{2A}$

12. The equation of the locus of a point which moves so as to be at equal distances from the point $(a, 0)$ and the y-axis is
    (a) $y^2 - 2ax + a^2 = 0$ (b) $y^2 + 2ax + a^2 = 0$
    (c) $x^2 - 2ay + a^2 = 0$ (d) $x^2 + 2ay + a^2 = 0$

13. PQ is a double ordinate of the parabola $y^2 = 4ax$. The locus of the points of trisection of PQ is
    (a) $9y^2 = 4ax$ (b) $9x^2 = 4ay$
    (c) $9y^2 + 4ax = 0$ (d) $9x^2 + 4ay = 0$

PARAMETRIC FORM

14. The parametric equation of the curve $y^2 = 8x$ are
    (a) $x = t^2, y = 2t$ (b) $x = 2t^2, y = 4t$
    (c) $x = 2t, y = 4t^2$ (d) $x = 4t, y = 2t^2$

Focal chord and chord passing through fixed point

15. If 'a' and 'c' are the segments of a focal chord of a parabola and b the semi-latus rectum, then
    (a) a, b, c are in $A.P.$ (b) a, b, c are in $G.P.$
    (c) a, b, c are in $H.P.$ (d) a, b, c are in $A.G.P.$

Tangent

16. The equation of the common tangent of the parabolas $x^2 = 108y$ and $y^2 = 32x$, is
    (a) $2x + 3y = 36$ (b) $2x + 3y + 36 = 0$
    (c) $3x + 2y = 36$ (d) $3x + 2y + 36 = 0$

17. The equation of a tangent to the parabola $y^2 = 4ax$ making an angle θ with x-axis is

(a) $y = x\cot\theta + a\tan\theta$ (b) $x = y\tan\theta + a\cot\theta$

(c) $y = x\tan\theta + a\cot\theta$ (d) $x = y\cot\theta + a\tan\theta$

18. The equation of the tangent to the parabola $y^2 = 4x + 5$ parallel to the line $y = 2x + 7$ is

(a) $2x - y - 3 = 0$ (b) $2x - y + 3 = 0$

(c) $2x + y + 3 = 0$ (d) $x - 2y + 6 = 0$

19. The point of the contact of the tangent to the parabola $y^2 = 4ax$ which makes an angle of $60°$ with x-axis, is

(a) $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$ (b) $\left(\frac{2a}{\sqrt{3}}, \frac{a}{3}\right)$

(c) $\left(\frac{a}{\sqrt{3}}, \frac{2a}{3}\right)$ (d) $\left(\frac{2a}{3}, \frac{a}{\sqrt{3}}\right)$

20. The equation of the tangent at a point $P(t)$ where 't' is any parameter to the parabola $y^2 = 4ax$, is

(a) $yt = x + at^2$ (b) $y = xt + at^2$

(c) $y = xt + \frac{a}{t}$ (d) $y = tx$

Normal

21. The point on the parabola $y^2 = 8x$ at which the normal is inclined at $60°$ to the x-axis has the co-ordinates

(a) $(6, -4\sqrt{3})$ (b) $(6, 4\sqrt{3})$

(c) $(-6, -4\sqrt{3})$ (d) $(-6, 4\sqrt{3})$

22. The equation of normal to the parabola $y^2 = 4ax$ at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, is

(a) $y = m^2x - 2mx - am^3$

(b) $m^3y = m^2x - 2am^2 - a$

(c) $m^3y = 2am^2 - m^2x + a$

(d) $y = m^2x + 2mx + am^2$

23. Equation of any normal to the parabola $y^2 = 4a(x - a)$ is

(a) $y = mx - 2am - am^3$

(b) $y = m(x + a) - 2am - am^3$

(c) $y = m(x - a) + \frac{a}{m}$

(d) $y = m(x - a) - 2am - am^3$

24. The normal to the parabola $y^2 = 8x$ at the point **(2, 4)** meets the parabola again at the point

(a) $\{18, 12\}$ (b) $\{18, 12\}$

(c) $\{18, 12\}$ (d) $(18, -12)$

Tangents from external point

25. From the point $(-1, -60)$ two tangents are drawn to the parabola $y^2 = 4x$. Then the angle between the two tangents is

(a) $30°$ (b) $45°$

(c) $60°$ (d) $90°$

26. Tangent to the parabola $y = x^2 + 6$ at $(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point

(a) $(-6, -9)$

(b) $(-13, -9)$

(c) $(-6, -7)$

(d) $(13, 7)$

Chord of contact and common chord

27. The length of chord of contact of the tangents drawn from the point $(2, 5)$ to the parabola $y^2 = 8x$, is

(a) $\frac{1}{2}\sqrt{41}$ (b) $\sqrt{41}$

(c) $\frac{3}{2}\sqrt{41}$ (d) $2\sqrt{41}$

28. If the segment intercepted by the parabola $y^2 = 4ax$ with the line $lx + my + n = 0$ subtends a right angle at the vertex, then

(a) $4al + n = 0$

(b) $4al + 4am + n = 0$

(c) $4am + n = 0$

(d) $al + n = 0$

29. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible value of the slope of this chord, are

(a) $\{-1, 1\}$ (b) $\{-2, 2\}$

(c) $\{-2, 1/2\}$ (d) $\{2, -1/2\}$

Chord with given mid point

30. The locus of the middle points of the chords of the parabola $y^2 = 4ax$ which passes through the origin

(a) $y^2 = ax$

(b) $y^2 = 2ax$

(c) $y^2 = 4ax$

(d) $x^2 = 4ay$

Learning Plus

1. The equation of the parabola whose focus is $(-3, 0)$ and the directrix is, $x + 5 = 0$ is:
   (a) $y^2 = 4(x-4)$ (b) $y^2 = 2(x+4)$
   (c) $y^2 = 4(x-3)$ (d) $y^2 = 4(x+4)$
2. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x-4y+7)^2$ is:
   (a) 4 (b) 2
   (c) 1/5 (d) 2/5
3. A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is:
   (a) $x^2 - 6x - 8y + 25 = 0$ (b) $y^2 - 8x - 6y + 25 = 0$
   (c) $x^2 - 6x + 8y - 25 = 0$ (d) $x^2 + 6x - 8y - 25 = 0$
4. The points on the parabola $y^2 = 12x$ whose focal distance is 4, are
   (a) $(2, \sqrt{3}), (2, -\sqrt{3})$ (b) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$
   (c) (1, 2) (d) None of these
5. The point of intersection of the curves whose parametric equations are $x = t^2 + 1, y = 2t$ and $x = 2s, y = 2/s$ is given by
   (a) (1, –3) (b) (2, 2)
   (c) (–2, 4) (d) (1, 2)
6. The length of the chord of the parabola, $y^2 = 12x$ passing through the vertex & making an angle of 60° with the axis of x is:
   (a) 8 (b) 4
   (c) 16/3 (d) None
7. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be
   (a) $\left(t + \frac{1}{t}\right)^2$ (b) $\left(t + \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$
   (c) $\left(t - \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$ (d) None
8. The normal chord of a parabola $y^2 = 4ax$ at the point $P(x_1, x_1)$ subtends a right angle at the
   (a) focus
   (b) vertex
   (c) end of the latus rectum
   (d) none of these
9. The latus rectum of a parabola whose focal chord is PSQ such that $SP = 3$ and $SQ = 2$ is given by:
   (a) 24/5 (b) 12/5
   (c) 6/5 (d) None
10. The equation of the tangent to the parabola $y = (x-3)^2$ parallel to the chord joining the points (3, 0) and (4, 1) is:
    (a) $2x - 2y + 6 = 0$ (b) $2y - 2x + 6 = 0$
    (c) $4y - 4x + 11 = 0$ (d) $4x - 4y = 13$
11. An equation of a tangent common to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is
    (a) $x - y + 1 = 0$ (b) $x + y - 1 = 0$
    (c) $x + y + 1 = 0$ (d) $y = 0$
12. At what point on the parabola $y^2 = 4x$ the normal makes equal angles with the axes?
    (a) (4, 4) (b) (9, 6)
    (c) (4, – 1) (d) (1, 2)
13. The line $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is
    (a) 12 (b) – 12
    (c) 24 (d) – 24
14. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is
    (a) 8 (b) $8\sqrt{2}$
    (c) 4 (d) $4\sqrt{2}$
15. The combined equation of pair of tangents from $(-2, 0)$ to the parabola, $y^2 = 8x$ is:
    (a) $y = \pm(x-2)$ (b) $x = \pm(y-2)$
    (c) $y = \pm(x+2)$ (d) $x = \pm(y+2)$
16. The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A & B. The co-ordinates of the point of intersection of the tangents drawn at the points A & B are:
    (a) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (b) $\left(-\frac{5}{2}, -\frac{7}{2}\right)$
    (c) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (d) $\left(-\frac{7}{2}, -\frac{5}{2}\right)$
17. The tangents to the parabola $x = y^2 + c$ from origin are perpendicular then c is equal to
    (a) 1/2 (b) 1
    (c) 2 (d) 1/4
18. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is
    (a) $y^2 = \frac{9}{2}ax$ (b) $y^2 = \frac{9}{4}ax$
    (c) $y^2 = 9ax$ (d) $x^2 = 4ay$

19. Locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$ is

(a) $2x - 1 = 0$ (b) $2x + 3 = 0$

(c) $2y + 3 = 0$ (d) $2x + 5 = 0$

20. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are

(a) (3, 2) (b) (2, 4)

(c) (3, 4) (d) (4, 1)

21. TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point $(-a, b)$ then the locus of T is

(a) $ay = 2b(x - b)$

(b) $bx = 2a(y - a)$

(c) $by = 2a(x - a)$

(d) $ax = 2b(y - b)$

22. If the tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q & R, then the mid point of QR is

(a) $(x_1 + b, y_1 + b)$

(b) $(x_1 - b, y_1 - b)$

(c) (x_1, y_1)

(d) $(x_1 + b, y_1)$

23. From the focus of the parabola $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex and focus of the parabola. The equation of the circle is

(a) $(x - 2)^2 + y^2 = 3$ (b) $(x - 2)^2 + y^2 = 9$

(c) $(x + 2)^2 + y^2 = 9$ (d) None

24. The locus of the middle points of the focal chords of the parabola, $y^2 = 4x$ is:

(a) $y^2 = x - 1$ (b) $y^2 = 2(x - 1)$

(c) $y^2 = 2(1 - x)$ (d) None

25. The feet of the perpendicular drawn from focus upon any tangent to the parabola, $y = x^2 - 2x - 3$ lies on

(a) $y + 4 = 0$ (b) $y = 0$

(c) $y = -2$ (d) $y + 1 = 0$

26. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is

(a) $x^2 + y^2 - 4y = 0$ (b) $x^2 + y^2 - 4y + 1 = 0$

(c) $x^2 + y^2 - 2x - 4y = 0$ (d) $x^2 + y^2 - 2x - 4y + 1 = 0$

27. Two parabolas $y^2 = 4a(x - l_1)$ and $x^2 = 4a(y - l_2)$ always touch one another, the quantities l_1 and l_2 are both variable. Locus of their point of contact has the equation

(a) $xy = a^2$ (b) $xy = 2a^2$

(c) $xy = 4a^2$ (d) None

Advanced Level Multiconcept Questions

1. Let A be the vertex and L the length of the latus rectum of the parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with A as vertex $2L$ the length of the latus rectum and the axis at right angles to that of the given curve is:

(a) $x^2 + 4x + 8y - 4 = 0$ (b) $x^2 + 4x - 8y + 12 = 0$

(c) $x^2 + 4x + 8y + 12 = 0$ (d) $x^2 + 8x - 4y + 8 = 0$

2. The focus of the parabola is (1, 1) and the tangent at the vertex has the equation $x + y = 1$. Then:

(a) equation of the parabola is $(x - y)^2 = 2(x + y - 1)$

(b) equation of the parabola is $(x - y)^2 = 4(x + y - 1)$

(c) the co-ordinates of the vertex are (1/2, 1/2)

(d) length of the latus rectum is $2\sqrt{2}$

3. P is a point on the parabola $y^2 = 4x$ where abscissa and ordinate are equal. Equation of a circle passing through the focus and touching the parabola at P is:

(a) $x^2 + y^2 - 13x + 2y + 12 = 0$

(b) $x^2 + y^2 - 3x - 18y + 2 = 0$

(c) $x^2 + y^2 + 13x - 2y - 14 = 0$

(d) $x^2 + y^2 - x = 0$

4. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(a) Latus rectum is half the latus rectum of the original parabola

(b) Vertex is $(a/2, 0)$

(c) Directrix is y-axis

(d) Focus has the co-ordinates $(a, 0)$

5. If one end of a focal chord of the parabola $y^2 = 4x$ is (1, 2), the other end lies on

(a) $x^2 y + 2 = 0$ (b) $xy + 2 = 0$

(c) $xy - 2 = 0$ (d) $x^2 + xy - y - 1 = 0$

6. Equation of a tangent to the parabola $y^2 = 12x$ which make an angle of 45° with line $y = 3x + 77$ is

(a) $2x - 4y + 3 = 0$

(b) $x - 2y + 12 = 0$

(c) $4x + 2y + 3 = 0$

(d) $2x + y - 12 = 0$

7. Two parabolas have the same focus. If their directrices are the $x-$axis & the $y-$axis respectively, then the slope of their common chord is:

(*a*) 1 (*b*) –1

(*c*) 4/3 (*d*) 3/4

8. Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is:

(*a*) $\left(\frac{p}{2}, p\right)$ (*b*) $\left(\frac{p}{2}, -p\right)$

(*c*) $\left(-\frac{p}{2}, p\right)$ (*d*) $\left(-\frac{p}{2}, -p\right)$

9. P is a point on the parabola $y^2 = 4ax$ $(a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the $x-$axis at a point whose co–ordinates are:

(*a*) $(-3a, 0)$

(*b*) $(-a, 0)$

(*c*) $(-2a, 0)$

(*d*) $(a, 0)$

Comprehension – 1 (Q. No. 10 to 12)

Let $L_1 : 5x - y - 3 = 0$ and $L_2 : x + 5y - 11 = 0$ are tangents to a parabola which meets the parabola at A and B. Also normals at A and B intersect at $M(3, 4)$, where M lies on axis of the parabola.

10. The equation of the directrix of the parabola is

(*a*) $x + 2y = 5$

(*b*) $x + y = 3$

(*c*) $2x - y = 0$

(*d*) $3x - y = 1$

11. The length of latus rectum of the parabola is

(*a*) $2\sqrt{2}$ (*b*) 4

(*c*) $4\sqrt{2}$ (*d*) 8

12. The equation of the circle circumscribing the ΔAMB is

(*a*) $x^2 + y^2 - 3x - 5y + 8 = 0$

(*b*) $x^2 + y^2 - 5x - 6y + 14 = 0$

(*c*) $x^2 + y^2 - 4x - 6y + 11 = 0$

(*d*) $x^2 + y^2 - 6x - 7y + 20 = 0$

13. AB is a chord of the parabola $y^2 = 4ax$ joining $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$. Match the following

Column – I	**Column – II**
(*a*) AB is a normal chord	(*p*) $t_2 = -t_1 + 2$
(*b*) AB is a focal chord	(*q*) $t_2 = -\frac{4}{t_1}$
(*c*) AB subtends 90° at (0, 0)	(*r*) $t_2 = -\frac{1}{t_1}$
(*d*) AB is inclined at 45° to the axis of parabola	(*s*) $t_2 = -t_1 - \frac{2}{t_1}$

14. The locus of midpoint of chord of the parabola $y^2 = 4ax$

Column – I

(*a*) which passes through focus, is

(*b*) which is normal, is

(*c*) for which line joining origin to the extremeties of chord are equally inclined to the axis of the parabola is

(*d*) whose length is 2, is

Column – II

(*p*) $y = 0$

(*q*) $4a^2 = (4ax - y^2)(y^2 + 4a^2)$

(*r*) $y^2 = 2a(x - a)$

(*s*) $y^4 + 2a(2a - x)y^2 + 8a^4 = 0$

NUMERICAL BASED QUESTIONS

15. The focal distance of a point P on the parabola $y^2 = 12x$, if the ordinate of P is 6, is

16. The point $P(9/2, 6)$ lies on the parabola $y^2 = 4ax$, then parameter of the point P is

17. The equation of the directrix of parabola $y^2 + 4y + 4x + 2 = 0$ is $x = a$ then find the value of a

18. If the vertex of the parabola $y = x^2 - 16x + k$ lies on x-axis, then the value of k is

19. The number of normal drawn to the parabola $y^2 = 4x$ from the point (1,0) is

20. Find the value of λ for which the equation $\lambda x^2 + 4xy + y^2 + \lambda x + 3y + 2 = 0$ represents a parabola

21. Find the latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7 + 5\sqrt{2}$.

22. Through the vertex O of the parabola $y^2 = 8x$, a perpendicular is drawn to any tangent meeting it at P & the parabola at Q. Find the value of $OP \cdot OQ$.

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is: [JEE Main-2014]

(a) $\frac{1}{8}$ (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

2. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is [JEE Main-2015]

(a) $x^2 = y$ (b) $y^2 = x$

(c) $y^2 = 2x$ (d) $x^2 = 2y$

3. Tangent and normal are drawn at $P(16, 16)$ *sq.* units the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ [JEE Main-2018]

(a) 2 (b) 3

(c) $\frac{4}{3}$ (d) $\frac{1}{2}$

4. Axis of a parabola lies along x-axis. If its vertex and focus are at distance 2 and 4 respectively from the origin, on the positive x-axis then which of the following points does not lie on it? [JEE Main-2019 (January)]

(a) $(5, 2\sqrt{6})$ (b) $(8, 6)$

(c) $(6, 4\sqrt{2})$ (d) $(4, -4)$

5. Equation of a common tangent to the circle, $x^2 + y^2 - 6x = 0$ and the parabola, $y^2 = 4x$, is : [JEE Main-2019 (January)]

(a) $2\sqrt{3}y = 12x + 1$ (b) $\sqrt{3}y = x + 3$

(c) $2\sqrt{3}y = -x - 12$ (d) $\sqrt{3}y = 3x + 1$

6. Let $A(4, -4)$ and $B(9, 6)$ be points on the parabola $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is: [JEE Main-2019 (January)]

(a) $31\frac{3}{4}$ (b) 32

(c) $30\frac{1}{2}$ (d) $31\frac{1}{4}$

7. The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}\,(x > 0)$, is: [JEE Main-2019 (January)]

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{3}{2}$ (d) $\frac{5}{4}$

8. If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have three common normals, then which one of the following is a valid choice for the ordered triad (a, b, c)? [JEE Main-2019 (January)]

(a) $\left(\frac{1}{2}, 2, 3\right)$ (b) $(1, 1, 3)$

(c) $\left(\frac{1}{2}, 2, 0\right)$ (d) $(1, 1, 0)$

9. The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is : [JEE Main-2019 (January)]

(a) $3\sqrt{2}$ (b) $2\sqrt{11}$

(c) $8\sqrt{2}$ (d) $6\sqrt{3}$

10. If the area of the triangle whose one vertex is the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and y – axis, is 250 sq. units, then a value of 'a' is: [JEE Main-2019 (January)]

(a) $5\sqrt{5}$ (b) $5(2^{1/3})$

(c) $(10)^{2/3}$ (d) 5

11. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbola $xy = 2$ is :- [JEE Main-2019 (January)]

(a) $x + y + 1 = 0$ (b) $x - 2y + 4 = 0$

(c) $x + 2y + 4 = 0$ (d) $4x + 2y + 1 = 0$

12. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of x-axis, is [JEE Main-2019 (January)]

(a) $y = x\tan\theta + 2\cot\theta$ (b) $y = x\tan\theta - 2\cot\theta$

(c) $x = y\cot\theta + 2\tan\theta$ (d) $x = y\cot\theta - 2\tan\theta$

13. The shortest distance between the line $y = x$ and the curve $y^2 = x - 2$ is : [JEE Main-2019 (April)]

(a) $\frac{7}{4\sqrt{2}}$ (b) $\frac{7}{8}$

(c) $\frac{11}{4\sqrt{2}}$ (d) 2

14. The tangent to the parabola $y^2 = 4x$ at the point where it intersects the circle $x^2 + y^2 = 5$ in the first quadrant, passes through the point : [JEE Main-2019 (April)]

(a) $\left(-\frac{1}{3}, \frac{4}{3}\right)$ (b) $\left(-\frac{1}{4}, \frac{1}{2}\right)$

(c) $\left(\frac{3}{4}, \frac{7}{4}\right)$ (d) $\left(\frac{1}{4}, \frac{3}{4}\right)$

15. If one end of a focal chord of the parabola, $y^2 = 16x$ is at (1, 4), then the length of this focal chord is [JEE Main-2019 (April)]

(a) 25 (b) 24

(c) 20 (d) 22

16. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then a is equal to: [JEE Main-2019 (April)]

(a) $2\sqrt{2}+1$ (b) $\sqrt{2}-1$

(c) $\sqrt{2}+1$ (d) $2\sqrt{2}-1$

17. The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is :- [JEE Main-2019 (April)]

(a) $4\pi(2-\sqrt{2})$ (b) $8\pi(3-2\sqrt{2})$

(c) $4\pi(3+\sqrt{2})$ (d) $8\pi(2-\sqrt{2})$

18. If the line $ax + y = c$, touches both the curves $x^2 + y^2 = 1$ and $y^2 = 4\sqrt{2}x$, then $|c|$ is equal to: [JEE Main-2019 (April)]

(a) 1/2 (b) 2

(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

19. Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and denote the foci of the hyperbola where S lies on the positive x-axis then P divides SS' in a ratio: [JEE Main-2019 (April)]

(a) 5 : 4 (b) 14 : 13

(c) 2 : 1 (d) 13 : 11

20. The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point : [JEE Main-2019 (April)]

(a) $\left(-\frac{5}{2}, 1\right)$ (b) $\left(-\frac{5}{2}, 1\right)$

(c) $\left(\frac{5}{2}, -1\right)$ (d) $\left(\frac{5}{2}, 1\right)$

21. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is : [JEE Main-2019 (April)]

(a) $x + y + 4 = 0$

(b) $x - 2y + 16 = 0$

(c) $2x - y + 2 = 0$

(d) $x - y + 4 = 0$

22. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to [JEE Main-2020 (January)]

(a) -64 (b) -32

(c) -128 (d) 128

23. The locus of a point which divides the line joining the point $(0, -1)$ and a point on the parabola, $x^2 = 4y$, internally in the ratio 1:2, is [JEE Main-2020 (January)]

(a) $9x^2 - 12y = 8$

(b) $x^2 - 3y = 2$

(c) $9x^2 - 3y = 2$

(d) $4x^2 - 3y = 2$

24. Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q. If area $(\Delta OPQ) = 4$ sq. units, then m is equal to [JEE Main-2020 (January)]

25. If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2}, -2\right)$, then the equation of the tangent to it at B is: [JEE Main-2020 (January)]

(a) $x - 2y + 8 = 0$ (b) $x + 2y + 8 = 0$

(c) $2x - y - 24 = 0$ (d) $2x + y - 24 = 0$

26. The area (in sq. units) of an equilateral triangle inscribed in the parabola $y^2 = 8x$, with one of its vertices on the vertex of this parabola, is : [JEE Main-2020 (September)]

(a) $64\sqrt{3}$ (b) $256\sqrt{3}$

(c) $128\sqrt{3}$ (d) $192\sqrt{3}$

27. If the tangent to the curve, $y = e^x$ at a point (c, e^c) and the normal to the parabola, $y^2 = 4x$ at the point (1, 2) intersect at the same point on the x-axis, then the value of c is _____. [JEE Main-2020 (September)]

28. Let P be a point on the parabola, $y^2 = 12x$ and N be the foot of the perpendicular drawn from P on the axis of the parabola. A line is now drawn through the mid-point M of PN, parallel to its axis which meets the parabola at Q. If the y-intercept of the line NQ is 4/3 then: [JEE Main-2020 (September)]

(a) $MQ = \frac{1}{4}$ (b) $PN = 3$

(c) $PN = 4$ (d) $MQ = \frac{1}{3}$

29. If the common tangent to the parabolas, $y^2 = 4x$ and $x^2 = 4y$ also touches the circle, $x^2 + y^2 = c^2$, then c is equal to : [JEE Main-2020 (September)]

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

30. The centre of the circle passing through the point (0, 1) and touching the parabola $y = x^2$ at the point (2, 4) is :

[JEE Main-2020 (September)]

(a) $\left(\frac{-53}{10}, \frac{16}{5}\right)$ (b) $\left(\frac{6}{5}, \frac{53}{10}\right)$

(c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$ (d) $\left(\frac{3}{10}, \frac{16}{5}\right)$

31. Let L_1 be a tangent to the parabola $y^2 = 4(x + 1)$ and L_2 be a tangent to the parabola $y^2 = 8(x + 2)$ such that L_1 and L_2 intersect at right angles. Then L_1 and L_2 meet on the straight line:

[JEE Main-2020 (September)]

(a) $2x + 1 = 0$ (b) $x + 3 = 0$

(c) $x + 2y = 0$ (d) $x + 2 = 0$

32. The locus of the mid-point of the line segment joining the focus of the parabola $y^2 = 4ax$ to a moving point of the parabola, is another parabola whose directrix is:

[JEE Main-2021]

(a) $x = a$ (b) $x = 0$

(c) $x = -\frac{a}{2}$ (d) $\frac{a}{2}$

33. If P is a point on the parabola $y = x^2 + 4$ which is closest to the straight line $y = 4x - 1$, then the co-ordinates of P are:

[JEE Main-2021]

(a) (–2, 8) (b) (1, 5)

(c) (3, 13) (d) (2, 8)

34. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line $2x + y = 1$. Which of the following points does NOT lie on it?

[JEE Main-2021]

(a) (0, 3) (b) (–6, 0)

(c) (4, 5) (d) (5, 4)

35. A line is a common tangent to the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$. If the two points of contact (a, b) and (c, d) are distinct and lie in the first quadrant, then $2(a + c)$ is equal to

[JEE Main-2021]

36. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0) a \neq 0$, then 'a' must be greater than:

[JEE Main-2021]

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) –1 (d) 1

JEE-ADVANCED PREVIOUS YEARS

1. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is.

[IIT JEE-2012]

Comprehension – 1 (Q. No. 2 to 4)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

[JEE Advanced-2013]

2. Length of chord PQ is

(a) $7a$ (b) $5a$

(c) $2a$ (d) $3a$

3. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

(a) $\frac{2}{3}\sqrt{7}$

(b) $\frac{-2}{3}\sqrt{7}$

(c) $\frac{2}{3}\sqrt{5}$

(d) $\frac{-2}{3}\sqrt{5}$

4. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is

[JEE Advanced-2014]

(a) 3 (b) 6

(c) 9 (d) 15

Comprehension – 2 (Q. No. 5 & 6)

Let a, r, s, t be nonzero real numbers. Let $P(at^2, 2at)$, Q, R $(ar^2, 2ar)$ and $(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$

[JEE Advanced-2014]

5. The value of r is

(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$

(c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

6. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$

(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

7. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k) then which of the following is (are) possible value(s) of p, h and k?

[JEE Advanced -2017]

(a) $p = -1, h = 1, k = -3$

(b) $p = 2, h = 3, k = -4$

(c) $p = -2, h = 2, k = -4$

(d) $p = 5, h = 4, k = -3$

Give the answer Q.8, Q.9 and Q.10 by appropriately matching the information given in the three columns of the following table.

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, repectively.

[JEE Advanced-2017]

Column-1	Column-2	Column-3
(I) $x^2+y^2=a^2$	*(i)* $my=m^2x+a$	*(P)* $\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II) $x^2+a^2y^2=a^2$	*(ii)* $y=mx+a\sqrt{m^2+1}$	*(Q)* $\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2=4ax$	*(iii)* $y=mx+\sqrt{a^2m^2-1}$	*(R)* $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2-a^2y^2=a^2$	*(iv)* $y=mx+\sqrt{a^2m^2+1}$	*(S)* $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

8. For $a=\sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only CORRET combination for obtaining its equation ?

(*a*) (I) (ii) (Q)

(*b*) (I) (i) (P)

(*c*) (III) (i) (P)

(*d*) (II) (ii) (Q)

9. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only CORRECT combination?

(*a*) (IV) (iv) (S) (*b*) (II) (iv) (R)

(*c*) (IV) (iii) (S) (*d*) (II) (iii) (R)

10. If a tangent to a suitable conic (Column 1) is found to be $y=x+8$ and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(*a*) (III) (i) (P) (*b*) (I) (ii) (Q)

(*c*) (II) (iv) (R) (*d*) (III) (ii) (Q)

11. Let E denote the parabola $y^2=8x$. Let $P=(-2, 4)$ and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is(are) True?

[JEE Advance-2021]

(*a*) The triangle PFQ is a right-angled triangle

(*b*) The triangle QPQ' is a right-angle triangle

(*c*) The distance between P and F is $5\sqrt{2}$

(*d*) F lies on the line joining Q and Q'

ANSWER KEY

Topicwise Questions

1. (*d*) **2.** (*c*) **3.** (*b*) **4.** (*c*) **5.** (*c*) **6.** (*c*) **7.** (*a*) **8.** (*d*) **9.** (*a*) **10.** (*d*) **11.** (*c*) **12.** (*a*)
13. (*a*) **14.** (*b*) **15.** (*c*) **16.** (*b*) **17.** (*c*) **18.** (*b*) **19.** (*a*) **20.** (*a*) **21.** (*a*) **22.** (*c*) **23.** (*d*) **24.** (*d*)
25. (*d*) **26.** (*c*) **27.** (*c*) **28.** (*a*) **29.** (*a*) **30.** (*b*)

Learning Plus

1. (*d*) **2.** (*d*) **3.** (*a*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*a*) **8.** (*a*) **9.** (*a*) **10.** (*d*) **11.** (*c*) **12.** (*d*)
13. (*c*) **14.** (*b*) **15.** (*c*) **16.** (*c*) **17.** (*d*) **18.** (*a*) **19.** (*d*) **20.** (*c*) **21.** (*c*) **22.** (*c*) **23.** (*b*) **24.** (*b*)
25. (*a*) **26.** (*d*) **27.** (*c*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a*, *b*) **2.** (*b*, *c*, *d*) **3.** (*a*, *d*) **4.** (*a*, *b*, *c*, *d*) **5.** (*a*, *b*, *d*) **6.** (*b*, *c*)
7. (*a*, *b*) **8.** (*a*, *b*) **9.** (*a*, *d*) **10.** (*b*) **11.** (*a*) **12.** (*c*)
13. (*a*) → (s), (*b*) → (r), (*c*) → (q), (*d*) → (p) **14.** (*a*) → (r), (*b*) → (s), (*c*) → (p), (*d*) → (q)
15. [6] **16.** [1.5] **17.** [1.5] **18.** [64] **19.** [1] **20.** [4] **21.** [20]
22. [16]

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*c*) **2.** (*d*) **3.** (*a*) **4.** (*b*) **5.** (*b*) **6.** (*d*) **7.** (*a*) **8.** (*b*) **9.** (*d*) **10.** (*d*) **11.** (*c*) **12.** (*c*)
13. (*a*) **14.** (*c*) **15.** (*a*) **16.** (*c*) **17.** (*b*) **18.** (*c*) **19.** (*a*) **20.** (*c*) **21.** (*d*) **22.** (*c*) **23.** (*a*) **24.** [0.5]
25. (*a*) **26.** (*d*) **27.** (4) **28.** (*a*) **29.** (*d*) **30.** (*c*) **31.** (*b*) **32.** (*b*) **33.** (*d*) **34.** (*d*) **35.** (9) **36.** (*d*)

JEE-ADVANCED PREVIOUS YEARS

1. (*d*) **2.** (*b*) **3.** (*d*) **4.** (*d*) **5.** (*d*) **6.** (*b*) **7.** (*b*) **8.** (*a*) **9.** (*b*) **10.** (*a*) **11.** (*a*, *b*, *d*)

PRAYAS 2.0 t.me/realprayasians2

17 HYPERBOLA

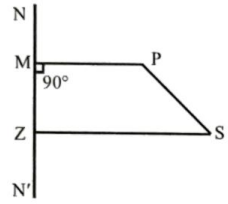
PRMAS 1.0 time/real progressions

DEFINITION

A hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point and a given straight line is always constant and is greater than 1.

In the given figure, S is the focus and $N'N$ the directrix.

Let P be any point on the hyperbola, then



$$\frac{PS}{PM} = e, e > 1.$$

Equation of a hyperbola can be obtained if the coordinates of its focus, equation of its directrix and eccentricity are given.

GENERAL EQUATION OF HYPERBOLA

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let $P(x, y)$ be any point on the hyperbola. Then by defination.

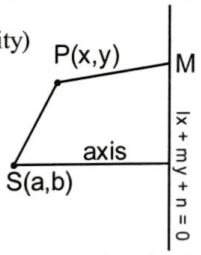
$\Rightarrow SP = e\,PM (e > 1$, is the eccentricity)

$\Rightarrow (x-a)^2 + (y-b)^2$

$$= e^2 \frac{(lx+my+n)^2}{(l^2+m^2)}$$

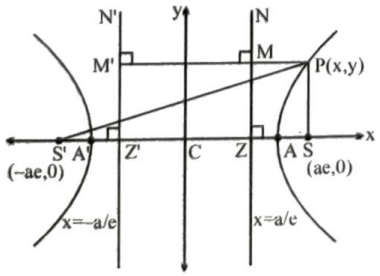
$\Rightarrow (l^2+m^2)\{(x-a)^2 + (y-b)^2\}$

$= e^2\{lx+my+n\}^2$



reduces to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, in which $\Delta \neq 0$, $h^2 > ab$.

STANDARD EQUATION & DEFINITION(S)



By definition of hyperbola.

$PS = ePM$

$\Rightarrow (PS)^2 = e^2 (PM)^2$

$$\Rightarrow (x-ae)^2 + (y-0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

or $$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

where $b^2 = a^2(e^2-1)$

Hence equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where $b^2 = a^2(e^2-1)$

FACTS ABOUT THE HYPERBOLA: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

1. If focus is $(ae, 0)$, then corresponding directrix is $x = \frac{a}{e}$ and if focus is $(-ae, 0)$, then corresponding directrix is $x = -\frac{a}{e}$.
2. By definition of hyperbola, the distance of any point P on the hyperbola from focus $= e$(the distance of P from the corresponding directrix)
3. Distance between foci $SS' = 2ae$ & distance between directrix $ZZ' = 2\frac{a}{e}$.
4. Two hyperbola are said to be similar if they have same eccentricity.
5. The hyperbola has two branches neither of them cut the y-axis (conjugate axis).
6. Since the fundamental equation to the hyperbola only differs from that to the ellipse is having $-b^2$ instead of b^2. It is observed that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.
7. Eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}}$. Also, $b^2 = a^2(e^2-1)$.

8. Equation of hyperbola when its transverse and conjugate axes are x and y-axes respectively is

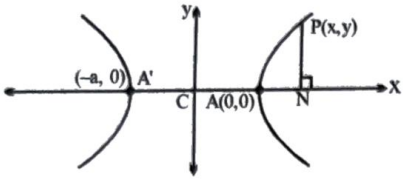
$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1 \quad \text{or} \quad x^2-\frac{y^2}{(e^2-1)}=a^2$$

If e kept constant and $a \to 0$, then hyperbola will tend to pair of straight lines $x^2-\frac{y^2}{(e^2-1)}=0$, both passing through the origin.

Thus in this situation limiting case of a hyperbola is a pair of straight lines.

9. Since $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\therefore \quad \frac{y^2}{b^2}=\frac{x^2}{a^2}-1=\frac{(x-a)(x+a)}{a^2}$$



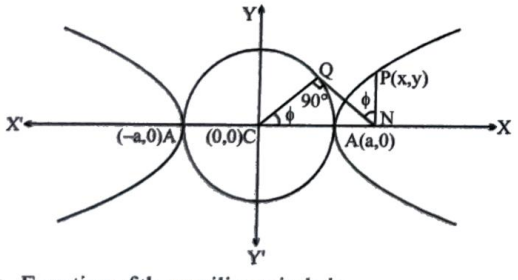
From figure, $AN = CN - CA = x - a$

$A'N = CN + CA = x + a$ and $PN = y$

$$\therefore \quad \frac{(PN)^2}{A'N \cdot AN^2}=\frac{b^2}{a^2}.$$

AUXILIARY CIRCLE & PARAMETRIC EQUATIONS OF HYPERBOLA

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola.



$\therefore$ Equation of the auxiliary circle is

$$x^2+y^2=a^2$$

Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The equations of $x = a \sec \phi$ and $y = \tan \phi$ are known as the parametric equations of the hyperbola.

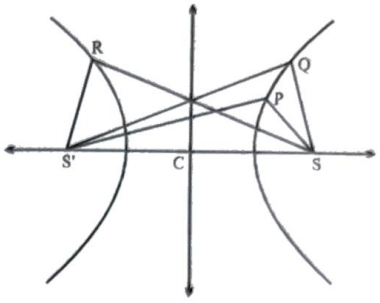
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

FOCAL DISTANCE OF A POINT ON HYPERBOLA

The hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

Let $P(x_1, y_1)$ be any point on (1).



$$\therefore \quad S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a$$

$$= AA' = \text{Transverse axis}$$

Hence, in the given figure $PS' - PS = QS' - QS = RS - RS'$ = length of transverse axis.

TRAIN YOUR BRAIN

Example 1: Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Sol. Equation can be rewritten as

$$\frac{(x-4)^2}{4^2}-\frac{(y-3)^2}{3^2}=1 \text{ so } a=4, b=3$$

$$b^2 = a^2(e^2-1) \text{ given } e=\frac{5}{4}$$

Foci : $X = \pm ae$, $Y = 0$ gives the foci as $(9, 3), (-1, 3)$

Centre : $X = 0$, $Y = 0$ i.e. $(4, 3)$

Directrices: $X = \pm \frac{a}{e}$ i.e. $x-4=-\frac{16}{5}$

$\therefore$ directrices are $5x - 36 = 0$; $5x - 4 = 0$

$$\text{Latus-rectum} = \frac{2b^2}{a}=2\cdot\frac{9}{4}=\frac{9}{2}$$

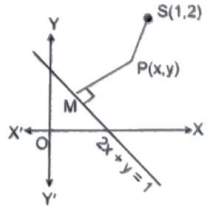
Example 2: Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Sol. Let $P(x,y)$ be any point on the hyperbola.

Draw PM perpendicular from P on the directrix.

Then by definition $SP = e\,PM$

$\Rightarrow \quad (SP)^2 = e^2\,(PM)^2$



$\Rightarrow (x-1)^2+(y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow 5(x^2+y^2-2x-4y+5\}$
$= 3(4x^2+y^2+1+4xy-2y-4x)$

$\Rightarrow 7x^2-2y^2+12xy-2x+14y-22=0$

which is the required hyperbola.

CHORD OF HYPERBOLA

Chord joining two points with eccentric angles α and β is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2} \quad ...(1)$$

if (1) passes through (d, 0) then

$$\frac{d}{a}\cos\frac{\alpha-\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

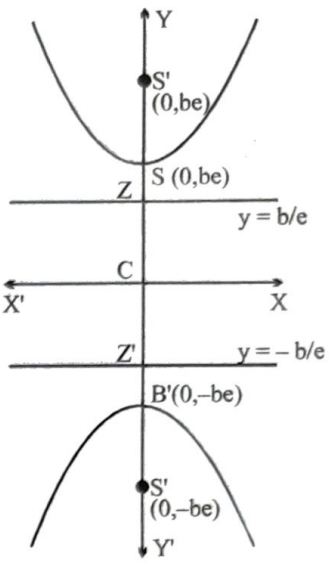
$$\frac{d}{a}=\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\frac{d+a}{d-a}=-\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{a-d}{a+d}$$

CONJUGATE HYPERBOLA

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.



Hence for the hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(1)$$

the conjugate hyperbola is,

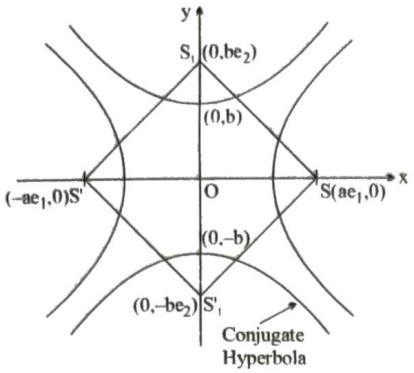
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad ...(2)$$

Comparison between hyperbola and its conjugate hyperbola

Basic Elements	Hyperbola	Conjugate Hyperbola
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	$(0,0)$	$(0,0)$
Length of transeverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Eccentricity	$b^2=a^2(e^2-1)$	$a^2=b^2(e^2-1)$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinate	$(a\sec\phi, b\tan\phi)$ $0<\phi<2\pi$	$(a\tan\phi, b\sec\phi)$ $0\le\phi<2\pi$
Focal distances	$ex_1 \pm a$	$ey_1 \pm b$
Diffference of focal distances	$2a$	$2b$
Equation of transverse axis	$y=0$	$x=0$
Equation of conjugates	$x=0$	$y=0$
Tangent at vertices	$x=\pm a$	$y=\pm b$

Note:

1. The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.



2. If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

TRAIN YOUR BRAIN

Example 3: The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

Sol. Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is $-x^2 + 3y^2 = 1 \Rightarrow \frac{-x^2}{1} + \frac{y^2}{1/3} = 1$

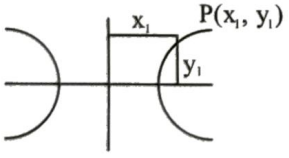
Here $a^2 = 1, b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$

MEANING OF $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

x_1 = perpendicular distance of P to TA

y_1 = perpendicular distance of P to CA



$\therefore \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$$\frac{\{\text{Perpendicular distance of P from conjugate axis}\}^2}{\{\text{Length of semi transverse axis}\}^2} - \frac{\{\text{Perpendicular distance of P from transverse axis}\}^2}{\{\text{Length of semi conjugate axis}\}^2} = 1$$

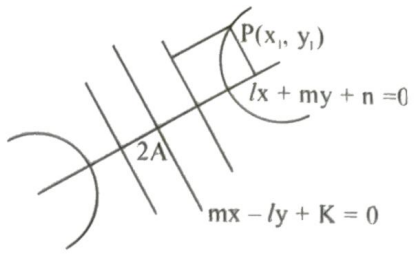
OBLIQUE HYPERBOLA

Let $lx + my + n = 0$ is Transverse axis

$my - ly + K = 0$ is conjugate axis

Length of transvese axis is $2A$

Length of conjugate axis is $2B$



$$\frac{\left(\frac{mx_1 - ly_1 + K}{\sqrt{m^2 + l^2}}\right)^2}{A^2} - \frac{\left(\frac{lx_1 + my_1 + n}{\sqrt{l^2 + m^2}}\right)^2}{B^2} = 1$$

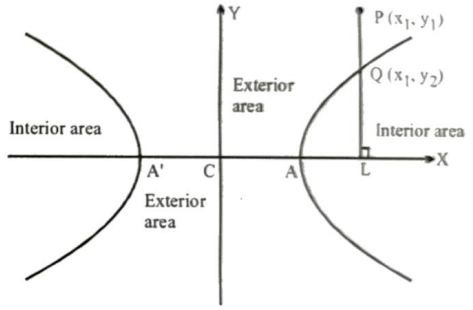
POSITION OF A POINT 'P' W.R.T. A HYPERBOLA

Let $S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the given hyperbola and $P(x_1, y_1)$ is the given point.

(*i*) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie inside the ellipse.

(*ii*) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie outside the ellipse.

(*iii*) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the ellipse.



TRAIN YOUR BRAIN

General Note: Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

Note: The **equation to the chord** of the hyperbola joining the two points $P(\alpha)$ & $Q(\beta)$ is given by

$$\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$$

Remark

(*i*) If chord passes through $(d, 0)$ then

$$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} = \frac{a - d}{a + d}$$

(*ii*) If $d = ae$ i.e. PQ is a focal chord, then

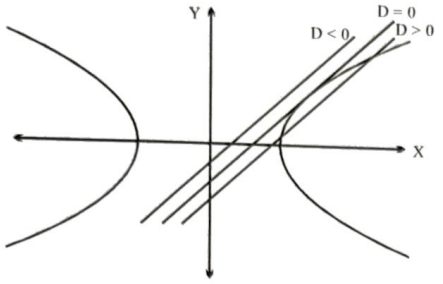
$$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} = \frac{1 - e}{1 + e}$$

(*iii*) If $d = -ae$ i.e. PQ is a focal chord, then

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}=\frac{1+e}{1-e}$$

LINE AND A HYPERBOLA

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)



and the given line be $y = mx + c$ ...(2)

The point of intersection of line and hyperbola can be obtained by solving (1) and (2), therefore

Eliminating y from (1) and (2), we get $\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$

$\Rightarrow\ b^2x^2 - a^2(mx + c)^2 = a^2b^2$

$\Rightarrow\ (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$ ...(3)

Above equation is a quadratic in x and gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary according as discriminant of (3) $>, =, < 0$.

i.e., $4m^2c^2a^4 - 4(a^2m^2 - b^2)a^2(b^2 + c^2) >, =, < 0$

or $-a^2m^2 + b^2 + c^2 >, =, < 0$

or $c^2 >, =, < a^2m^2 - b^2$

TRAIN YOUR BRAIN

Example 4: Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore\ m \times 1 = -1 \Rightarrow\ m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36}-\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore\ a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$y = (-1)x \pm \sqrt{36\times(-1)^2 - 9}$

$y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0$

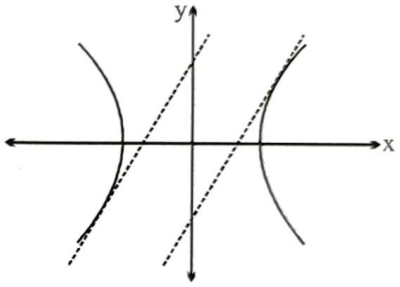
TANGENT TO THE HYPERBOLA : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(*a*) **Point form:** Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note: In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1\,m + y_1^2 + m^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1\ y_1)$ to the hyperbola.

(*b*) **Slope form:** Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

and the given line $y = mx + c$ touches hyperbola then



$c^2 = a^2m^2 - b^2$ or $c = \pm\sqrt{a^2m^2 - b^2}$. This is the required **condition of tangency**

Note:

(*i*) Equation of any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ may be taken as $y = mx + \sqrt{a^2m^2 - b^2}$ or $y = mx - \sqrt{a^2m^2 - b^2}$. The co-ordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2 - b^2)}}\right).$$

(*ii*) The equation of any tangent to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ may be taken as $(y - k) = m(x - h) \pm \sqrt{a^2m^2 - b^2}$.

(*c*) **Parametric form:** Equation of the tangent to the given hyperbola at the point

$(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note:

(*i*) Point of intersection of the tangents at θ_1 & θ_2 is

$$x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\ y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$$

(*ii*) Slope of tangent at $P(\theta)$ is $\frac{b}{a}\operatorname{cosec}\theta$.

Example 5: A common tangent to $9x^2 - 16y^2 = 144$ and $x^2+y^2=9$ is

Sol. $\frac{x^2}{16}-\frac{y^2}{9}=1, x^2+y^2=9$

Equation of tangent $y = mx + \sqrt{16m^2-9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1+m'^2}$ (circle)

For common tangent $m = m'$ and

$$3\sqrt{1+m'^2} = \sqrt{16m^2-9} \text{ or } 9+9m^2 = 16m^2-9$$

or $7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

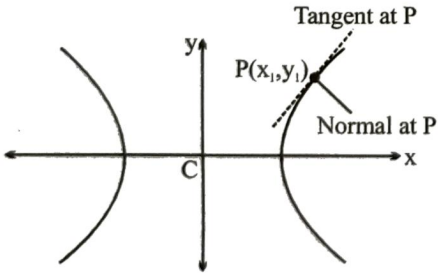
$\therefore$ required equation is $y = \pm 3\sqrt{\frac{2}{7}}\ x \pm 3\sqrt{1+\frac{18}{7}}$

or $y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$

Normal to the hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(*a*) **Point form:** Equation of normal to a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$



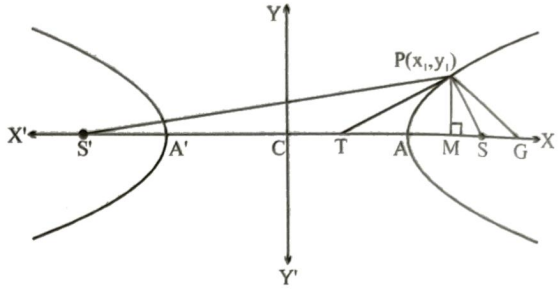
(b) **Slope form:** The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

(*c*) **Parametric form:** The equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ to the given hyperbola is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

Note: In general, maximum four normals can be drawn to a hyperbola from a point and if $\alpha, \beta, \gamma, \delta$ are the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π i.e. $\alpha+\beta+\gamma+\delta=(2n+1)\pi$.

PROPERTIES

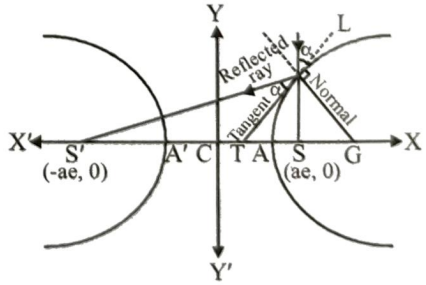
1. Normal and tangent at any point $P(x_1, y_1)$ meet the transverse axis at G and T respectively.



The tangent & normal at any point of a hyperbola bisect the angle between the focal radii.

Note: that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a>k>b>0)$ are confocal and therefore orthogonal.

2. The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

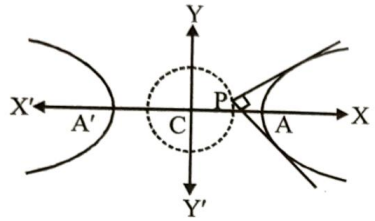


3. Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2=a^2$ & the product of the feet of these perpendiculars is b^2.
i.e., square of semi-conjugate axis.
Similar property exists in ellipse also.

4. The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.
5. The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

DIRECTOR CIRCLE

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which are perpendicular to each other, is called the director circle.



Two tangents (real or imaginary) can be drawn from $P(h, k)$.

If pair of perpendicular tangents are drawn from $P(h, k)$ then $m_1 m_2=\frac{k^2+b^2}{h^2-a^2}=-1$.

$\Rightarrow\ h^2+k^2=a^2-b^2$

$\therefore$ locus of $P(h, k)$ is $x^2+y^2=a^2-b^2$.

Note:

(*i*) If $b^2<a^2$ this circle is real

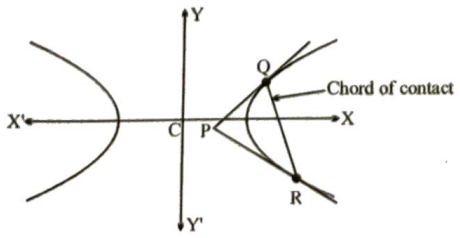
(*ii*) If $b^2=a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

(*iii*) If $b^2>a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

COMMON ARTICLES

(*i*) **Chord of contact:** If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.



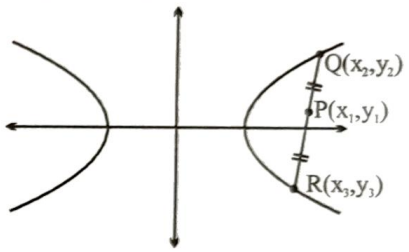
(*ii*) **Pair of tangents:** The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

or $SS_1=T^2$

(*iii*) **Chord with a given middle point (x_1, y_1):** The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$



or $T=S_1$

Example 6: Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

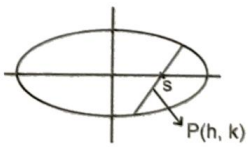
Sol. Let $P=(h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given is $T=S_1$

i.e. $\frac{xh}{a^2}-\frac{yk}{b^2}-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$

since it is a focal chord,

$\therefore$ it passes through focus, either $(ae, 0)$ or $(-ae, 0)$



If it passes through $(ae, 0)$

$\therefore$ locus is $\frac{ex}{a}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$.

DIAMETER

The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2 m}x$

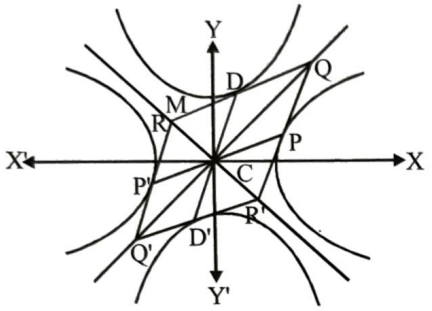
CONJUGATE DIAMETERS

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y = mx, y = m_1x$ be conjugate diameters, then $mm_1 = \frac{b^2}{a^2}$

Prop. 1: If a pair of diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.

Prop. 2: The parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of constant area.



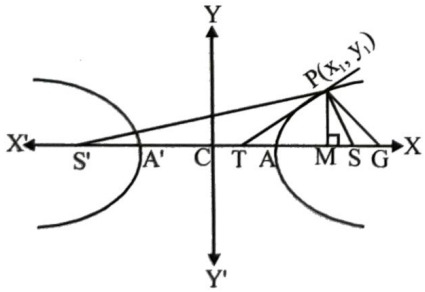
Area of parallelogram $QRQ'R' = 4$ (Area of parallelogram $QDCP$)

$= 4 \cdot CP \times$ (Perpendicular length from C on QD)

$$= 4 \cdot \sqrt{\left(a^2 \sec^2\phi + b^2 \tan^2\phi\right)} \times \frac{1}{\sqrt{\frac{\tan^2\phi}{a^2} + \frac{\sec^2\phi}{b^2}}}$$

$= 4ab =$ Constant

Prop. 3 : If the normal at P meets the transverse axis in G, then $SG = e \cdot SP$. Prove also that the tangent and normal bisect the angle between the focal distances of P.



Prop. 4: If a pair of conjugate diameters meet the hyperbola in P, P' and its conjugate in D, D' then the asymptotes bisect PD, PD', $P'D$ and $P'D'$.

Note: If CD is the conjugate diameter of a diameter CP of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where P is $(a \sec\phi, b \tan\phi)$ then equationof CP where C is $(0, 0)$ is

$$y - 0 = \frac{b\tan\phi - 0}{a\sec\phi - 0}(x - 0)$$

$$\Rightarrow \quad y = \frac{b}{a}\sin\phi \cdot x$$

Comparing this with $y = mx$

$$m = \frac{b}{a}\sin\phi$$

$$\because \quad mm' = \frac{b^2}{a^2} \quad \Rightarrow \quad \frac{b}{a}\sin\phi \times m' = \frac{b^2}{a^2}$$

$$\therefore \quad m' = \left(\frac{b}{a}\right)\text{cosec}\,\phi$$

$\therefore$ Equation of conjugate diameter is $y = m'x = (b/a)$ cosec $\phi . x$ on solving

$$y = (b/a)\ \text{cosec}\ \phi . x \text{ and } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then we get co-ordinates of D

i.e, $\quad D \equiv (a\tan\phi, b\sec\phi)$

ASYMPTOTES

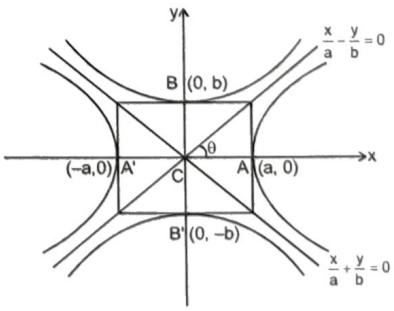
Definition: If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola.**

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore$ Equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

Combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.



PARTICULAR CASE

When $b = a$ the asymptotes of the rectangular hyperbola. $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angles.

Note:

(*i*) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(*ii*) If a hyperbola is equilateral then the conjugate *hyper*-bola is also equilateral

(*iii*) A hyperbola and its conjugate have the same asymptote.

(*iv*) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

$$H: \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0, \quad A: \frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$$

$$C: \frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0,$$

So, $A-H=C-A$.

(*v*) The asymptotes passes through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(*vi*) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(*vii*) Asymptotes are the tangent to the hyperbola from the centre.

(*viii*) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $f(x, y)=0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

TRAIN YOUR BRAIN

Example 7: Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Sol. Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes.

This will represent two straight line so $4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$

$\Rightarrow \lambda = 2$

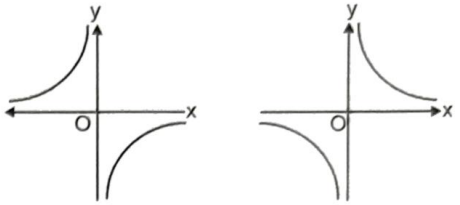
$\Rightarrow 2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow (2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

RECTANGULAR HYPERBOLA

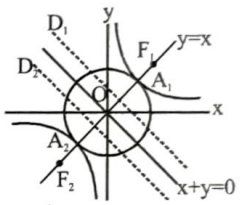
A hyperbola is said to be rectangular hyperbola if its transverse axis is equal to its conjugate axis i.e. $b = a \Rightarrow e = \sqrt{2}$

In rectangular hyperbola asymptotes are perpendicular to each other then $x^2 - y^2 = a^2$, its asymptotes are $y = \pm x$. Rotating the axes by an angle $-\frac{\pi}{4}$ or $\frac{\pi}{4}$ about the origin, equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is reduced to $xy = \frac{a^2}{2}$, $(xy = c^2)$ or $xy = -\frac{a^2}{2}$, $(xy = -c^2)$ respectively. In $xy = c^2$ or $xy = -c^2$ asymptotes are coordinates axes.



$xy = -c^2$ $\qquad$ $xy = c^2$

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(*a*) Equation is $xy = c^2$ with parametric representation $x = ct, y = c/t,\ t \in R - \{0\}$.

(*b*) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1 t_2 y = c(t_1 + t_2)$ with slope $m = \frac{-1}{t_1 t_2}$

(*c*) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ and at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(*d*) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(*e*) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Note: For the hyperbola $xy = c^2$

(*i*) Vertices : (c, c) & $(-c, -c)$.

(*ii*) Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

(*iii*) Directrices : $x + y = \pm\sqrt{2}c$

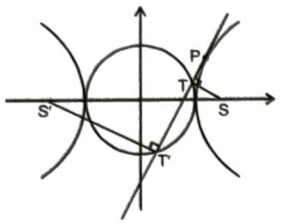
(*iv*) Latus rectum : $\ell = 2\sqrt{2}c = T.A = C.A$

(*v*) Transvers axis : $y = x$, Conjugate axis : $y = -x$

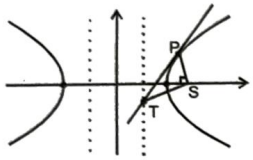
IMPORTANT HIGHLIGHTS OF HYPERBOLA

- Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$
- Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any

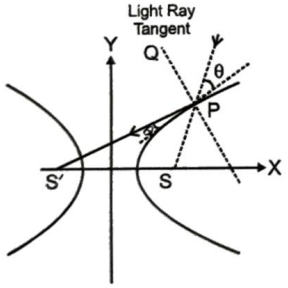
tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



- The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

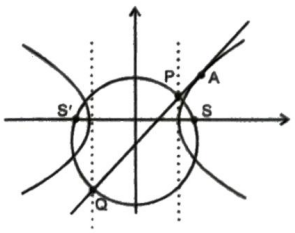


- The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

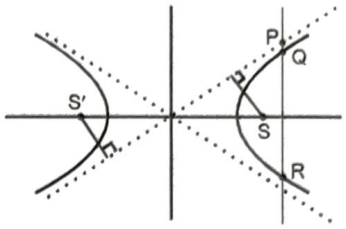


Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1\ (a > k > b > 0)$ are confocal and therefore orthogonal.

- The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



- If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.



$(PQ)\,(PR) = b^2$

- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.
- The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a $\Delta\, CQR$ of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the $\Delta\, CQR$ in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.
- If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.
- A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ $i = 1, 2, 3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\ t_2\ t_3}, -c t_1\ t_2\ t_3\right)$.
- If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then
  - (*a*) $t_1\ t_2\ t_3\ t_4 = 1$
  - (*b*) the centre of the mean position of the four points bisects the distance between the centres of the two curves.
  - (*c*) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1\ t_2\ t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1\ t_2\ t_3\right)\right\}$$

Topicwise Questions

STANDARD FORM AND EQUATION OF HYPERBOLA

1. A directrix of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is:

(a) $x=9/\sqrt{13}$ (b) $y=9/\sqrt{13}$

(c) $x=6/\sqrt{13}$ (d) $y=6/\sqrt{13}$

2. The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13, can be

(a) $25x^2-144y^2=900$

(b) $144x^2-25y^2=900$

(c) $144x^2+25y^2=900$

(d) $25x^2+144y^2=900$

3. If (4, 0) and (–4, 0) be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is

(a) 5/2 (b) 2

(c) 3/2 (d) $\sqrt{2}$

4. The equation of the transverse and conjugate axis of the hyperbola $16x^2-y^2+64x+4y+44=0$ are

(a) $x=2, y+2=0$ (b) $x=2, y=2$

(c) $y=2, x+2=0$ (d) $x+2=0, y+2=0$

5. If $(0,\pm4)$ and $(0,\pm2)$ be the foci and vertices of a hyperbola, then its equation is

(a) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (b) $\frac{x^2}{12}-\frac{y^2}{4}=1$

(c) $\frac{y^2}{4}-\frac{x^2}{12}=1$ (d) $\frac{y^2}{12}-\frac{x^2}{4}=1$

6. If the eccentricities of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ be e and e_1, then $\frac{1}{e^2}+\frac{1}{e_1^2}=$

(a) 1 (b) 2

(c) 3 (d) 5

7. The equation of the hyperbola whose directrix is $x+2y-1=0$, focus (2, 1) and eccentricity 2 will be

(a) $x^2-16xy-11y^2-12x+6y+21=0$

(b) $3x^2+16xy+15y^2-4x-14y-1=0$

(c) $x^2+16xy+11y^2-12x-6y+21=0$

(d) $3x^2-16xy+15y^2-4x+14y+1=0$

POSITION OF POINT & AUXILIARY CIRCLE

8. The equation of auxiliary circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(a) $x^2+y^2=a^2$ (b) $x^2+y^2=b^2$

(c) $x^2+y^2=a^2+b^2$ (d) $x^2+y^2=a^2-b^2$

LINE & HYPERBOLA, TANGENT & NORMAL

9. If the line $y=2x+\lambda$ be a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{144}=1$, then λ equal to -

(a) 16 (b) -16

(c) ±16 (d) ±4

10. The line $3x-4y=5$ is a tangent to the hyperbola $x^2-4y^2=5$. The point of contact is

(a) (3, 1) (b) (2, 1/4)

(c) (1, 3) (d) (1, 2)

11. The equation of the tangents to the conic $3x^2-y^2=3$ perpendicular to the line $x+3y=2$ is

(a) $y=3x\pm\sqrt{6}$ (b) $y=6x\pm\sqrt{3}$

(c) $y=x\pm\sqrt{6}$ (d) $y=3x\pm6$

12. If the straight line $x\cos\alpha+y\sin\alpha=p$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

(a) $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$

(b) $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

(c) $a^2\sin^2\alpha+b^2\cos^2\alpha=p^2$

(d) $a^2\sin^2\alpha-b^2\cos^2\alpha=p^2$

13. The equation of the normal at the point $(a\sec\theta, b\tan\theta)$ of the curve $b^2x^2-a^2y^2=a^2b^2$ is

(a) $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2+b^2$

(b) $\frac{ax}{\tan\theta}+\frac{by}{\sec\theta}=a^2+b^2$

(c) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

(d) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2-b^2$

14. The condition that the straight line $lx + my = n$ may be a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ is given by

(a) $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

(b) $\frac{l^2}{a^2} - \frac{m^2}{b^2} = \frac{(a^2+b^2)^2}{n^2}$

(c) $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$

(d) $\frac{l^2}{a^2} + \frac{m^2}{b^2} = \frac{(a^2-b^2)^2}{n^2}$

15. The equation of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $(8, 3\sqrt{3})$ is

(a) $\sqrt{3}x + 2y = 25$ (b) $x + y = 25$

(c) $y + 2x = 25$ (d) $2x + \sqrt{3}y = 25$

16. The value of m, for which the line $y = mx + \frac{25\sqrt{3}}{3}$, is a normal to the conic $\frac{x^2}{16} - \frac{y^2}{9} = 1$, is

(a) $\sqrt{3}$ (b) $-\frac{2}{\sqrt{3}}$

(c) $-\frac{\sqrt{3}}{2}$ (d) 1

17. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

18. The equation of common tangents to the parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$, is

(a) $2x \pm y + 1 = 0$ (b) $2x \pm y - 1 = 0$

(c) $x \pm 2y + 1 = 0$ (d) $x \pm 2y - 1 = 0$

CHORD OF CONTACTS, CHORD WITH GIVEN MID POINT.

19. What will be equation of that chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid point is (5, 3)

(a) $115x - 117y = 17$ (b) $125x - 48y = 481$

(c) $127x + 33y = 341$ (d) $15x + 121y = 105$

ASYMPTOTES

20. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

(a) $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

(b) $2x^2 + 5xy + 2y^2 = 0$

(c) $2x^2 + 5xy + 2y^2 - 4x + 5y + 2 = 0$

(d) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

RECTANGULAR HYPERBOLA

21. The coordinates of the foci of the rectangular hyperbola $xy = c^2$ are

(a) $(\pm c, \pm c)$

(b) $(\pm c\sqrt{2}, \pm c\sqrt{2})$

(c) $\left(\pm\frac{c}{\sqrt{2}}, \pm\frac{c}{\sqrt{2}}\right)$

(d) $\left(\pm\frac{c}{2}, \pm\frac{c}{2}\right)$

22. The locus of the point of intersection of lines $(x + y)t = a$ and $x - y = at$, where t is the parameter, is

(a) A circle

(b) An ellipse

(c) A rectangular hyperbola

(d) A parabola

23. A tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepts a length of unity from each of the co-ordinate axes, then the point (a, b) lies on the rectangular hyperbola

(a) $x^2 - y^2 = 2$ (b) $x^2 - y^2 = 1$

(c) $x^2 - y^2 = -1$ (d) $x^2 - y^2 = -2$

24. If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ equals

(a) 5 (b) 4

(c) –5 (d) –4

25. The distance between the directrices of a rectangular hyperbola is 10 units, then distance between its foci is

(a) $10\sqrt{2}$ (b) 5

(c) $5\sqrt{2}$ (d) 20

26. If a circle cuts a rectangular hyperbola $xy = c^2$ at A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then

(a) $t_1t_2 = t_3t_4$ (b) $t_1t_2t_3t_4 = 1$

(c) $t_1 = t_2$ (d) $t_3 = t_4$

Learning Plus

1. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

   (*a*) 1 (*b*) $\sqrt{2}$

   (*c*) 2 (*d*) 1/2

2. Which of the following pair, may represent the eccentricities of two conjugate hyperbolas, for all $\alpha \in (0, \pi/2)$?

   (*a*) $\sin\alpha, \cos\alpha$ (*b*) $\tan\alpha, \cot\alpha$

   (*c*) $\sec\alpha, \operatorname{cosec}\alpha$ (*d*) $1+\sin\alpha, 1+\cos\alpha$

3. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci, is:

   (*a*) $\frac{4}{3}$ (*b*) $\frac{4}{\sqrt{3}}$

   (*c*) $\frac{2}{\sqrt{3}}$ (*d*) None of these

4. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k is

   (*a*) ellipse (*b*) parabola

   (*c*) circle (*d*) hyperbola

5. If the latus rectum of an hyperbola be 8 and eccentricity be $\frac{3}{\sqrt{5}}$ then the equation of the hyperbola (with axes along co-ordinate axes) can be

   (*a*) $4x^2 - 5y^2 = 100$ (*b*) $5x^2 - 4y^2 = 100$

   (*c*) $4x^2 + 5y^2 = 100$ (*d*) $5x^2 + 4y^2 = 100$

6. If the centre, vertex and focus of a hyperbola be (0, 0), (4, 0) and (6, 0) respectively, then the equation of the hyperbola is

   (*a*) $4x^2 - 5y^2 = 8$ (*b*) $4x^2 - 5y^2 = 80$

   (*c*) $5x^2 - 4y^2 = 80$ (*d*) $5x^2 - 4y^2 = 8$

7. The equation of the hyperbola whose foci are (6, 5), (–4, 5) and eccentricity 5/4 is

   (*a*) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$

   (*b*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

   (*c*) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = -1$

   (*d*) None of these

8. The vertices of a hyperbola are at (0, 0) and (10, 0) and one of its foci is at (18, 0). The equation of the hyperbola is

   (*a*) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (*b*) $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$

   (*c*) $\frac{x^2}{25} - \frac{(y-5)^2}{144} = 1$ (*d*) $\frac{(x-5)^2}{25} - \frac{(y-5)^2}{144} = 1$

9. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, –2). The equation of the hyperbola (if axes are along co-ordinate axes)is

   (*a*) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ (*b*) $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$

   (*c*) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ (*d*) None of these

10. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then a value of α is

    (*a*) $\pi/6$ (*b*) $\pi/4$

    (*c*) $\pi/3$ (*d*) $\pi/2$

11. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci then the equation of the circle through the points of intersection of two conics is

    (*a*) $x^2 + y^2 = 5$

    (*b*) $\sqrt{5}(x^2+y^2) - 3x - 4y = 0$

    (*c*) $\sqrt{5}(x^2+y^2) + 3x + 4y = 0$

    (*d*) $x^2 + y^2 = 25$

12. If $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point on the hyperbola whose distance from the origin is $\sqrt{6}$ where P is in the first quadrant then $\theta =$

    (*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{3}$

    (*c*) $\frac{\pi}{6}$ (*d*) None

13. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

    (*a*) $e > \sqrt{3}$ (*b*) $1 < e < \frac{2}{\sqrt{3}}$

    (*c*) $e = \frac{2}{\sqrt{3}}$ (*d*) $e > \frac{2}{\sqrt{3}}$

14. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is:

(*a*) zero (*b*) 1

(*c*) 2 (*d*) 4

15. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to:

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{4}$

(*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{2}$

16. The equation to the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are

(*a*) $y = \pm x \pm \sqrt{b^2 - a^2}$ (*b*) $y = \pm x \pm (a^2 - b^2)$

(*c*) $y = \pm x \pm \sqrt{a^2 - b^2}$ (*d*) $y = \pm x \pm \sqrt{a^2 + b^2}$

17. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(*a*) $x^2 + y^2 = 9$ (*b*) $x^2 + y^2 = 1/9$

(*c*) $x^2 + y^2 = 7/144$ (*d*) $x^2 + y^2 = 1/16$

18. The equation of the common tangent to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$ is

(*a*) $2x \pm y + 1 = 0$ (*b*) $x \pm y + 1 = 0$

(*c*) $x \pm 2y + 1 = 0$ (*d*) $x \pm y + 2 = 0$

19. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(*a*) $3x - 4y = 4$ (*b*) $3y - 4x + 4 = 0$

(*c*) $4x - 4y = 3$ (*d*) $3x - 4y = 2$

20. The chords passing through $L(2, 1)$ intersects the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at P and Q. If the tangents at P and Q intersects at R then Locus of R is

(*a*) $x - y = 1$ (*b*) $9x - 8y = 72$

(*c*) $x + y = 3$ (*d*) None of these

21. Equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point $(6, 2)$ is

(*a*) $16x - 75y = 418$ (*b*) $75x - 16y = 418$

(*c*) $25x - 4y = 400$ (*d*) None of these

22. The asymptotes of the hyperbola $xy = hx + ky$ are :

(*a*) $x - k = 0$ & $y - h = 0$

(*b*) $x + h = 0$ & $y + k = 0$

(*c*) $x - k = 0$ & $y + h = 0$

(*d*) $x + k = 0$ & $y - h = 0$

23. The asymptotes of the hyperbola $xy - 3x - 2y = 0$ are

(*a*) $x - 2 = 0$ and $y - 3 = 0$

(*b*) $x - 3 = 0$ and $y - 2 = 0$

(*c*) $x + 2 = 0$ and $y + 3 = 0$

(*d*) $x + 3 = 0$ and $y + 2 = 0$

24. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of orthocentre of the ΔPQR are

(*a*) (x_4, y_4) (*b*) $(x_4, -y_4)$

(*c*) $(-x_4, -x_4)$ (*d*) $(-x_4, -y_4)$

25. A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its

(*a*) orthocenter

(*b*) circum centre

(*c*) centroid

(*d*) incentre

26. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t', then

(*a*) $t' = -\frac{1}{t^3}$ (*b*) $t' = \frac{1}{t}$

(*c*) $t' = \frac{1}{t^2}$ (*d*) $t'^2 = -\frac{1}{t^2}$

27. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is:

(*a*) $y + mx = 0$ (*b*) $y - mx = 0$

(*c*) $my - x = 0$ (*d*) $my + x = 0$

28. If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then $t^3 t_1$ has the value equal to

(*a*) 1 (*b*) -1

(*c*) 0 (*d*) None

Advanced Level Multiconcept Questions

JEE-ADVANCED
MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. The co-ordinates of a focus of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ are

(a) $(-1, 1)$ (b) $(6, 1)$

(c) $(4, 1)$ (d) $(-6, 1)$

2. If 'e' is the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is angle between the asymptotes, then $\cos(\theta/2)$ is:

(a) $\frac{1-e}{e}$ (b) $\frac{\sqrt{e^2-1}}{e}$

(c) $\frac{1}{e}$ (d) $\frac{1}{1+e}$

3. If $(5, 12)$ and $(24, 7)$ are the foci of a conic passing through the origin then the eccentricity of conic is

(a) $\sqrt{386}/12$ (b) $\sqrt{386}/13$

(c) $\sqrt{386}/25$ (d) $\sqrt{386}/38$

4. If θ is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity e, then $\sec\theta/2$ can be

(a) e (b) $e/2$

(c) $e/3$ (d) $1/e$

5. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $(\sqrt{3}, 0)$ when associated with two asymptotes constitutes.

(a) isosceles triangle

(b) an equilateral triangle

(c) a triangle whose area is $\sqrt{3}$ *sq.* unit

(d) a right isosceles triangle

6. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then

(a) $PG = PC$ (b) $Pg = PC$

(c) $PG = Pg$ (d) $Gg = PC$

7. The point of contact of $5x + 12y = 19$ and $x^2 - 9y^2 = 9$ will lie on

(a) $4x + 15y = 0$ (b) $7x + 12y = 19$

(c) $4x + 15y + 1 = 0$ (d) $7x - 12y = 19$

8. Equation $(2+\lambda)x^2 - 2\lambda xy + (\lambda - 1)y^2 - 4x - 2 = 0$ represents a hyperbola if

(a) $\lambda = 4$ (b) $\lambda = 1$

(c) $\lambda = 4/3$ (d) $\lambda = -1$

9. Equation of a tangent passing through $(2, 8)$ to the hyperbola $5x^2 - y^2 = 5$ is:

(a) $3x - y + 2 = 0$

(b) $3x + y - 14 = 0$

(c) $23x - 3y - 22 = 0$

(d) $3x - 23y + 178 = 0$

10. The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular to the line $y = x$ are:

(a) $y = -x + 7$ (b) $y = -x + 3$

(c) $y = -x - 4$ (d) $y = -x - 3$

Comprehension – 1 (Q. No. 11 to 13)

The general second degree equation represent a hyperbola if $h^2 - ab > 0$ and $\Delta \neq 0$ where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. Let H and H' be two hyperbolas. They are said to be conjugate to one another if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other. If A is the pair of its asymptotes, then equation of H, H', A differ only by constant term. Also $H - A = A - H'$.

Now answer the following questions.

11. Let P be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and PN be the perpendicular on transverse axis. Let A and A' be the vertices of hyperbola, then $\frac{PN^2}{NA \cdot NA'} =$

(a) $\frac{a^2}{b^2}$ (b) $\frac{b^2}{a^2}$

(c) a^2 (d) b^2

12. The director circle of the hyperbola $x^2 - y^2 = a^2$ is

(a) $x^2 + y^2 = ax^2$ (b) $x^2 + y^2 = \frac{a^2}{2}$

(c) $x^2 + y^2 = a^2$ (d) None of these

13. A double ordinate PNP' of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the asymptote at Q and Q' respectively where Q is nearer to P. Then $PQ \cdot PQ' =$
(a) square of semitransverse axis
(b) square of semiconjugate axis
(c) square of transverse axis
(d) square of conjugate axis

Comprehension – 2 (Q. No. 14 to 16)

A line is drawn through the point $P(-1, 2)$ meets the hyperbola $xy = c^2$ at the points A and B (Points A and B lie on the same side of P) and Q is a point on the line segment AB.

14. If the point Q is choosen such that PA, PQ and PB are in AP, then locus of point Q is
(a) $x = y(1 + 2x)$ (b) $x = y(1 + x)$
(c) $2x = y(1 + 2x)$ (d) $2x = y(1 + x)$

15. If the point Q is choosen such that PA, PQ and PB are in $G.P$, then locus of point Q is
(a) $xy - y + 2x - c^2 = 0$ (b) $xy + y - 2x + c^2 = 0$
(c) $xy + y + 2x + c^2 = 0$ (d) $xy - y - 2x - c^2 = 0$

16. If the point Q is choosen such that PA, PQ, PB are in $H.P$, then locus of Q is
(a) $2x - y = 2c^2$ (b) $x - 2y = 2c^2$
(c) $2x + y + 2c^2 = 0$ (d) $x + 2y = 2c^2$

17. **Match the column:**

Column - I	**Column - II**
(a) Number of integral values of b for which tangent parallel to 16 line $y = x + 1$ can be drawn to hyperbola $\frac{x^2}{5} - \frac{y^2}{b^2} = 1$ is	(p) 6
(b) The equation of the hyperbola with vertices (3, 0) and (–3, 0) and semi-latusrectum 4, is given by is $4x^2 - 3y^2 = 4k$, then $k =$	(q) 4
(c) The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{3} = 1$ is $\sqrt{k}$, then k is	(r) 3
(d) An equation of a tangent to the hyperbola, $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\frac{\pi}{4}$ with the transverse axis is $y = x + l$, $(l > 0)$, then $2l$ is	(s) 9

18. **Match the column**

Column - I	**Column - II**
(a) The foci of the hyperbola $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ are	(p) (10, 5)
(b) The foci of the hyperbola $8x^2 - y^2 - 64x + 10y + 71 = 0$ are	(q) (2, –2)
(c) The foci of the hyperbola $9x^2 - 16y^2 - 36x + 96y + 36 = 0$ are	(r) (–4, 4) (s) (–2, 5)

NUMERICAL BASED QUESTIONS

19. If the product of the length of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes is $\frac{k}{3}$ then find k.

20. A hyperbola passes through the points (3, 2) and (–17, 12) and has its centre at origin and transverse axis is along x-axis. Then find the length of its transverse axis.

21. If e_1 and e_2 are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola, prove that $e_1^{-2} + e_2^{-2} = k$, then find k

22. Total number of tangents of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, that are perpendicular to the line $5x + 2y - 3 = 0$, is/are

23. Tangents are drawn to $3x^2 - 2y^2 = 6$ from a point P. If these tangents intersect the coordinate axes at concyclic points then the locus of P is $x^2 - y^2 = q$. Find the value of $q/2$

24. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Find the value of b^2.

25. If e and e' are the eccentricities of the ellipse $5x^2 + 9y^2 = 45$ and the hyperbola $5x^2 - 4y^2 = 45$ respectively. Find the value of ee'.

26. Find the area of triangle formed by tangent to the hyperbola $xy = 16$ *at* (16, 1) and co-ordinate axes equals.

27. If m_1 and m_2 are slopes of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ which passes through the point of contact of $3x - 4y = 5$ and $x^2 - 4y^2 = 5$ then $32(m_1 + m_2 - m_1m_2) =$ .....

28. Tangents are drawn from the point $(\alpha, 2)$ to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ & ϕ to the x –axis . If $\tan\theta \cdot \tan\phi = 2$, then the value of $2\alpha^2 - 7$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of conjugate axis is equal to half of the distance between its foci, is

[*JEE Main-2016*]

(*a*) $\frac{4}{3}$ (*b*) $\frac{4}{\sqrt{3}}$

(*c*) $\frac{2}{\sqrt{3}}$ (*d*) $\sqrt{3}$

2. A hyperbola passes through the point $p(\sqrt{2}, \sqrt{3})$ and has foci *at* $(\pm 2, 0)$. The nth tangent to this hyperbola at P also passes through the point. [*JEE Main-2017*]

(*a*) $(-\sqrt{2}, -\sqrt{3})$ (*b*) $(3\sqrt{2}, 2\sqrt{3})$

(*c*) $(2\sqrt{2}, 3\sqrt{3})$ (*d*) $(\sqrt{3}, \sqrt{2})$

3. Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval:

[*JEE Main-2019 (January)*]

(*a*) $(3, \infty)$ (*b*) $\left(\frac{3}{2}, 2\right]$

(*c*) $(2, 3]$ (*d*) $\left(1, \frac{3}{2}\right]$

4. A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse of axis of length 4 along the x-axis. Then the eccentricity of the hyperbola is: [*JEE Main-2019 (January)*]

(*a*) $\frac{2}{\sqrt{3}}$ (*b*) $\frac{3}{2}$

(*c*) $\sqrt{3}$ (*d*) 2

5. The equation of a tengent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is: [*JEE Main-2019 (January)*]

(*a*) $x - y + 1 = 0$ (*b*) $x - y + 7 = 0$

(*c*) $x - y + 9 = 0$ (*d*) $x - y - 3 = 0$

6. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is:

[*JEE Main-2019 (January)*]

(*a*) 13/12 (*b*) 2

(*c*) 13/6 (*d*) 13/8

7. If the vertices of a hyperbola be at (–2, 0) and (2, 0) and one of its foci be *at* (–3, 0), then which one of the following points does not lie on this hyperbola?

[*JEE Main-2019 (January)*]

(*a*) $(-6, 2\sqrt{10})$ (*b*) $(2\sqrt{6}, 5)$

(*c*) $(4, \sqrt{15})$ (*d*) $(6, 5\sqrt{2})$

8. If the eccentricity of the standard hyperbola passing through the point (4,6) is 2, then the equation of the tangent of the hyperbola at (4,6) is

[*JEE Main-2019 (April)*]

(*a*) $2x - y - 2 = 0$ (*b*) $3x - 2y = 0$

(*c*) $2x - 3y + 10 = 0$ (*d*) $x - 2y + 8 = 0$

9. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is

[*JEE Main-2019 (April)*]

(*a*) $\frac{\sqrt{5}}{2}$ (*b*) $\frac{3}{\sqrt{5}}$

(*c*) $\frac{2}{\sqrt{5}}$ (*d*) $\frac{\sqrt{15}}{2}$

10. If $5x+9=0$ is the directrix of the hyperbola $16x^2-9y^2=144$, then its corresponding focus is: [JEE Main-2019 (April)]

(a) $\left(-\frac{5}{3},0\right)$ (b) $(5,0)$

(c) $(-5,0)$ (d) $\left(\frac{5}{3},0\right)$

11. If a directrix of a hyperbola centred at the origin and passing through the point $(4,-2\sqrt{3})$ is $5x=4\sqrt{5}$ and its eccentricity is e, then: [JEE Main-2019 (April)]

(a) $4e^4-24e^2+35=0$ (b) $4e^4-8e^2-35=0$

(c) $4e^4-12e^2-27=0$ (d) $4e^4-24e^2+27=0$

12. If a hyperbola passes through the point $P(10, 16)$ and it has vertices at $(\pm6, 0)$, then the equation of the normal to it at P is [JEE Main-2020 (January)]

(a) $x+3y=58$ (b) $x+2y=42$

(c) $3x+4y=94$ (d) $2x+5y=100$

13. If e_1 and e_2 are the eccentricities of the ellipse, $\frac{x^2}{18}+\frac{y^2}{4}=1$ and the hyperbola, $\frac{x^2}{9}-\frac{y^2}{4}=1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2+3y^2=k$, then k is equal to [JEE Main-2020 (January)]

(a) 14 (b) 15

(c) 16 (d) 17

14. For some $\theta\in\left(0,\frac{\pi}{2}\right)$, if the eccentricity of the hyperbola, $x^2-y^2\sec^2\theta=10$ is $\sqrt{5}$ times the eccentricity of the ellipse, $x^2\sec^2\theta+y^2=5$, then the length of the latus rectum of the ellipse, is : [JEE Main-2020 (September)]

(a) $2\sqrt{6}$ (b) $\frac{2\sqrt{5}}{3}$

(c) $\frac{4\sqrt{5}}{3}$ (d) $\sqrt{30}$

15. A line parallel to the straight line $2x-y=0$ is tangent to the hyperbola $\frac{x^2}{4}-\frac{y^2}{2}=1$ at the point (x_1, y_1). Then $x_1^2+5y_1^2$ is equal to: [JEE Main-2020 (September)]

(a) 8 (b) 6

(c) 10 (d) 5

16. Let e_1 and e_2 be the eccentricities of the ellipse, $\frac{x^2}{25}+\frac{y^2}{b^2}=1(b<5)$ and the hyperbola, $\frac{x^2}{16}-\frac{y^2}{b^2}=1$ respectively satisfying $e_1e_2=1$. If α and β are the dist ances between the foci of the ellipse and the foci of the hyperbola respectively, then the ordered pair (α, β) is equal to: [JEE Main-2020 (September)]

(a) (8, 10) (b) $\left(\frac{24}{5},10\right)$

(c) $\left(\frac{20}{3},12\right)$ (d) (8,12)

17. A hyperbol a having the transverse axis of length $\sqrt{2}$ has the same foci as that of the ellipse of $3x^2+4y^2=12$, then this hyperbola does not pass through which of the following points? [JEE Main-2020 (September)]

(a) $\left(-\sqrt{\frac{3}{2}},1\right)$ (b) $\left(\sqrt{\frac{3}{2}},\frac{1}{\sqrt{2}}\right)$

(c) $\left(\frac{1}{\sqrt{2}},0\right)$ (d) $\left(1,-\frac{1}{\sqrt{2}}\right)$

18. Let $P(3, 3)$ be a point on the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal to it at P intersects the x-axis at $(9, 0)$ and e is its eccentricity, then the ordered pair (a^2, e^2) is equal to: [JEE Main-2020 (September)]

(a) (9,3) (b) $\left(\frac{9}{2},3\right)$

(c) $\left(\frac{3}{2},2\right)$ (d) $\left(\frac{9}{2},2\right)$

19. If the line $y=mx+c$ is a common tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{64}=1$ and the circle $x^2+y^2=36$, then which one of the following is true ? [JEE Main-2020 (September)]

(a) $5m=4$

(b) $8m+5=0$

(c) $c^2=369$

(d) $4c^2=369$

20. The locus of the midpoints of the chord of the circle, $x^2+y^2=25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$, is : [JEE Main-2021 (March)]

(a) $\left(x^2+y^2\right)^2-9x^2+16y^2=0$

(b) $\left(x^2+y^2\right)^2-9x^2+144y^2=0$

(c) $\left(x^2+y^2\right)^2-16x^2+9y^2=0$

(d) $\left(x^2+y^2\right)^2-9x^2-16y^2=0$

21. The locus of the mid points of the chords of the hyperbola $x^2 - y^2 = 4$, which touch the parabola $y^2 = 8x$, is:

(a) $y^3(x-2) = x^2$ (b) $x^3(x-2) = y^2$

(c) $y^2(x-2) = x^3$ (d) $x^2(x-2) = y^3$

[JEE Main-2021 (August)]

22. The locus of the centroid of the triangle formed by any point P on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, and its foci is: [JEE Main-2021 (July)]

(a) $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

(b) $9x^2 - 16y^2 + 36x + 32y - 144 = 0$

(c) $16x^2 - 9y^2 + 32x + 36y - 144 = 0$

(d) $9x^2 - 16y^2 + 36x + 32y - 36 = 0$

23. Let a line $L: 2x + y = k, k > 0$ be a tangent to the hyperbola $x^2 - y^2 = 3$. If L is also a tangent to the parabola $y^2 = \alpha x$, then α is equal to: [JEE Main-2021 (July)]

(a) 12 (b) –12

(c) 24 (d) –24

24. Consider a hyperbola $H: x^2 - 2y^2 = 4$. Let the tangent at a point $P(4, \sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1, y_1), x_1 \geq 0$. If F is a focus of H which is nearer to the point P, then the area of ΔQFR is equal to :

(a) $\sqrt{6} - 1$ (b) $\frac{7}{\sqrt{6}} - 2$

(c) $4\sqrt{6} - 1$ [JEE Main-2021 (March)]

JEE-ADVANCED PREVIOUS YEARS

1. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contacts of the tangents on the hyperbola are

[IIT JEE-2012]

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $(3\sqrt{3}, -2\sqrt{2})$

(d) $(-3\sqrt{3}, 2\sqrt{2})$

2. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle? [JEE Advanced-2017]

(a) $a, 4, 1$ (b) $2a, 4, 1$

(c) $a, 4, 2$ (d) $2a, 8, 1$

3. Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of $60°$ at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

[JEE Advanced-2018]

LIST-I	LIST-II
(P) The length of the conjugate	(1) 8 axis of H is
(Q) The eccentricity of H is	(2) $\frac{4}{\sqrt{3}}$
(R) The distance between	(3) $\frac{2}{\sqrt{3}}$ the foci of H is
(S) The length of the latus	(4) 4 rectum of H is

The correct option is :

(a) $P \to 4$; $Q \to 2$; $R \to 1$; $S \to 3$

(b) $P \to 4$; $Q \to 3$; $R \to 1$; $S \to 2$

(c) $P \to 4$; $Q \to 1$; $R \to 3$; $S \to 2$

(d) $P \to 3$; $Q \to 4$; $R \to 2$; $S \to 1$

4. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point $(1,0)$, and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE? [JEE Advanced-2020]

(a) $1 < e < \sqrt{2}$ (b) $\sqrt{2} < e < 2$

(c) $\Delta = a^4$ (d) $\Delta = b^4$

ANSWER KEY

Topicwise Questions

1. (*a*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** (*c*) **6.** (*a*) **7.** (*a*) **8.** (*a*) **9.** (*c*) **10.** (*a*) **11.** (*a*) **12.** (*b*)
13. (*c*) **14.** (*a*) **15.** (*d*) **16.** (*b*) **17.** (*b*) **18.** (*a*) **19.** (*b*) **20.** (*d*) **21.** (*b*) **22.** (*c*) **23.** (*b*) **24.** (*c*)
25. (*d*) **26.** (*b*)

Learning Plus

1. (*b*) **2.** (*c*) **3.** (*c*) **4.** (*d*) **5.** (*a*) **6.** (*c*) **7.** (*a*) **8.** (*b*) **9.** (*c*) **10.** (*b*) **11.** (*a*) **12.** (*a*)
13. (*d*) **14.** (*a*) **15.** (*d*) **16.** (*c*) **17.** (*d*) **18.** (*a*) **19.** (*a*) **20.** (*b*) **21.** (*b*) **22.** (*a*) **23.** (*a*) **24.** (*d*)
25. (*a*) **26.** (*a*) **27.** (*a*) **28.** (*b*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*c*, *d*) **2.** (*b*, *c*) **3.** (*a*, *d*) **4.** (*a*, *d*) **5.** (*b*, *c*) **6.** (*a*, *b*, *c*) **7.** (*a*) **8.** (*b*, *d*) **9.** (*a*, *c*) **10.** (*b*, *d*) **11.** (*b*) **12.** (*d*)
13. (*b*) **14.** (*c*) **15.** (*b*) **16.** (*a*) **17.** (*a*) - (q), (*b*) - (s), (*c*) - (s), (*d*) - (q) **18.** (*a*) → (r, t); (*b*) → (p, s); (*c*) → (s)
19. [0002] **20.** [0002] **21.** [0001] **22.** [0000] **23.** [2.5] **24.** [0007] **25.** [1] **26.** [32]
27. [22] **28.** [4]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. (*c*) **2.** (*c*) **3.** (*a*) **4.** (*a*) **5.** (*a*) **6.** (*a*) **7.** (*d*) **8.** (*a*) **9.** (*c*) **10.** (*c*) **11.** (*a*) **12.** (*d*)
13. (*c*) **14.** (*c*) **15.** (*b*) **16.** (*a*) **17.** (*b*) **18.** (*b*) **19.** (*d*) **20.** (*a*) **21.** (*c*) **22.** (*a*) **23.** (*d*) **24.** (*b*)

JEE-ADVANCED
PREVIOUS YEARS

1. (*a*, *b*) **2.** (*a*, *c*, *d*) **3.** (*b*) **4.** (*a*, *d*)

PRAYAS 2.0 t.me/realprayasians2

18 SETS

PRAYAS 1.0 t.me/realprayasians

SET

A set is a well defined collection of distinct objects. Sets are usually denoted by capital letters $A, B, \ldots, X, Y, Z$. The elements of the set are denoted by small letters. There are two methods for representing a set.

(i) Tabulation method or Roster form

All the elements belonging to the set are written in curly brackets.

If A is the set of days of a week, then

A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

(ii) Set Builder Method or Set rule method

In this method, we use the definition, which is satisfied by all the elements of set.

In above example set A may be written as

$A = \{x : x \text{ is a day of week}\}$

(a) Some well known sets and their notations

(i) Set of all natural numbers $N = \{1, 2, 3, \ldots\}$

(ii) Set of all integers Z or $I = \{0, \pm1, \pm2, \ldots\}$

(iii) Set of non zero integers Z_0 or $I_0 = \{\pm1, \pm2, \pm3, \ldots\}$

(iv) Set of all rational numbers

$Q = \{x : x = p/q$, where p and q relatively prime integers and $q \neq 0\}$

Q_0 denotes the set of all non–zero rational numbers.

(v) Set of real numbers is denoted by R

(vi) Set of complex numbers is denoted by C

(b) Subset & Superset

If A and B are two sets such that every element of A is also an element of B, then A is a subset of B and B is superset of A. We write $A \subseteq B$.

- If $A \subseteq B$ & $B \subseteq A$ then A and B said to be equal sets, i.e. $A = B$.
- If $A \subseteq B$ and $A \neq B$, there A is called as proper subset of B and denoted by $A \subset B$.
- If a set A has n elements, then the number of subsets of $A = 2^n$.

(c) Universal Set

The universal set is the superset for all the sets under the consideration.

The set of complex numbers is the universal set for all possible sets related numbers.

(d) Null set or Void set

A set having no element is called as null set or empty set or void set. It is denoted by ϕ or $\{\ \}$. The null set is unique and is the subset of every set.

(e) Disjoint sets

Two sets A & B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A & B are said to be intersecting or overlapping sets.

(f) Singleton Set

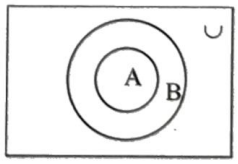
A set having one and only one element is called singleton set or unit set.

(g) Power Set

Power set of a set A is the collection of all subsets of A and is denoted by $P(A)$ or 2^n.

VENN - EULER DIAGRAMS

Venn – diagram is a systematic representation of sets in pictorial form. A set is represented by circle inside the universal set which itself represented by rectangular region.



A⊂B

BASIC OPERATIONS

(a) Union of sets

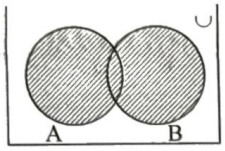
If A and B are two sets, then the union of two sets is denoted by $A \cup B$ and defined as

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

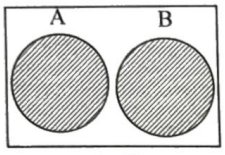
Union of n sets $A_1, A_2, \ldots, A_n$ is denoted and defined as

$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for at least one } i\}$

Union is also known as join or "logical sum" of A & B.



$A \cup B$

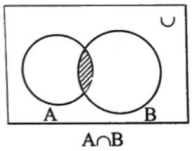


$A \cup B$

(b) Intersection of sets

The intersection of sets A & B is denoted by $A \cap B$ & defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Intersection of n sets $A_1, A_2, \ldots, A_n$ is denoted & defined by $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } i\}$



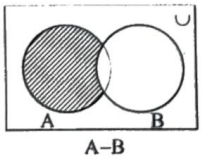
$A \cap B$

(c) Complement

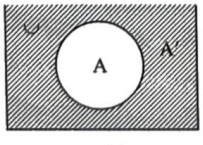
The complement of a set B relative to another set A is denoted & defined as $A - B = \{x : x \in A \text{ and } x \notin B\}$ i.e. the set of all points (elements) which belong to A but which do not belong to B.

The complement of set A relative to universal set U is denoted by $U - A$ or by $\sim A$ or A^c or by A'.

Thus $A' = \{x : x \in \cup \text{ and } x \notin A\}$ OR $A' = \{x : x \notin A\}$



$A - B$

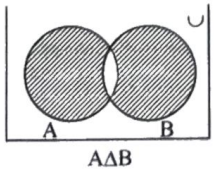


A'

(d) Symmetric difference of two sets

The symmetric difference of sets A & B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$

Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$



$A \Delta B$

CARTESIAN PRODUCT

The Cartesian product of two sets A and B is the set $\{(a, b): a \in A \text{ and } b \in B\}$ and is denoted by $A \times B$. If A has m elements and B has n elements, then $A \times B$ has mn elements.

- The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is the set of all ordered n tuples $(a_1, a_2, \ldots, a_n)$, $a_i \in A_i$, $i = 1, 2, \ldots n$ and is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or $\prod_{i=1}^{n} A_i$

CARDINAL NUMBER

The no. of distinct elements in a set A is denoted by $n(A)$ and known as cardinal number of the set A.

- Two finite sets A & B are equivalent if their cardinal number are same.

IMPORTANT PROPERTIES

(*i*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(*ii*) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A$ & B are disjoint non–void sets.

(*iii*) $n(A - B) = n(A) - n(A \cap B)$

(*iv*) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$

(*v*) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(*vi*) No. of elements in exactly two of the sets A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(*vii*) No. of elements in exactly one of the sets $A, B, C = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(*viii*) $n(A' \cup B') = n(U) - n(A \cap B)$

(*ix*) $n(A' \cap B') = n(U) - n(A \cup B)$

LAWS OF ALGEBRA OF SETS

(a) Idempotent laws

(*i*) $A \cup A = A$

(*ii*) $A \cap A = A$

(b) Identity laws

(*i*) $A \cup \phi = A$

(*ii*) $A \cap U = A$

i.e. ϕ & U are identity elements for union & intersection respectively.

(c) Commutative laws

(*i*) $A \cup B = B \cup A$

(*ii*) $A \cap B = B \cap A$

i.e. union & intersection are commutative

(d) Associative laws

(*i*) $(A \cup B) \cup C = A \cup (B \cup C)$

(*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$

i.e. union & intersection are associative

(e) Distributive laws

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

(f) De–Morgan's laws

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $(A')' = A$

TRAIN YOUR BRAIN

Example 1: If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

Sol. We have $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case 1

When $n(A \cap B)$ is minimum, i.e., $n(A \cap B) = 0$

This is possible only when $A \cap B = \phi$. In this case,

$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$.

So maximum number of elements in $A \cup B$ is 9.

Case 2

When $n(A \cap B)$ is maximum.

This is possible only when $A \subseteq B$.

In this case $n(A \cap B) = 3$.

$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6$

So, minimum number of elements in $A \cup B$ is 6.

Example 2: If A, B and C are three sets and U is the universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $\text{In}(A \cap B) = 100$. Find $n(A' \cap B')$.

Sol. We have $A' \cap B' = (A \cup B)'$

$$\therefore\quad n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$$
$$= n(U) - [n(A) + n(B) - n(A \cap B)]$$
$$= 700 - (200 + 300 - 100) = 300$$

Example 3: Prove that for non empty sets A and B.

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Sol. Consider

$$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$$
$$= (A \cap B) \cap (A' \cup B')$$
$$= [(A \cup B) \cap A'] \cup [(A \cup B) \cap B']$$
$$= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')]$$
$$= [\phi \cup (B \cap A')] \cup [(A \cap B') \cup \phi]$$
$$= (B \cap A') \cup (A \cap B')$$
$$= (B - A) \cup (A - B) \qquad \ldots(1)$$

Now R.H.S. $= (A - B) \cup (B - A) \cup (A \cap B)$

$= [(A \cup B) - (A \cap B)] \cup (A \cap B)$ using (1)

$= [(A \cup B) \cap (A \cap B')] \cup (A \cap B)$

$= [(A \cup B) \cup (A \cap B)] \cap [(A \cap B)' \cup (A \cap B)]$

$= (A \cup B) \cap U$, where U is universal set.

$= A \cup B$ = L.H.S.

Example 4: If $N = \{ax : x \in N\}$ describe the set $3N \cap 7N$.

Sol. $3N = \{3x : x \in N\} = \{3, 6, 9, \ldots\}$

$7N = \{7x : x \in N\} = \{7, 14, 21, 28, \ldots\}$

$\therefore\ 3N \cap 7N = \{y : y$ is a multiple of 3 and y is a multiple of $7\}$

$= \{y : y$ is a multiple of $21\}$

$= \{21, 42, 63, \ldots\} = 21N$

Example 5: Solve $3x^2 - 12x = 0$.

When (i) $x \in N$ (ii) $x \in Z$

(iii) $x \in S$ where $S = \{a + ib : b \neq 0;\ a, b \in R\}$

Sol. $3x^2 - 12x = 0 \Leftrightarrow 3x(x - 4) = 0 \Leftrightarrow x = 0$ or $x = 4$

(i) when $x \in N \Rightarrow x = 4$

(ii) when $x \in Z \Rightarrow x = 0$ or $x = 4$

(iii) when $x \in S \Rightarrow$ No solution

Example 6: In a certain city, only two newspapers A and B are published. It is known that 25% of the city population reads A and 20% reads B, while 8% reads both A and B. It is also known that 30% of those who read A but not B, look into advertisements and 40% of those who read B but not A, look into advertisements, while 50% of those who read both A and B, look into advertisements, What % of the population read an advertisement?

Sol. Let L = Set of people who read paper A

M = Set of people who read paper B

Then $n(L) = 25$, $n(M) = 20$, $n(L \cap M) = 8$

$n(L - M) = n(L) - n(L \cap M) = 25 - 8 = 17$

$n(M - L) = n(M) - n(L \cap M) = 20 - 8 = 12$

$\therefore$ % of people reading an advertisement

$= (30\%$ of $17) + (40\%$ of $12) + (50\%$ of $8)$

$$= \frac{51}{10} + \frac{24}{5} + 4 = 13.9\%$$

Topicwise Questions

DEFINITION AND TYPE OF SETS

1. The set of intelligent students in a class is
   (a) A null set
   (b) A singleton set
   (c) A finite set
   (d) Not a well defined collection
2. Which of the following is the empty set
   (a) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
   (b) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
   (c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
   (d) $\{x : x$ is a real number and $x^2 = x + 2\}$
3. The set $A = \{x : x \in R, x^2 = 16$ and $2x = 6\}$ equals
   (a) ϕ (b) [14, 3, 4]
   (c) [3] (d) [4]
4. If a set A has n elements, then the total number of subsets of A is
   (a) n (b) n^2
   (c) 2^n (d) $2n$
5. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The values of m and n are
   (a) 7, 6 (b) 6, 3
   (c) 5, 1 (d) 8, 7
6. The number of proper subsets of the set {1, 2, 3} is
   (a) 8 (b) 7
   (c) 6 (d) 5
7. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$, then
   (a) $X \subseteq Y$ (b) $Y \subseteq X$
   (c) $X = Y$ (d) None of these

OPERATIONS ON SETS

8. Given the sets $A = \{1, 2, 3\}, B = \{3, 4\}, C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is
   (a) {3} (b) {1, 2, 3, 4}
   (c) {1, 2, 4, 5} (d) {1, 2, 3, 4, 5, 6}
9. If $A \subseteq B$, then $A \cup B$ is equal to
   (a) A (b) $B \cap A$
   (c) B (d) None of these
10. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to
    (a) A (b) B
    (c) A^c (d) B^c
11. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to
    (a) A (b) B
    (c) ϕ (d) $A \cap B^c$
12. If $N_a = \{an : n \in N\}$ then $N_3 \cap N_4 =$
    (a) N_7 (b) N_{12}
    (c) $N_3 N_3$ (d) $N_4 N_4$
13. If $aN = \{ax : x \in N\}$ and $bN \cap cN = dN$, where $b, c \in N$ are relatively prime, then
    (a) $d = bc$ (b) $c = bd$
    (c) $b = cd$ (d) None of these
14. If the sets A and B are defined as
    $A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$
    $B = \{(x, y) : y = -x, x \in R\}$, then
    (a) $A \cap B = A$ (b) $A \cap B = B$
    (c) $A \cap B = \phi$ (d) None of these
15. Let $A\,[x : x \in R, |x| < 1]$; $B = [x : x \in R, |x - 1| \geq 1]$ and $A \cup B = R - D$, then the set D is
    (a) $[x : x \in R, 1 \leq x \leq 2]$ (b) $[x : x \in R, 1 \leq x < 2]$
    (c) $[x : x \in R, 1 < x \leq 2]$ (d) None of these
16. If the sets A and B are defined as
    $A = \{(x, y) : y = e^x, x \in R\}$
    $B = \{x, y\} : y = x, x \in R$, then
    (a) $B \subseteq A$ (b) $A \subseteq B$
    (c) $A \cap B = \phi$ (d) $A \cup B = A$
17. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$ then $X \cup Y$ is equal to
    (a) X (b) Y
    (c) N (d) None of these

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

18. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$
    (a) 3 (b) 6
    (c) 9 (d) 18
19. If A and B are two sets such that $n(A) = 70$, $n(B) = 60$, and $n(A \cup B) = 110$, then $n(A \cap B)$ is equal to
    (a) 240 (b) 50
    (c) 40 (d) 20

20. Let $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$ then $n(A^c \cap B^c)$

(a) 400 (b) 600

(c) 300 (d) 200

21. If $A = [(x, y) : x^2 + y^2 = 25]$ and $B = [(x, y) : x^2 + 9y^2 = 144]$, then $A \cap B$ contains

(a) One point (b) Three points

(c) Two points (d) Four points

22. In a town of 10,000 families it was found that 40% family buy newspaper A, 20% buy newspaper B and 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspapers, then number of families which buy A only is

(a) 3100 (b) 3300

(c) 2900 (d) 1400

23. In a city 20 percent of the population travels by car, 50 percent travels by bus and 10 percent travels by both car and bus. Then persons travelling by car or bus is

(a) 80 percent (b) 40 percent

(c) 60 percent (d) 70 percent

24. Suppose $A_1, A_2, A_3, \ldots\ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots\ldots, B_n$ are n sets each with 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each elements of S belongs to exactly 10 of the $A'_i s$ and exactly 9 of the $B'_j s$. Then n is equal to

(a) 15 (b) 3

(c) 45 (d) None of these

25. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is

(a) 6 (b) 9

(c) 7 (d) All of these

LAWS OF ALGEBRA OF SETS

26. If A, B and C are any three sets, then $A \times (B \cap C)$ is equal to

(a) $(A \times B) \cup (A \times C)$ (b) $(A \times B) \cap (A \times C)$

(c) $(A \cup B) \times (A \cup C)$ (d) $(A \cap B) \times (A \cap C)$

27. If A, B and C are any three sets, then $A \times (B \cup C)$ is equal to

(a) $(A \times B) \cup (A \times C)$ (b) $(A \cup B) \times (A \cup C)$

(c) $(A \times B) \cap (A \times C)$ (d) None of these

28. If A, B and C are any three sets, then $A - (B \cup C)$ is equal to

(a) $(A - B) \cup (A - C)$ (b) $(A - B) \cap (A - C)$

(c) $(A - B) \cup C$ (d) $(A - B) \cap C$

29. If $A = [x : x$ is a multiple of 3] and $B = [x : x$ is a multiple of 5], then $A - B$ is ($\overline{A}$ means complement of A)

(a) $\overline{A} \cap B$ (b) $A \cap \overline{B}$

(c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$

30. If A, B and C are non-empty sets, then $(A - B) \cup (B - A)$ equals

(a) $(A \cup B) - B$ (b) $A - (A \cap B)$

(c) $(A \cup B) - (A \cap B)$ (d) $(A \cap B) \cup (A \cup B)$

CARTESIAN PRODUCT OF SETS

31. If $A = \{0, 1)$, and $B = \{1, 0\}$, then $A \times B$ is equal to

(a) $\{0, 1, 1, 0\}$ (b) $\{(0, 1), (1, 0)\}$

(c) $\{0, 0\}$ (d) $\{(0,1),(0,0),(1,1),(1,0)\}$

32. If $A = \{2, 4, 5\}, B = \{7, 8, 9\}$ then $n(A \times B)$ is equal to

(a) 6 (b) 9

(c) 3 (d) 0

33. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is

(a) $p + q$ (b) $p + q + 1$

(c) pq (d) p^2

34. If $A = \{a, b\}, B = \{c, d\}, C = \{d, e\}$ then $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$ is equal to

(a) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$

(c) $A \times (B \cup C)$ (d) $A \times (B \cap C)$

35. If $A = \{x : x^2 - 5x + 6 = 0\}, B = \{2, 4\}, C = \{4, 5\}$ then $A \times (B \cap C)$ is

(a) $\{(2, 4), (3, 4)\}$ (b) $\{(4, 2), (4, 3)\}$

(c) $\{(2, 4), (3, 4), (4, 4)\}$ (d) $\{(2,2), (3,3), (4,4), (5,5)\}$

36. If P, Q and R are subsets of a set A, then $R \times (P^c \cup Q^c)^c$ =

(a) $(R \times P) \cap (R \times Q)$ (b) $(R \times Q) \cap (R \times P)$

(c) $(R \times P) \cup (R \times Q)$ (d) None of these

Learning Plus

1. Let $A = \{x : x \in R, x \geq 2\}$ and $B = \{x : x \in R, x < 4\}$. Then $A \cap B =$
   (a) $\{x : x \in R, 2 < x < 4\}$, (b) $\{x : x \in R, 2 \leq x < 4\}$
   (c) B (d) A
2. If $A = \{\phi, \{(\phi)\}\}$, then the power set $P(A)$ of A is
   (a) A (b) $\{\phi, \{\phi\}, A\}$
   (c) $\{\phi, \{\phi\}, \{\{\phi\}\}, A\}$ (d) $\{\phi, \{\phi\}, \{\{\phi\}\}\}$
3. If $A = \{2, 3\}$ and $B = \{x \mid x \in N \text{ and } x < 3\}$, then $A \times B$ is
   (a) $\{(2, 1), (2, 2), (3, 1), (3, 2)$
   (b) $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$
   (c) $\{(1, 2), (2, 2), (3, 3), (3, 2)\}$
   (d) $\{(1, 1), (2, 2), (3, 3), (3, 2)\}$
4. Let U be the universal set containing 700 elements. If A, B are sub–sets of U such that $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Then $n(A' \cap B') =$
   (a) 400 (b) 600
   (c) 300 (d) 200
5. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is
   (a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$
   (c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$
6. If $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{x \in N : 30 < x^2 < 70\}$, $B = \{x : x \text{ is a prime number less than } 10\}$, then which of the following is incorrect?
   (a) $A \cup B = \{2, 3, 5, 6, 7, 8\}$ (b) $A \cap B = \{7, 8\}$
   (c) $A - B = \{6, 8\}$ (d) $A \Delta B = \{2, 3, 5, 6, 8\}$
7. Let X be the universal set for sets A and B, if $n(A) = 200$, $n(B) = 300$ and $n(A \cup B) = 100$, then $n(A' \cap B')$ is equal to 300 provided $n(x)$ is equal j to
   (a) 600 (b) 700
   (c) 800 (d) 900
8. If two sets A and B are having 80 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are
   (a) 2^{80} (b) 80^2
   (c) 81 (d) 79
9. In a town of 10000 families it was found that 40% family buy newspaper A, 20% family buy newspaper B and 10% family buy newspaper C. 5% families buy A and B, 3% families buy B and C and 4% families buy A and C. If 2% families buy all the three newspapers. Then, number of families which buy A only is
   (a) 3100 (b) 3300
   (c) 2900 (d) 1400
10. If A and B be two universal sets and $A \cup B \cup C = U$. Then, $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to
    (a) $A \cup B \cup C$
    (b) $A \cup (B \cap C)$
    (c) $A \cap B \cap C$
    (d) $A \cap (B \cup C)$
11. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to
    (a) $B \cap C'$
    (b) $A \cap C$
    (c) $B' \cap C'$
    (d) None of these
12. If there are three athelitic teams in a school 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is
    (a) 42 (b) 43
    (c) 45 (d) None of these
13. If $P(A)$ denotes the power set of A and A is the void set, then what is number of elements in $P\{P\{P\{P(A)\}\}\}$?
    (a) 0 (b) 1
    (c) 4 (d) 16

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

SET

1. Two sets A and B are as under

$A = \{(a, b) \in R \times R : |a-5| < 1 \text{ and } |b-5| < 1\}$;

$B = \{(a, b) \in R \times R : 4(a-6)^2 + 9(b-5)^2 \le 36\}$

Then— **[JEE Main-2018]**

(a) $A \subset B$ (b) $A \notin B = \phi$ (an empty set)

(c) neither $A \subset B$ nor $B \subset A$

(d) $B \subset A$

2. In a class of 140 students numbered 1 to 140, all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is: **[JEE Main-2019 (January)]**

(a) 102 (b) 42

(c) 1 (d) 38

3. Let $S = \{1,2,3,\ldots\ldots, 100\}$. The number of non-empty subsets A of S such that the product of elements in A is even is : **[JEE Main-2019 (January)]**

(a) $2^{100} - 1$ (b) $2^{50}(2^{50} - 1)$

(c) $2^{50} - 1$ (d) $2^{50} + 1$

4. Let Z be the set of integers. If $A = \{X \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is : **[JEE Main-2019 (January)]**

(a) 2^{15} (b) 2^{18}

(c) 2^{12} (d) 2^{10}

5. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisement is: **[JEE Main-2019 (April)]**

(a) 12.8 (b) 13.5

(c) 13.9 (d) 13

6. Let A, B and C be sets such that $\phi \ne A \cap B \subseteq C$. Then which of the following statements is not true? **[JEE Main-2019 (April)]**

(a) If $(A - C) \subseteq B$ then $A \subseteq B$

(b) $(C \cup A) \cap (C \cup B) = C$

(c) If $(A - B) \subseteq C$, then $A \subseteq C$

(d) $B \cap C \ne \phi$

7. Let $X = \{n \in N : 1 \le n \le 50\}$. If $A = \{n \in X$: n is a multiple of 2$\}$; $B = \{n \in X$: n is a multiple of 7$\}$, then the number of elements in the smallest subset of X containing both A and B is _____. **[JEE Main-2020 (January)]**

8. If $A = \{x \in R : |x| < 2\}$ and $B = \{|x - 2| \ge 3\}$: then: **[JEE Main-2020 (January)]**

(a) $A - B = [-1, 2)$

(b) $A \cup B = R - (2, 5)$

(c) $B - A = R - (-2, 5)$

(d) $A \cap B = (-2, -1)$

9. Let S be the set of all integer solutions, (x, y, z), of the system of equations **[JEE Main-2020 (September)]**

$x - 2y + 5z = 0$

$-2x + 4y + z = 0$

$-7x + 14y + 9z = 0$

such that $15 \le x^2 + y^2 + z^2 \le 150$. Then, the number of elements in the set S is equal to _____.

10. Consider the two sets: **[JEE Main-2020 (September)]**

$A = \{m \in R$: both the roots of $x^2 - (m+1)x + m + 4 = 0$ are real$\}$ and $B = [-3, 5)$.

Which of the following is not true?

(a) $A \cap B = \{-3\}$ (b) $B - A = (-3, 5)$

(c) $A \cup B = R$ (d) $A - B = (-\infty, -3) \cup (5, \infty)$

11. Let $\bigcup_{i=1}^{50} X_i = \bigcup_{i=1}^{n} Y_i = T$ where each X_i contains 10 elements and each Y_i contains 5 elements. If each element of the set T is an element of exactly 20 of sets X_i's and exactly 6 of sets Y_i's, then n is equal to **[JEE Main-2020 (September)]**

(a) 50 (b) 15

(c) 30 (d) 45

12. A survey shows that 63% of the people in a city read newspaper A whereas 76% read newspaper B. If x% of the people read both the newspapers, then a possible value of x can be : [*JEE Main-2020 (September)*]

(*a*) 37 (*b*) 55

(*c*) 29 (*d*) 65

13. A survey shows that 73% of the persons working in an office like coffee, whereas 65% like tea. If x denotes the percentage of them, who like both coffee and tea, then x cannot be : [*JEE Main-2020 (September)*]

(*a*) 63

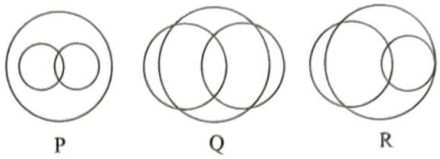
(*b*) 36

(*c*) 38

(*d*) 54

14. Set A has m elements and Set B has n elements. If the total number of subsets of A is 112 more than total number of subsets of B, then the value of $m \cdot n$ is —. [*JEE Main-2020 (September)*]

15. In a school, there are three types of games to be played. Some of the students play two types of games, but none play all the three games. Which Venn diagrams can justify the above statement? [*JEE Main-2021*]



(*a*) P and Q (*b*) P and R

(*c*) None of these (*d*) Q and R

ANSWER KEY

Topicwise Questions

1. (*d*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*c*)	**10.** (*a*)	**11.** (*d*)	**12.** (*b*)
13. (*a*)	**14.** (*c*)	**15.** (*b*)	**16.** (*c*)	**17.** (*b*)	**18.** (*b*)	**19.** (*d*)	**20.** (*c*)	**21.** (*d*)	**22.** (*b*)	**23.** (*c*)	**24.** (*c*)
25. (*d*)	**26.** (*b*)	**27.** (*a*)	**28.** (*b*)	**29.** (*b*)	**30.** (*c*)	**31.** (*d*)	**32.** (*b*)	**33.** (*c*)	**34.** (*c*)	**35.** (*a*)	**36.** (*b*)

Learning Plus

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*c*)	**6.** (*d*)	**7.** (*b*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)	**11.** (*a*)	**12.** (*b*)
13. (*d*)											

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*a*)	**2.** (*d*)	**3.** (*b*)	**4.** (*a*)	**5.** (*c*)	**6.** (*a*)	**7.** [29]	**8.** (*c*)	**9.** [8]	**10.** (*d*)	**11.** (*c*)	**12.** (*b*)
13. (*b*)	**14.** [28.00]		**15.** (*c*)							

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1 Relations and Function

RELATIONS

A relation R from a non-empty set A to a non empty set B is a subset of the cartersian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element.

DOMAIN OF A RELATION

The set of all first elements of the ordered pairs in a relation R form a set A to a set B is called the domain of the relation R.

RANGE OF A RELATION

The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R, Note that range $\subseteq$ codomain.

Note : The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

TYPES OF RELATION:

In this section, we intend to define various types of relations on a given set A.

1. **Empty / Void Relation:** Let A be a set, then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.
2. **Universal Relation:** Let A be a set, then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.
   Note : The void and the universal relations on a set A are respectively the smallest and the largest relations on A.
3. **Identity Relation:** Let A be a set, then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.
   In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.
   Example : The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to element 1 and 3.
4. **Reflexive Relation:** A relation R on set A is said to be reflexive if every element of A is related to itself.
   Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$. A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.
   Example : Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.
   Example : A relation R on N defined by $(x, y) \in R \Leftrightarrow x \geq y$ is a reflexive relation on N, because every natural number is greater than or equal to itself.
   Note :
   (i) The identity relation on a non-void set A is always reflexiverelation on A. However, a reflexive on A is not necessarily the identity relation on A. For example, the relation $R = \{(a, a), (b, b), (c, c), (a, b)\}$ is a reflexive relation on set $A = \{a, b, c\}$ but is not the identity relation on A.
   (ii) The universal relation on a non-void set A is reflexive.
5. **Symmetric Relation:** A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$ i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$
   Example: Let L be the set of all lines in plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$ i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.
   Example: Let $A = \{1, 2, 3, 4\}$ and let R_1 and R_2 be relation on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2) (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.
   Note:
   (i) The identity and the Universal relations on a non-void set are symmetric relations.
   (ii) A relation R on a set A is not a symmetric relation if there are at least two elements $a,b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.
   (iii) A reflexive relation on a set A is not necessarily symmetric. For example, the relation $R = \{(1, 1), (2, 2), (3,3),(1, 3)\}$ is a reflexive relation on set $A = \{1, 2, 3\}$ but it is not symmetric.
6. **Transitive Relation:** Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ i.e. aRb & $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Note : The identity and the universal relations on a non-void set are transitive.

7. **Equivalence Relation:** A relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

TRAIN YOUR BRAIN

Example 1: Check the following relations for being reflexive, symmetric, transitive and thus choose the equivalence relations if any.

(i) a R b iff $|a| \le b$; a, b $\in$ set of real numbers.
(ii) a R b iff a < b; a, b $\in$ N.
(iii) a R b iff $|a-b| > \frac{1}{2}$; a, b $\in$ R.
(iv) a R b iff a divides b; a, b $\in$ N.

Sol. :

(i) Not reflexive, not symmetric but transitive
Let a = –2 and b = 3; (–2, 3) $\in$ R. Since $|-2| \le 3$ is true
Since $|-2| = 2 \not\le -2$ hence relation is not Reflexive
Since $|3| \le -2$ is wrong hence relation is not symmetric
Now Let a, b, c be three real numbers such that $|a| \le b$ and $|b| \le c$
$|a| \le b \Rightarrow b \ge 0$, so $|b| \le c \Rightarrow b \le c$
Hence $|a| \le c$ is true so the given relation is transitive.

(ii) Not reflexive, not symmetric but transitive.
Since no natural number is less than itself hence not reflexive,
If a < b then b < a is false. Hence not symmetric.
If a < b then b < c clearly a < c. Hence transitive

(iii) Not reflexive, symmetric, not transitive.

$|a-a| = 0 \not> \frac{1}{2}$ hence it is not reflexive.

$|a-b| > \frac{1}{2} \Rightarrow |b-a| = |a-b| > \frac{1}{2}$

Hence symmetric.

Let a = 1, b = – 1 and c = $\frac{3}{2}$,
$|a-b| = 2 > \frac{1}{2}$ so (a, b) $\in$ R; $|b-c| = \frac{5}{2} > \frac{1}{2}$
So (b, c) $\in$ R

But $|a-c| = \left|1-\frac{3}{2}\right| = \frac{1}{2} \not> \frac{1}{2}$ so $(a,c) \notin R$.

Hence R is not a transitive relation.

(iv) Reflexive, not symmetric, transitive

Since $\frac{a}{a} = 1$ i.e. every number divides itself, hence R is reflexive.

If a divides b then b does not divide a (unless (a = b) hence the relation is not symmetric.

If a divides b and b divides c then it is clear that a will divide c. Hence transitive.

Example 2: Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2 3)} is reflexive but neither symmetric nor transitive.

Sol. R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as (1, 2) $\in$ R but (2,1) $\notin$ R. Similarly, R is not transitive, as (1,2) $\in$ R and (2,3) $\in$ R but (1, 3) $\notin$ R.

Example 3: Show that the relation R in the set Z of integers given by R = {(a, b): 2 divides a - b} is an equivalence relation.

Sol. R is reflexive, as 2 divides (a – a) for all a $\in$ Z. Further, if (a, b) $\in$ R, then 2 divides a – b. Therefore, 2 divides b – a. Hence, (b, a) $\in$ R, which shows that R is symmetric. Similarly, if (a, b) $\in$ R and (b, c) $\in$ R, then a – b and b – c are divisible by 2. Now, a – c = (a – b) + (b – c) is even. So, (a – c) is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in Z.

DEFINITION

"A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y", where X and Y are two non empty sets.

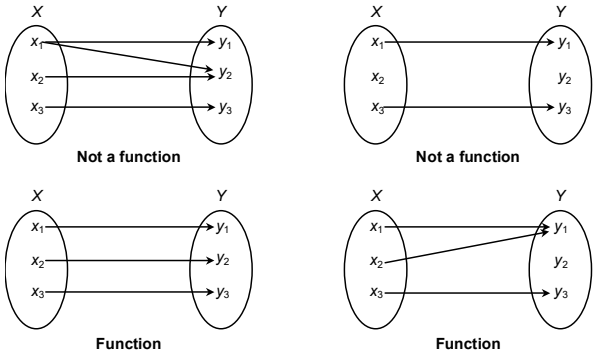
Let the correspondence be 'f' then mathematically we write f : X $\to$ Y where y = f(x), x $\in$ X and y $\in$ Y. We say that 'y' is the image of 'x' under 'f' (or x is the pre image of y).

So we can say that for a function, following two conditions must be satisfied:

(i) A mapping f: X $\to$ Y is said to be a function if each element in the set X has its image in set Y. It is possible that a few elements in the set Y are present which are not the images of any element in set X.

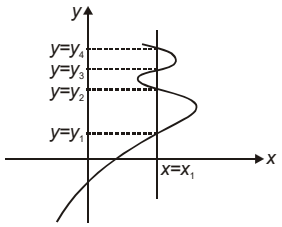
(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can't be multi-valued.

For y = f(x) (y is a function of x), x is called independent variable and y is called dependent variable.

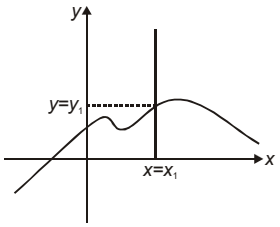


Let us take an example of $y^2 = x$. Here y cannot be a function of x as $y = \pm\sqrt{x}$ and for each value of x $\in$ **R**, we get two values of y $\in$ **R**, which is in contradiction to the above specified conditions, for a relation to be a function. So $y = \pm\sqrt{x}$ is not a function but is a relation. Let us take an another example of f : **R** $\to$ **R**, $y = x^3$. Here for each and every value of x $\in$ **R**, we have unique value of y $\in$ **R**. As both the conditions are satisfied, $y = x^3$ is said to be a function of x. Graphically, if a line parallel to

y-axis cuts the graph of the relation between x and y, in not more then one point, the relation is said to be a function in x or in other words, y is said to be a function of x.



y is not a function of x



y is a function of x

TRAIN YOUR BRAIN

EXAMPLE 4 : For which of the following, y can be a function of x. ($x \in R, y \in R$)

(i) $(x-h)^2+(y-k)^2=r^2$

(ii) $y^2=4ax$

(iii) $x^4=y^2$ (iv) $x^6=y^3$ (v) $3y=(\log x)^2$

Sol. (i) $(x-h)^2+(y-k)^2=r^2$

$\Rightarrow$ y = $y=k\pm\sqrt{r^2-(x-h)^2}$

Here for each value of $x \in \mathbf{R}$, we get two values of $y \in \mathbf{R}$. So this is not a function.

(ii) $y^2=4ax \Rightarrow y=\pm\sqrt{4ax}$

This is also not a function with the same reason mentioned above.

(iii) $y=\pm x^2$, not a function with same reason mentioned above.

(iv) $x^6=y^3 \Rightarrow y=x^2$

Here for each and every value of $x \in \mathbf{R}$, we get a unique value of $y \in \mathbf{R}$, so this relation is a function in x.

(v) $3y=(\log x)^2 \Rightarrow y=\dfrac{(\log x)^2}{3}$

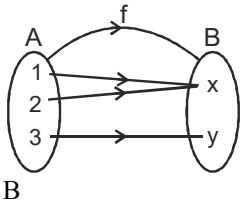
Here for all values of $x \in \mathbf{R}$ we can't find $y \in \mathbf{R}$, as logx is not defined for $x \in \mathbf{R}^- \cup \{0\}$.

So this is also not a function.

"Function" as an ordered pair :

$f: A \to B$

$f: \{(1,x),(2,x),(3,y)\}$



$\Rightarrow f \subset A \times B$

where $A \times B$ is the cartesian product of two set A & B

$= \{(1,x),(2,x),(3,x),(1,y),(2,y),(3,y)\}$

Remark : Every function from A to B satisfied the following Relation :

(1) $f \subset A \times B$

(2) Notation : $\forall\ a \in A\ \exists\ b \in B \mid b=f(a)$

(For all) (there exist) (such that)

(3) $(a,b) \in f$ & $(a,c) \in f \Rightarrow b=c$

(4) In a graphical representation of a function $y=f(x)$.

If vertical line cuts the curve more than once then it is not a function. It is called as vertical line test

DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION

Let $f: A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. If a member 'a' of A is associated to the member 'b' of B, then 'b' is called the **f-image** of 'a' and we write b = f(a). Further 'a' is called a **pre-image** of 'b'. The set $\{f(a): \forall\ a \in A\}$ is called the **range** of f. Clearly $f(A) \subseteq B$.

If only expression of f(x) is given (domain and codomain are not mentioned), then domain is set of those values of 'x' for which f (x) is real, while codomain is considered to be $(-\infty, \infty)$ (except in ITFs)

A function whose domain and range are both subsets of real numbers is called a **real function.**

Algebraic Operations on Functions : If f & g are real valued functions of x with domains A and B respectively, then both f & g are defined in $A \cap B$. Now we define $f+g$, $f-g$, $(f.g)$ & (f/g) as follows:

(i) $(f \pm g)(x)=f(x) \pm g(x)$

(ii) $(f.g)(x)=f(x).g(x)$

] domain in each case is $A \cap B$

(iii) $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B0\}$.

TRAIN YOUR BRAIN

Example 5 : Find the domain of the following functions:

(i) $f(x)=\sqrt{x^2-x-6}+\sqrt{6-x}$

(ii) $f(x)=\sqrt{3x-x^3}$

(iii) $f(x)=\sqrt{\dfrac{x-2}{x+2}}+\sqrt{\dfrac{1-x}{1+x}}$

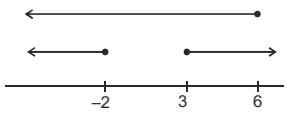
(iv) $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$

(v) $y=\log_{(x-4)}(x^2-11x+24)$

(vi) $y=\sqrt{\log_3(\cos(\sin x))}$

Sol. (i) $x^2-x-6 \geq 0$ and $6-x \geq 0$

$(x-3)(x+2) \geq 0$ and $x \leq 6$



$x \geq 3$ or $x \leq -2$ $\quad x \in (-\infty, 6]$

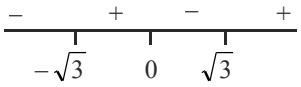
$x \in (-\infty, -2] \cup [3, \infty)$

$\therefore x \in (-\infty, -2] \cup [3, 6]$

(ii) $3x - x^3 \geq 0 \Rightarrow x^3 - 3x \leq 0$

$\Rightarrow x(x^2 - 3) \leq 0 \Rightarrow x(x - \sqrt{3})(x + \sqrt{3}) \leq 0$

$\therefore x \in (-\infty, -\sqrt{3}\,] \cup [0, \sqrt{3}\,]$



(iii) $\dfrac{x-2}{x+2} \geq 0$ and $\dfrac{1-x}{1+x} \geq 0$

Now $\dfrac{x-2}{x+2} \geq 0 \quad \Rightarrow x \geq 2$ or $x < -2$

Also $\dfrac{1-x}{1+x} \geq 0 \quad \Rightarrow -1 < x \leq 1$

Hence, the given function has empty domain i.e. domain of function = ϕ.

(iv) $\sin x \geq 0$ and $16 - x^2 \geq 0$

$\Rightarrow 2n\pi \leq x \leq (2n+1)\pi$

and $\quad -4 \leq x \leq 4$

$\therefore$ Domain is $[-4, -\pi] \cup [0, \pi]$

(v) $y = \log_{(x-4)} (x^2 - 11x + 24)$

Here 'y' would assume real value, if

$x - 4 > 0$ and $\neq 1, x^2 - 11x + 24 > 0$

$\Rightarrow x > 4$ and $\neq 5, (x-3)(x-8) > 0$

$\Rightarrow x > 4$ and $\neq 5, x < 3$ or $x > 8$

$\Rightarrow x > 8 \Rightarrow$ Domain (y) = $(8, \infty)$.

(vi) $y = \sqrt{\log_3 (\cos(\sin x))}$

Here 'y' would assume real value if,

$\log_3 (\cos(\sin x)) \geq 0 \Rightarrow \cos(\sin x) \geq 1$

$\Rightarrow \cos(\sin x) = 1$ as $-1 \leq \cos\theta \leq 1$

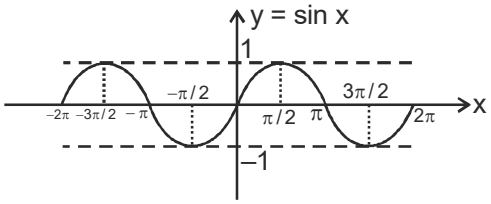
$\Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in \mathbf{I}$

Hence domain consists of the multiples of π i.e. Domain = $\{n\pi : n \in \mathbf{I}\}$.

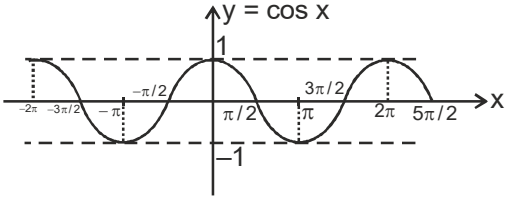
IMPORTANT TYPE OF FUNCTIONS

(1) Trigonometric function :

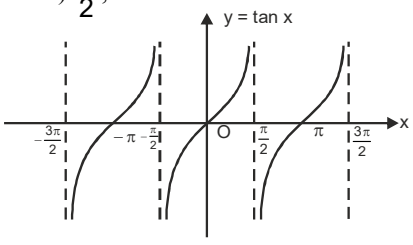
	Function	Domain	Curve	Range
(i)	$f(x) = \sin x$	$x \in R$		$y \in [-1, 1]$



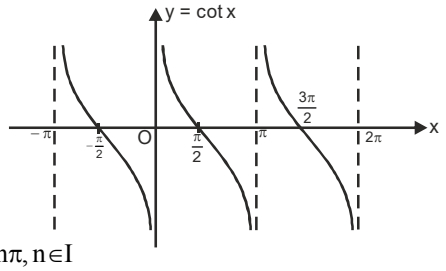
(ii)	$f(x) = \cos x$	$x \in R$	$y \in [-1, 1]$



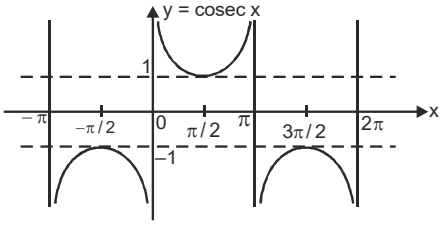
(iii)	$f(x) = \tan x$	$x \in R - (2n+1)\dfrac{\pi}{2}, n \in I$	$y \in R$



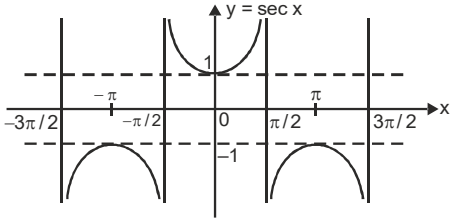
(iv) $f(x) = \cot x$ $\quad x \in R - n\pi, n \in I$ $\quad y \in R$



(v) $f(x) = \text{cosec } x$ $\quad x \in R - n\pi, n \in I$ $\quad y \in (-\infty, -1] \cup [1, \infty)$



(vi) $f(x) = \sec x$ $\quad x \in R - (2n+1)\frac{\pi}{2}, n \in I$ $\quad y \in (-\infty, -1] \cup [1, \infty)$



(2) Polynomial Function : If a function f defined by $(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n$ where $a_0, a_1, a_2 \ldots\ldots a_n \in R, n \in W$ If $a_0 \neq 0$, then f(x) is called n^{th} degree polynomial function and Domain $x \in R$.

Note :

(i) If n is odd, then polynomial is of odd degree. Its range is always R.

(ii) If n is even, then polynomial is of even degree. Its range is never R.

(iii) A polynomial function of degree one with no constant term is called odd linear function.
i.e. $f(x) = ax, a \neq 0$

(iv) $f(x) = ax + b, a \neq 0$ is a linear polynomial function.

(v) $f(x) = c, (c \neq 0)$, is a non linear function (its degree is zero).

(vi) $f(x) = 0$ is a polynomial function whose degree is not defined

(vii) There are only two polynomial functions, satisfying the relation $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. They are :

(*a*) $f(x) = x^n + 1$ &

(*b*) $f(x) = 1 - x^n$, where n is a positive integer.

(3) Algebraic Function : A function is called an algebraic function, if it can be constructed using algebraic operations such as additions, subtractions, multiplication, division taking roots etc. on algebraic expressions

(i) $f(x) = \sqrt{x^4 + 5x^2} + x + (x^3 + 5)^{3/5} \rightarrow$ algebraic function.

(ii) $f(x) = \frac{(x^5 + 5x^2)^{3/5}}{x^3} + 3\sqrt{x^2 + 5x + 6} + \ell n\ x \rightarrow$ transcendental function

(iii) $f(x) = \sqrt{x^2 + 7} + e^{\ell n\, x} + \frac{x + 7}{\sqrt{x^2 + 7}} \rightarrow$ algebraic function. y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \ldots. + P_{n-1}(x).y + P_n(x) = 0$. Where n is a positive integer and $P_0(x)$, $P_1(x)$ ..... are polynomials in x, e.g. $x^3 + y^3 - 3xy = 0$.

Note :

(i) All polynomial functions are algebraic but converse is not true.

(ii) Function which are not algebraic are called as transcendental function.
e.g. exponential function, logarithmic function.

(4) Rational Function : It is a function of form $f(x) = \frac{g(x)}{h(x)}$, where g(x) & h(x) are polynomial function and $h(x) \neq 0$

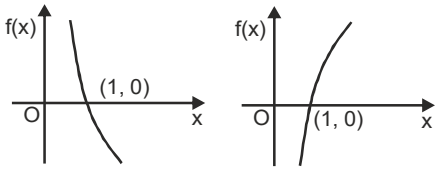
e.g. $f(x) = \frac{x^4 - 3x^2 + 2}{x^2 - 4}$

(5) Logarithmic function : $f(x) = \log_a x$, where $x > 0, a > 0, a \neq 1$

$a \rightarrow$ base, $x \rightarrow$ number or argument of log.

Case–I : $0 < a < 1$ **Case–II :** $a > 1$

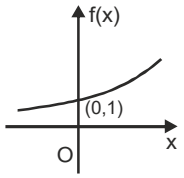
$f(x) = \log_a x$ $f(x) = \log_a x$



Domain : $x \in (0, \infty)$ Range : $y \in R$

(6) Exponential function : $f(x) = a^x$, where $a > 0, a \neq 1$ $a \rightarrow$ Base $x \rightarrow$ Exponent

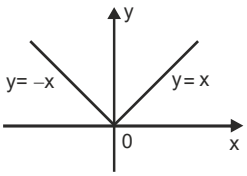
Case–I : $0 < a < 1$; e.g. $a = 1/2$, $f(x) = \left(\frac{1}{2}\right)^x$ **Case–II :** $a > 1$ e.g. $a = 2$, $f(x) = 2^x$



Domain : $x \in R$, Range : $y \in (0, \infty)$

(7) Absolute value function (Modulus function) :

$y = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$



Domain : $x \in R$; Range : $y \in R^+ \cup \{0\}$

Basic properties of | x |

- $\|x\| = |x|$
- Geometrical meaning of $|x - y|$ is the distance between x and y.
- $|x| > a \Rightarrow x > a$ or $x < -a$ if $a \in \mathbf{R}^+$ and $x \in \mathbf{R}$
- $|x| < a \Rightarrow -a < x < a$ if $a \in \mathbf{R}^+$ and $x \in \phi$
- $|xy| = |x||y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$
- $|x + y| \leq |x| + |y|$

It is a very useful and interesting property. Here the equality sign holds if x and y either both are non-negative or non-positive (i.e. x. $y \geq 0$). ($|x| + |y|$) represents the sum of distances of numbers x and y from the origin and $|x + y|$ represents the distance of number $x + y$ from the origin (or distance between 'x' and '–y' measured along the number line).

- $|x - y| \geq |x| - |y|$

Here again the equality sign holds if x and y either both are non-negative or non-positive (i.e. x. $y \geq 0$). ($|x| - |y|$) represents the difference of distances of numbers x and y from the origin and $|x - y|$ represents the distance between 'x' and 'y' measured along the number line. The last two properties can be put in one compact form i.e., $|x| - |y| \leq |x \pm y| \leq |x| + |y|$.

TRAIN YOUR BRAIN

Example 6 : Solve the following inequalities for real values of x:

(i) $|x - 1| < 2$

(ii) $|x - 3| > 5$

(iii) $0 < |x - 1| \leq 3$

(iv) $|x - 1| + |2x - 3| = |3x - 4|$ (v) $\left|\frac{x - 3}{x^2 - 4}\right| \leq 1$

Sol. (i) Since both the sides of the given inequality are non-negative for all x's, when squaring them, we get the inequality $(x - 1)^2 < 4$.

Which is equivalent to the given inequality.

We then have: $x^2 - 2x - 3 < 0 \Rightarrow (x + 1)(x - 3) < 0$

$\Rightarrow \quad x \in (-1, 3)$.

Alternate method (i)

Since $|x - 1| = \begin{cases} x - 1 & \text{if } x - 1 \geq 0 \\ -(x - 1) & \text{if } x - 1 < 0 \end{cases}$

$\Rightarrow \quad x - 1 < 2$, if $x - 1 \geq 0$

or $\quad -(x - 1) < 2$, if $x - 1 < 0$

$\Rightarrow \quad x \in [1, 3) \cup (-1, 1)$

$\Rightarrow \quad x \in (-1, 3)$.

Alternate method (ii)

We may regard $|x - 1|$ as the distance on the number line between points x and 1.

Hence, we have to indicate on the number line all such points x which are at a distance less than 2 from the point 1. Desired solution is (–1, 3).



(ii) $|x - 3| > 5$

$\Rightarrow \quad x - 3 < -5$ or $x - 3 > 5$

$\Rightarrow \quad x < -2$ or $x > 8$

$\Rightarrow \quad x \in (-\infty, -2) \cup (8, \infty)$

(iii) $0 < |x - 1| \leq 3$

Here $|x - 1| > 0 \Rightarrow x \neq 1$

Also, $|x - 1| \leq 3 \Rightarrow -3 \leq x - 1 \leq 3$,

$\Rightarrow \quad -2 \leq x \leq 4, x \neq 1 \Rightarrow x \in [-2, 1) \cup (1, 4]$.

(iv) Since $3x - 4 = x - 1 + 2x - 3$, $|3x - 4| = |x - 1| + |2x - 3|$

$\Rightarrow \quad (x - 1)(2x - 3) \geq 0$

$\Rightarrow \quad x \in (-\infty, 1] \cup [3/2, +\infty)$.

(v) $\left|\frac{x-3}{x^2-4}\right| \le 1$ It is clear that $x^2 - 4 \ne 0$

$\Rightarrow \quad x \ne 2, -2$

Now, $\left|\frac{x-3}{x^2-4}\right| \le 1 \Rightarrow -1 \le \frac{x-3}{x^2-4} \le 1$

Let us take up, $\frac{x-3}{x^2-4} \ge -1$

$\Rightarrow$

$$\frac{(x+2)(x-2)\left(x-\frac{-1-\sqrt{29}}{2}\right)\left(x-\frac{-1+\sqrt{29}}{2}\right)}{(x+2)^2(x-2)^2} \ge 0$$

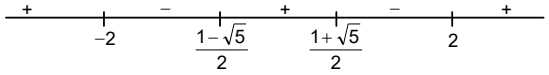
$$\Rightarrow x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup (-2, 2) \cup \left[\frac{-1+\sqrt{29}}{2}, \infty\right) \quad ...(i)$$

Now let us take up, $\frac{x-3}{x^2-4} \le 1$

$$\Rightarrow \quad \frac{x-3-x^2+4}{(x^2-4)} \le 0 \Rightarrow \frac{-x^2+x+1}{(x^2-4)} \le 0$$

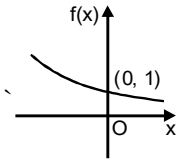
$$\Rightarrow \frac{x^2-x-1}{(x^2-4)} \ge 0$$

$$\Rightarrow \frac{(x+2)(x-2)\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right)}{(x+2)^2(x-2)^2} \ge 0$$



$$\Rightarrow x \in (-\infty, -2) \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup (2, \infty) \quad ...(ii)$$

From (i) and (ii)

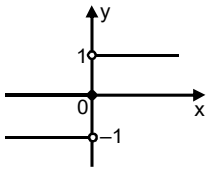


$$\Rightarrow x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{29}}{2}, +\infty\right)$$

(8) Signum function : $y = \text{sgn}(x) = \begin{cases} 1; x > 0 \\ 0; x = 0 \\ -1; x < 0 \end{cases}$

Generally, we can also write

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$



Domain : $x \in R$; Range : $y \in \{-1, 0, 1\}$

(9) Greatest integer function (step-up function) : The function $f(x) = [x]$ is called the greatest integer function and is defined as follows:

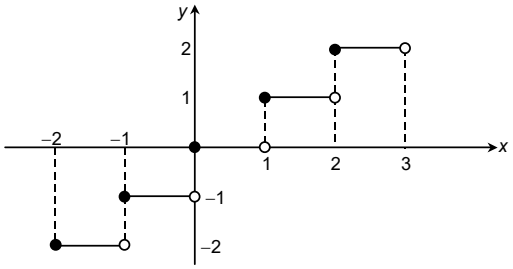
$[x]$ is the greatest integer less than or equal to x.

Then $[x] = x$ if x is an integer = integer just less than x if x is not an integer.

Examples: $[3] = 3, [2.7] = 2, [-7.8] = -8, [0.8] = 0$

In other words if we list all the integers less than or equal to x, then the integer greatest among them is called greatest integer of x. Greatest integer of x is also called integral part of x. $y = f(x) = [x]$

Domain : **R** ; Range : **I** ;



Properties :

(i) $[x] \le x < [x] + 1$

(ii) $[x + m] = [x] + m \; ; \; m \in I$

(iii) $[x] + [-x] = \begin{cases} 0 \; ; & x \in I \\ -1 \; ; & x \notin I \end{cases}$

(10) Fractional part function :

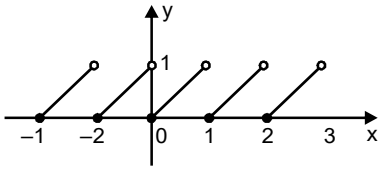
$y = f(x) = \{x\} = x - [x]$

Domain : $x \in R$; Range : $[0, 1)$

Eg. $2.3 = 2 + 0.3 \rightarrow$ fractional part

$\downarrow$

Integer part



Properties :

(i) Fractional part of any integer is zero.

(ii) $\{x+n\}=\{x\}, n\in I$

(iii) $\{x\}+\{-x\}=\begin{cases}0\ ; & x\in I\\ 1\ ; & \text{otherwise}\end{cases}$

Important properties of greatest integer function and fractional part of x.

(i) $[x]\ge n\Rightarrow x\ge n \text{ or } x\in[n,\infty),\ n\in \mathbf{I}$

$[x]>n\Rightarrow x\ge n+1 \text{ or } x\in[n+1,\infty),\ n\in \mathbf{I}$

$[x]\le n\Rightarrow x<n+1 \text{ or } x\in(-\infty,n+1),\ n\in \mathbf{I}$

$[x]<n\Rightarrow x<n \text{ or } x\in(-\infty,n),\ n\in \mathbf{I}$

(ii) $[[x]]=[x], [\{x\}]=0, \{[x]\}=0$

(iii) $[n+x]=n+[x]$ where $n\in \mathbf{I}$

(iv) $[x]+[-x]=\begin{cases}0, & \text{if } x\in \text{integer}\\ -1, & \text{if } x\notin \text{integer}\end{cases}$

(v) $\{x\}+\{-x\}=\begin{cases}0, & \text{if } x\in \text{integer}\\ 1, & \text{if } x\notin \text{integer}\end{cases}$

(vi) $[x+y]=\begin{cases}[x]+[y], & \text{if } \{x\}+\{y\}<1\\ [x]+[y]+1, & \text{if } \{x\}+\{y\}\ge 1\end{cases}$.

(vii) $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right], n\in \mathbf{N}, x\in \mathbf{R}$

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Example 7 : Solve the inequality $[x]^2-3[x]+2\le 0$.

Sol. $[x]^2-3[x]+2\le 0$

$\Rightarrow ([x]-1)([x]-2)\le 0$

$\Rightarrow 1\le[x]\le 2$

$\Rightarrow 1\le x<3 \Rightarrow x\in[1,3)$.

Example 8 : Solve the equation $4[x]=x+\{x\}$.

Sol. We are given that, $4[x]=x+\{x\}=[x]+2\{x\}$

$\Rightarrow 3[x]=2\{x\} \Rightarrow \{x\}=\frac{3}{2}[x]$ ...(i)

As $0\le\{x\}<1 \Rightarrow 0\le\frac{3}{2}[x]<1$

$\Rightarrow [x]=0 \Rightarrow 0\le x<1$

$\Rightarrow$ from (i), $\{x\}=\frac{3}{2}.(0)=0$

Hence x = 0 is the only solution.

Example 9 : Solve the equation, $x^2-4x+[x]+3=0$.

Sol. We are given that, $x^2-4x+[x]+3=0$

$\Rightarrow x^2-4x+x-\{x\}+3=0$

$\Rightarrow x^2-3x+3=\{x\} \Rightarrow 0\le x^2-3x+3<1$

Now x^2-3x+3

$=\left(x-\frac{3}{2}\right)^2+\frac{3}{4}>0$

$\Rightarrow x^2-3x+3>0$ for all $x\in \mathbf{R}$

Now consider $x^2-3x+3<1$

$\Rightarrow x^2-3x+2<0 \Rightarrow (x-1)(x-2)<0$

$\Rightarrow 1<x<2 \Rightarrow [x]=1$

Now from the original equation we have $x^2-4x+4=0$

$\Rightarrow (x-2)^2=0$

$\Rightarrow x=2,$ which does not satisfy $1<x<2$

Thus the given equation does not have any solution.

Example 10 : Find the domain of following functions.

(i) $f(x)=\ln(-2+3x-x^2)$

(ii) $f(x)=e^{-\frac{1}{x^2-1}}$

(iii) $f(x)=\sin(2x+a-b), (a,b\in R)$

Sol. (i) $f(x)=\ln(-2+3x-x^2)$

for f(x) to define, $-2+3x-x^2>0$

$\Rightarrow x^2-3x+2<0 \Rightarrow (x-1)(x-2)<0$

$\Rightarrow x\in(1,2)$

So domain (f) : $x\in(1,2)$

(ii) $f(x)=e^{-\frac{1}{x^2-1}}$

for f(x) to define, $x^2-1\ne 0$

$\Rightarrow x\ne\pm 1$

so domain (f) $\to x\in \mathbf{R}-\{1,-1\}$

(iii) $f(x)=\sin(2x+a-b)$

for f(x) to define, $2x+a-b\in \mathbf{R}$

i.e. $x\in \mathbf{R}$

so domain (f) $\to x\in \mathbf{R}$

Range: Range of y = f(x) is the collection of all outputs corresponding to each real number of the domain. To find the range of function:

(i) First of all find the domain of y = f(x).

(ii) If domain is a set having only finite number of points, then range is the set of corresponding f(x) values.

(iii) If domain of y = f(x) is R or R– {Some finite points}, then express x in terms of y. From this find y for x to be defined or real or form an equation in terms of x & apply the condition for real roots.

Example 11 : Find the range of the following functions

(i) $f(x)=\log_e(\sin x^{\sin x}+1)$ where $0<x<\pi/2$.

(ii) $f(x)=\log_e(2\sin x+\tan x-3x+1)$ where $\frac{\pi}{6}\le x\le\frac{\pi}{3}$

Sol. **(i)** $0<x<\pi/2 \Rightarrow 0<\sin x<1$

$\therefore$ Range of $\log_e(\sin x^{\sin x}+1)$ for $0<x<\pi/2$

= Range of $\log_e(x^x+1)$ for $0<x<1$

Let $h(x)=x^x+1=e^{x\log_e x}+1$

$\therefore h'(x)=e^{x\log_e x}(1+\log_e x) \Rightarrow h'(x)>0$ for $x>1/e$ and $h'(x)<0$ for $x<1/e$

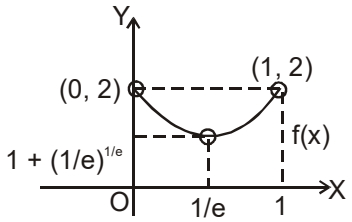
$\therefore$ h(x) has a minima at x = 1/e

Also $\lim\limits_{x\to 0^+} h(x)=1+e^{\lim\limits_{x\to 0}\left(\frac{\ln x}{1/x}\right)}=1+e^{\lim\limits_{x\to 0}\left(\frac{1/x}{-1/x^2}\right)}$

$=1+e^0=2$ and $\lim\limits_{x\to 1^-} h(x)=2$

$\therefore 0<x<1 \Rightarrow 1+\left(\frac{1}{e}\right)^{\frac{1}{e}}<(x^x+1)<2$

$\Rightarrow \log_e\left(1+\left(\frac{1}{e}\right)^{\frac{1}{e}}\right)<\log_e(x^x+1)<\log_e 2$



$\therefore$ Range of $f(x) = \left(\log_e\left(1+e^{\frac{-1}{e}}\right), \log_e 2 \right)$

(ii) Let $h(x) = (2\sin x + \tan x - 3x + 1)$

$\Rightarrow h'(x) = (2\cos x + \sec^2 x - 3)$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$\therefore h'(x) > 0 \Rightarrow 2\cos^3 x - 3\cos^2 x + 1 > 0$

$$(\cos x - 1)^2\left(\cos x + \frac{1}{2}\right) > 0 \ \forall\ x \in [\pi/6, \pi/3]$$

$\Rightarrow$ h(x) is an increasing function of x

$\Rightarrow h(\pi/6) \le h(x) \le h(\pi/3)$

$$\Rightarrow \log_e\left(2+\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right) \le \log_e h(x) \le \log_e(1+2\sqrt{3}-\pi)$$

$\therefore$ Range of f(x) is

$$\left[\log_e\left(2+\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right), \log(1+2\sqrt{3}-\pi)\right]$$

(11) Equal or Identical Functions : Two functions f & g are said to be equal if :

(i) The domain of f = The domain of g $\Rightarrow D_f = D_g$

(ii) The range of f = The range of g $\Rightarrow R_f = R_g$

(iii) $f(x) = g(x),\ \forall\ x \in$ their common domain.

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Example 12: Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ then $D_f : R - \{0\}$

and $D_g : R - \{0\}$ $\therefore D_f = D_g$

$\Rightarrow$ Hence both functions are identical

Example 13 : Let $f(x) = \sin x$ and $g(x) = \frac{1}{\text{cosec}\, x}$ then $D_f : x \in R$

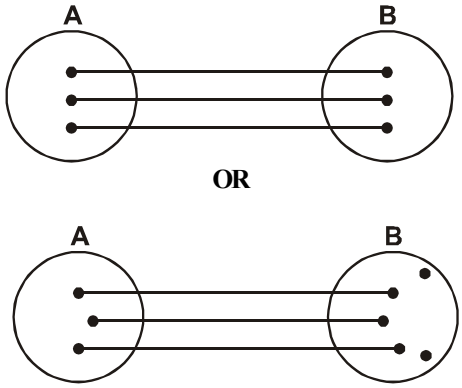
and $D_g : x \in R - \{n\pi\}$ $D_f \ne D_g$

$\Rightarrow$ Hence both functions are non-identical

CLASSIFICATION OF FUNCTIONS

(1) One – One Function (Injective mapping): A function f: $A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.

Diagrammatically an injective mapping can be shown as



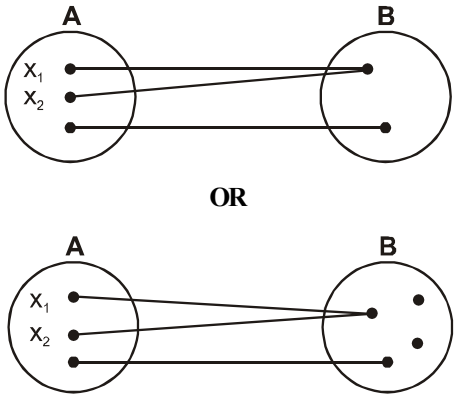
Remark :

(i) Any function which is entirely increasing or decreasing in its domain, is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one.

(2) Many–One function : A function f: $A \to B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus f: $A \to B$ is many one if for $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \ne x_2$.

Diagrammatically a many one mapping can be shown as

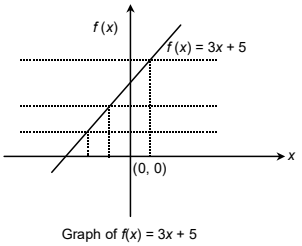


Remark :

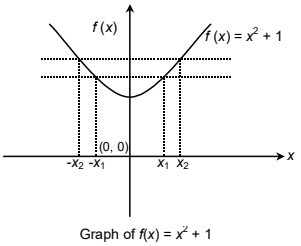
(i) A continuous function f(x) which has atleast one local maximum or local minimum, is many–one. In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one.

(ii) If a function is one–one, it cannot be many–one and vice versa.

(iii) If f and g both are one-one, then fog and gof would also be one-one (if they exist).



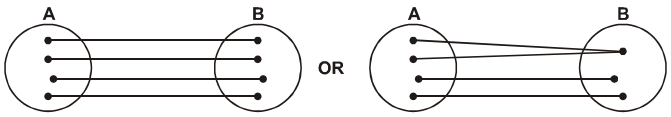
Graph of $f(x) = 3x + 5$



Graph of $f(x) = x^2 + 1$

(3) Onto-function (Surjective mapping) : If the function $f: A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B . Thus $f: A \to B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

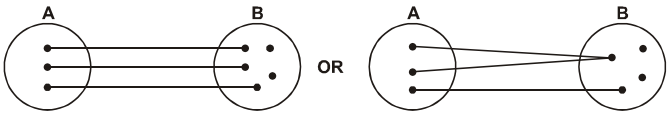
Diagramatically surjective mapping can be shown as



Note that : if range ≡ co–domain, then f(x) is onto.

(4) Into function : If $f: A \to B$ is such that there exists atleast one element in co–domain which is not the f image of any element in domain, then f(x) is into.

Diagramatically into function can be shown as



Note that : a many-one function can be made one-one by redefining the domain of the original function.

Methods to determine one-one and many-one:

- Let $x_1, x_2 \in$ domain of f and if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- Conversely if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many- one.
- If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.
- Any continuous function f(x), which has at least one local maxima or local minima is many-one.
- All even functions are many one.
- All polynomial functions of even degree defined in R have at least one local maxima or minima and hence are many one in the domain R. Polynomial functions of odd degree can be one-one or many- one.
- If f is a rational function then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence we can write $f(x_1) - f(x_2) = (x_1 - x_2)\ g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-one else one-one.
- Draw the graph of y = f(x) and determine whether f(x) is one-one or many-one.

Methods to determine whether a function is onto or into :

- If range = codomain, then f is onto. If range is a proper subset of codomain, then f is into.
- Solve f(x) = y for x, say x = g(y).
  Now, if g(y) is defined for each $y \in$ codomain and $g(y) \in$ domain of f for all $y \in$ codomain, then f(x) is onto. If this requirement is not met by at least one value of y in codomain, then f(x) is into.

Note: Any polynomial function $f: \mathbf{R} \to \mathbf{R}$ is onto, if degree is odd; into, if degree of f is even.

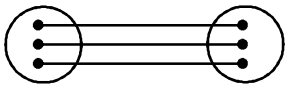
Remark :

(i) If a function is onto, it cannot be into and vice versa.

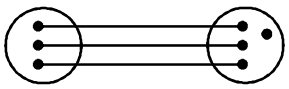
(ii) If f and g are both onto, then gof or fog may or may not be onto.

Thus a function can be one of these four types :

(a) one–one onto (injective & surjective)



(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



(iii) If f is both injective & surjective, then it is called a **Bijective** function. Bijective functions are also named as invertible, non singular or biuniform functions.

(iv) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one-one.

(v) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

TRAIN YOUR BRAIN

Example 14 : Show that the function $f(x) = \dfrac{x^2 - 8x + 18}{x^2 + 4x + 30}$ is not one-one.

Sol. A function is one-one if $f(x_1)=f(x_2) \Rightarrow x_1 = x_2$ (only)

Now

$f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2 - 8x_1 + 18}{x_1^2 - 4x_1 + 30} = \frac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$$

$$\Rightarrow 12x_1^2x_2 - 12x_1x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$$

$$\Rightarrow (x_1 - x_2)\{12x_1x_2 + 12(x_1 + x_2) - 312\} = 0$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 = \frac{26 - x_2}{1 + x_2}$$

Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ alone, f(x) is not a one-one function.

Example 15 : If $f: X \to [1, \infty)$ be a function defined as $f(x) = 1 + 3x^3$. Find the super set of all the sets X such that f(x) is one-one.

Sol. Note that $f(x) \geq 1$

$\Rightarrow 1 + 3x^3 \geq 1 \Rightarrow x^3 \geq 0 \Rightarrow x \in [0, \infty)$

Moreover for $x_1, x_2 \in [0, \infty)$, $x_1 \neq x_2$

$\Rightarrow 1 + 3x_1^3 \neq 1 + 3x_2^3 \Rightarrow f(x_1) \neq f(x_2)$

Thus $f : [1, \infty)$ is one-one for $X \in [0, \infty)$.

Example 16 : Let $f : N \to I$ be a function defined as $f(x) = x - 1000$. Show that f is an into function.

Sol. Let $f(x) = y = x - 1000$

$\Rightarrow x = y + 1000 = g(y)$ (say)

Here g(y) is defined for each $y \in I$, but $g(y) \notin N$ for $y \leq -1000$.

Hence f is into.

BIJECTIVE FUNCTION

The function which is one-one and onto both is called bijective function.

TRAIN YOUR BRAIN

Example 17 : Show that $f : R \to R$, $f(x) = \dfrac{x}{1+|x|}$ is not bijective.

Sol. Let x_1 and x_2 be both positive.

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

$$\Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$$

$\Rightarrow x_1 = x_2$

If x_1 and x_2 are both negative, then

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 = x_2$$

If x_1 and x_2 are of opposite sign, then $\dfrac{x_1}{1+|x_1|} = \dfrac{x_2}{1+|x_2|}$ cannot be true since the denominators are positive. i.e. if x_1 and x_2 are of opposite sign, then they cannot have the same image.

Hence the function is one-one But if $y = \dfrac{x}{1+|x|}$, then y is numerically less than 1.Hence the function cannot take values greater than or equal to 1.

Therefore the function is not onto, hence not bijective.

Example 18 : Let f be an injective function such that $f(x)\, f(y) + 2 = f(x) + f(y) + f(xy)\ \forall\ x, y \in R$. If $f(4) = 65$ and $f(0) \neq 2$, then show that $f(x) - 1 = x^3\ \forall\ x \in R$.

Sol. Given that $f(x)\, f(y) + 2 = f(x) + f(y) + f(xy)$ ....(i)

Putting $x = y = 0$ in equation (i), we get $f(0)\, f(0) + 2 = f(0) + f(0) + f(0)$

or $(f(0))^2 + 2 = 3f(0)$ or $(f(0) - 2)(f(0) - 1) = 0$ or $f(0) = 1$ ($\because f(0) \neq 2$) ....(ii)

Again putting $x = y = 1$ in equation (i) and repeating the above steps, we get

$(f(1) - 2)(f(1) - 1) = 0$

But $f(1) \neq 1$ as f(x) is injective.

$\therefore\ f(1) = 2$....(iii)

Now putting $y = 1/x$ in equation (i), we get

$$f(x)\, f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1) \text{ or } f(x)\, f\left(\frac{1}{x}\right) + 2$$

$$= f(x) + f\left(\frac{1}{x}\right) + 2$$

or $f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

or $f(x)\, f\left(\frac{1}{x}\right) - f(x) - f\left(\frac{1}{x}\right) - 1 + 1 = 0$

or $f(x)\left\{f\left(\frac{1}{x}\right) - 1\right\} - 1.\left\{f\left(\frac{1}{x}\right) - 1\right\} = 1$

or $\{f(x) - 1\}\left\{f\left(\frac{1}{x}\right) - 1\right\} = 1$ ...(iv)

Let $f(x) - 1 = g(x) \Rightarrow f\left(\frac{1}{x}\right) - 1 = g\left(\frac{1}{x}\right)$

$\therefore$ from equation (iv), we get $g(x)\, g\left(\frac{1}{x}\right) = 1$ which is only possible when $g(x) = \pm x^n$

$\therefore\ f(x) = \pm x^n + 1$ or $f(x) = \pm x^n + 1$ or $65 = \pm 4^n + 1$

or $4^n = 64 = (4)^3$

$\therefore\ n = 3$

$\therefore\ f(x) = x^3 + 1$ or $f(x) - 1 = x^3$

(neglecting negative sign)

FUNCTIONAL EQUATION

Functional Equation is an equation where the unknown is a function. On solving such an equation we obtain one or more functions as solutions. If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x)\,.\,f(y) \Rightarrow f(x) = x^n,\ n \in R$

(iii) $f(x + y) = f(x)\,.\,f(y) \Rightarrow f(x) = a^{kx},\ a > 0$

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

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Example 19: If $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ and $f(0) = 1\ \forall\ x, y \in R$. Determine f(x).

Sol. Given $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$

Putting $x = y = 0$; then $f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1 + 1)^2 = 2^2$

Again putting $x = 0, y = 1$ Then $f(2) = (\sqrt{f(0)} + \sqrt{f(0)})^2$

$= (1 + 2)^2 = 3^2$

and for $x = 1, y = 1$; $f(3) = (\sqrt{f(1)} + \sqrt{f(1)})^2$

$= (2 + 2)^2 = 4^2$

Similarly, $f(x) = (x + 1)^2$

Example 20: Let $f: R - \{2\} \to R$ function satisfying the following functional equation,

$$2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80, \forall\, x \in R - \{2\}.$$

Determine $f(x)$.

Sol. We have, $f(x) = -\frac{3}{2} f\left(\frac{2x+29}{x-2}\right) + 50x + 40$ ...(i)

Replacing x by $\frac{2x+29}{x-2}$ in the given functional equation we get,

$$f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2} f\left(\frac{2\left(\frac{2x+29}{x-2}\right)+29}{\left(\frac{2x+29}{x-2}\right)-2}\right) + 50\left(\frac{2x+29}{x-2}\right) + 40$$

$$\Rightarrow f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2} f(x) + 50\left(\frac{2x+29}{x-2}\right) + 40 \ldots$$

(ii) putting (ii) in (i), we get,

$$f(x) = \frac{9}{4} f(x) - 75\left(\frac{2x+29}{x-2}\right) - 60 + 50x + 40$$

$$\Rightarrow \frac{9}{4} f(x) - f(x) = 20 - 50x + 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow \frac{5}{4} f(x) = 20 - 50x + 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow f(x) = 16 - 40x + 60\frac{(2x+29)}{(x-2)}$$

COMPOSITE FUNCTIONS

Let $f: X \to Y_1$ and $g: Y_2 \to Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Remark : Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Properties of composite functions :

(i) The composite of functions is not commutative i.e. gof ≠ fog.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

Example 21 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Sol (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.
Domain of g is R, range of g is $[1, \infty)$.

For gof(x)

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$ {equal to the domain of f}

$gof(x) = g\{f(x)\} = g(\sqrt{x+3}) = 1 + (x+3) = x + 4$. Range of gof is $[1, \infty)$.

For fog(x)

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

$fog(x) = f\{g(x)\} = f(1 + x^2) = \sqrt{x^2 + 4}$

Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.
Domain of g is R, range of g is $[-1, \infty)$.

For gof(x)

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1, \infty)$

For fog(x)

Since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of $g : g(x) \in [0, \infty)$, the domain of f$\}$.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{x \in R: 0 \le x^2 - 1\} = \{x \in R: x \le -1 \text{ or } x \ge 1\}$

$= (-\infty, -1] \cup [1, \infty)$

$fog(x) = f\{g(x)\} = f(x^2 - 1) = \sqrt{x^2 - 1}$ Its range is $[0, \infty)$.

Example 22 : Two functions are defined as under,

$$f(x) = \begin{cases} x+1 & x \le 1 \\ 2x+1 & 1 < x \le 2 \end{cases}, \quad g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$$

Find fog and gof.

Sol. $(fog)(x) = f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Let us consider, $g(x) \le 1$:

(i) $x^2 \le 1$, $-1 \le x < 2$

$\Rightarrow -1 \le x \le 1, -1 \le x < 2 \Rightarrow -1 \le x \le 1$

(ii) $x + 2 \le 1, 2 \le x \le 3 \Rightarrow x \le -1, 2 \le x \le 3 \Rightarrow x \in \phi$

Let us consider, $1 < g(x) \le 2$,

(iii) $1 < x^2 \le 2, -1 \le x < 2$

$\Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}]$,

$-1 \le x < 2 \Rightarrow 1 < x \le \sqrt{2}$

(iv) $1 < x + 2 \le 2,\ 2 \le x \le 3$

$\Rightarrow -1 < x \le 0, 2 \le x \le 3, x \in \phi$

$$\text{Thus } f(g(x)) = \begin{cases} x^2 + 1, & -1 \le x \le 1 \\ 2x^2 + 1, & 1 < x \le \sqrt{2} \end{cases}$$

Now, let us consider gof :

$$\text{gof} = g(f(x)) = \begin{cases} f^2(x), & -1 \le f(x) < 2 \\ f(x) + 2, & 2 \le f(x) \le 3 \end{cases}$$

Let us consider $-1 \le f(x) < 2$:

(i) $-1 \le x + 1 < 2,\ x \le 1 \Rightarrow -2 \le x < 1,\ x \le 1$

$\Rightarrow -2 \le x < 1$

(ii) $-1 \le 2x + 1 < 2, 1 < x \le 2$

$\Rightarrow -1 \le x < 1/2,\ 1 < x \le 2 \Rightarrow x \in \phi$

Let us consider $2 \le f(x) \le 3$:

(iii) $2 \le x + 1 \le 3, x \le 1 \Rightarrow 1 \le x \le 2, x \le 1 \Rightarrow x = 1$

(iv) $2 \le 2x + 1 \le 3,\ 1 < x \le 2 \Rightarrow 1 \le 2x \le 2,\ 1 < x \le 2$

$\Rightarrow 1/2 \le x \le 1,\ 1 < x \le 2 \Rightarrow x \in \phi$

$$g(f(x)) = \begin{cases} (x+1)^2, & -2 \le x < 1 \\ x + 3, & x = 1 \end{cases}$$

If can also be written as $g(f(x)) = (x + 1)^2, -2 \le x \le 1$.

Graphical method : $$f(g(x)) = \begin{cases} g(x) + 1, & g(x) \le 1 \\ 2g(x) + 1, & 1 < g(x) \le 2 \end{cases}$$

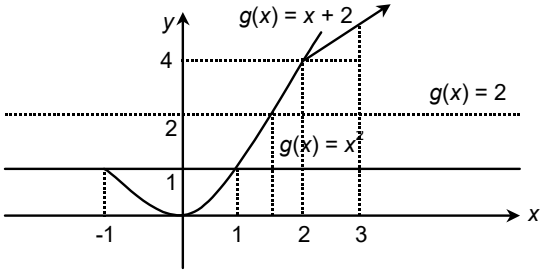
Here g(x) becomes the variable that means we would draw the graph g(x). It is clear that

$g(x) \le 1\ \forall\ x \in [-1, 1]$

and $1 < g(x) \le 2\ \forall\ x \in (1, \sqrt{2}]$

$$\Rightarrow f(g(x)) = \begin{cases} x^2 + 1, & -1 \le x \le 1 \\ 2x^2 + 1, & 1 < x \le \sqrt{2} \end{cases}$$

$$\text{Now, } g(f(x)) = \begin{cases} f^2(x), & -1 \le f(x) < 2 \\ f(x) + 2, & 2 \le f(x) \le 3 \end{cases}$$



In this case f(x) becomes the variable and we will draw the graph of f(x).

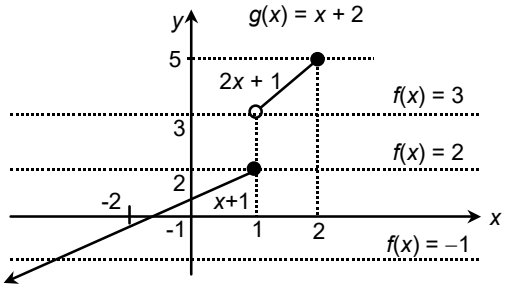
From the graph we observe that

$-1 \le f(x) < 2 \Rightarrow x \in [-2, 1)$ and $f(x) = x + 1$

$2 \le f(x)) \le 3 \Rightarrow x = 1$ and $f(x) = x + 1$

$$g(f(x)) = \begin{cases} (x+1)^2, & -2 \le x < 1 \\ x + 3 = 4, & x = 1 \end{cases}$$

i.e. $g(f(x)) = (x + 1)^2, -2 \le x \le 1$



GENERAL DEFINITION

(1) Identity function : A function $f : A \to A$ defined by $f(x) = x$ $\forall\ x \in A$ is called the identity of A & denoted by I_A. **Ex :** $f : R^+ \to R^+$; $f(x) = e^{\ell n x}$ and $f : R \to R$; $f(x) = \ell n\, e^x$

Every Identity function is a bijection.

(2) Constant function : A function $f : A \to B$ is said to be constant function. If every element of set A has the same functional image in set B i.e. $f : A \to B$; $f(x) = c\ \forall\ x \in A$ & $c \in B$ is called constant function.

(3) Homogeneous function : A function is said to be homogeneous w.r.t. any set of variables when each of its term is of the same degree w.r.t. those variables.

Ex. (i) $f(x, y) = x^2 + y^2 + 5xy$

Symbolically if, $f(tx, ty) = t^n \cdot f(x, y)$, then f(x, y) is a homogenous function of degree n.

(4) Bounded Function : A function y = f(x) is said to be bounded if it can be express is the formof $a \le f(x) \le b$ where a and b are finite quantities.

Ex : $-1 \le \sin x \le 1$; $0 \le \{x\} < 1$; $-1 \le \text{sgn}(x) \le 1$ but e^x is not bounded.

Ex : Any function having singleton range like constant function.

(5) Implicit function & Explicit function : If y has been expressed entirely in terms of 'x' then it is called an explicit function. Otherwise the function is non as implicit function Corresponding to each implicit equation there can be one, two or more explicit function satisfying it

Ex : $y = x^3 + 4x^2 + 5x \to$ Explicit function

Ex : $x + y = \sin y \to$ Implicit equation

Ex : $y = 1 - x \to$ Explicit function

EVEN & ODD FUNCTIONS

If function be defined in symmetric interval [–x, x]

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Remark :

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .

(b) A function may be neither even nor odd.

(c) Inverse of an even function is not defined.

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin .

(e) A function (whose domain is symmetric about origin) can be expressed as a sum of an even & an odd function. e.g.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd and other even then f.g will be odd.

f(x)	g(x)	f(x) + g(x)	f(x) – g(x)	f(x).g(x)	f(x)/g(x)	(gof)(x)	(f o g)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	odd	odd
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

TRAIN YOUR BRAIN

Example 23 : Which of the following functions is odd ?

(*a*) $\text{sgn}\, x + x^{2000}$ (*b*) $|x| - \tan x$

(*c*) $x^3 \cot x$ (*d*) $\text{cosec}\, x^{55}$

Sol. Let's name the function of the parts (A), (B), (C) and (D) as f(x), g(x), h(x) & $\phi(x)$ respectively. Now

(A) $f(-x) = \text{sgn}(-x) + (-x)^{2000} = -\text{sgn}\, x + x^{2000} \neq f(x) \;\&\; \neq -f(x)$

$\therefore$ f is neither even nor odd.

(B) $g(-x) = |-x| - \tan(-x) = |x| + \tan x$

$\therefore$ g is neither even nor odd.

(C) $h(-x) = (-x)^3 \cot(-x) = -x^3(-\cot x) = x^3 \cot x = h(x)$

$\therefore$ h is an even function

(D) $\phi(-x) = \text{cosec}\,(-x)^{55} = \text{cosec}\,(-x^{55}) = -\text{cosec}\, x^{55} = -\phi(x)$

$\therefore$ ϕ is an odd function.

Example 24 : $f(x) = (\tan x^5)\, e^{x^3 \text{sgn}\, x^7}$ is

(*a*) an even function

(*b*) an odd function

(*c*) neither even nor odd function(D) none of these

Sol. $f(x) = (\tan(x^5))\, e^{x^3 \text{ sgn } (x^7)}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$O \quad (O) \quad e^{O \times O \,(O)} = O \times e^{O \times O}$

$= O \times e^E = O \times E = O$

PERIODIC FUNCTION

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x + T) = f(x)$, for all values of x and x + T within the domain of f(x). The least positive period is called the principal or fundamental period of f.

e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π.

Remark :

(a) A constant function is always periodic, with no fundamental period.

(b) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(c) if f(x) has a period T then f(ax + b) has a period $T/|a|$.

(d) If f(x) has a period T_1 & g(x) also has a period T_2 then period of $f(x) \pm g(x)$ or $\frac{f(x)}{g(x)}$ is L.C.M of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ or $\frac{f(x)}{g(x)}$ is nonperiodic e.g. $|\sin x|$ has the period π, $|\cos x|$ also has the period π

$\therefore$ $|\sin x| + |\cos x|$ also has a period π. But the fundamental period of $|\sin x| + |\cos x|$ is $\pi/2$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if

g is one-one, then T is the period of gof

g is also periodic with T′ as the period and the range of f is a subset of [0, T′], then T is the period of gof

(f) Inverse of a periodic function does not exist.

TRAIN YOUR BRAIN

Example 25 : Find period of the following functions

(i) $f(x) = \sin \frac{x}{2} + \cos \frac{x}{3}$

(ii) $f(x) = \{x\} + \sin x$

(iii) $f(x) = \cos x \,.\, \cos 3x$

(iv) $f(x) = \sin \frac{3x}{2} \,.\, (-\cos \frac{x}{3})\,(-\tan \frac{2x}{3})$.

Sol.

(i) Period of sin x/2 is 4π while period of cos x/3 is 6π. Hence period of sin x/2 + cos x/3 is 12π {L.C.M. or 4 & 6 is 12}

(ii) Period of $\sin x = 2\pi$; Period of $\{x\} = 1$; but L.C.M. of 2π & 1 is not possible $\therefore$ it is aperiodic

(iii) $f(x) = \cos x \cdot \cos 3x$; Period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) =$ 2π but 2π may or may not be the fundamental period. The fundamental period can be $\frac{2\pi}{n}$, where $n \in N$. Hence cross-checking for $n = 1, 2, 3.....$ we find π to be fundamental period

$f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{3/2}$

$= \text{L.C.M. of } \frac{4\pi}{3}, 6\pi, \frac{2\pi}{3} = 12\pi$

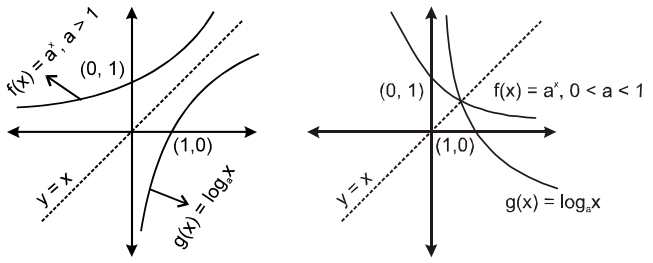
INVERSE OF A FUNCTION

Let $f: A \to B$ be a one–one & onto function, then their exists a unique function

$g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall\ x \in A$ & $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

PROPERTIES OF INVERSE FUNCTION :

(a) The graphs of f and g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.



(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g $f(x) = 1/x$

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverse of each other. e.g. $f(x) = x + 1, g(x) = x + 2$. However if $fog(x) = gof(x) = x$, then $g(x) = f^{-1}(x)$

(d) If f and g are two bijections $f: A \to B, g: B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

(e) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \frac{1}{g'(x)}$

TRAIN YOUR BRAIN

Example 26 : Find the inverse of the function

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$$

Sol. Given $f(x) = f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ....(1)

$$\therefore\ x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ y^2/64, & y^2/64 > 4 \end{cases}$$

$$= \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

$$f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$$

[From (1)].

$$\text{Hence } f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ x^2/64, & x > 16 \end{cases}.$$

Example 27: A function $f: \left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$ defined as, $f(x) = x^2 - 3x + 4$. Then compute $f^{-1}(x)$ and find the solution of the equation, $f(x) = f^{-1}(x)$.

Sol. $f(x) = y = x^2 - 3x + 4 \Rightarrow x^2 - 3x + (4 - y) = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(4 - y)}}{2} = \frac{3 + \sqrt{4y - 7}}{2}$$

$$f^{-1}(y) = \frac{3 + \sqrt{4x - 7}}{2}$$

graphs of $f^{-1}(x)$ & f(x) intersect each other at y = x

$\Rightarrow$ $f(x) = x \Rightarrow x^2 - 3x + y = x \Rightarrow x = 2$

Example 28: Determine $f^{-1}(x)$, if given function is invertible

(i) $f: (-\infty, -1) \to (-\infty, -2)$ defined by $f(x) = -(x + 1)^2 - 2$

(ii) $f: \left[\frac{\pi}{6}, \frac{7\pi}{6}\right] \to [-1, 1]$ defined by

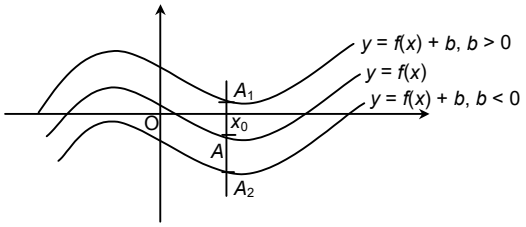
$$f(x) = \sin\left(x + \frac{\pi}{3}\right)$$

Sol. (7) (i) $-1 - \sqrt{-x - 2}$

(ii) $\frac{2\pi}{3} + \sin^{-1}x$

TRANSFORMATIONS OF GRAPHS

- **Drawing the graph of y = f(x) + b, b ∈ R, from the known graph of y = f(x)**



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x).

$y|_{x=x_0} = f(x_0)$

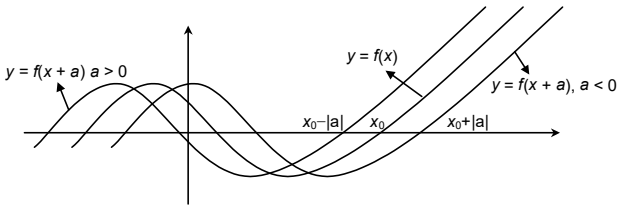
The corresponding point on f(x) + b would be $f(x_0) + b$

$b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

$b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

- **Drawing the graph of y = f(x + a), a ∈ R from the known graph of y = f(x)**

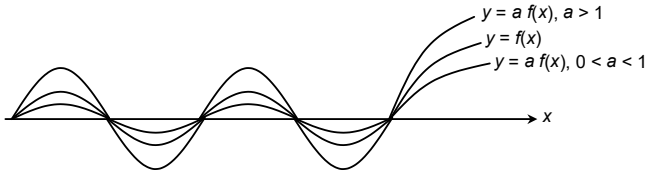


Let us take any point $x_0 \in$ domain of f(x), and set $x + a = x_0$ or $x = x_0 - a$. $a > 0 \Rightarrow x < x_0$, and $a < 0$

$\Rightarrow x > x_0$. That means x_0 and $x_0 - a$ would give us same abscissa for f(x) and f(x + a) respectively.

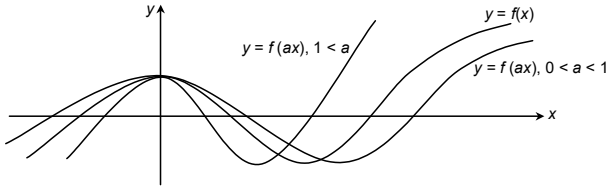
As such for a > 0, graph of f(x + a) can be obtained simply by translating the graph of f(x) in the negative x-direction through a distance 'a' units. If a < 0, graph of f(x + a) can be obtained by translating the graph of f(x) in the positive x-direction through a distance 'a' units.

- **Drawing the graph of y = a. f(x) from the known graph of y = f(x)**



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

- **Drawing the graph of y = f (ax) from the known graph of y = f(x)**



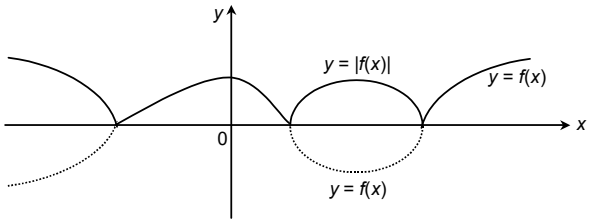
Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$

Clearly if $0 < a < 1$ then $x > x_0$ and f(x) will stretch by 1/a units against y-axis, and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units against y-axis.

Note that the point of maxima and minima are on the line parallel to the x-axis for both the curves above.

- **Drawing the graph of y = | f (x)| from the known graph of y = f(x)**

| f(x) | = f(x) if f(x) ≥ 0 and | f(x) | = –f(x) if f(x) < 0. It means that the graph of f(x) and | f(x) | would coincide if f(x) ≥ 0 and the portions where f(x) < 0 would get inverted in the upward direction.



The figure would make the procedure clear.

Topicwise Questions

RELATIONS

1. The relation R defined in N as aRb $\Leftrightarrow$ b is divisible by a is
(a) Reflexive but not symmetric
(b) Symmetric but not transitive
(c) Symmetric and transitive
(d) None of these

2. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is
(a) 2^9 (b) 9^2
(c) 3^2 (d) 2^{9-1}

3. Let R be a reflexive relation on a set A and I be the identity relation on A. Then
(a) $R \subset I$ (b) $I \subset R$
(c) $R = I$ (d) $R = 2I$

4. The relation "is subset of" on the power set P(a) of a set A is
(a) Symmetric (b) Anti-symmetric
(c) Equivalency relation (d) Reflexive

5. Let R be a relation over the set N × N and it is defined by $(a, b)R(c, d) \Rightarrow a + d = b + c$. Then R is
(a) Reflexive only (b) Symmetric only
(c) Transitive only (d) An equivalence relation

6. The number of reflexive relations of a set with four elements is equal to
(a) 2^{16} (b) 2^{12}
(c) 2^8 (d) 2^4

7. Let R be a relation on the set N of natural numbers defined by nRm $\Leftrightarrow$ n is a factor of m (i.e., n|m). Then R is
(a) Reflexive and symmetric
(b) Transitive and symmetric
(c) Equivalence
(d) Reflexive, transitive but not symmetric

FUNCTION DOMAIN AND CO-DOMAIN AND RANGE

8. The domain of $\sin^{-1}(\log_3 x)$ is
(a) [0, 1] (b) [0, ∞]
(c) R (d) [1/3, 3]

9. Domain of the function $\sqrt{2-x} - \frac{1}{\sqrt{9-x^2}}$ is
(a) (–3, 1) (b) [–3, 1]
(c) (–3, 2] (d) [–3, 1)

10. Range of the function $f(x) = \frac{x^2+x+2}{x^2+x+1}; x \in R$ is
(a) $(1, \infty)$ (b) $(1, 11/7]$
(c) $(1, 7/3]$ (d) $(1, 7/5]$

TYPES OF FUNCTION

11. Mapping $f : R \to R$ which is defined as $f(x) = \cos x, x \in R$ will be
(a) Neither one-one nor onto
(b) One-one
(c) Onto
(d) One-one onto

12. Let $f : R - \{n\} \to R$ be a function defined by $f(x) = \frac{x-m}{x-n}$ where $m \neq n$. Then
(a) f is one-one onto (b) f is one-one into
(c) f is many one onto (d) f is many one into

13. Let $f(x) = \frac{x^2-4}{x^2+4}$ for $|x| > 2$, then the function $f : (-\infty, -2] \cup [2, \infty) \to (-1, 1)$ is
(a) One-one into (b) One-one onto
(c) Many one into (d) Many one onto

14. If $f : R \to S$ defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then the interval of S is
(a) [–1, 3] (b) [1, 1]
(c) [0, 1] (d) [0, –1]

15. If R denotes the set of all real numbers then the function $f : R \to R$ defined $f(x) = [x]$
(a) One-one only
(b) Onto only
(c) Both one-one and onto
(d) Neither one-one nor onto

IDENTICAL FUNCTION & FUNCTIONAL EQUATION.

16. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$-
(a) ≥ 0 only when $\theta \geq 0$
(b) ≤ 0 for all real θ
(c) ≥ 0 for all real θ
(d) ≤ 0 only when $\theta \leq 0$

17. If $f(x) = \frac{x}{x-1}$, then $\frac{f(a)}{f(a+1)} =$
(a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$
(c) $f(a^2)$ (d) $f\left(\frac{-a}{a-1}\right)$

18. If $f(x)=\cos(\log x)$,

then $f(x^2)f(y^2)-\frac{1}{2}\left[f\left(\frac{x^2}{2}\right)+f\left(\frac{x^2}{y^2}\right)\right]$ has the value

(*a*) $\frac{1}{2}\left[\cos\log y^2-\cos\log\frac{1}{2}\right]$

(*b*) $\frac{1}{2}\left[\cos\log x^2y^2+\cos\log\frac{x^2}{2}\right]$

(*c*) $\cos\log x^2y^2-\cos\log\frac{x^2}{2}$

(*d*) $\frac{1}{2}\left[\cos\log x^2y^2-\cos\log\frac{x^2}{2}\right]$

19. If $\phi(x)=a^x$, then $\{\phi(p)\}^3$ is equal to

(*a*) $\phi(3p)$ (*b*) $3\phi(p)$

(*c*) $6\phi(p)$ (*d*) $2\phi(p)$

COMPOSITE FUNCTION

20. If $f(x)=\frac{\alpha x}{x+1}$, $x\neq -1$. Then, for what value of α is

$f(f(x))=x$

(*a*) $\sqrt{2}$ (*b*) $-\sqrt{2}$

(*c*) 1 (*d*) -1

21. Let $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, & \text{If } x<0\\ 0, & \text{If } x=0\\ 1, & \text{if } x>0\end{cases}$, then for

all values of x the value of fog(x)

(*a*) x (*b*) 1

(*c*) f(x) (*d*) g(x)

EVEN AND ODD FUNCTION

22. Let the function $f(x)=3x^2-4x+8\log(1+|x|)$ be defined on the interval [0, 1]. The even extension of f(x) to the interval [–1, 0] is -

(*a*) $3x^2+4x+8\log(1+|x|)$

(*b*) $3x^2-4x+8\log(1+|x|)$

(*c*) $3x^2+4x-8\log(1+|x|)$

(*d*) $3x^2-4x-8\log(1+|x|)$

23. If $f(x)=2x^6+3x^4+4x^2$ then $f'(x)$ is

(*a*) Even function

(*b*) An odd function

(*c*) Neither even nor odd

(*d*) both even-odd

24. Which of the following function is even function

(*a*) $f(x)=\frac{a^x+1}{a^x-1}$ (*b*) $f(x)=x\left(\frac{a^x-1}{a^x+1}\right)$

(*c*) $f(x)=\frac{a^x-a^{-x}}{a^x+a^{-x}}$ (*d*) $f(x)=\sin x$

25. If $f(x)=\log\frac{1+x}{1-x}$, then $f(x)$ is

(*a*) Even function

(*b*) $f(x_1)f(x_2)=f(x_1+x_2)$

(*c*) $\frac{f(x_1)}{f(x_2)}=f(x_1-x_2)$

(*d*) Odd function

26. The function $f(x)=\log(x+\sqrt{x^2+1})$, is

(*a*) An even function

(*b*) An odd function

(*c*) A Periodic function

(*d*) Neither an even nor odd function

PERIODIC FUNCTION

27. If $f: R\to R$ is a function satisfying the property $f(x+1)+f(x+3)=2\ \forall\ x\in R$ then the period (may not be fundamental period) of f(x) is

(*a*) 3 (*b*) 4

(*c*) 7 (*d*) 6

28. If f(x) = cos (ax) + sin (bx) is periodic, then which of the followings is false -

(*a*) a and b both are rational

(*b*) non-periodic if a is rational but b is irrational

(*c*) non-periodic if a is irrational but b is rational

(*d*) none of these

29. Function $f(x)=\sin x+\tan x+\text{sgn}(x^2-6x+10)$ is

(*a*) periodic with period 2π

(*b*) periodic with period π

(*c*) non-periodic

(*d*) periodic with period 4π

30. The period of $f(x)=x-[x]$, if it is periodic, is

(*a*) f(x) is not periodic (*b*) $\frac{1}{2}$

(*c*) 1 (*d*) 2

INVERSE OF FUNCTION

31. If $f(x)=x^3-1$ and domain of f= {0,1, 2, 3}, then domain of f^{-1} is -

(*a*) {0, 1, 2, 3} (*b*) {1, 0, –7, –26}

(*c*) {–1, 0, 7, 26} (*d*) {0, –1, – 2, –3}

32. The inverse of the function $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is

(*a*) $\frac{1}{2}\log\frac{1+x}{1-x}$ (*b*) $\frac{1}{2}\log\frac{2+x}{2-x}$

(*c*) $\frac{1}{2}\log\frac{1-x}{1+x}$ (*d*) $2\log(1+x)$

MISCELLONEOUS

33. The function $f(x) = \sqrt{\log_{10}\cos(2\pi x)}$ exists -

(*a*) for any rational x

(*b*) only when x is a positive integer

(*c*) only when x is fractional

(*d*) for any integer value of x including zero

34. Let A = N × N and * be the binary operation on A defined by (a, b) * (c, d) = (a + c, b + d). Find the identity element for * on A, if any.

(*a*) 1

(*b*) 2

(*c*) 3

(*d*) does not exists

35. Number of binary operations on the set {a, b} are

(*a*) 2

(*b*) 2^2

(*c*) 2^3

(*d*) 2^4

Learning Plus

RELATIONS

1. Let R be a reflexive relation on a set A and *I* be the identity relation on A. Then

(*a*) $R \subset I$ (*b*) $I \subset R$

(*c*) $R = I$ (*d*) $R = 2I$

2. R is a relation over the set of real numbers and it is given by $nRm \Leftrightarrow nm \geq 0$. Then R is

(*a*) Symmetric and transitive

(*b*) Reflexive and symmetric

(*c*) A partial order relation

(*d*) An equivalence relation

3. Let S be the set of all real numbers. Then the relation $R = \{(a, b) : 1 + ab > 0\}$ on S is

(*a*) Reflexive and symmetric but not transitive

(*b*) Reflexive and transitive but not symmetric

(*c*) Symmetric, transitive but not reflexive

(*d*) Reflexive, transitive and symmetric

4. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$, be relation on the set $A = \{3, 6, 9, 12\}$. The relation is

(*a*) reflexive and transitive only

(*b*) reflexive only

(*c*) an equivalence relation

(*d*) reflexive and symmetric only

5. Let W denote the words in the English dictionary. Define the relation Rby : $R = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$. Then R is –

(*a*) reflexive, symmetric and not transitive

(*b*) reflexive, symmetric and transitive

(*c*) reflexive, not symmetric and transitive

(*d*) not reflexive, symmetric and transitive

6. Let R and S be two non-void relations on a set A. Which of the following statements is false

(*a*) R and S are transitive $\Rightarrow R \cup S$ is transitive

(*b*) R and S are transitive $\Rightarrow R \cap S$ is transitive

(*c*) R and S are symmetric $\Rightarrow R \cup S$ is symmetric

(*d*) R and S are reflexive $\Rightarrow R \cap S$ is reflexive

7. Let *R* be a relation on the set *N* be defined by $\{(x, y) \mid x, y \in N, 2x + y = 41\}$. Then *R* is

(*a*) Reflexive

(*b*) Symmetric

(*c*) Transitive

(*d*) Not Reflexive, symmetric, transitive

8. Let R_1 be a relation defined by

$R_1 = \{(a, b) \mid a \geq b, a, b \in R\}$. Then R_1 is

(*a*) An equivalence relation on R

(*b*) Reflexive, transitive but not symmetric

(*c*) Symmetric, Transitive but not reflexive

(*d*) Neither transitive not reflexive but symmetric

9. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(*a*) [1, 2) (*b*) [2, 3)

(*c*) [1, 2] (*d*) [2, 3]

10. Range of $f(x) = \ell n\,(3x^2 - 4x + 5)$ is

(*a*) $\left[\ell n\frac{11}{3}, \infty\right)$ (*b*) $[\ell n\,10, \infty)$

(*c*) $\left[\ell n\frac{11}{6}, \infty\right)$ (*d*) $\left[\ell n\frac{11}{12}, \infty\right)$

11. Which of the following functions are not injective mapping-

(*a*) $f(x) = |x+1|, x \in [-1, \infty)$

(*b*) $g(x) = x + \frac{1}{x};\ x \in (0, \infty)$

(*c*) $h(x) = x^2 + 4x - 5;\ x \in (0, \infty)$

(*d*) $k(x) = e^{-x};\ x \in [0, \infty)$

12. The function $f: [2, \infty) \to Y$, defined by $f(x) = x^2 - 4x + 5$ is both one–one and onto if

(*a*) $Y = R$ (*b*) $Y = [1, \infty)$

(*c*) $Y = [4, \infty)$ (*d*) $Y = [5, \infty)$

13. Let $f: (e, \infty) \to R$ be defined by $f(x) = \ell n\,(\ell n(\ell n\, x))$, then

(*a*) f is one one but not onto

(*b*) f is onto but not one – one

(*c*) f is one–one and onto

(*d*) f is neither one–one nor onto

14. The function $f: R \to R$ defined by $f(x) = 6^x + 6^{|x|}$ is

(*a*) one-one and onto

(*b*) many-one and onto

(*c*) one-one and into

(*d*) many-one and into

15. Let 'f' be a function from R to R given by $f(x) = \frac{x^2 - 4}{x^2 + 1}$. Then f(x) is

(*a*) one-one and into (*b*) one-one and onto

(*c*) many-one and into (*d*) many-one and onto

16. If $f(x) = \cos\left[\frac{\pi^2}{2}\right]x + \sin\left[-\frac{\pi^2}{2}\right]x$, where [.] denotes the greatest integer function, then which of the following is not correct?

(*a*) $f(0) = 1$ (*b*) $f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}+1}$

(*c*) $f\left(\frac{\pi}{2}\right) = 0$ (*d*) $f(\pi) = 0$

17. Which of the following pair of functions are identical?

(*a*) $f(x) = \sin^{-1} x + \cos^{-1} x$ and $g(x) = \frac{\pi}{2}$

(*b*) $f(x) = \tan^{-1} x + \cot^{-1} x$ and $g(x) = \frac{\pi}{2}$

(*c*) $f(x) = \sec^{-1} x + \text{cosec}^{-1} x$ and $g(x) = \frac{\pi}{2}$

(*d*) All of these

18. If $f(x) = 2\sin^2\theta + 4\cos(x+\theta)\sin x.\sin\theta + \cos(2x+2\theta)$ then value of $f^2(x) + f^2\left(\frac{\pi}{4} - x\right)$ is

(*a*) 0 (*b*) 1

(*c*) –1 (*d*) x^2

19. Let $f(x) = ax^2 + bx + c$, where a, b, c are rational and $f: Z \to Z$, where Z is the set of integers. Then a + b is

(*a*) a negative integer

(*b*) an integer

(*c*) non-integral rational number

(*d*) None of these

20. Which one of the following pair of functions are identical ?

(*a*) $e^{(\ell n\, x)^{1/2}}$ and $\sqrt{x}$

(*b*) $\tan^{-1}(\tan x)$ & $\cot^{-1}(\cot x)$

(*c*) $\cos^2 x + \sin^4 x$ and $\sin^2 x + \cos^4 x$

(*d*) $\frac{|x|}{x}$ and sgn (x) where sgn(x) stands for signum function.

21. If $f(x) = \cos(\ell n\, x)$, then $f(x)\, f(y) - \frac{1}{2}\left(f\left(\frac{x}{y}\right) + f(xy)\right)$ has the value

(*a*) –1 (*b*) $\frac{1}{2}$

(*c*) –2 (*d*) None of these

22. If $y = f(x)$ satisfies the condition

$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ $(x \neq 0)$, then f(x) is equal to

(*a*) $-x^2 + 2$ (*b*) $-x^2 - 2$

(*c*) $x^2 - 2,\ x \in R - \{0\}$ (*d*) $x^2 - 2,\ |x| \in [2, \infty)$

23. If $f(a) = 1$ and $f(n+1) = 2f(n) + 1$ if $n \geq 1$, then f(n) is equal to

(*a*) $2^n + 1$ (*b*) 2^n

(*c*) $2^n - 1$ (*d*) $2^{n-1} - 1$

24. A function $f: R \to R$ satisfies the condition $x^2 f(x) + f(1-x) = 2x - x^4$. Then f(x) is:

(*a*) $-x^2 - 1$ (*b*) $-x^2 + 1$

(*c*) $x^2 - 1$ (*d*) $-x^4 + 1$

25. If x and y satisfy the equation $y = 2[x] + 3$ and $y = 3[x - 2]$ simultaneously, where [.] denotes the greatest integer function, then [x + y] is equal to

(*a*) 21 (*b*) 9

(*c*) 30 (*d*) 12

26. The function f(x) is defined in [0, 1] then the domain of definition of the function $f[\ell n\,(1-x^2)]$ is given by :

(*a*) $x \in \{0\}$

(*b*) $x \in [-\sqrt{1+e}\,-1] \cup [1 + \sqrt{1+e}\,]$

(*c*) $x \in (-\infty, \infty)$

(*d*) None of these

27. If 'f' and 'g' are bijective functions and fog is defined, then fog must be:

(*a*) injective (*b*) surjective

(*c*) bijective (*d*) into only

28. The function $f(x) = [x] + \frac{1}{2}$, $x \notin I$ is a/an (where [.] denotes greatest integer function)

(*a*) Even (*b*) odd

(*c*) neither even nor odd (*d*) none of these

29. It is given that f(x) is an even function and satisfy the relation $f(x) = \frac{xf(x^2)}{2 + \tan^2 x.f(x^2)}$ then the value of f(10) is

(*a*) 10 (*b*) 100

(*c*) 50 (*d*) None of these

30. The period of $\sin \frac{\pi}{4}[x] + \cos \frac{\pi x}{2} + \cos \frac{\pi}{3}[x]$, where [x] denotes the integral part of x is

(*a*) 8 (*b*) 12

(*c*) 24 (*d*) Non–periodic

31. Let f(x) = x (2 – x), $0 \le x \le 2$. If the definition of 'f' is extended over the set, R – [0, 2] by f(x + 2) = f(x), then 'f' is a

(*a*) periodic function of period 1

(*b*) non-periodic function

(*c*) periodic function of period 2

(*d*) periodic function of period 1/2

32. Let $f : N \to N$ where $f(x) = x + (-1)^{x-1}$ then f is-

(*a*) Inverse of itself

(*b*) even function

(*c*) periodic

(*d*) identity

33. If [x] stands for the greatest integer function, then the value of

$$\left[\frac{1}{2} + \frac{1}{1000}\right] + \left[\frac{1}{2} + \frac{2}{1000}\right] + \ldots\ldots\ldots + \left[\frac{1}{2} + \frac{999}{1000}\right]$$

(*a*) 498 (*b*) 499

(*c*) 500 (*d*) 501

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. For the function $f(x) = \ell n (\sin^{-1} \ell og_2 x)$,

(*a*) Domain is $\left[\frac{1}{2}, 2\right]$ (*b*) Range is $\left(-\infty, \ell n \frac{\pi}{2}\right]$

(*c*) Domain is (1, 2] (*d*) Range is R

2. Let $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective?

(*a*) $f(x) = x^2$ (*b*) $g(x) = x^3$

(*c*) $h(x) = \sin 2x$ (*d*) $k(x) = \sin(\pi x/2)$

3. Let $f : I \to R$ (where I is the set of positive integers) be a function defined by $f(x) = \sqrt{x}$, then f is:

(*a*) one – one (*b*) many one

(*c*) onto (*d*) into

4. If f: $R \to [-1, 1]$, where $f(x) = \sin\left(\frac{\pi}{2}[x]\right)$, (where [.] denotes the greatest integer function), then

(*a*) f(x) is onto (*b*) f(x) is into

(*c*) f(x) is periodic (*d*) f(x) is many one

5. Let $f : [-1, 1] \to [0, 2]$ be a linear function which is onto, then f(x) is/are

(*a*) $1 - x$ (*b*) $1 + x$

(*c*) $x - 1$ (*d*) $x + 2$

6. If 'f' and 'g' are bijective functions and gof is defined then gof must be

(*a*) injective (*b*) surjective

(*c*) bijective (*d*) into only

7. Let $f : R - \left\{\frac{1}{5\sqrt{2}}\right\} \to R - \left\{\frac{1}{5\sqrt{2}}\right\}$ be a function defined as $f(x) = \left(\frac{x - 2\sqrt{5}}{5\sqrt{2}x - 1}\right)$ then which of the following is(are) CORRECT?

(*a*) $f\left(f\left(f\left(f(2017)\right)\right)\right) = 2017$

(*b*) $f(x) = f^{-1}(x)$ has more than 3 real roots

(*c*) f(x) is a bijective function

(*d*) $f\left(f\left(f(x)\right)\right) = f\left(f\left(f\left(f\left(f(x)\right)\right)\right)\right)\ \forall\ x \ne \frac{1}{5\sqrt{2}}$

8. Let $f(x)=\left(\frac{1-x}{1+x}\right), 0\le x\le 1$ and $g(x)=4x(1-x), 0\le x\le 1$, then

(*a*) $fog=\frac{1-4x+4x^2}{1+4x-4x^2}, 0\le x\le 1$

(*b*) $fog=\frac{1-4x-4x^2}{1+4x-4x^2}, \frac{1}{2}\le x\le 1$

(*c*) $gof=\frac{8x(1-x)}{(1+x)^2}, 0\le x\le 1$

(*d*) $gof=\frac{8x(1+x)}{(1+x)^2}, 0\le x\le 1$

9. Which of following pairs of functions are identical.

(*a*) $f(x)=e^{\ell n \sec^{-1}x}$ and $g(x)=\sec^{-1}x$

(*b*) $f(x)=\tan(\tan^{-1}x)$ and $g(x)=\cot(\cot^{-1}x)$

(*c*) $f(x)=\text{sgn}(x)$ and $g(x)=\text{sgn}(\text{sgn}(x))$

(*d*) $f(x)=\cot^2 x.\cos^2 x$ and $g(x)=\cot^2 x-\cos^2 x$

Comprehension–1 (No.10 TO 12)

Let $f(x)=\frac{x^3}{3}+\frac{x^2}{2}+ax+b \quad \forall x\in R$

10. Least value of 'a' for which f(x) is injective function, is

(*a*) $\frac{1}{4}$ (*b*) 1

(*c*) $\frac{1}{2}$ (*d*) $\frac{1}{8}$

11. If $a=-1$, then f(x) is

(*a*) bijective (*b*) many–one and onto

(*c*) one–one and into (*d*) many– one and into

12. f(x) is invertible iff

(*a*) $a\in\left[\frac{1}{4},\infty\right), b\in R$ (*b*) $a\in\left[\frac{1}{8},\infty\right), b\in R$

(*c*) $a\in\left(-\infty,\frac{1}{4}\right], b\in R$ (*d*) $a\in\left(-\infty,\frac{1}{4}\right), b\in R$

Comprehension–2 (NO. 13 TO 15)

Consider the function $f(x)=\begin{cases}x^2-1, & -1\le x\le 1\\ \ell nx, & 1<x\le e\end{cases}$

Let $f_1(x)=f(|x|)$

$f_2(x)=|f(|x|)|$

$f_3(x)=f(-x)$

Now answer the following questions.

13. Number of positive solution of the equation $2f_2(x)-1=0$ is (are)

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) 1

14. Number of integral solution of the equation $f_1(x)=f_2(x)$ is (are)

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

15. If $f_4(x)=\log_{27}(f_3(x)+2)$, then range of $f_4(x)$ is

(*a*) [1, 9] (*b*) $\left[\frac{1}{3},\infty\right)$

(*c*) $\left[0,\frac{1}{3}\right]$ (*d*) [1, 27]

16. Match the column :

	Column – I	**Column – II**
(*a*)	The period of the function $y=\sin(2\pi t+\pi/3)+2\sin(3\pi t+\pi/4)+3\sin 5\pi t$	(p) 1/2
(*b*)	$y=\{\sin(\pi x)\}$ is a many one function for $x\in(0,a)$, where $\{x\}$ denotes fractional part of x, then a may be	(q) 8
(*c*)	The fundamental period of the function $y=\frac{1}{2}\left(\frac{\lvert\sin(\pi/4)x\rvert}{\cos(\pi/4)x}+\frac{\sin(\pi/4)x}{\lvert\cos(\pi/4)x\rvert}\right)$	(r) 2
(*d*)	If $f:[0,2]\to[0,2]$ is bijective function defined by $f(x)=ax^2+bx+c$, where a, b, c are non-zero real numbers, then $f(b)$ is equal to	(s) 0

17. Let $f(x)=\sin^{-1}x$, $g(x)=\cos^{-1}x$ and $h(x)=\tan^{-1}x$. For what interval of variation of x the following are true.

	Column – I		**Column – II**
(*a*)	$f\left(\sqrt{x}\right)+g\left(\sqrt{x}\right)=\pi/2$	(p)	$[0,\infty)$
(*b*)	$f(x)+g\left(\sqrt{1-x^2}\right)=0$	(q)	$[0,1]$
(*c*)	$g\left(\frac{1-x^2}{1+x^2}\right)=2h(x)$	(r)	$(-\infty,1)$
(*d*)	$h(x)+h(a)=h\left(\frac{1+x}{1-x}\right)$	(s)	$[-1,0]$

NUMERICAL VALUE BASED

18. Let f be a function satisfying $2f(x)-3f(1/x)=x^2$ for any $x\neq 0$, then absolute value of $f(b)$ is

19. If $f(x)+f(x+4)=f(x+2)+f(x+6)\ \forall x\in R$, and $f(5)=10$, then $\sum_{r=1}^{100} f(5+8r)$ equal to

20. Absolute value of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is

21. If $f^2(x) = -f(x)$ and $g(x) = f\not{c}(x)$ and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then $F(10)$ is equal to

22. If f(x) is a function that is odd and even simultaneously, then $f(c) - f(b)$ is equal to

23. If f is a function such that $f(0) = 2$, $f(a) = 3$ and $f(x + 2) = 2f(x) - f(x + 1)$ for every real x then f(5) is

24. An odd function is symmetric about the vertical line $x = a\ (a > 0)$ and if $\sum_{r=0}^{\infty}[f(1+4r)]^r = 8$, then find the value of $f(a)$.

25. Let $f : R \to R$ such that
$f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1\ \forall\ x, y \in R$.
Then find the value of $|(f(16))| - 125$.

26. The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of f (7) is equal to

27. The total number of solutions of $[x]^2 = x + 2\{x\}$, where [.] and {.} denote the greatest integer and the fractional part functions, respectively, is equal to

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

RELATIONS

1. If $R = \{(x, y) ; x, y \in Z, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers Z, then the domain of R^{-1} is :

[*JEE Main-2020 (September)*]

(*a*) {0, 1} (*b*) {–2, –1, 1, 2}

(*c*) {–1, 0, 1} (*d*) {–2, –1, 0, 1, 2}

2. Let R_1 and R_2 be two relation defined as follows :

[*JEE Main-2020 (September)*]

$R_1 = \{(a, b) \in R^2 : a^2 + b^2 \in Q\}$ and

$R_2 = \{(a,b) \in R^2 : a^2 + b^2 \in Q\}$, where Q is the set of the rational numbers. Then :

(*a*) Neither R_1 nor R_2 is transitive.

(*b*) R_2 is transitive but R_1 is not transitive

(*c*) R_1 and R_2 are both transitive.

(*d*) R_1 is transitive but R_2 is not transitive.

FUNCTION

3. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$ and

$S = \{x \in R : f(x) = f(-x)\}$; then S:

[*JEE Main -2016*]

(*a*) is an empty set.

(*b*) contains exactly one element

(*c*) contains exactly two elements

(*d*) contains more than two elements

4. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is:

[*JEE Main - 2017*]

(*a*) neither injective nor surjective.

(*b*) invertible.

(*c*) injective but not surjective.

(*d*) surjective but not injective

5. Let $S = \{x \in R : x \geq 0$ and $2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S :

[*JEE Main - 2018*]

(*a*) contains exactly one element.

(*b*) contains exactly two elements.

(*c*) contains exactly four elements.

(*d*) is an empty set.

6. For $x \in R - [0, 1]$, let $f_1(x) = \frac{1}{x}, f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$ be three given functions. If a function, J(x) satisfies $(f_2 o J o f_1)(x) = f_3(x)$ then J(x) is equal to :

[*JEE Main - 2019 (January)*]

(*a*) $f_3(x)$ (*b*) $\frac{1}{x}f_3(x)$

(*c*) $f_2(x)$ (*d*) $f_1(x)$

7. Let $A = \{x \in R : x$ is not a positive integer$\}$ Define a fuction $f : A \to R$ as $f(x) = \frac{2x}{x-1}$ then f is

[*JEE Main - 2019 (January)*]

(*a*) injection but nor surjective

(*b*) not injective

(*c*) surjective but not injective

(*d*) neither injective nor surjective

8. Let N be the set of natural numbers and two functions f and g be defined as f , g : N $\to$ N such that

$$f(n)=\begin{cases}\dfrac{n+1}{2} & \text{if} \quad n \text{ is odd}\\ \dfrac{n}{2} & \text{if} \quad n \text{ is even}\end{cases}$$

and $g(n)=n-(-1)^n$. Then fog is:

[*JEE Main - 2019 (January)*]

(*a*) onto but not one-one.
(*b*) one-one but not onto.
(*c*) both one-one and onto.
(*d*) neither one-one nor onto.

9. Let a function f : (0, ∞) $\to$ (0, ∞) be defined by $f(x)=\left|1-\dfrac{1}{x}\right|$. Then f is: **[*JEE Main - 2019 (January)*]**

(*a*) not injective but it is surjective
(*b*) injective only
(*c*) neither injective nor surjective
(*d*) both injective as well as surjective

10. Let $f_k(x)=\dfrac{1}{k}\left(\sin^k x+\cos^k x\right)$ for k = 1, 2, 3....Then for all x

11. Let $f(x)=a^x\,(a>0)$ be written as $f(x)=f_1(x)+f_2(x)$, where $f_1(x)$ is an even function of $f_2(x)$ is an odd function. Then $f_1(x+y)+f_1(x-y)$ equals **[*JEE Main - 2019(April)*]**

(*a*) $2f_1(x)f_1(y)$ (*b*) $2f_1(x)f_2(y)$
(*c*) $2f_1(x+y)\,f_2(x-y)$ (*d*) $2\,f_1(x+y)\,f_1(x-y)$

12. Let $\sum_{k=1}^{10} f(a+k)=16\left(2^{10}-1\right)$, where the function f satisfies $f(x+y)=f(x)f(y)$ for all natural numbers x, y and $f(a)=2$ then the natural number 'a' is

[*JEE Main - 2019(April)*]

(*a*) 4 (*b*) 3
(*c*) 16 (*d*) 2

13. If the function f : R–{1,–1} $\to$ A defined by

$f(x)=\dfrac{x^2}{1-x^2}$, is surjective, then A is equal to

[*JEE Main - 2019(April)*]

(*a*) R–[–1,0) (*b*) R–(–1,0)
(*c*) R–{–1} (*d*) [0, ∞)

14. Let $f(x)=x^2, x\in R$. For any $A\subseteq R$, define $g(a)=\{x\in R, f(x)\in A\}$. If S = [0, 4], then which one of the following statements is not true ? **[*JEE Main - 2019(April)*]**

(*a*) $f(g(S))\neq f(S)$ (*b*) $f(g(S))=S$
(*c*) $g(f(S))=g(S)$ (*d*) $g(f(S))\neq S$

15. Let $f(x)=\log_e(\sin x)$, $(0<x<\pi)$ and $g(x)=\sin^{-1}(e^{-x})$, $(x\geq 0)$. If α is a positive real number such that $a=(fog)'(\alpha)$ and $b=(fog)(\alpha)$, then : **[*JEE Main - 2019(April)*]**

(*a*) $a\alpha^2-b\alpha-a=0$ (*b*) $a\alpha^2+b\alpha-a=-2\alpha^2$
(*c*) $a\alpha^2+b\alpha+a=0$ (*d*) $a\alpha^2-b\alpha-a=1$

16. For $x\in\left(0,\dfrac{3}{2}\right)$, let $f(x)=\sqrt{x}$, $g(x)=\tan x$ and $h(x)=\dfrac{1-x^2}{1+x^2}$. If $\phi(x)=((hof)og)(x)$, then $\phi=\left(\dfrac{\pi}{3}\right)$ is equal to : **[*JEE Main - 2019(April)*]**

(*a*) $\tan\dfrac{\pi}{12}$ (*b*) $\tan\dfrac{7\pi}{12}$

(*c*) $\tan\dfrac{11\pi}{12}$ (*d*) $\tan\dfrac{5\pi}{12}$

17. If $g(x)=x^2+x-1$ and $(gof)(x)=4x^2-10x+5$, then $f\left(\dfrac{5}{4}\right)$ is equal to **[*JEE Main-2020 (January)*]**

(*a*) $\dfrac{1}{2}$ (*b*) $\dfrac{-3}{2}$

(*c*) $\dfrac{-1}{2}$ (*d*) $\dfrac{3}{2}$

18. The inverse function of $f(x)=\dfrac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}$, $x\in(-1,1)$ is

[*JEE Main-2020 (January)*]

(*a*) $\dfrac{1}{4}\log_e\left(\dfrac{1+x}{1-x}\right)$

(*b*) $\dfrac{1}{4}\log_e\left(\dfrac{1-x}{1+x}\right)$

(*c*) $\dfrac{\log_8 e}{4}\log_e\left(\dfrac{1+x}{1-x}\right)$

(*d*) $\dfrac{\log_8 e}{4}\log_e\left(\dfrac{1-x}{1+x}\right)$

19. Let f : R $\to$ R be a function which satisfies $f(x+y)=f(x)+f(y)\ \forall\ x, y\in R$. If $f(a)=2$ and $g(n)=\sum_{k=1}^{(n-1)} f(k), n\in N$ then the value of n, for which g(n) = 20, is :

[*JEE Main-2020 (September)*]

20. The domain of the function

$f(x)=\sin^{-1}\left(\dfrac{|x|+5}{x^2+1}\right)$ is **[*JEE Main-2020 (September)*]**

$(-\infty,-a]\cup[a,\infty)$. Then a is equal to:

21. Let A = {a, b, c} and B = {1, 2, 3, 4}. Then the number of elements in the set C = {f : A → B | 2 ∈ f(*a*) and f is not one-one} is _____.

[*JEE Main-2020* (*September*)]

22. For a suitably chosen real constant a, let a function, f : R – {–a} → R be defined by $f(x) = \frac{a-x}{a+x}$. Further suppose that for any real number $x \neq -a$ and $f(x) \neq -a$, (fof)(x) = x. Then $f\left(-\frac{1}{2}\right)$ is equal to :

[*JEE Main-2020* (*September*)]

(*a*) -3 (*b*) $\frac{1}{3}$

(*c*) $-\frac{1}{3}$ (*d*) 3

23. If {p} denotes the fractional part of the number p, then $\left\{\frac{3^{200}}{8}\right\}$, is equal to :

[*JEE Main-2020* (*September*)]

(*a*) $\frac{5}{8}$ (*b*) $\frac{1}{8}$

(*c*) $\frac{7}{8}$ (*d*) $\frac{3}{8}$

24. If $f(x+y) = f(x)\,f(y)$ and $\sum_{x=1}^{\infty} f(x) = 2$, x, y ∈ N, Where N is the set of all natural numbers, then the value of $\frac{f(4)}{f(2)}$ is :

[*JEE Main-2020* (*September*)]

(*a*) $\frac{1}{9}$ (*b*) $\frac{4}{9}$

(*c*) $\frac{1}{3}$ (*d*) $\frac{2}{3}$

25. Let $f : N \to N$ be a function such that $f(m+n) = f(m) + f(n)$ for every $m, n \in N$. If $f(6) = 18$, then $f(2).f(3)$ is equal to: **[*JEE Main-2021* (*August*)]**

(*a*) 6 (*b*) 54

(*c*) 18 (*d*) 36

26. The domain of the function **[*JEE Main-2021* (*August*)]**

$f(x) = \sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\frac{x-1}{x+1}\right)$ is :

(*a*) $\left[0, \frac{1}{4}\right]$ (*b*) $[-2, 0] \cup \left[\frac{1}{4}, \frac{1}{2}\right]$

(*c*) $\left[\frac{1}{4}, \frac{1}{2}\right] \cup \{0\}$ (*d*) $\left[0, \frac{1}{2}\right]$

27. The domain of the function $\text{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is:

(*a*) $\left(-1, -\frac{1}{2}\right] \cup (0, \infty)$ (*b*) $\left[-\frac{1}{2}, 0\right) \cup [1, \infty)$

(*c*) $\left(-\frac{1}{2}, \infty\right) - \{0\}$ (*d*) $\left[-\frac{1}{2}, \infty\right) - \{0\}$

28. Consider function $f : A \to B$ and $g : B \to C (A, B, C \subseteq R)$ such that $(gof)^{-1}$ exists, then :

(*a*) f and g both are one-one

(*b*) f and g both are onto

(*c*) f is one-one and g is onto

(*d*) f is onto and g is one-one

29. Let N be the set of natural numbers and a relation R on N be defined by $R = \left\{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\right\}$. Then the relation R is :

(*a*) symmetric but neither reflexive nor transitive

(*b*) reflexive but neither symmetric nor transitive

(*c*) reflexive and symmetric, but not transitive

(*d*) an equivalence relation

30. If the domain of the function $f(x) = \frac{\cos^{-1}\sqrt{x^2-x+1}}{\sqrt{\sin^{-1}\left(\frac{2x-1}{2}\right)}}$ is the interval $(\alpha, \beta]$, then $\alpha + \beta$ is equal to:

(*a*) $\frac{3}{2}$ (*b*) 2

(*c*) $\frac{1}{2}$ (*d*) 1

JEE-ADVANCED PREVIOUS YEAR'S

1. Let f(x) = x^2 and g(x) = sin x for all x ∈ R. Then the set of all x satisfying (f o g o g o f) (x) = (g o g o f) (x), where (f o g) (x) = f(g(x)), is **[*IIT JEE - 2012*]**

(*a*) $\pm\sqrt{n\pi}$, n ∈ {0, 1, 2,....}

(*b*) $\pm\sqrt{n\pi}$, n ∈ {1, 2,....}

(*c*) $\frac{\pi}{2} + 2n\pi$, n ∈ {.....–2, –1, 0, 1, 2,....}

(*d*) 2nπ, n ∈ {...., –2, –1, 0, 1, 2,....}

2. The function f : [0, 3] $\rightarrow$ [1, 29], defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is **[IIT JEE-2012]**

(*a*) one-one and onto

(*b*) onto but not one-one

(*c*) one-one but not onto

(*d*) neither one-one nor onto

3. Let $f: (-1, 1) \rightarrow IR$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2 \theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ iσ (are) **[IIT JEE-2012]**

(*a*) $1 - \sqrt{\frac{3}{2}}$

(*b*) $1 + \sqrt{\frac{3}{2}}$

(*c*) $1 - \sqrt{\frac{2}{3}}$

(*d*) $1 + \sqrt{\frac{2}{3}}$

4. Let f: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ be given by

$f(x) = (\log(\sec x + \tan x))^3$. Then **[JEE Advanced-2014]**

(*a*) f(x) is an odd function

(*b*) f(x) is a one-one function

(*c*) f(x) is an onto function

(*d*) f(x) is an even function

5. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

[JEE Advanced- 2015]

(*a*) $\cos\beta > 0$ (*b*) $\sin\beta < 0$

(*c*) $\cos(\alpha + \beta) > 0$ (*d*) $\cos\alpha < 0$

6. Let the function $f: [0,1] \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{4^x}{4^x + 2}$

Then the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + ... + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right) \text{ is } ....$$

[JEE(Advanced) - 2020]

ANSWER KEY

Topicwise Questions

1. (*a*)	**2.** (*a*)	**3.** (*b*)	**4.** (*b*)	**5.** (*d*)	**6.** (*b*)	**7.** (*d*)	**8.** (*d*)	**9.** (*c*)	**10.** (*c*)
11. (*a*)	**12.** (*b*)	**13.** (*c*)	**14.** (*a*)	**15.** (*d*)	**16.** (*c*)	**17.** (*c*)	**18.** (*d*)	**19.** (*a*)	**20.** (*d*)
21. (*b*)	**22.** (*a*)	**23.** (*b*)	**24.** (*b*)	**25.** (*d*)	**26.** (*b*)	**27.** (*b*)	**28.** (*d*)	**29.** (*a*)	**30.** (*c*)
31. (*c*)	**32.** (*a*)	**33.** (*d*)	**34.** (*d*)	**35.** (*d*)					

Learning Plus

1. (*b*)	**2.** (*b*)	**3.** (*a*)	**4.** (*a*)	**5.** (*a*)	**6.** (*a*)	**7.** (*d*)	**8.** (*b*)	**9.** (*b*)	**10.** (*a*)
11. (*b*)	**12.** (*b*)	**13.** (*c*)	**14.** (*d*)	**15.** (*c*)	**16.** (*d*)	**17.** (*b*)	**18.** (*b*)	**19.** (*b*)	**20.** (*c*)
21. (*d*)	**22.** (*d*)	**23.** (*c*)	**24.** (*b*)	**25.** (*c*)	**26.** (*a*)	**27.** (*a*)	**28.** (*b*)	**29.** (*d*)	**30.** (*c*)
31. (*c*)	**32.** (*a*)	**33.** (*c*)							

Advanced Level Multiconcept

QuePREHENSION/MATCHING/NUMERICAL

1. (*a,c*)	**2.** (*b,c,d*)	**3.** (*a,d*)	**4.** (*b,d*)	**5.** (*a,b*)	**6.** (*a,b,c*)	**7.** (*a,b,c,d*)	**8.** (*a,c*)	**9.** (*b,c,d*)	**10.** (*a*)
11. (*a*)	**12.** (*a*)	**13.** (*a*)		**15.** (*c*)					

16. $(a) \rightarrow$ q, r, $(b) \rightarrow$ q, r, $(c) \rightarrow$ q, $(d) \rightarrow$ s **17.** $(a) \rightarrow$ q, $\rightarrow$ $(b) \rightarrow$ s, $(c) \rightarrow$ p, $(d) \rightarrow$ r

18. ([1.75] **19.** [1000] **20.** [0001] **21.**[0005] **22.**[0000] **23.** [0013] **24.** [0.875] **25.** [0002] **26.** [0004] **27.** [0004]

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*c*)	**2.** (*a*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)	**6.** (*a*)	**7.** (*a*)	**8.** (*a*)	**9.** (*c*)	**10.** (*a*)
11. (*a*)	**12.** (*b*)	**13.** (*a*)	**14.** (*d*)	**15.** (*d*)	**16.** (*c*)	**17.** (*c*)	**18.** (*c*)	**19.** (*c*)	**20.** (*a*)
21. (19)	**22.** (*d*)	**23.** (*b*)	**24.** (*b*)	**25.** (*b*)	**26.** (*c*)	**27.** (*d*)	**28.** (*c*)	**29.** (*b*)	**30.** (*a*)

JEE-ADVANCED PREVIOUS YEAR'S

1. (*a*)	**2.** (*b*)	**3.** (*ab*)	**4.** (*a,b,c*)	**5.** (*b,c,d*)	**6.** (19.00)

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

2 Inverse Trigonometric Functions

INTRODUCTION

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. represents angles or real numbers whose values of sine, cosine and tangent is 'x', provided that the value in numerical form is smallest. These can be written as arc sin x, arc cos x etc. If two angles whose modulus is equal, in which one is positive and other is negative then we take positive sign.

It is worthwhile noting that the functions sinx, cosx etc. are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-odomain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

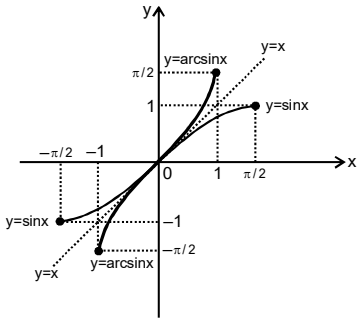
DOMAIN & PRINCIPLE VALUE RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

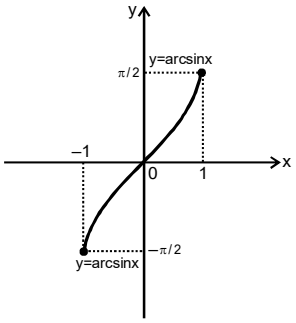
S.No.	Function	Domain	Principle value range (PVR)
1.	$y = \sin^{-1}x$	$x \in [-1, 1]$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1}x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1}x$	$x \in R$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1}x$	$x \in R$	$y \in (0, \pi)$
5.	$y = \sec^{-1}x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \operatorname{cosec}^{-1}x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS

(a) $f: [-\pi/2, \pi/2] \to [-1, 1]$
$f(x) = \sin x$

$f^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$
$f^{-1}(x) = \sin^{-1}x$





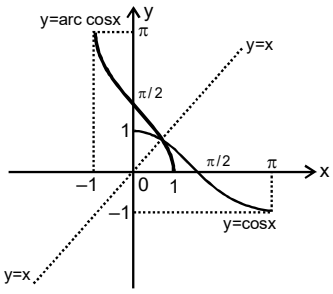
Note : $\sin^{-1}x$ is θ, then θ is the numerically smallest angle, either negative or positive whose sine is equal to x.

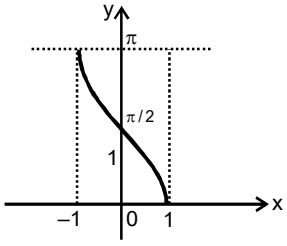
Remarks :

(i) $\sin^{-1}x$ is monotonic increasing in its domain
(ii) It is a bounded function
(iii) It is an odd function
(iv) It is aperiodic
(v) It is continuous

(b) $f: [0, \pi] \to [-1, 1]$
$f(x) = \cos x$

$f^{-1}: [-1, 1] \to [0, \pi]$
$f^{-1}(x) = \cos^{-1}x$





Note : $\cos^{-1}x$ is θ, then θ is the angle $\in [0, \pi]$, whose cosine is equal to x.

Remarks :

(i) $\cos^{-1} x$ is monotonic decreasing in its domain

(ii) It is a bounded function

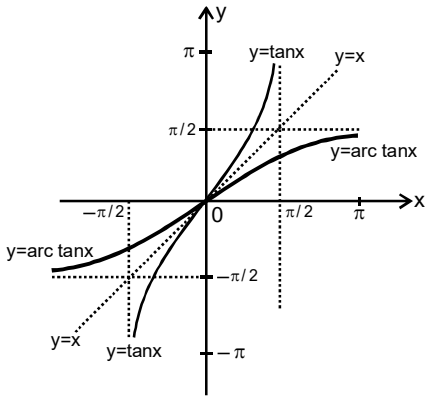
(iii) It is aperiodic

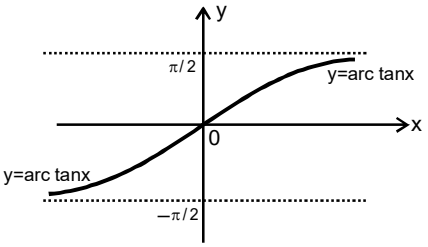
(iv) It is neither even nor odd

(v) It is continuous

(c) $f: (-\pi/2, \pi/2) \to R$ $\qquad$ $f^{-1} : R \to (-\pi/2, \pi/2)$

$f(x) = \tan x$ $\qquad$ $f^{-1}(x) = \tan^{-1} x$





Note : $\tan^{-1}x$ is θ, then θ is the angle $\in (-\pi/2, \pi/2)$, whose tangent is equal to x.

Remarks :

(i) $\tan^{-1} x$ is monotonic increasing in its domain

(ii) It is an odd function

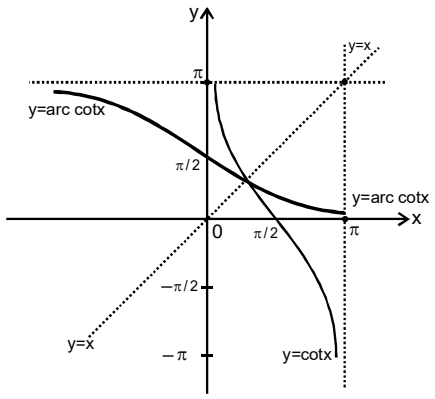
(iii) It is a continuous function

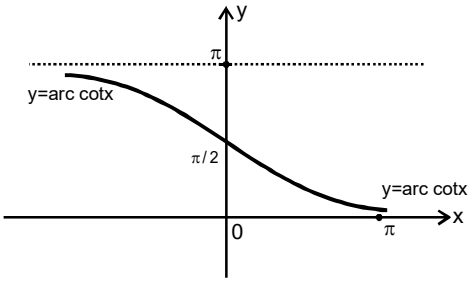
(iv) It is aperiodic

(v) It is a bounded function

(d) $f: (0, \pi) \to R$ $\qquad$ $f^{-1} : R \to (0, \pi)$

$f(x) = \cot x$ $\qquad$ $f^{-1}(x) = \cot^{-1} x$





Note : $\cot^{-1}x$ is θ, then θ is the angle $\in (0, \pi)$, whose cotangent is equal to x.

Remarks :

(i) $\cot^{-1} x$ is monotonic decreasing in its domain

(ii) It is neither even nor odd function

(iii) It is a continuous function

(iv) It is aperiodic

(v) It is a bounded function

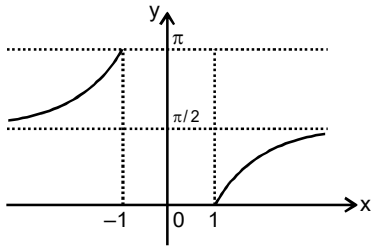
(e) $f: [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1}: (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

Note : $\sec^{-1}x$ is θ, then θ is the angle $\in [0, \pi/2) \cup (\pi/2, \pi]$, whose secant is equal to x.



Remarks :

(i) $\sec^{-1} x$ is increasing whenever it is defined

(ii) It is neither even nor odd function

(iii) It is aperiodic

(iv) It is a bounded function

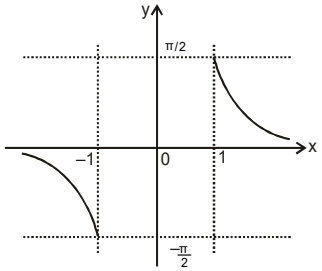
(f) $f: [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1}: (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$

Note : $\text{cosec}^{-1} x$ is θ, then θ is the angle $\in [-\pi/2, 0) \cup (0, \pi/2]$, whose cosecant is equal to x.



Remarks :

(i) $\text{cosec}^{-1} x$ is monotonic decreasing

(ii) It is an odd function

(iii) It is aperiodic

(iv) It is a bounded function

Note : Basis on the above discussion we get following results :

(i) All inverse trigonometric functions shows angle.

(ii) If $x \geq 0$ then all six trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$, $\cot^{-1} x$ shows acute angle.

(iii) If $x < 0$ then $\sin^{-1} x$, $\tan^{-1} x$ and $\text{cosec}^{-1} x$, shows angle between $-\pi/2$ to 0 (IV quadrant)

(iv) If $x < 0$ then $\cos^{-1} x$, $\cot^{-1} x$ and $\sec^{-1} x$ shows obtuse angle (II quadrant)

(v) III quadrant never used in inverse trigonometric functions.

TRAIN YOUR BRAIN

Example 1: $(a) \tan^{-1}(-1)$, $(b) \cot^{-1}(-1)$, $(c) \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Sol. $(a) \tan\left(-\frac{\pi}{4}\right) = -1$,

$\therefore \tan^{-1}(-1) = -\frac{\pi}{4} \left\{\because -\frac{\pi}{4} \in \text{range of } \tan^{-1} x\right\}$

(b) $\cot\left(\frac{3\pi}{4}\right) = -1$

$\therefore \cot^{-1}(-1) = \frac{3\pi}{4} \left\{\because \frac{3\pi}{4} \in \text{range of } \cot^{-1} x\right\}$

(c) $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}$,

$\therefore \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3} \left\{\because \frac{-\pi}{3} \in \text{range of } \sin^{-1} x\right\}$

Example 2: Simplify

$$\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) - \tan^{-1}(-\sqrt{3}) + \cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

Sol. The value $= \frac{-\pi}{4} + \frac{2\pi}{3} - \left(\frac{-\pi}{3}\right) + \left(\frac{2\pi}{3}\right)$

$= -\frac{\pi}{4} + \frac{2\pi}{3} + \frac{\pi}{3} + \frac{2\pi}{3} = \frac{17\pi}{12}$

Example 3: Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Sol. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

$= \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Self Practice Problems

(1) Find the value of $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$

(2) Find the value of $\operatorname{cosec}[\sec^{-1}(-\sqrt{2})+\cot^{-1}(-1)]$

(3) Find the domain of $y=\sec^{-1}(x^2+3x+1)$

(4) Find the domain of $y=\cos^{-1}\left(\frac{x^2}{1+x^2}\right)$

(5) Find the domain of $y=\tan^{-1}(\sqrt{x^2-1})$

Answers :

(1) 1

(2) -1

(3) $(-\infty,-3]\cup[-2,-1]\cup[0,\infty)$

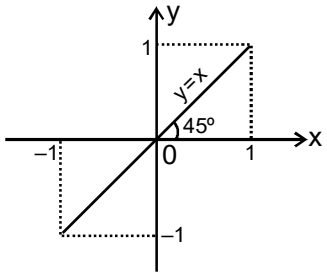
(4) R

(5) $(-\infty,-1]\cup[1,\infty)$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

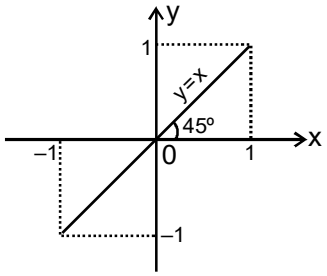
P-1 **(i)** $y=\sin(\sin^{-1}x)=x$

$x\in[-1,1], y\in[-1,1]$



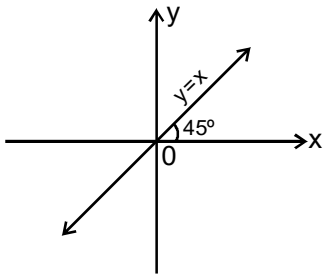
(ii) $y=\cos(\cos^{-1}x)=x$

$x\in[-1,1], y\in[-1,1]$



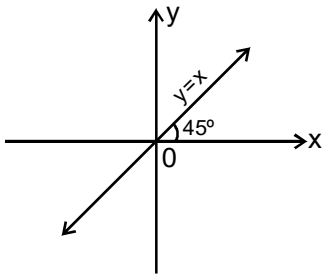
(iii) $y=\tan(\tan^{-1}x)=x$

$x\in R, y\in R$



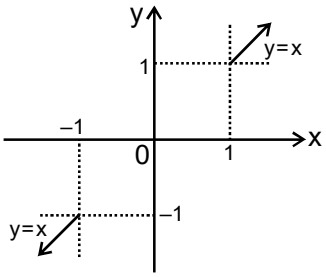
(iv) $y=\cot(\cot^{-1}x)=x$

$x\in R, y\in R$



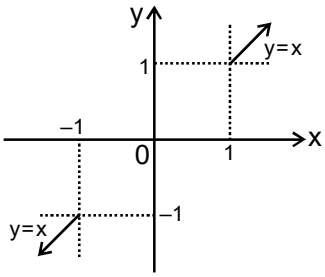
(v) $y=\operatorname{cosec}(\operatorname{cosec}^{-1}x)=x$

$|x|\geq 1, |y|\geq 1$



(vi) $y=\sec(\sec^{-1}x)=x$

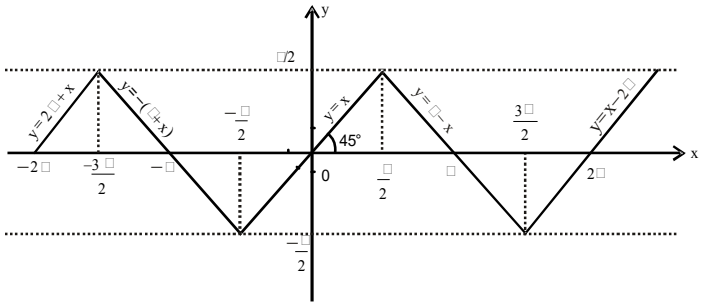
$|x|\geq 1 ; |y|\geq 1$



P-2

(i) $y=\sin^{-1}(\sin x), x\in R, y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, is periodic function with period 2π and it is an odd function

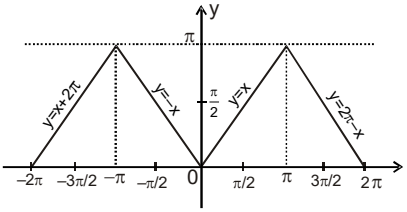
$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x & , -\pi\leq x\leq -\frac{\pi}{2}\\ x & , -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x & , \frac{\pi}{2}\leq x\leq\pi\end{cases}$$



e.g. $\sin^{-1}(\sin 1) = 1$; $\sin^{-1}(\sin 2) = \pi - 2$; $\sin^{-1}(\sin 3) = \pi - 3$; $\sin^{-1}(\sin 4) = \pi - 4$; $\sin^{-1}(\sin 5) = 5 - 2\pi$

(ii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, is periodic function with period 2π and it is an even function

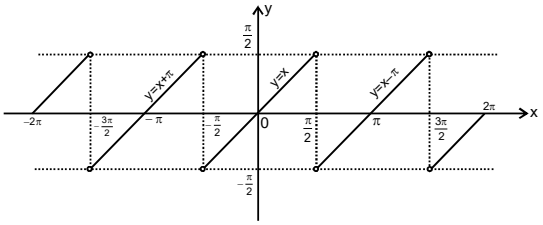
$$\cos^{-1}(\cos x) = \begin{cases} -x\ , & -\pi \le x \le 0 \\ x\ , & 0 < x \le \pi \\ 2\pi - x\ , & \pi < x \le 2\pi \\ x - 2\pi\ , & 2\pi < x \le 3\pi \end{cases}$$



e.g. $\cos^{-1}(\cos 1) = 1$; $\cos^{-1}(\cos 2) = 2$; $\cos^{-1}(\cos 3) = 3$; $\cos^{-1}(\cos 4) = 2\pi - 4$; $\cos^{-1}(\cos 5) = 2\pi - 5$

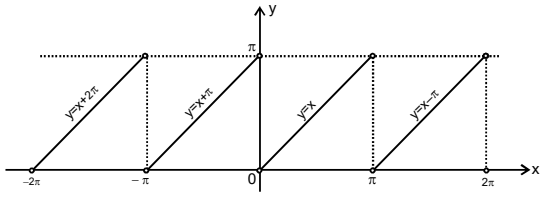
(iii) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is periodic function with period π and it is an odd function

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi & ; -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x & ; -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & ; \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

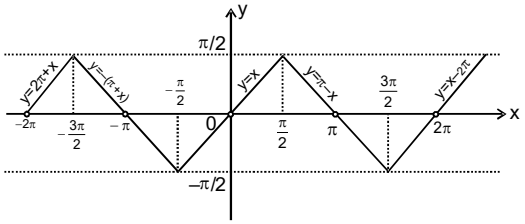


(iv) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi, n \in I\}$, $y \in [0, \pi]$, is periodic function with period π and it is neither an even nor odd function

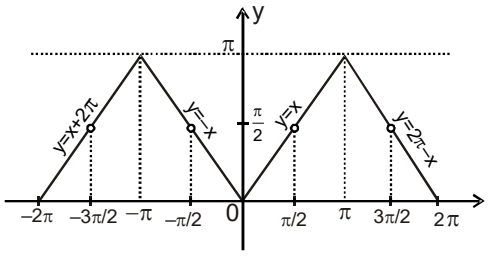
$$\cot^{-1}(\cot x) = \begin{cases} x + \pi\ ; & -\pi < x < 0 \\ x\ ; & 0 < x < \pi \\ x - \pi\ ; & \pi < x < 2\pi \end{cases}$$



(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is periodic function with period 2π and it is an odd function



(vi) $y = \sec^{-1}(\sec x)$, $x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is periodic function with period 2π and it is an even function



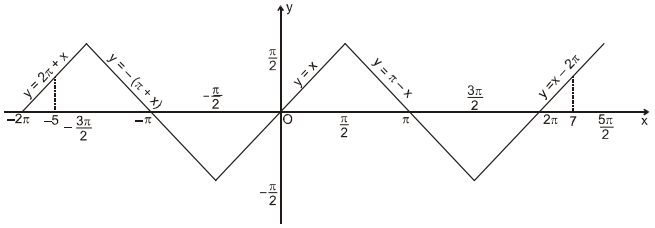
TRAIN YOUR BRAIN

Example 4 Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Sol. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \because 2\pi < 7 < \frac{5\pi}{2}$

aph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore \sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because -2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that $\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example 5: Evaluate following

(i) $\sin(\cos^{-1} 3/5)$ **(ii)** $\cos(\tan^{-1} 3/4)$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Sol. **(i)** Let $\cos^{-1} 3/5 = \theta$ then $\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$ then $\tan\theta = 3/4 \Rightarrow \cos\theta = 4/5$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right)$

$= \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

TRAIN YOUR BRAIN

Example 6: Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol. Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ .....(i)

$\because \cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

(i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because \cot^{-1} x = \tan^{-1}\frac{1}{x}$ $\quad$ if $x > 0$

$\therefore y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \Rightarrow y = -\frac{3}{2}$

Example 7: Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Sol. $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Example 8: Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$.

If y simplifies to $a\pi + b$ then find $(a - b)$.

Sol. $\sin^{-1}(\sin 8) = \sin^{-1}(\sin(3\pi - 8)) = 3\pi - 8$

$\tan^{-1}(\tan 10) = \tan^{-1}(\tan(10 - 3\pi)) = 10 - 3\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

$\sec^{-1}(\sec 9) = \sec^{-1}(\sec(9 - 2\pi)) = 9 - 2\pi$

$\cot^{-1}(\cot 6) = \cot^{-1}(\cot(6 - \pi)) = 6 - \pi$

$\text{cosec}^{-1}(\text{cosec } 7) = \text{cosec}^{-1}(\text{cosec}(7 - 2\pi)) = 7 - 2\pi$

$y = (3\pi - 8) + (3\pi - 10) + (4\pi - 12) + (2\pi - 9) + (-\pi + 6) + (2\pi - 7) = 13\pi - 40$

$\Rightarrow a = 13$ and $b = -40 \Rightarrow a - b = 13 - (-40) = 53$

P-3 **(i)** $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}; |x| \geq 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}; |x| \geq 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x}, & x > 0 \\ \pi + \tan^{-1}\frac{1}{x}, & x < 0 \end{cases}$

P-4 **(i)** $\sin^{-1}(-x) = -\sin^{-1}x \; ; \; -1 \leq x \leq 1$

(ii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \; ; \; |x| \geq 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x \; ; \; x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x \; ; \; x \in R$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1}x \; ; \; -1 \leq x \leq 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x \; ; \; |x| \geq 1$

P-5 **(i)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \; ; \; |x| \leq 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \; ; \; x \in R$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2} \; ; \; |x| \geq 1$

Note : $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$

Example 9: Find the value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$.

Sol. $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$ and $\cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6)$

$= -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$

P-6 (i) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0 \text{ and } xy < 1 \text{ (acute angle)} \\ \pi + \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0 \text{ and } xy > 1 \text{ (obtuse angle)} \end{cases}$$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \quad x > 0, y > 0$

Note : (i) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ (remember)

(ii) $\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 = \frac{\pi}{2}$ (remember)

(iii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z$

$$= \begin{cases} \tan^{-1}\left[\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right], \text{where } x > 0, y > 0, z > 0 \text{ and } xy + yz + zx < 1 \\ \pi + \tan^{-1}\left[\frac{x+y+z+xyz}{1-(xy+yz+zx)}\right], \text{where } x > 0, y > 0, z > 0 \text{ and } xy + yz + zx > 1 \end{cases}$$

(I) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(II) Parallelly we have $\sin^{-1}x = \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), x > 0; y > 0$

and $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right), x > 0, y > 0, x < y$

Example 10: Prove that,

$$\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3\sqrt{11}} + \sin^{-1}\frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

Sol. $\tan^{-1}\frac{1}{2\sqrt{2}} + \tan^{-1}\frac{1}{7\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}}$

$$= \tan^{-1}\frac{\frac{1}{2\sqrt{2}} + \frac{1}{7\sqrt{2}}}{1 - \frac{1}{28}} + \tan^{-1}\frac{3}{\sqrt{2}}$$

$$= \tan^{-1}\frac{9\sqrt{2}}{27} + \tan^{-1}\frac{3}{\sqrt{2}}$$

$$= \tan^{-1}\frac{\sqrt{2}}{3} + \tan^{-1}\frac{3}{\sqrt{2}}$$

$$= \cot^{-1}\frac{3}{\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}} = \frac{\pi}{2}$$

Example 11: Find the value of $\sin^{-1}\frac{3}{\sqrt{73}} + \cos^{-1}\frac{11}{\sqrt{146}} + \cot^{-1}\sqrt{3}$.

Sol. $\tan^{-1}\frac{3}{8} + \tan^{-1}\frac{5}{11} + \cot^{-1}\sqrt{3} = \tan^{-1}\left[\frac{\frac{3}{8} + \frac{5}{11}}{1 - \frac{3}{8}\cdot\frac{5}{11}}\right]$

$$+ \cot^{-1}\sqrt{3} = \tan^{-1}(1) + \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

Example 12: If $\cos^{-1}x/2 + \cos^{-1}y/3 = \theta$, prove that $9x^2 + 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Sol. Let $\cos^{-1}x/2 = \alpha$, and $\cos^{-1}y/3 = \beta$

$\therefore \cos\alpha = x/2$ and $\cos\beta = y/3$.

Given, $\alpha + \beta = \theta \therefore \cos(\alpha + \beta) = \cos\theta$

or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \theta$

or $\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1 - \frac{x^2}{4}}\sqrt{1 - \frac{y^2}{9}} = \cos\theta$

$$\text{or } \frac{xy}{6} - \frac{\sqrt{4-x^2}.\sqrt{9-y^2}}{6} = \cos\theta$$

$$\text{or } (xy - 6\cos\theta)^2 = (4-x^2)(9-y^2)$$

$$\text{or } x^2y^2 + 36\cos^2\theta - 12xy\cos\theta = 36 - 9x^2 - 4y^2 + x^2y^2$$

$$\text{or } 9x^2 - 12y\cos\theta + 4y^2 = 36(1-\cos^2\theta)$$

$$\text{or } 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta.$$

Example 13: Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$.

Sol. Let $\cos^{-1}\frac{a}{b} = \theta \therefore \cos\theta = \frac{a}{b}$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}} = \frac{2}{\cos\theta} = \frac{2b}{a} = \text{R.H.S.}$$

Example 14: Evaluate $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

Sol. Let $z = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

$\because \sin^{-1}\frac{4}{5} = \frac{\pi}{2} - \cos^{-1}\frac{4}{5}$

$$\therefore z = \cos^{-1}\frac{12}{13} + \left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right) - \tan^{-1}\frac{63}{16}.$$

$$z = \frac{\pi}{2} - \left(\cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}\right) - \tan^{-1}\frac{63}{16} \quad \text{.....(i)}$$

$\because \frac{4}{5} > 0, \frac{12}{13} > 0$ and $\frac{4}{5} < \frac{12}{13}$

$$\therefore \cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}$$

$$= \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right]$$

$$= \cos^{-1}\left(\frac{63}{65}\right)$$

$\therefore$ equation (i) can be written as

$$z = \frac{\pi}{2} - \cos^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$$

$$z = \sin^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right) \quad \text{.....(ii)}$$

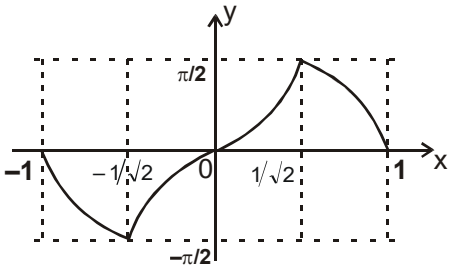
$\because \sin^{-1}\left(\frac{63}{65}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

$\therefore$ from equation (ii), we get

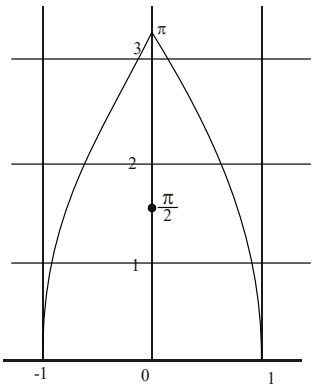
$$\therefore z = \tan^{-1}\left(\frac{63}{16}\right) - \tan^{-1}\left(\frac{63}{16}\right) \Rightarrow z = 0$$

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

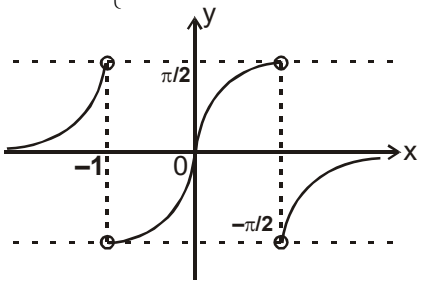
(i) $\sin^{-1}2x\sqrt{1-x^2} = \begin{cases} -\pi - 2\sin^{-1}x & ; -1 \le x \le \frac{-1}{\sqrt{2}} \\ 2\sin^{-1}x & ; \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & ; \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$



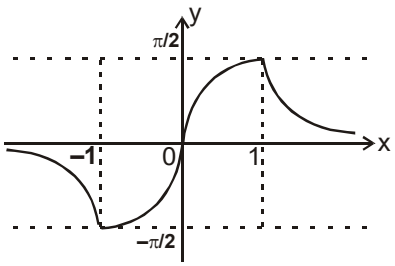
(ii) $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x & ; \ 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & ; -1 \le x \le 0 \end{cases}$



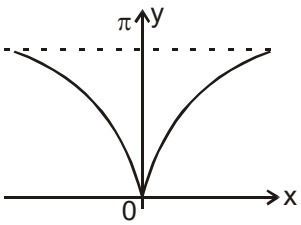
(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & ; \ x < -1 \\ 2\tan^{-1}x & ; \ -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; \ x > 1 \end{cases}$



(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1}x & ; \ x < -1 \\ 2\tan^{-1}x & ; \ -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & ; \ x > 1 \end{cases}$



(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & ; \ x \ge 0 \\ -2\tan^{-1}x & ; \ x \le 0 \end{cases}$



Example 15: Show that $\cos^{-1}\dfrac{1}{7} + 2\cot^{-1}\dfrac{1}{3} = \dfrac{5\pi}{4}$

Sol. $\cot^{-1}\dfrac{1}{7} + 2\cot^{-1}\dfrac{1}{3}$

$= \dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{7} + 2\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{3}\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + 2\tan^{-1}\dfrac{1}{3}\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + 2\tan^{-1}\dfrac{2.1/3}{1-(1/3)^2}\right)\left(\because \dfrac{1}{3} < 1\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{3}{4}\right)$

$= \dfrac{3\pi}{2} - \tan^{-1}\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}.\frac{3}{4}}\left(\because \dfrac{1}{7}.\dfrac{3}{4} < 1\right)$

$= \dfrac{3\pi}{2} - \tan^{-1}1 = \dfrac{3\pi}{2} - \dfrac{\pi}{4} = \dfrac{5\pi}{4}$

Example 16: Evaluate **(i)** $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{2}{3}\right)$ **(ii)** $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right)$

Sol. **(i)** Let $\cos^{-1}\dfrac{2}{3} = \theta$. Then $\cos\theta = \dfrac{2}{3}$ and $0 < \theta < \dfrac{\pi}{2}$

Now $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{2}{3}\right) = \tan\left(\dfrac{1}{2}\theta\right)$

$= \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} = \dfrac{1-2/3}{\sqrt{1-(2/3)^2}} = \dfrac{1/3}{\frac{\sqrt{9-4}}{3}} = \dfrac{1}{\sqrt{5}}$

Alter : $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-2/3}{1+2/3} = \dfrac{1}{5}$

$\therefore \ \tan\dfrac{\theta}{2} = \dfrac{1}{\sqrt{5}} . \left[\text{As} \quad 0 < \dfrac{\theta}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}\right]$

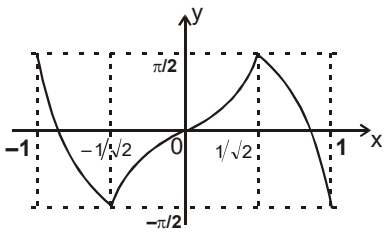
(ii) Let $\tan^{-1}\dfrac{1}{5} = \theta$. Then $\tan\theta = \dfrac{1}{5}$ and $0 < \theta < \dfrac{\pi}{2}$

Now $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right) = \tan\left(2\theta - \dfrac{\pi}{4}\right)$

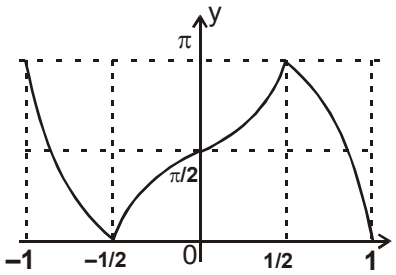
$= \dfrac{\tan 2\theta - 1}{1 + \tan 2\theta} = \dfrac{\frac{2\tan\theta}{1-\tan^2\theta}}{1 + \frac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{2\tan\theta - 1 + \tan^2\theta}{1 - \tan 2\theta + 2\tan\theta} = \dfrac{\frac{2}{5} - 1 + \frac{1}{25}}{1 - \frac{1}{25} + \frac{2}{5}} = \dfrac{10 - 25 + 1}{25 - 1 + 10} = \dfrac{-14}{34} = \dfrac{-7}{17}$.

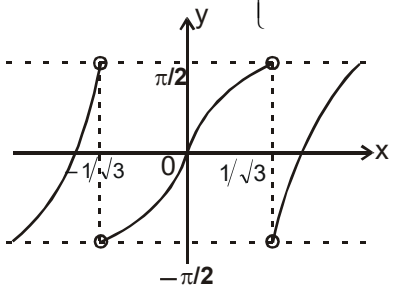
(vi) $\sin^{-1}(3x - 4x^3) = \begin{cases} -\pi - 3\sin^{-1}x & ; \ -1 \le x \le \dfrac{-1}{2} \\ 3\sin^{-1}x & ; \ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & ; \ \dfrac{1}{2} \le x \le 1 \end{cases}$



(vii) $\cos^{-1}(4x^3-3x) = \begin{cases} -2\pi + 3\cos^{-1}x & ; \ -1 \le x \le \dfrac{-1}{2} \\ 2\pi - 3\cos^{-1}x & ; \ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x & ; \ \dfrac{1}{2} \le x \le 1 \end{cases}$



(viii) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \begin{cases} \pi + 3\tan^{-1}x & ; \ x < \dfrac{-1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \ \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \ x > \dfrac{1}{\sqrt{3}} \end{cases}$



EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

TRAIN YOUR BRAIN

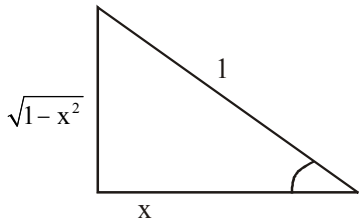
Example 17: Solve $\cos^{-1}x\sqrt{3} + \cos^{-1}x = \pi/2$.

Sol. Given, $\cos^{-1}x\sqrt{3} + \cos^{-1}x = \pi/2$ ..(1)

or $\cos^{-1}x\sqrt{3} = \pi/2 - \cos^{-1}x$

or $\cos\cos^{-1}x\sqrt{3} = \cos(\pi/2 - \cos^{-1}x)$

or $x\sqrt{3} = \sin\cos^{-1}x$ or $x\sqrt{3} = \sin\sin^{-1}\sqrt{1-x^2}$



or $x\sqrt{3} = \sqrt{1-x^2}$ Squaring we get $3x^2 = 1 - x^2$

or $4x^2 = 1 \Rightarrow x = \pm 1/2$

Verification : When $x = 1/2$

L.H.S. of equation $= \cos^{1}(\sqrt{3}/2) + \cos^{-1}(1/2)$

$= \pi/6 + \pi/3 + \pi/2 =$ R.H.S. of equation

When $x = -1/2$.

L.H.S. of equation $= \cos^{-1}(-\sqrt{3}/2) + \cos^{-1}(-1/2)$

$= \pi - \cos^{-1}(\sqrt{3}/2) + \pi - \cos^{-1}(1/2)$

$= \pi - \pi/6 + \pi - \pi/3 = 3\pi/2 \neq$ R.H.S. of equation

$\therefore$ $x = 1/2$ is the only solution

Example 18: Solve for x : $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$.

Sol. We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x.\cot^{-1}x = \dfrac{5\pi^2}{8}$

$\Rightarrow \left(\dfrac{\pi}{2}\right)^2 - 2\tan^{-1}x.(\pi/2 - \tan^{-1}x) = \dfrac{5\pi^2}{8}$

$\Rightarrow \dfrac{\pi^2}{4} - 2.\dfrac{\pi}{2}.\tan^{-1}x + 2(\tan^{-1}x)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \dfrac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1}x = -\pi/4, 3\pi/4 \Rightarrow \tan^{-1}x = -\pi/4;\ x = -1$ {neglecting $\tan^{-1}x = 3\pi/4$ as $\tan^{-1}x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ }

INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

TRAIN YOUR BRAIN

Example 19: Find the interval in which $\cos^{-1}x > \sin^{-1}x$.

Sol. We have, $\cos^{-1}x > \sin^{-1}x$ {for $\cos^{-1}x$ to be real; $x \in [-1, 1]$}

$\Rightarrow 2\cos^{-1}x > \pi/2$

$\Rightarrow \cos^{-1}x > \pi/4$ or $\cos(\cos^{-1}x) < \cos\pi/4$

$\Rightarrow x < \dfrac{1}{\sqrt{2}}$ $\therefore x \in \left(-1, \dfrac{1}{\sqrt{2}}\right)$

Example 20: Find the solution set of the inequation $\sin^{-1}(\sin 5) > x^2 - 4x$

Sol. $\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

$\Rightarrow x \in \left(2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi}\right)$

TRAIN YOUR BRAIN

Example 21: Sum the series, $\tan^{-1}\frac{4}{1+3\cdot 4}+\tan^{-1}\frac{6}{1+8\cdot 9}+\tan^{-1}\frac{8}{1+15\cdot 16}+$ ..... to 'n' terms.

Sol. $T_n=\tan^{-1}\frac{2(n+1)}{1+\{(n+1)^2-1\}\{(n+1)^2\}}$

$=\tan^{-1}\frac{2n+2}{1+\left(n^2+2n\right)(n+1)^2}$

$=\tan^{-1}\left[\frac{2n+2}{1+n(n+2)(n+1)(n+1)}\right]$

$=\tan^{-1}\left[\frac{(n+1)(n+2)-n(n+1)}{1+\{n(n+1)\}\{(n+1)(n+2)\}}\right]$

$=\tan^{-1}(n+1)(n+2)-\tan^{-1}n(n+1)$

Put $n=1, 2, 3,$ ..... , n and add, we get $S_n=\tan^{-1}(n+1)(n+2)-\tan^{-1}2$

Example 22: Sum the series upto 'n' terms,

$\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{2}{16}+\tan^{-1}\frac{2}{25}+$..... to n terms.

Also show that, $S_\infty=\tan^{-1}3$.

Sol. $T_n=\tan^{-1}\frac{2}{n^2+2n+1}$

$=\tan^{-1}\frac{(n+2)-n}{1+n(n+2)}$

$=\tan^{-1}(n+2)-\tan^{-1}(n)$

Hence,

$S_n=\tan^{-1}(n+2)+\tan^{-1}(n+1)-(\tan^{-1}1+\tan^{-1}2)$

$S_\infty=\lim_{n\to\infty}S_n=\frac{\pi}{2}+\frac{\pi}{2}-\left(\pi+\tan^{-1}\left(\frac{1+2}{1-2}\right)\right)=\tan^{-1}3$

Topicwise Questions

GRAPHS, DOMAIN & PRINCIPLE VALUE RANGE OF ITF

1. The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

(*a*) $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ (*b*) $\frac{-\pi}{3}$

(*c*) $\frac{-2\pi}{3}$ (*d*) $\frac{5\pi}{3}$

PROPERTIES OF ITF (I - VI)

2. The value of $\cos^{-1}(\cos 12)-\sin^{-1}(\sin 14)$ is

(*a*) -2 (*b*) $8\pi-26$

(*c*) $4\pi+2$ (4) 2

3. If $\tan^{-1}x-\tan^{-1}y=\tan^{-1}A$, then $A=$

(*a*) $x-y$ (*b*) $x+y$

(*c*) $\frac{x-y}{1+xy}$ (*d*) $\frac{x+y}{1-xy}$

4. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=$

(*a*) $\frac{7\pi}{6}$ (*b*) $\frac{5\pi}{6}$

(*c*) $\frac{\pi}{6}$ (*d*) $\frac{3\pi}{2}$

5. If $\sin^{-1}x=\theta+\beta$ and $\sin^{-1}y=\theta-\beta$, then $1+xy=$

(*a*) $\sin^2\theta+\sin^2\beta$ (*b*) $\sin^2\theta+\cos^2\beta$

(*c*) $\cos^2\theta+\cos^2\beta$ (*d*) $\cos^2\theta+\sin^2\beta$

6. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]=$

(*a*) $\frac{\pi}{4}-\frac{x}{2}$ (*b*) $\frac{\pi}{4}+\frac{x}{2}$

(*c*) $\frac{x}{2}$ (*d*) $\frac{\pi}{4}-x$

7. $\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)=$

(*a*) 5 (*b*) 13

(*c*) 15 (*d*) 6

8. The value of x which satisfies the equation $\tan^{-1}x=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(*a*) 3 (*b*) –3

(*c*) $\frac{1}{3}$ (*d*) $-\frac{1}{3}$

9. $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=$

(*a*) $\frac{\sqrt{3}}{2}$ (*b*) $-\frac{\sqrt{3}}{2}$

(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{2}$

10. $\cot^{-1}3+\text{cosec}^{-1}\sqrt{5}=$

(*a*) $\frac{\pi}{3}$ (*b*) $\frac{\pi}{4}$

(*c*) $\frac{\pi}{6}$ (*d*) $\frac{\pi}{2}$

11. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\cos\frac{5\pi}{3}\right)$ is

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{5\pi}{3}$

(*c*) $\frac{10\pi}{3}$ (*d*) 0

12. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$ is

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{2}$

(*c*) $\frac{\pi}{12}$ (*d*) $\frac{\pi}{6}$

13. The value of $\tan\left(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}\right)$ is

(*a*) 5/6 (*b*) 7/6

(*c*) 1/6 (*d*) 1/7

14. $\cos\left[2\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}\right]=$

(*a*) $\frac{2\sqrt{6}}{5}$ (*b*) $-\frac{2\sqrt{6}}{5}$

(*c*) $\frac{1}{5}$ (*d*) $-\frac{1}{5}$

15. $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}=\ \pi$

(*a*) π (*b*) $\frac{\pi}{2}$

(*c*) $\frac{3\pi}{2}$ (*d*) 2π

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

16. $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{4}$ (*d*) π

17. $\tan(\cos^{-1}x)$ is equal to

(*a*) $\frac{\sqrt{1-x^2}}{x}$ (*b*) $\frac{x}{1+x^2}$

(*c*) $\frac{\sqrt{1+x^2}}{x}$ (*d*) $\sqrt{1-x^2}$

18. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=$

(*a*) $\frac{17}{7}$ (*b*) $-\frac{17}{7}$

(*c*) $\frac{7}{17}$ (*d*) $-\frac{7}{17}$

19. $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=$

(*a*) $\cot^{-1}\sqrt{x}$ (*b*) $\tan^{-1}\sqrt{x}$

(*c*) $\tan^{-1}x$ (*d*) $\cot^{-1}x$

20. $\sin\left(4\tan^{-1}\frac{1}{3}\right)=$

(*a*) $\frac{12}{25}$ (*b*) $\frac{24}{25}$

(*c*) $\frac{1}{5}$ (*d*) $\frac{25}{24}$

21. The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)+\cos(\tan^{-1}2\sqrt{2})=\frac{16}{15}$

(*a*) $\frac{16}{15}$ (*b*) $\frac{14}{15}$

(*c*) $\frac{12}{15}$ (*d*) $\frac{11}{15}$

22. The solution of $\sin^{-1}x-\sin^{-1}2x=\pm\frac{\pi}{3}$ is

(*a*) $\pm\frac{1}{3}$ (*b*) $\pm\frac{1}{4}$

(*c*) $\pm\frac{\sqrt{3}}{2}$ (*d*) $\pm\frac{1}{2}$

23. $\sin\left[3\sin^{-1}\left(\frac{1}{5}\right)\right]=$

(*a*) 71/125 (*b*) 74/125

(*c*) 3/5 (*d*) 1/2

EQUATIONS, INEQUATIONS INVOLVING ITF, SUMMATION OF SERIES

24. A solution of the equation $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$ is

(*a*) $x=1$ (*b*) $x=-1$

(*c*) $x=0$ (*d*) $x=\pi$

25. If $\tan(x+y)=33$ and $x=\tan^{-1}3$, then y will be

(*a*) 0.3 (*b*) $\tan^{-1}(1.3)$

(*c*) $\tan^{-1}(0.3)$ (*d*) $\tan^{-1}\left(\frac{1}{18}\right)$

26. If $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$, then $x=$

(*a*) $\sqrt{2}$ (*b*) 3

(*c*) $\sqrt{3}$ (*d*) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

27. $\sin^{-1}x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$,

(*a*) 0 (*b*) $\frac{1}{\sqrt{5}}$

(*c*) $\frac{2}{\sqrt{5}}$ (*d*) $\frac{\sqrt{3}}{2}$

28. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=$

(*a*) 0 (*b*) 1

(*c*) $\frac{1}{xyz}$ (*d*) xyz

29. For the equation $\cos^{-1}x+\cos^{-1}2x+\pi=0$, the number of real solution is

(*a*) 1 (*b*) 2

(*c*) 0 (*d*) ∞

30. The equation $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$ has

(*a*) No solution (*b*) Only one solution

(*c*) Two solutions (*d*) Three solutions

Learning Plus

1. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is equal to

(*a*) $\frac{5\pi}{6}$ (*b*) $\frac{7\pi}{12}$

(*c*) $\frac{5\pi}{12}$ (*d*) $\frac{3\pi}{5}$

2. Domain of $f(x) = \cos^{-1}x + \cot^{-1}x + \text{cosec}^{-1}x$ is

(*a*) $[-1, 1]$ (*b*) R

(*c*) $(-\infty, -1] \cup [1, \infty)$ (*d*) $\{-1, 1\}$

3. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is

(*a*) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (*b*) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

(*c*) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (*d*) none of these

4. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}] = y$, then

(*a*) $y = \frac{4}{5}$ (*b*) $y = \frac{2}{\sqrt{5}}$

(*c*) $y = -\frac{2}{\sqrt{5}}$ (*d*) $y^2 = \frac{10}{11}$

5. If $\sum_{i=1}^{n} \cos^{-1}\alpha_i = 0$, then $\sum_{i=1}^{n} \alpha_i$

(*a*) n (*b*) –n

(*c*) 0 (*d*) None of these

6. If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is

(*a*) 1 (*b*) 5

(*c*) 9 (*d*) None of these

7. $\text{cosec}^{-1}(\cos x)$ is real if

(*a*) $x \in [-1, 1]$

(*b*) $x \in R$

(*c*) x is an odd multiple of $\frac{\pi}{2}$

(*d*) x is an integral multiple of π

8. If $\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is equal to

(*a*) x (*b*) $\pi - x$

(*c*) $2\pi + x$ (*d*) $2\pi - x$

9 The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to

(*a*) 3/4 (*b*) –3/4

(*c*) 1/16 (*d*) 1/4

10. The value of $\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$, where $x \in \left(\frac{\pi}{2}, \pi\right)$ is

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{\pi}{4}$

(*c*) $-\frac{\pi}{4}$ (*d*) $-\frac{\pi}{2}$

11. If $x \ge 0$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then

(*a*) $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$ (*b*) $0 < \theta \le \frac{\pi}{4}$

(*c*) $0 < \theta \le \frac{\pi}{2}$ (*d*) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

12. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(*a*) $\frac{2\pi}{3}$ (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{6}$ (*d*) π

13. The equation $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has :

(*a*) no solution
(*b*) unique solution
(*c*) infinite number of solutions
(*d*) none of these

14. The solution of the equation $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) - \frac{\pi}{6} = 0$ is

(1) $x = 2$ (*b*) $x = -4$

(*c*) $x = 4$ (*d*) none of these

15. If $x < 0$ then value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is equal to

(*a*) $\frac{\pi}{2}$ (*b*) $-\frac{\pi}{2}$

(*c*) 0 (*d*) none of these

16. The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(*a*) $\frac{6}{17}$ (*b*) $\frac{7}{16}$

(*c*) $\frac{5}{7}$ (*d*) $\frac{17}{6}$

17. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$, $x\neq 0$ is equal to

(*a*) x (*b*) 2x

(*c*) $\frac{2}{x}$ (*d*) $\frac{x}{2}$

18. $\tan^{-1}a+\tan^{-1}b$, where $a>0, b>0, ab>1$, is equal to

(*a*) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (*b*) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)-\pi$

(*c*) $\pi+\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (*d*) $\pi-\tan^{-1}\left(\frac{a+b}{1-ab}\right)$

19. $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$ is equal to

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{2}$

(*c*) $\frac{\pi}{3}$ (*d*) none of these

20. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, $\frac{\pi}{2}<x<\pi$, is

(*a*) $\pi-\frac{x}{2}$ (*b*) $\frac{\pi}{2}+\frac{x}{2}$

(*c*) $\frac{x}{2}$ (*d*) $2\pi-\frac{x}{2}$

21. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $0\le x\le 1$ are

(*a*) $0,\pi$ (*b*) $0,\frac{\pi}{4}$

(*c*) $-\frac{\pi}{4},\frac{\pi}{4}$ (*d*) $\frac{\pi}{4},\frac{\pi}{2}$

22. Which one of the following correct ?

(*a*) $\tan 1>\tan^{-1}1$ (*b*) $\tan 1<\tan^{-1}1$

(*c*) $\tan 1=\tan^{-1}1$ (*d*) None of these

23. The number of solutions of the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

24. $\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{5}{13}\right)$ is equal to

(*a*) $\cos^{-1}\left(\frac{33}{65}\right)$ (*b*) $\cos^{-1}\left(-\frac{33}{65}\right)$

(*c*) $\cos^{-1}\left(\frac{64}{65}\right)$ (*d*) none of these

25. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then $\tan\theta$ is equal to

(*a*) 1/3 (*b*) 3

(*c*) 1 (*d*) –1

26. The set of values of 'x' for which the formula $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$ is true, is

(*a*) $(-1,0)$ (*b*) $[0,1]\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$

(*c*) $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ (*d*) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

27. $\cos^{-1}\left\{\frac{1}{2}x^2+\sqrt{1-x^2}\cdot\sqrt{1-\frac{x^2}{4}}\right\}=\cos^{-1}\frac{x}{2}-\cos^{-1}x$ holds for

(*a*) $|x|\le 1$ (*b*) $x\in R$

(*c*) $0\le x\le 1$ (*d*) $-1\le x\le 0$

28 All possible values of p and q for which $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}$ holds, is

(*a*) $p=1, q=\frac{1}{2}$ (*b*) $q>1, p=\frac{1}{2}$

(*c*) $0\le p\le 1, q=\frac{1}{2}$ (*d*) None of these

29. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then a value of x is

(*a*) 1 (*b*) 3

(*c*) 4 (*d*) 5

30. The value of x satisfying $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$ are

(*a*) $0,\frac{1}{2}$ (*b*) 0

(*c*) $1,-1$ (*d*) None of these

31. The number of solution(s)of the equation $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, is / are

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) more than two

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for

(*a*) $\sin^{-1}\alpha$ (*b*) $\cos^{-1}\alpha$
(*c*) $\sec^{-1}\alpha$ (*d*) $\operatorname{cosec}^{-1}\alpha$

2 α, β and γ are three angles given by $\alpha = 2\tan^{-1}(\sqrt{2}-1)$, $\beta = 3\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right)$ and $\gamma = \cos^{-1}\frac{1}{3}$. Then

(*a*) $\alpha > \beta$ (*b*) $\beta > \gamma$
(*c*) $\alpha < \gamma$ (*d*) $\alpha > \gamma$

3. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then

(*a*) $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$

(*b*) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(*c*) $x^{50} + y^{25} + z^5 = 0$

(*d*) $\frac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

4. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(-\frac{14\pi}{5}\right)\right\}\right]$ is:

(*a*) $\cos\left(-\frac{7\pi}{5}\right)$ (*b*) $\sin\left(\frac{\pi}{10}\right)$

(*c*) $\cos\left(\frac{2\pi}{5}\right)$ (*d*) $-\cos\left(\frac{3\pi}{5}\right)$

5. If $6\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \pi$, then

(*a*) $x = 1$ (*b*) $x = 2$
(*c*) $x = 3$ (*d*) $x = 4$

6. $\sin^{-1}x > \cos^{-1}x$ holds for

(*a*) all values of x (*b*) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

(*c*) $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ (*d*) $x = 0.75$

7. Let $f(x) = \sin^{-1}x + \cos^{-1}x$. Then $\frac{\pi}{2}$ is equal to

(*a*) $f\left(\frac{1}{2}\right)$ (*b*) $f(k^2 - 2k + 3), k \in R$

(*c*) $f\left(\frac{1}{1+k^2}\right), k \in R$ (*d*) $f(-2)$

8. If $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, then x may be

(*a*) 1 (*b*) $-\frac{1}{2}$

(*c*) $\frac{3}{2}$ (*d*) $-\frac{3}{2}$

9. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to:

(*a*) $\frac{1}{2}\cos^{-1}x$ (*b*) $\cos^{-1}\sqrt{\frac{1+x}{2}}$

(*c*) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (*d*) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

10. If the numerical value of $\tan(\cos^{-1}(4/5) + \tan^{-1}(2/3))$ is a/b in the simplest form then

(*a*) $a + b = 23$ (*b*) $a - b = 11$
(*c*) $3b = a + 1$ (*d*) $2a = 3b$

11. $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4 - 2n^2 + 2}$ is equal to:

(*a*) $\tan^{-1}2 + \tan^{-1}3$ (*b*) $4\tan^{-1}1$

(*c*) $\pi/2$ (*d*) $\sec^{-1}\left(-\sqrt{2}\right)$

12. For the equation $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$, which of the following is invalid ?

(*a*) $a^2x + 2a = x$ (*b*) $a^2 + 2ax + 1 = 0$
(*c*) $a \neq 0$ (*d*) $a \neq -1, 1$

13 If $a = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ and $b = \tan^{-1}(-\sqrt{3}) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$, then

(*a*) $a - b = 17\pi/12$ (*b*) $a + b = 17\pi/12$
(*c*) $a + b = -7\pi/12$ (*d*) $a - b = \pi/12$

14. If $\cos^{-1}x = \tan^{-1}x$, then

(*a*) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$

(*b*) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(*c*) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

(*d*) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

Comprehension – 1 (No. 15 to 17)

$$\tan^{-1}(\tan\theta)=\begin{cases}\pi+\theta & , \quad -\frac{3\pi}{2}<\theta<-\frac{\pi}{2}\\ \theta & , \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}\\ -\pi+\theta & , \quad \frac{\pi}{2}<\theta<\frac{3\pi}{2}\end{cases},$$

$$\sin^{-1}(\sin\theta)=\begin{cases}-\pi-\theta & , \quad -\frac{3\pi}{2}\le\theta<-\frac{\pi}{2}\\ \theta & , \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ \pi-\theta & , \quad \frac{\pi}{2}<\theta\le\frac{3\pi}{2}\end{cases},$$

$$\cos^{-1}(\cos\theta)=\begin{cases}-\theta & , \quad -\pi\le\theta<0\\ \theta & , \quad 0\le\theta\le\pi\\ 2\pi-\theta & , \quad \pi<\theta\le 2\pi\end{cases}$$

Based on the above results, answer each of the following :

15. $\cos^{-1}x$ is equal to

(*a*) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<1$

(*b*) $-\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*c*) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*d*) $\sin^{-1}\sqrt{1-x^2}$ if $0<x<1$

16. $\sin^{-1}x$ is equal to

(*a*) $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*b*) $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<1$

(*c*) $\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$

(*d*) $-\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$

17. $\cos^{-1}x$ is equal to

(*a*) $-\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

(*b*) $\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

(*c*) $-\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $0<x<1$

(*d*) $\pi+\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

Comprehension – 2 (No. 18 to 20)

The domain and range of inverse circular function are defined as follows

	Domain	**Range**
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-(\pi)$
$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$

18. $\sin^{-1}x<\frac{3\pi}{4}$ then solution set of x is

(*a*) $\left(\frac{1}{\sqrt{2}},1\right]$ (*b*) $\left(-\frac{1}{\sqrt{2}},-1\right]$

(*c*) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ (*d*) None of these

19. $\sin^{-1}x+\text{cosec}^{-1}x$ at $x=-1$ is

(*a*) π (*b*) 2π

(*c*) 3π (*d*) $-\pi$

20. If $x\in[-1, 1]$, then range of $\tan^{-1}(-x)$ is

(*a*) $\left[\frac{3\pi}{4},\frac{7\pi}{4}\right]$ (*b*) $\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$

(*c*) $[-\pi, 0]$ (*d*) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

21. Match the column

Column - I	**Column - II**
(*a*) $\sin^{-1}x-\cos^{-1}x$ is maximum at	(p) $x=-1$
(*b*) $(\sin^{-1}x)^2+(\cos^{-1}x)^2$ is minimum at	(q) $x=-\frac{1}{\sqrt{2}}$
(*c*) $(\tan^{-1}x)^2+(\cot^{-1}x)^2$ is minimum at	(r) $x=0$
(*d*) $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ is maximum at	(s) $x=\frac{1}{\sqrt{2}}$
	(t) $x=1$

22. Match the Column

Column - I	Column - II
(*a*) The value of $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$ is	(p) $\frac{17}{6}$
(*b*) The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ is	(q) $\frac{-7}{17}$
(*c*) The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is	(r) $\frac{3}{4}$
(*d*) The value of $\cos\tan^{-1}\sin\cot^{-1}\left(\frac{1}{2}\right)$ is	(s) $\frac{\sqrt{5}}{3}$

23. Match the column

Column – I	Column – II
(*a*) $f(x)=\sin^{-1}\left(\frac{2}{\lvert\sin x-1\rvert+\lvert\sin x+1\rvert}\right)$	(P) f(x) is many one
(*b*) $f(x)=\cos^{-1}(\lvert x-1\rvert-\lvert x-2\rvert)$	(Q) Domain of f(x) is R
(*c*) $f(x)=\sin^{-1}\left(\frac{\pi}{\lvert\sin^{-1}x-(\pi/2)\rvert+\lvert\sin^{-1}x+(\pi/2)\rvert}\right)$	(R) Range contain only irrational number
(*d*) $f(x)=\cos(\cos^{-1}\lvert x\rvert)+\sin^{-1}(\sin x)-\text{cosec}^{-1}(\text{cosec}x)+\text{cosec}^{-1}\lvert x\rvert$	(S) f(x) is even.

NUMERICAL BASED QUESTIONS

24. If $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$, then x is equal to____.

25. The sum of roots of the function

$$\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{1+4x}=\tan^{-1}\frac{2}{x^2}$$

26. Solution of equation $\cot^{-1}x+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$ is

27. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals

28. The value of $\left\{\tan\left(\cos^{-1}\left(-\frac{2}{7}\right)-\pi/2\right)\right\}$ is $\frac{a}{b\sqrt{c}}$ then find value of a + b + c

29. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. If $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}, x>\frac{3}{4}$ then x is equal to :

(*a*) $\frac{\sqrt{145}}{12}$ (*b*) $\frac{\sqrt{145}}{10}$

(*c*) $\frac{\sqrt{146}}{12}$ (*d*) $\frac{\sqrt{145}}{11}$

[*JEE Main - 2019 (January)*]

2. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to : **[JEE Main - 2019 (January)]**

(*a*) π (*b*) 7π

(*c*) 0 (*d*) 10

3. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is : **[JEE Main - 2019 (January)]**

(*a*) $\frac{21}{19}$ (*b*) $\frac{19}{21}$

(*c*) $\frac{22}{23}$ (*d*) $\frac{23}{22}$

4. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval: **[JEE Main - 2019 (January)]**

(*a*) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

(*b*) $(\cot 2, \infty)$

(*c*) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

(*d*) $(\cot 5, \cot 4)$

5. Considering only the principal values of inverse functions, the set

$$A = \left\{x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}\right\}$$

[JEE Main - 2019 (January)]

(*a*) contains two elements

(*b*) contains more than two elements

(*c*) is a singleton

(*d*) is an empty set

6. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, **[JEE Main - 2019(April)]**

where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to :

(*a*) $\sin^{-}\left(\frac{9}{5\sqrt{10}}\right)1$ (*b*) $\tan^{-}\left(\frac{9}{14}\right)1$

(*c*) $\cos^{-}\left(\frac{9}{5\sqrt{10}}\right)1$ (*d*) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

7. The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to : **[JEE Main - 2019(April)]**

(*a*) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$ (*b*) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

(*c*) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$ (*d*) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$

8. If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \leq x \leq 1, -2 \leq y \leq 2$, $x \leq \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[JEE Main - 2019(April)]**

(*a*) $4\sin^2\alpha - 2x^2y2$ (*b*) $4\cos^2\alpha + 2x^2y2$

(*c*) $4\sin^2\alpha$ (*d*) $2\sin^2\alpha$

9. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ is equal to : **[JEE Main-2020 (September)]**

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{7\pi}{4}$

(*c*) $\frac{3\pi}{2}$ (*d*) $\frac{5\pi}{4}$

10. If S is the sum of the first 10 terms of the series **[JEE Main-2020 (September)]**

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \ldots,$$

then tan (S) is equal to :

(*a*) $-\frac{6}{5}$ (*b*) $\frac{5}{11}$

(*c*) $\frac{10}{11}$ (*d*) $\frac{5}{6}$

11. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy **[JEE Main-2021 (March)]**

$$\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$$ is equal to :

(*a*) 2 (*b*) 0

(*c*) 3 (*d*) 1

12. The sum of possible values of x for

$$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$$ is : **[JEE Main-2021 (March)]**

(*a*) $-\frac{32}{4}$ (*b*) $-\frac{33}{4}$

(*c*) $-\frac{31}{4}$ (*d*) $-\frac{30}{4}$

13. If $\cot^{-1}(\alpha) = \cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \ldots\ldots$ upto 100 terms, then α is : **[*JEE Main-2021 (March)*]**

(*a*) 1.02 (*b*) 1.03

(*c*) 1.01 (*d*) 1.00

14. The number of solutions of the equation

$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$, and [x] denotes the greatest integer less than or equal to x, is : **[*JEE Main-2021 (March)*]**

(*a*) 0 (*b*) Infinite

(*c*) 2 (*d*) 4

JEE-ADVANCED PREVIOUS YEAR'S

1. The value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$ is **[*JEE Advanced-2013*]**

(*a*) $\frac{23}{25}$

(*b*) $\frac{25}{23}$

(*c*) $\frac{23}{24}$

(*d*) $\frac{24}{23}$

2. Match List I with List II and select the correct answer using the code given below the lists : **[*JEE Advanced- 2013*]**

List - I	**List - II**
P. $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right)^{1/2}$ takes value	**1.** $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q. If cos x + cos y + cox z = 0 = sin x + sin y + sin z then possible value of cos $\frac{x-y}{2}$ is	**2.** $\sqrt{2}$
R. If cos $\left(\frac{\pi}{4} - x\right)$ cos 2x + sin x sin 2x sec x = cos xsin 2x sec x + cos $\left(\frac{\pi}{4} + x\right)$ cos 2x then possible value of sec x is	**3.** $\frac{1}{2}$
S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then possible value of x is	**4.** 1

Codes :

	P	**Q**	**R**	**S**
(*a*)	4	3	1	2
(*b*)	4	3	2	1
(*c*)	3	4	2	1
(*d*)	3	4	1	2

3. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10 - x}{10}$ is **[*JEE Advanced-2014*]**

4. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s)is(are) **[*JEE Advanced -2015*]**

(*a*) $\cos\beta > 0$ (*b*) $\sin\beta < 0$

(*c*) $\cos(\alpha + \beta) > 0$ (*d*) $\cos\alpha < 0$

5. The number of real solutions of the equation

$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is _____ (Here, the inverse trigonometric functions $\sin^{-1} x$ and $\cos^{-1} x$ assume value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

[*JEE Advanced - 2018*]

6. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$f_n(x) = \Sigma_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ for all $x \in (0,\infty)$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TURE? **[*JEE Advanced - 2018*]**

(*a*) $\Sigma_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$)

(*b*) $\Sigma_{j=1}^{10}\left(1 + f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(*c*) For any fixed positive integer n, $\lim_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(*d*) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

7. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{and} \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes value $\sin\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$ and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$. **[*JEE Advanced - 2018*]**

List-I	**List-II**
(P) The range of f is	(1) $\left(-\infty, \frac{1}{1-e}\right] \cup \leq \left[\frac{e}{e-1}, \infty\right)$
(Q) The range of g contains	(2) $(0, 1)$
(R) The domain of f contains	(3) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(S) The domain of g is	(4) $(-\infty, 0) \cup (0, \infty)$
	(5) $\left(-\infty, \frac{e}{e-1}\right]$
	(6) $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

(*a*) (P) $\to$ 4 ; (Q) $\to$ 2 ; (R) $\to$ 1 ; (S) $\to$ 1

(*b*) (P) $\to$ 3 ; (Q) $\to$ 3 ; (R) $\to$ 6 ; (S) $\to$ 5

(*c*) (P) $\to$ 4 ; (Q) $\to$ 2 ; (R) $\to$ 1 ; (S) $\to$ 6

(*d*) (P) $\to$ 4 ; (Q) $\to$ 3 ; (R) $\to$ 6 ; (S) $\to$ 5

8. The value of

$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals

[*JEE Advanced-2019*]

ANSWER KEY

Topicwise Questions

1. (*b*) **2.** (*b*) **3.** (*c*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*c*) **8.** (*a*) **9.** (*c*) **10.** (*b*)
11. (*a*) **12.** (*d*) **13.** (*d*) **14.** (*b*) **15.** (*a*) **16.** (*a*) **17.** (*a*) **18.** (*d*) **19.** (*b*) **20.** (*b*)
21. (*b*) **22.** (*d*) **23.** (*a*) **24.** (*c*) **25.** (*c*) **26.** (*c*) **27.** (*b*) **28.** (*b*) **29.** (*c*) **30.** (*a*)

Learning Plus

1. (*b*) **2.** (*d*) **3.** (*c*) **4.** (*b*) **5.** (*a*) **6.** (*b*) **7.** (*d*) **8.** (*d*) **9.** (*a*) **10.** (*d*)
11. (*d*) **12.** (*b*) **13.** (*b*) **14.** (*c*) **15.** (*b*) **16.** (*d*) **17.** (*c*) **18.** (*c*) **19.** (*a*) **20 .** (*b*)
21. (*b*) **22.** (*a*) **23.** (*c*) **24.** (*b*) **25 .** (*b*) **26.** (*d*) **27 .** (*c*) **28.** (*c*) **29.** (*b*) **30.** (*a*)
31. (*b*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*c*,*d*) **2 .** (*b*,*c*) **3.** (*a*,*b*) **4.** (*b*, *c*, *d*) **5.** (*b*, *d*) **6.** (*c*, *d*) **7.** (*a*,*c*) **8.** (*a*,*c*,*d*) **9.** (*a*, *b*, *c*) **10.** (*a*,*b*,*c*)
11. (*a*, *d*) **12.** (*b*,*c*) **13.** (*a*,*c*) **14.** (*a*, *c*) **15.** (*d*) **16.** (*c*) **17.** (*d*) **18.** (*d*) **19.** (*a*) **20.** (*b*)
21. (*a*) $\rightarrow$(t),(*b*) $\rightarrow$ (s),(*c*) $\rightarrow$(t),(*d*) $\rightarrow$(p) **22.** (*a*) $\rightarrow$(p),(*b*) $\rightarrow$(q),(*c*) $\rightarrow$(r)(*d*) $\rightarrow$(s)
23. (*a*) $\rightarrow$(p, q, r, s);(*b*) $\rightarrow$(p);(*c*) $\rightarrow$(P,R,S); (*d*) $\rightarrow$(p) **24.** (13) **25.** (*c*) **26.** (*c*) **27.** (*a*) **28.** (10)
29. (*b*)

JEE Mains & Advanced Past Years Questions

JEE-MAINS
PREVIOUS YEAR'S

1. (*a*) **2.** (*a*) **3.** (*a*) **4.** (*b*) **5.** (*c*) **6.** (*a*) **7.** (*c*) **8.** (*c*) **9.** (*c*) **10.** (*d*)
11. (*c*) **12.** (*a*) **13.** (*c*) **14.** (*a*)

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*b*) **2.** (*b*) **3.** [3] **4.** (*bcd*) **5.** (2) **6.** (*d*) **7.** (*a*) **8.** [0.01]

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

3 Limit of Function

DEFINITION

The concept of limit is used to discuss the behaviour of a function close to a certain point.

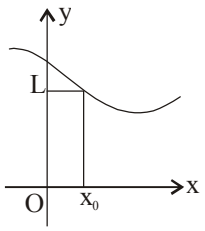
Meaning of $x \to x_0$: Let x is variable and x_0 is a constant, when x takes the values very close to x_0 we say that x approaches to x_0, but x is not equal to x_0. We denotes it by $\mathbf{x \to x_0}$

If x approach x_0 from right hand side then we denote it by $\mathbf{x \to x_0^+}$ and if x approach x_0 from left hand side then we denote it by $\mathbf{x \to x_0^-}$

For e.g. : $x \to 5^- \Rightarrow$ x can take values like 4.999, 4.9999, 4.999999 ..... similarly when $x \to 5^+ \Rightarrow$ x can take values like 5.01, 5.0001, 5.0000001 .....

Informal Definition of limits

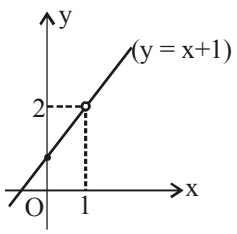
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that function approaches the limit L as x approaches x_0, and we write $\lim\limits_{x \to x_0} f(x) = L$.



This definition is "informal" because phrases like arbitrarily close and sufficiently close are imprecise, their meaning depends on the context.

The definition is clear enough and enables us to recognize and evaluate limits of specific function.

e.g., $f(x) = \dfrac{x^2 - 1}{x - 1}$



Clearly the function is not defined at $x = 1$, but for values close to $x = 1$ the function can be written as

$$f(x) = x + 1$$

As x approaches 1 (written as $x \to 1$), $f(x)$ approaches the value 2 (i.e., $f(x) \to 2$) we write this as

$$\lim_{x \to 1} f(x) = 2$$

Left hand limit and Right Hand limit

LHL : It is the value to which function f(x) as x approaches to x_0 from left hand side and we write it as $\lim\limits_{x \to x_0^-} f(x)$.

RHL : It is the value to which function f(x) as x approaches to x_0 from right hand side and we write it as $\lim\limits_{x \to x_0^+} f(x)$.

Existence of limit : $\lim\limits_{x \to x_0} f(x)$ is said to be exist and equal to L and if LHL = RHL = L

i.e. $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = L$ (a finite quantity).

e.g. f(x) = [x](greatest integer function)

For any integer n.

$\lim\limits_{x \to n^-} f(x) = n - 1$

$\lim\limits_{x \to n^+} f(x) = n$

In such cases we say that $\lim\limits_{x \to n} f(x)$ does not exist.

Formal definition of limit

Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself, we say that $f(x)$ approaches the limit L as x approaches x_0 and write $\lim\limits_{x \to x_0} f(x) = L$, if for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \in.$$

FUNDAMENTAL THEOREMS ON LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits. $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = K$

1. **Sum Rule :** $\lim_{x\to c}[f(x)+g(x)] = L+K$
2. **Difference Rule :** $\lim_{x\to c}[f(x)-g(x)] = L-K$
3. **Product Rule :** $\lim_{x\to c}[f(x)\,g(x)] = LK$
4. **Quotient Rule :** $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. **Constant Multiplication Rule :** $\lim_{x\to c}[b\,f(x)] = bL$
6. **Composition Rule :** $\underset{x\to a}{\text{Limit}}\ f(g(x)) = f\left(\underset{x\to a}{\text{Limit}}\ g(x)\right) = f(m)$; provided f is continuous at $x = m$.

   For example $\underset{x\to a}{\text{Limit}}\ \ell n(f(x)) = \ell n\left(\underset{x\to a}{\text{Limit}}\ f(x)\right)$
   $= \ell n\, l (l > 0)$.

TRAIN YOUR BRAIN

Example 1: Evaluate the following limits:

(i) $\ell\lim_{x\to 2}(x+2)$ (ii) $\ell\lim_{x\to 0}\cos(\sin x)$

Sol. (i) $x+2$ being a polynomial in x, its limit as $x \to 2$ is given by $\ell\lim_{x\to 2}(x+2) = 2+2 = 4$

(ii) $\ell\lim_{x\to 0}\cos(\sin x) = \cos\left(\ell\lim_{x\to 0}\sin x\right) = \cos 0 = 1$

Example 2: Evaluate $\lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right]$ (where [*] denotes the greatest integer function)

Sol. Let $P = \lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right]$

for $x > 0$, $\sin^{-1}x > x$

$\Rightarrow \frac{\sin^{-1}x}{x} > 1$

$\therefore$ R.H.L. $= \lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right] = 1$

for $x < 0$, $\sin^{-1}x > x$

$\Rightarrow \frac{\sin^{-1}x}{x} > 1$

$\therefore$ L.H.L. $= \lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right] = 1$

Hence $P = 1$.

INDETERMINATE FORMS

If on putting $x = a$ in f(x), any one of $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, \infty^0, 0^0, 1^\infty$ form is obtained, then the limit has an indeterminate form. All the above forms are interchangeable, i.e. we can change one form to other by suitable substitutions etc.

In such cases $\ell\lim_{x\to a} f(x)$ may exist.

Consider $f(x) = \frac{x^2-4}{x-2}$. Here $\ell\lim_{x\to 2} x^2-4 = 0$ and $\ell\lim_{x\to 2} x-2 = 0$

$\therefore$ $\ell\lim_{x\to 2} f(x)$ has an indeterminate form of the type $\frac{0}{0}$.

$\ell\lim_{x\to\infty}\frac{\ell n\,x}{x}$ has an indeterminate form of type $\frac{\infty}{\infty}$.

$\ell\lim_{x\to 0}(1+x)^{1/x}$ is an indeterminate form of the type 1^∞

Example 3: Which of the following limits are in indeterminant forms. Also indicate the form

(i) $\lim_{x\to 0}\frac{1}{x}$ (ii) $\lim_{x\to 1}\frac{1-x}{1-x^2}$

(iii) $\lim_{x\to 0} x\,\ell n\,x$ (iv) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2}\right)$

(v) $\lim_{x\to 0}(\sin x)^x$ (vi) $\lim_{x\to 0}(\ell n\,x)^x$

(vii) $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$ (viii) $\lim_{x\to 0}(1)^{\frac{1}{x}}$

Sol. (i) No (ii) Yes $\frac{0}{0}$ form
(iii) Yes $0\times\infty$ form (iv) Yes $(\infty-\infty)$ form
(v) Yes, 0^0 form (vi) Yes ∞^0 form
(vii) Yes 1^∞ form (viii) No

Remark:

(i) '0' doesn't means exact zero but represent a value approaching towards zero similarly to '1' and infinity.

(ii) $\infty+\infty = \infty$

(iii) $\infty\times\infty = \infty$

(iv) $(a/\infty) = 0$ if a is finite

(v) $\frac{a}{0}$ is not defined for any $a \in R$.

(vi) $a\,b = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

(vii) $\ell\lim_{x\to 0}\frac{x}{x}$ is an indeterminate form whereas $\ell\lim_{x\to 0}\frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function)

Students may remember these forms alongwith the prefix 'tending to'

i.e. $\frac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form whereas $\frac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

similarly (tending to one)$^{\text{tending to } \infty}$ is indeterminate form whereas (exactly one)$^{\text{tending to } \infty}$ is not an indeterminate form, its value is one.

To evaluate a limit, we must always put the value where 'x' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminate form comes, we have to remove the indeterminancy, once the indeterminancy is removed the limit can be evaluated by putting the value of x, where it is approaching.

METHODS OF REMOVING INDETERMINANCY

Basic methods of removing indeterminancy are

(*a*) Factorisation (*b*) Rationalisation
(*c*) Using standard limits (*d*) Substitution
(*e*) Expansion of functions.

Factorisation Method

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

TRAIN YOUR BRAIN

Example 4:

(i) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

(ii) $\lim\limits_{x\to 2} \left[\dfrac{1}{x-2}-\dfrac{2\,(2x-3)}{x^3-3x^2+2x}\right]$

Sol. (i) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3} = \lim\limits_{x\to 3} \dfrac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

(ii) $\lim\limits_{x\to 2} \left[\dfrac{1}{x-2}-\dfrac{2\,(2x-3)}{x^3-3x^2+2x}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{1}{x-2}-\dfrac{2\,(2x-3)}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{x^2-5x+6}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{x-3}{x(x-1)}\right] = -\dfrac{1}{2}$

Rationalisation Method

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Example 5: Evaluate:

(i) $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

(ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Sol. (i) $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

$= \lim\limits_{x\to 1} \dfrac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$

$= \lim\limits_{x\to 1} \dfrac{(15-15x)}{(3-3x)} \times \dfrac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}} = \dfrac{5}{2}$

(ii) The form of the given limit is $\dfrac{0}{0}$ when $x \to 0$. Rationalising the numerator, we get

$\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

$= \lim\limits_{x\to 0} \left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{(1+x)-(1-x)}{x\,(\sqrt{1+x}+\sqrt{1-x})}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{2x}{x\,(\sqrt{1+x}+\sqrt{1-x})}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \dfrac{2}{2} = 1$

Example 6: Evaluate $\lim\limits_{x\to 2} \dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$

Sol. $L = \lim\limits_{x\to 2} \dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}} \cdot \dfrac{(\sqrt{x^2-4}-\sqrt{x-2})}{(\sqrt{x^2-4}-\sqrt{x-2})}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x^2-4)-(x-2)}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{x^2-x-2}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x-2)(x+1)}$

$= \lim\limits_{x\to 2} \dfrac{\sqrt{x^2-4}-\sqrt{x-2}}{x+1} = 0.$

Example 7: Evaluate : $\lim_{x\to 0}\frac{\sqrt{(1+\sin 3x)}-1}{\ln(1+\tan 2x)}$

Sol. $\lim_{x\to 0}\frac{\sqrt{(1+\sin 3x)}-1}{\ln(1+\tan 2x)}\left(\text{form}\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{(\sqrt{(1+\sin 3x)}-1)(\sqrt{(1+\sin 3x)}+1)}{\ln(1+\tan 2x)(\sqrt{(1+\sin 3x)}+1)}$

$= \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{\tan 2x}{\ln(1+\tan 2x)}\cdot\lim_{x\to 0}\frac{3x}{\tan 2x}\cdot$

$\lim_{x\to 0}\frac{\tan 2x}{(\sqrt{1+\sin 3x)}+1)} = 1\,.\,1\,.\,\frac{3}{2}\cdot\frac{1}{(1+1)} = \frac{3}{4}.$

Example 8: Find $\lim_{n\to\infty}\left(\frac{2n^3}{2n^2+3}+\frac{1+5n^2}{5n+1}\right)$

Sol. $\lim_{n\to\infty}\left(\frac{2n^3-13n^2+3}{10n^3+2n^2+15n+3}\right)$

$\lim_{n\to\infty}\left(\frac{2-13/n+3/n^3}{10+2/n+15/n^2+3/n^3}\right) = \frac{1}{5}$

Example 9: Find $\lim_{n\to\infty}\frac{\sqrt{n^2+1}+\sqrt{n}}{\sqrt[4]{n^4+1}-\sqrt{n}}$

Sol. $\lim_{n\to\infty}\frac{(n)\left[\sqrt{1+1/n^2}+\sqrt{1/n}\right]}{n\left(\sqrt{1+1/n^4}-\sqrt{1/n}\right)}$

$= \lim_{n\to\infty}\frac{\left[\sqrt{1+1/n^2}+\sqrt{1/n}\right]}{\left(\sqrt{1+1/n^4}-\sqrt{1/n}\right)} = \frac{1+0}{1+0} = 1$

Example 10: Find $\lim_{x\to\infty}\left(\frac{1-2x}{(1+8x^3)^{1/3}}+2^{-x^7}\right)$

Sol. $\lim_{\substack{x\to\infty\\ 1/x\to 0}}\left(\frac{x\left(\frac{1}{x}-2\right)}{x\left(\frac{1}{x^3}+8\right)^{1/3}}+\frac{1}{2^{x^7}}\right) = \frac{-2}{(8)^{1/3}} = -1$

Standard Limits

(a) (i) $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

[Where x is measured in radians]

(ii) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$

(iii) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$; $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a$

(iv) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$; $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

(v) $\lim_{x\to 0}\frac{e^x-1}{x} = 1$; $\lim_{x\to 0}\frac{a^x-1}{x}$

$= \log_e a = \ell na \quad , a > 0$

(vi) $\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

(vii) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

(b) If $f(x)\to 0$, when $x\to a$, then

(i) $\lim_{x\to a}\frac{\sin f(x)}{f(x)} = 1$

(ii) $\lim_{x\to a}\cos f(x) = 1$

(iii) $\lim_{x\to a}\frac{\tan f(x)}{f(x)} = 1$

(iv) $\lim_{x\to a}\frac{e^{f(x)}-1}{f(x)} = 1$

(v) $\lim_{x\to a}\frac{b^{f(x)}-1}{f(x)} = \ell n\, b,\ (b>0)$

(vi) $\lim_{x\to a}\frac{\ell n(1+f(x))}{f(x)} = 1$

(vii) $\lim_{x\to a}(1+f(x))^{\frac{1}{f(x)}} = e$

(c) $\lim_{x\to a} f(x) = A > 0$ and $\lim_{x\to a}\phi(x) = B$(a finite quantity), then $\lim_{x\to a}[f(x)]^{\phi(x)} = A^B$.

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Example 11: Evaluate: $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$

Sol. $\lim_{x\to 0}\frac{(1+x)^n-1}{x} = \lim_{x\to 0}\frac{(1+x)^n-1}{(1+x)-1} = n$

Example 12: Evaluate : $\lim_{x\to 0}\frac{e^{3x}-1}{x/2}$

Sol. $\lim_{x\to 0}\frac{e^{3x}-1}{x/2} = \lim_{x\to 0}\ 2\times 3\ \frac{e^{3x}-1}{3x} = 6.$

Example 13: Evaluate : $\lim\limits_{x\to0}\ \dfrac{\tan x-\sin x}{x^3}$

Sol. $\lim\limits_{x\to0}\ \dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to0}\ \dfrac{\tan x(1-\cos x)}{x^3}$

$=\lim\limits_{x\to0}\ \dfrac{\tan x\,.\,2\sin^2\dfrac{x}{2}}{x^3}$

$=\lim\limits_{x\to0}\ \dfrac{1}{2}\cdot\dfrac{\tan x}{x}\cdot\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2=\dfrac{1}{2}.$

Example 14: Evaluate: $\lim\limits_{x\to0}\ \dfrac{\sin 2x}{\sin 3x}$

Sol. $\lim\limits_{x\to0}\ \dfrac{\sin 2x}{\sin 3x}=\lim\limits_{x\to0}\left[\dfrac{\sin 2x}{2x}\cdot\dfrac{2x}{3x}\cdot\dfrac{3x}{\sin 3x}\right]$

$=\left[\lim\limits_{2x\to0}\ \dfrac{\sin 2x}{2x}\right]\cdot\dfrac{2}{3}\cdot\left[\lim\limits_{3x\to0}\ \dfrac{3x}{\sin 3x}\right]$

$=1\,.\,\dfrac{2}{3}\times\left[\lim\limits_{3x\to0}\ \dfrac{\sin 3x}{3x}\right]=\dfrac{2}{3}\times1=\dfrac{2}{3}$

Example 15: Evaluate : $\lim\limits_{x\to\infty}\ \left(1+\dfrac{2}{x}\right)^x$

Sol. $\lim\limits_{x\to\infty}\ \left(1+\dfrac{2}{x}\right)^x=e^{\lim\limits_{x\to\infty}\frac{2}{x}.x}=e^2.$

Example 16: Show that $\lim\limits_{x\to0}\ \dfrac{\sin x(1-\cos x)}{x^3}=\dfrac{1}{2}$

Sol. $\lim\limits_{x\to0}\ \dfrac{\sin x(1-\cos x)}{x^3}=\lim\limits_{x\to0}\ \dfrac{\sin x\cdot2\sin^2\dfrac{x}{2}}{x^3}$

$=\lim\limits_{x\to0}\ 4\cos\dfrac{x}{2}\cdot\dfrac{\sin^3\dfrac{x}{2}}{x^3}=\lim\limits_{x\to0}\ 4(1)\left(\dfrac{1}{2}\right)^3=\dfrac{1}{2}$

Example 17: If $f(x)=\dfrac{3x-\sin^{-1}x}{4x-\tan^{-1}x}$, then $\lim\limits_{x\to0}\ f(x)$ is equal to

Sol. $\lim\limits_{x\to0}\ \dfrac{3x-\sin^{-1}x}{4x-\tan^{-1}x}=\lim\limits_{x\to0}\ \dfrac{3-\dfrac{\sin^{-1}x}{x}}{4-\dfrac{\tan^{-1}x}{x}}=\dfrac{3-1}{4-1}=\dfrac{2}{3}$

Use of Substitution in Solving Limit Problems

Sometimes in solving limit problem we convert $\lim\limits_{x\to a}\ f(x)$ into $\lim\limits_{h\to0}\ f(a+h)$ or $\lim\limits_{h\to0}\ f(a-h)$ according as need of the problem. (here h is approaching to zero.)

Limits Using Expansion

(a) $a^x=1+\dfrac{x\ \ell n\ a}{1!}+\dfrac{x^2\ \ell n^2a}{2!}+\dfrac{x^3\ \ell n^3a}{3!}+\ldots\ldots,\ a\ >\ 0$

(b) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\ldots$

(c) $\ell n\,(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots\ldots,\ \text{for}-1<x\le1$

(d) $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots\ldots$

(e) $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots\ldots$

(f) $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\ldots$

(g) $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots.$

(h) $\sin^{-1}x=x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+\ldots\ldots$

(i) $\sec^{-1}x=1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+\ldots\ldots$

(j) for $|x|<1,\ n\in R;\ (1+x)^n$

$=1+nx+\dfrac{n(n-1)}{1\,.\,2}x^2+\dfrac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3+\ldots\ldots\ldots\infty$

(k) $(1+x)^{\frac{1}{x}}=e\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-\ldots\ldots\ldots\right)$

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Example 18: Evaluate $\lim\limits_{x\to0}\ \dfrac{e^x-1-x}{x^2}$

Sol. $\lim\limits_{x\to0}\ \dfrac{e^x-1-x}{x^2}$

$=\lim\limits_{x\to0}\ \dfrac{\left(1+x+\dfrac{x^2}{2!}+\ldots\ldots\right)-1-x}{x^2}=\dfrac{1}{2}$

Example 19: Evaluate $\lim\limits_{x\to0}\ \dfrac{\tan x-\sin x}{x^3}$

Sol. $\lim\limits_{x\to0}\ \dfrac{\tan x-\sin x}{x^3}$

$=\lim\limits_{x\to0}\ \dfrac{\left(x+\dfrac{x^3}{3}+\ldots\ldots\right)-\left(x-\dfrac{x^3}{3!}+\ldots\ldots\right)}{x^3}$

$=\dfrac{1}{3}+\dfrac{1}{6}=\dfrac{1}{2}.$

Example 20: Evaluate $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)-\sin x+\frac{x^2}{2}}{x\tan x\sin x}$

Sol. $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)-\sin x+\frac{x^2}{2}}{x\tan x\sin x}$

$= \lim\limits_{x\to 0} \dfrac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-.....\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right)+\frac{x^2}{2}}{x^3\cdot\frac{\tan x}{x}\cdot\frac{\sin x}{x}}$

$= \dfrac{1}{3}+\dfrac{1}{6}=\dfrac{1}{2}$

Example 21: Evaluate $\lim\limits_{x\to 0} \dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}$

Sol. $\lim\limits_{x\to 0} \dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x} = \lim\limits_{x\to 0} \dfrac{e-e\left(1-\frac{x}{2}+.....\right)}{\tan x}$

$= \lim\limits_{x\to 0} \dfrac{e}{2}\times\dfrac{x}{\tan x}=\dfrac{e}{2}$

Example 22: Find $\underset{x\to 0}{\text{Lim}}\dfrac{e^{\sin x}-\sin x-1}{x^2}$

Sol. $\underset{x\to 0}{\text{Lim}}\dfrac{e^{\sin x}-\sin x-1}{x^2}$

$= \underset{x\to 0}{\text{Lim}}\dfrac{\left(1+\frac{\sin x}{\lfloor 1}+\frac{\sin^2 x}{\lfloor 2}+\frac{\sin^3 x}{\lfloor 3}+.....\right)-\sin x-1}{x^2}$

$= \underset{x\to 0}{\text{Lim}}\left(\dfrac{\sin^2 x}{2!(x^2)}+\dfrac{\sin^3 x}{3!(x^2)}+...\right)$

$= \dfrac{1}{2}$

Example 23: Evaluate $\underset{x\to 0}{\text{Limit}}\dfrac{3\sin x-\sin 3x}{x-\sin x}$

Sol. $\underset{x\to 0}{\text{Limit}}\dfrac{4\sin^3 x}{x-\sin x} = \underset{x\to 0}{\text{Limit}}\dfrac{4x^3}{x-\sin x} = \underset{x\to 0}{\text{Limit}}$

$\dfrac{4x^3}{x-(x-\frac{x^3}{3!}....)}=24.$

Example 24: Evaluate $\underset{x\to 0}{\text{Lim}}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$.

Sol. Let $L=\underset{x\to 0}{\text{Lim}}\dfrac{e^x-e^{-x}-2x}{x^3}\cdot\dfrac{x-\sin x}{x^3}$

Now $\ell_1=\underset{x\to 0}{\text{Lim}}\dfrac{x-\sin x}{x^3}=\dfrac{1}{6}$ (use $x=3t$)

$\ell_2=\underset{x\to 0}{\text{Lim}}\dfrac{e^x-e^{-x}-2x}{x^3}$ put $x=3y$

$= \underset{y\to 0}{\text{Limit}}\dfrac{e^{3y}-e^{-3y}-6y}{27\,y^3}$

$= \underset{y\to 0}{\text{Limit}}\dfrac{(e^y-e^{-y})^3+3(e^y-e^{-y})-6y}{27\,y^3}$

$\left[\underset{y\to 0}{\text{Limit}}\dfrac{8}{27}\dfrac{(e^{2y}-1)^3}{(2y)^8}+\dfrac{1}{9}\ell_2\right]$

$\ell_2=\dfrac{8}{27}+\dfrac{\ell_2}{9}\Rightarrow\ell_2=\dfrac{1}{3}$

Hence $L=\dfrac{\ell_1}{\ell_2}=1/2$

Limit when $x\to\infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x\to\infty$, we use to substitute $y=\dfrac{1}{x}$ and in this case $y\to 0^+$.

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Example 25: Evaluate $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x}$

Sol. $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x}=\lim\limits_{x\to\infty}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}=1$

Example 26: Evaluate $\lim\limits_{x\to\infty}\dfrac{x-2}{2x-3}$

Sol. $\lim\limits_{x\to\infty}\dfrac{x-2}{2x-3}=\lim\limits_{x\to\infty}\dfrac{1-\frac{2}{x}}{2-\frac{3}{x}}=\dfrac{1}{2}.$

Example 27: Evaluate $\lim\limits_{x\to\infty}\dfrac{x^2-4x+5}{3x^2-x^3+2}$

Sol. $\lim\limits_{x\to\infty} \dfrac{x^2-4x+5}{3x^2-x^3+2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}-\frac{4}{x^2}+\frac{5}{x^3}}{\frac{3}{x}-1+\frac{2}{x^3}} = 0$

Example 28: Evaluate $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Sol. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$ (Put $x=-\dfrac{1}{t}$, $x\to-\infty \Rightarrow t\to 0^+$)

$= \lim\limits_{t\to 0^+} \dfrac{\sqrt{3+2t^2}.\frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim\limits_{t\to 0^+} \dfrac{\sqrt{3+2t^2}}{-(1+2t)} \dfrac{t}{|t|}$

$= \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$.

Some Important Notes

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$

(ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$

(iii) $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$ (iv) $\lim\limits_{x\to\infty} \dfrac{(\ell nx)^n}{x} = 0$

(v) $\lim\limits_{x\to 0^+} x(\ell nx)^n = 0$

As $x\to\infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ and $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h\to 0^+$.

Example 29: Evaluate $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x}$

Sol. $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x} = 0$

Limits of form 1^∞, 0^0, ∞^0

(A) All these forms can be converted into $\dfrac{0}{0}$ form in the following ways

(a) If $x\to 1, y\to\infty$, then $z=(x)^y$ is of 1^∞ form
$\Rightarrow \ell n\, z = y\,\ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{\ell nx}{\frac{1}{y}}$ $\left(\dfrac{0}{0}\text{ form}\right)$

As $y\to\infty \Rightarrow \dfrac{1}{y}\to 0$ and $x\to 1 \Rightarrow \ell nx\to 0$

(b) If $x\to 0, y\to 0$, then $z=x^y$ is of (0^0) form
$\Rightarrow \ell n\, z = y\,\ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}}$ $\left(\dfrac{0}{0}\text{ form}\right)$

(c) If $x\to\infty, y\to 0$, then $z=x^y$ is of $(\infty)^0$ form
$\Rightarrow \ell n\, z = y\,\ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}}$ $\left(\dfrac{0}{0}\text{ form}\right)$

(B) $(1)^\infty$ type of problems can be solved by the following method

(a) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\lim\limits_{x\to a} [f(x)]^{g(x)}$; where $f(x)\to 1$; $g(x)\to\infty$ as $x\to a$

$= \lim\limits_{x\to a} [1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)}$

$= \lim\limits_{x\to a} \left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)g(x)}$

$= e^{\lim\limits_{x\to a}[f(x)-1]g(x)}$

Some More Standard forms

These standard forms are used in case $f(x)\to 0$ when $x\to a$.

(i) $\lim\limits_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

(ii) $\lim\limits_{x\to a} \dfrac{b^{f(x)}-1}{f(x)} = \log_e b \quad (b>0)$

(iii) $\lim\limits_{x\to a} \dfrac{\log(1+f(x))}{f(x)} = 1$

(iv) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}$

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Example 30: Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Sol. Since it is in the form of 1^∞

$\lim\limits_{x\to\infty} \left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

$= \exp\left(\lim\limits_{x\to\infty}\left(\dfrac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)\right) = e^{-8}$

Example 31: Evaluate $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Sol. Since it is in the form of 1^∞ so $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$

$= e^{\frac{2\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \dfrac{1}{e}$

Example 32: Evaluate $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Sol. $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put $x=a+h$

$$=\lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)}$$

$$=\lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)}=e^{\lim\limits_{h\to 0}-\cot\frac{\pi h}{2a}\cdot\left(1+\frac{h}{a+h}-1\right)}$$

$$=e^{\lim\limits_{h\to 0}-\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right)\cdot\frac{2a}{\pi}\cdot\frac{h}{a+h}}=e^{-\frac{2}{\pi}}$$

Example 33: Evaluate $\lim\limits_{x\to 0^+}x^x$

Sol. Let $y=\lim\limits_{x\to 0^+}x^x$

$$\Rightarrow \ell n\, y=\lim_{x\to 0^+}x\,\ell n\,x$$

$$=\lim_{x\to 0^+}-\frac{\ell n\frac{1}{x}}{\frac{1}{x}}=0,$$

as $\frac{1}{x}\to\infty\Rightarrow y=1$

Example 34: Evaluate $\lim\limits_{x\to 0}\frac{x\cdot 2^x-x}{1-\cos x}$

Sol. This is $\frac{0}{0}$ form

The given limit

$$=\lim_{x\to 0}\frac{x^2(2^x-1)}{\left(2\sin^2\frac{x}{2}\right)x}=\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\frac{1}{2}\left(\frac{x}{\sin\frac{x}{2}}\right)^2$$

$$=\log_e 2\times\frac{1}{2}\left(\frac{1}{1/2}\right)^2=2\log_e 2=\log_e 4$$

Example 35: Evaluate $\lim\limits_{t\to 0}\frac{\ell n\cos(\sin t)}{t^2}$

Sol. $\lim\limits_{t\to 0}\frac{\ell n\cos(\sin t)}{t^2}=\lim\limits_{t\to 0}\frac{\ell n\left(1-2\sin^2\frac{(\sin t)}{2}\right)}{-2\sin^2\left(\frac{\sin t}{2}\right)}\cdot\frac{\left(-2\sin^2\left(\frac{\sin t}{2}\right)\right)}{t^2}$

$$=\lim_{t\to 0}-2.\left(\frac{\sin\left(\frac{\sin t}{2}\right)}{\frac{\sin t}{2}}\right)^2\cdot\frac{\sin^2 t}{4.t^2}=-\frac{1}{2}$$

Example 36: Find $\lim\limits_{x\to\pi/4}\frac{\ln\tan x}{1-\cot x}$

Sol. Put $x=t+\pi/4\Rightarrow\lim\limits_{t\to 0}\frac{\ln\tan(t+\pi/4)}{1-\cot(t+\pi/4)}$

$$=\lim_{t\to 0}\frac{\ln\left[\frac{1+\tan t}{1-\tan t}\right]}{1-\left[\frac{\cot t\cot\pi/4-1}{\cot t+\cot\pi/4}\right]}$$

$$=\lim_{t\to 0}\frac{\ln(1+\tan t)}{\frac{2\tan t}{t+\tan t}}+\lim_{t\to 0}\frac{\ln(1-\tan t)(1+\tan t)}{-2\tan t}$$

$$=\frac{1}{2}[1.1+1.1]=\frac{2}{2}=1.$$

Example 37: Evaluate $\lim\limits_{x\to 0}\frac{\ell n\left(1+x^2+x^4\right)}{\left(e^x-1\right).x}$.

Sol. $\lim\limits_{x\to 0}\frac{\ell n\left(1+x^2+x^4\right)}{x^2}=\lim\limits_{x\to 0}\ell n\,(1+x^2+x^4)^{\frac{1}{x^2}}$

$$=\ell n\lim_{x\to 0}e^{\frac{1}{x^2}\left(1+x^2+x^4-1\right)}=\lim_{x\to 0}\frac{x^2+x^4}{x^2}$$

$$=\lim_{x\to 0}(1+x^2)=1$$

Example 38: Evaluate $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)$

Sol. Here, $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1-r^2+r^4}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1+(r^2-r)(r^2+r)}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r^2+r)-(r^2-r)}{1+(r^2+r)(r^2-r)}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\{\tan^{-1}(r^2+r)-\tan^{-1}(r^2-r)\}$$

$$=\lim_{n\to\infty}[(\tan^{-1}2-\tan^{-1}0)+(\tan^{-1}6-\tan^{-1}2)+(\tan^{-1}12-$$

$$\tan^{-1}6)+.....+\tan^{-1}(n^2+n)-\tan^{-1}(n^2-n)]$$

$$=\lim_{n\to\infty}\{\tan^{-1}(n^2+n)-\tan^{-1}(0)\}$$

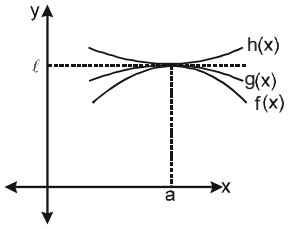
$$=\tan^{-1}(\infty)-\tan^{-1}(0)=\frac{\pi}{2}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)=\frac{\pi}{2}$$

SANDWITCH THEOREM OR SQUEEZE PLAY THEOREM

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that

$\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$, Then $\lim_{x\to a} g(x) = \ell$.



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Example 39: Evaluate $\lim_{n\to\infty}\frac{[x]+[2x]+[3x]+....+[nx]}{n^2}$, where [.] denotes greatest integer function.

Sol. We know that, $x-1 \le [x] \le x$; $2x-1 \le [2x] \le 2x$;
$3x-1 \le [3x] \le 3x$; ..... $nx-1 \le [nx] \le nx$

$\therefore (x+2x+3x+....+nx)-n<[x]+[2x]+.....+[nx]\le(x+2x+....+nx)$

$$\Rightarrow \frac{xn(n+1)}{2}-n \le \sum_{r=1}^{n}[r\,x] \le \frac{x.n(n+1)}{2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n}$$

$$\le \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2}\le \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)$$

$$\Rightarrow \frac{x}{2} \le \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2}\le\frac{x}{2}$$

$$\therefore \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2}=\frac{x}{2}$$

2nd Method

We know that $[x] = x - \{x\}$

$$\sum_{r=1}^{n}[r\,x] = [x]+[2x]+....+[nx]$$

$$=(x+2x+3x+...+nx)-(\{x\}+\{2x\}+....+\{nx\})$$

$$=\frac{xn(n+1)}{2}-(\{x\}+\{2x\}+..+\{nx\})$$

$$\therefore \frac{1}{n^2}\sum_{r=1}^{n}[r\,x]$$

$$=\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{\{x\}+\{2x\}+....+\{nx\}}{n^2}$$

Since, $0 \le \{rx\} < 1, \therefore 0 \le \sum_{r=1}^{n}\{r\,x\} < n$

$$\Rightarrow \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2}=0$$

$$\therefore \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}$$

$$=\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=\frac{x}{2}$$

TRAIN YOUR BRAIN

Example 40: Let a function f(x) be such that $|f(x)| \le M$ for any $x \ne 0$. Prove that $\lim_{x\to0} x\,f(x)=0$

Sol. We have $|x\,f(x)| \le |x|.M \Rightarrow -M|x| \le x\,f(x) \le M|x|$ for any $x \ne 0$.

Since $\lim_{x\to0} M|x| = 0$ and $\lim_{x\to0} -M|x| = 0$, by Sandwich theorem, $\lim_{x\to0} x\,f(x)=0$.

Topicwise Questions

Existence of Limit

1. $\lim_{x\to 0}\frac{|x|}{x}=$

(*a*) 1 (*b*) –1
(*c*) 0 (*d*) Does not exist

2. If *n* is an integer, then $\lim_{x\to n^+}(x-[n])=$

(*a*) 0 (*b*) 1
(*c*) –1 (*d*) Does not exist

Algebric Methods, Substitution, factorization & Rationalization

3. $\lim_{x\to 1}\frac{\log x}{x-1}=$

(*a*) 1 (*b*) –1
(*c*) 0 (*d*) ∞

4. If $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=80$, where *n* is a positive integer, then $n=$

(*a*) 3 (*b*) 5
(*c*) 2 (*d*) 1

5. $\lim_{x\to 0}\frac{1-\cos 2x}{x}=$

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 4

6. If $\lim_{x\to 0} kx\,\text{cosec}\,x=\lim_{x\to 0} x\,\text{cosec}\,kx$, then $k=$

(*a*) 1 (*b*) –1
(*c*) ± 1 (*d*) ± 2

7. $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}=$

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) –2

8. $\lim_{x\to 0}\frac{x\left(e^x-1\right)}{1-\cos x}=$

(*a*) 0 (*b*) ∞
(*c*) –2 (*d*) 2

Trigonometric limit and exponential limit and standard limits and 1^∞ form.

9. $\lim_{x\to 0}\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}=$

(*a*) 1/2 (*b*) 1/4
(*c*) 2 (*d*) 4

10. $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=$

(*a*) 1/2 (*b*) –1/2
(*c*) 1 (*d*) –1

11. $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=$

(*a*) –1 (*b*) 1
(*c*) 2 (*d*) –2

12. $\lim_{x\to \pi/2}\left(\sec\theta-\tan\theta\right)=$

(*a*) 0 (*b*) 1/2
(*c*) 2 (*d*) ∞

13. $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}=$

(*a*) $\frac{5}{3}(a+2)^{2/3}$ (*b*) $\frac{5}{3}(a+2)^{5/3}$

(*c*) $\frac{5}{3}a^{2/3}$ (*d*) $\frac{5}{3}a^{5/3}$

14. $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}=$

(*a*) $a\cos a+a^2\sin a$ (*b*) $a\sin a+a^2\cos a$

(*c*) $2a\sin a+a^2\cos a$ (*d*) $2a\cos a+a^2\sin a$

15. $\lim_{x\to 3}\left\{\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\right\}=$

(*a*) 1 (*b*) 2
(*c*) –1 (*d*) –2

16. $\lim_{x\to \pi/4}\frac{\sqrt{2}\cos x-1}{\cot x-1}=$

(*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\frac{1}{2}$

(*c*) $\frac{1}{2\sqrt{2}}$ (*d*) 1

17. $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=$

(*a*) 1 (*b*) –1
(*c*) 0 (*d*) limit does not exist.

18. $\lim_{x\to\infty}\left[1+\frac{1}{mx}\right]^x$ equal to

(*a*) $e^{1/m}$ (*b*) $e^{-1/m}$

(*c*) e^m (*d*) m^e

19. The value of the limit of $\frac{x^3-8}{x^2-4}$ as x tends to 2 is

(*a*) 3 (*b*) $\frac{3}{2}$

(*c*) 1 (*d*) 0

20. The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(*a*) 3 (*b*) 9

(*c*) 18 (*d*) 21

21. $\lim_{x\to\infty}\frac{2x^2+3x+4}{3x^2+3x+4}$ is equal to

(*a*) $\frac{2}{3}$ (*b*) 1

(*c*) 0 (*d*) ∞

22. The value of $\lim_{x\to 2}\frac{3^{x/2}-3}{3^x-9}$ is

(*a*) 0 (*b*) 1/3

(*c*) 1/6 (*d*) in 3

23. $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}, n\in N,$ ([x] denotes greatest integer less than or equal to x)

(*a*) Has value -1

(*b*) Has value 0

(*c*) Has value 1

(*d*) Does not exist

24. If $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k,$ then the value of k is

(*a*) 0 (*b*) $-\frac{1}{3}$

(*c*) $\frac{2}{3}$ (*d*) $-\frac{2}{3}$

25. $\lim_{\theta\to\frac{\pi}{2}}\frac{\frac{\pi}{2}-\theta}{\cot\theta}=$

(*a*) 0 (*b*) -1

(*c*) 1 (*d*) ∞

26. The value of $\lim_{n\to\infty}\frac{1-n^2}{\sum n}$ will be

(*a*) -2 (*b*) -1

(*c*) 2 (*d*) 1

Expansion

27. $\lim_{x\to 0}\frac{\log_e(1+x)}{3^x-1}$

(*a*) $\log_e 3$ (*b*) 0

(*c*) 1 (*d*) $\log_3 e$

28. $\lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{x+b}=$

(*a*) 1 (*b*) e^{b-a}

(*c*) e^{a-b} (*d*) e^b

L' Hospital Rule

29. If $f(a)=2, f'(a)=1, g(a)=-1; g'(a)=2$, then $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}=$

(*a*) 3 (*b*) 5

(*c*) 0 (*d*) -3

Sandwich Theorem

30. $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+.....+\frac{n}{1-n^2}\right]$ is equal to

(*a*) 0 (*b*) $-\frac{1}{2}$

(*c*) $\frac{1}{2}$ (*d*) 2

Learning Plus

1. $\ell\lim_{x\to 0}\sin^{-1}(\sec x)$ is equal to

(*a*) $\frac{\pi}{2}$ (*b*) 1

(*c*) zero (*d*) None of these

2. $\lim_{x\to\infty}\sec^{-1}\left(\frac{x}{x+1}\right)=$

(*a*) 0 (*b*) π

(*c*) $\frac{\pi}{2}$ (*d*) does not exist

3. $\lim_{x\to 0}\left(\cot\left(\frac{\pi}{4}+x\right)\right)^{\text{cosec } x}=$

(*a*) e^{-1} (*b*) e^2

(*c*) e^{-2} (*d*) e^1

4. $\ell\lim_{x\to 1}\frac{\left(\sum_{k=1}^{100}x^k\right)-100}{x-1}$ is equal to

(*a*) 0 (*b*) 5050

(*c*) 4550 (*d*) -5050

5. $\lim_{n\to\infty}$ $[(1+x)(1+x^2)(1+x^4).....(1+x^{2^n})]$ if $|x|<1$ has the value equal to

(*a*) 0 (*b*) 1

(*c*) $1-x$ (*d*) $(1-x)^{-1}$

6. $\lim_{x\to 3} \frac{(x^3+27)\,\ell n(x-2)}{(x^2-9)}$ is equal to

(*a*) -8 (*b*) 8

(*c*) 9 (*d*) -9

7. The value of $\underset{x\to 0}{\text{Limit}} \frac{\sin(l\,n(1+x))}{l\,n(1+\sin x)}$ is

(*a*) 0 (*b*) 1/2

(*c*) 1/4 (*d*) 1

8. $\underset{x\to\pi/2}{\text{Limit}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right]$ is (where [*] denotes greatest integer function)

(*a*) –1 (*b*) 0

(*c*) –2 (*d*) does not exist

9. $\underset{x\to\infty}{\text{Limit}} \frac{x^3.\sin\frac{1}{x}+x+1}{x^2+x+1}=$

(*a*) 0 (*b*) 1/2

(*c*) 1 (*d*) None of these

10. $\underset{n\to\infty}{\text{Limit}} \frac{-3n+(-1)^n}{4n-(-1)^n}$ is

(*a*) $-\frac{3}{4}$

(*b*) $-\frac{3}{4}$ if n is even; $\frac{3}{4}$ if n is odd

(*c*) not exist if n is even ; $-\frac{3}{4}$ if n is odd

(*d*) + 1 if n is even ; does not exist if n is odd

11. $\lim_{n\to\infty}$ $n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$ is equal to:

(*a*) $\frac{\pi}{3}$ (*b*) $\frac{\pi}{4}$

(*c*) $\frac{\pi}{6}$ (*d*) None of these

12. $\lim_{n\to\infty} \frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$ is equal to

(*a*) 5 (*b*) 3

(*c*) 1 (*d*) zero

13. $\lim_{x\to 0}\left[\frac{\sin[x-3]}{[x-3]}\right]$ (where [*] denotes greatest integer function)

(*a*) 0 (*b*) 1

(*c*) does not exist (*d*) sin 1

14. $\underset{x\to\infty}{\text{Limit}}\left(\frac{x+2}{x-2}\right)^{x+1}=$

(*a*) e^4 (*b*) e^{-4}

(*c*) e^2 (*d*) None of these

15. $\underset{x\to 0^+}{\text{Limit}}$ $(1+\tan^2\sqrt{x})^{5/x}=$

(*a*) e^5 (*b*) e^2

(*c*) e (*d*) None of these

16. $\lim_{x\to 0} \frac{\sin(6x^2)}{\ell n\cos(2x^2-x)}$ is equal to

(*a*) 12 (*b*) – 12

(*c*) 6 (*d*) – 6

17. $\lim_{x\to\infty}\left(x-x^2\,\ell n\left(1+\frac{1}{x}\right)\right)$ is equal to :

(*a*) $\frac{1}{2}$ (*b*) $\frac{3}{2}$

(*c*) $\frac{1}{3}$ (*d*) 1

18. $\underset{x\to\infty}{\text{Limit}}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x=$

(*a*) 1 (*b*) 2

(*c*) e^2 (*d*) e

19. The limiting value of $(\cos x)^{\frac{1}{\sin x}}$ as $x\to 0$ is:

(*a*) 1 (*b*) e

(*c*) 0 (*d*) none of these

20. Let $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$, then $\lim_{x\to\infty} f(x)$ equals

(*a*) 0 (*b*) –1/2

(*c*) 1 (*d*) None of these

21. $\underset{x\to\pi/2}{\text{Lim}} \frac{2^{-\cos x}-1}{x(x-\pi/2)}=$

(*a*) $\frac{2\ell n2}{\pi}$ (*b*) $\ell n2$

(*c*) $\frac{2}{\pi}$ (*d*) does not exists

22. $\lim_{x\to 0} (\cos mx)^{n/x^2}$

(*a*) $e^{-m^2n/4}$ (*b*) $e^{-m^2n/2}$

(*c*) $e^{-mn^2/2}$ (*d*) $e^{-mn^2/4}$

23. The limit $\lim_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}}{x^2}$ is equal to

(*a*) 1/2 (*b*) –1/2

(*c*) 3/2 (*d*) 1

24. The value of $\underset{x\to 0}{\text{Limit}} \frac{\cos(\sin x)-\cos x}{x^4}$ is equal to

(*a*) 1/5 (*b*) 1/6

(*c*) 1/4 (*d*) 1/2

25. If $\underset{x\to\infty}{\text{Lim}}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$, then the values of a & b are

(*a*) $a \in R, b \in R$ (*b*) $a = 1, b \in R$

(*c*) $a \in R, b = 2$ (*d*) $a = 1, b = 2$

26. $\underset{x\to\infty}{\text{Lim}}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$ is equal to

(*a*) e^4 (*b*) e^2

(*c*) e^3 (*d*) e

27. $\lim_{x\to 0} \frac{\sin[\cos x]}{1+[\cos x]}$ (where [*] denotes greatest integer function)

(*a*) equal to 1 (*b*) sin 1

(*c*) equal to zero (*d*) non existent

28. $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$ is

(*a*) e (*b*) e^2

(*c*) 1/e (*d*) does not exist

29. $\ell\underset{x\to 0}{\text{im}} \frac{e^{-\frac{x^2}{2}}-\cos x}{x^3\sin x}$ is equal to

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{6}$

(*c*) $\frac{1}{12}$ (*d*) $\frac{1}{8}$

30. Evaluate

$$\underset{n\to\infty}{\text{Limit}}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+....+\frac{1}{\sqrt{n^2+2n}}\right)$$

(*a*) 1 (*b*) 1/2

(*c*) 0 (*d*) 2

Advanced Level Multiconcept Questions

JEE-ADVANCED MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. Let $f(x) = \frac{x^2-9x+20}{x-[x]}$ (where [x] denotes greatest integer less than or equal to x), then

(*a*) $\ell\underset{x\to 5^-}{\text{im}}\ f(x) = 0$ (*b*) $\ell\underset{x\to 5^+}{\text{im}}\ f(x) = 1$

(*c*) $\ell\underset{x\to 5}{\text{im}}\ f(x)$ does not exist (*d*) None of these

2. If $f(x) = \frac{\sqrt{x^2+2}}{3x-6}$, then

(*a*) $\ell\underset{x\to-\infty}{\text{im}}\ f(x) = -\frac{1}{3}$ (*b*) $\ell\underset{x\to\infty}{\text{im}}\ f(x) = \frac{1}{3}$

(*c*) $\ell\underset{x\to-\infty}{\text{im}}\ f(x) = \frac{1}{3}$ (*d*) $\ell\underset{x\to\infty}{\text{im}}\ f(x) = -\frac{1}{3}$

3. $\ell\underset{x\to\infty}{\text{im}} \frac{(ax+1)^n}{x^n+A}$ is equal to

(*a*) a^n if $n \in N$

(*b*) ∞ if $n \in Z^-$ & $a = A = 0$

(*c*) $\frac{1}{1+A}$ if $n = 0$

(*d*) a^n if $n \in Z^-$, $A = 0$ & $a \neq 0$

4. Let $f(x) = \begin{cases} 1+\frac{2x}{a}, & 0 \le x < 1 \\ ax, & 1 \le x < 2 \end{cases}$, if $\ell\underset{x\to 1}{\text{im}}\ f(x)$ exists, then value of a is :

(*a*) 1 (*b*) – 1

(*c*) 2 (*d*) – 2

5. If $f(x) = \frac{\cos 2-\cos 2x}{x^2-|x|}$, then

(*a*) $\ell\underset{x\to -1}{\text{im}}\ f(x) = 2\sin 2$ (*b*) $\ell\underset{x\to 1}{\text{im}}\ f(x) = 2\sin 2$

(*c*) $\ell\underset{x\to -1}{\text{im}}\ f(x) = 2\cos 2$ (*d*) $\ell\underset{x\to 1}{\text{im}}\ f(x) = 2\cos 2$

6. If $\ell = \lim\limits_{x\to 0} \dfrac{x(1+a\cos x)-b\sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{1+a\cos x}{x^2} - \lim\limits_{x\to 0} \dfrac{b\sin x}{x^3}$, where $\ell \in R$, then

(*a*) (a, b) = (–1, 0)
(*b*) a & b are any real numbers
(*c*) $\ell = 0$
(*d*) $\ell = \dfrac{1}{2}$

7. If $\lim\limits_{x\to 0} (\cos x + a\sin bx)^{\frac{1}{x}} = e^2$, then the possible values of 'a' & 'b' are :

(*a*) a = 1, b = 2 (*b*) a = 2, b = 1
(*c*) a = 3, b = 2/3 (*d*) a = 2/3, b = 3

8. If $f(x) = |x|^{\sin x}$, then

(*a*) $\lim\limits_{x\to 0^-} f(x) = 1$ (*b*) $\lim\limits_{x\to 0^+} f(x) = 1$

(*c*) $\lim\limits_{x\to 0} f(x) = 1$ (*d*) limit does not exist

9. If $\lim\limits_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = p$ (finite), then

(*a*) a = – 2 (*b*) a = – 1
(*c*) p = – 2 (*d*) p = – 1

10. Let $f(x) = \dfrac{|x+\pi|}{\sin x}$, then

(*a*) $f(-\pi^+) = -1$
(*b*) $f(-\pi^-) = 1$
(*c*) $\lim\limits_{x\to -\pi} f(x)$ does not exist
(*d*) $\lim\limits_{x\to \pi} f(x)$ does not exist

Comprehension–01(11to 13)

Consider $f(x) = \dfrac{\sin x + ae^x + be^{-x} + c\,\ell n(1+x)}{x^3}$, where a, b, c are real numbers.

11. If $\lim\limits_{x\to 0^+} f(x)$ is finite, then the value of a + b + c is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) –2

12. If $\lim\limits_{x\to 0^+} f(x) = \ell$ (finite), then the value of ℓ is

(*a*) – 2 (*b*) $-\dfrac{1}{2}$

(*c*) – 1 (*d*) $-\dfrac{1}{3}$

13. Using the values of a, b, c as found above, the value of $\lim\limits_{x\to 0^+} x\,f(x)$ is

(*a*) 0 (*b*) $\dfrac{1}{2}$

(*c*) $-\dfrac{1}{2}$ (*d*) 2

Comprehension–02(14 to 16)

Let $f(x) = \lim\limits_{n\to\infty} \left(1 - \sin x + \sqrt[n]{e}\sin x\right)^n$, $n \in N$

$$a = \frac{2}{11}\lim_{x\to 0}\left(\left[\frac{\sin x}{x}\right]+\left[\frac{2\sin x}{x}\right]+\left[\frac{3\sin x}{x}\right]+\ldots\ldots+\left[\frac{11\sin x}{x}\right]\right)$$

$$b = \lim_{x\to 0}\left(\frac{x^2}{\left[\frac{\tan x}{x}\right]-\cos x}\right)$$

[**Note:** [k] denotes greatest integer function less than or equal to k.]

14. The value of (a + b) is equal to

(*a*) 2 (*b*) 6
(*c*) 10 (*d*) 12

15. Number of integral values of λ so that the equation $bx^2 - b^2x + \lambda = 0$ has roots α, β such that $1 < \alpha < 2$ and $2 < \beta < 3$ is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

16. If $\lim\limits_{u\to 0} [1 + u\,ln(1+k^2)]^{\frac{1}{u}} = 2k\ ln^2 f(x)$, $k > 0$ and $x \in (0, \pi)$, then the value of (x + k) is equal to

(*a*) $1 + \dfrac{\pi}{4}$ (*b*) $1 + \dfrac{\pi}{2}$

(*c*) 1 (*d*) $\dfrac{\pi}{2}$

17. Match the column:

Let $\phi(x) = \dfrac{a_0x^m + a_1x^{m+1} + \ldots.. + a_kx^{m+k}}{b_0x^n + b_1x^{n+1} + \ldots. + b_\ell x^{n+\ell}}$, where $a_0 \neq 0$, $b_0 \neq 0$ and m, n $\in$ N, then $\lim\limits_{x\to 0} \phi(x)$ is equal to

Column – I	**Column – II**
(*a*) m > n	(p) ∞
(*b*) m = n	(q) $-\infty$
(*c*) m < n and n – m is even, $\dfrac{a_0}{b_0} > 0$	(r) $\dfrac{a_0}{b_0}$
(*d*) m < n and n – m is even, $\dfrac{a_0}{b_0} < 0$	(s) 0

18. Match the column:

$\lim_{x\to 0} f(x)$, where f(x) is as in **COLUMN - I,** is

Column – I	**Column – II**
(*a*) $f(x)=\dfrac{\tan[e^2]x^2-\tan[-e^2]x^2}{\sin^2 x}$, where [.] represents greatest integer function.	(p) $\dfrac{\sqrt{2}}{8}$
(*b*) $f(x)=\left[\left(\min(t^2+4t+6)\right)\dfrac{\sin x}{x}\right]$, where [.] represents greatest integer function.	(q) 15
(*c*) $f(x)=\dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$	(r) 1
(*d*) $f(x)=\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$	(s) $\dfrac{1}{2}$

NUMERICAL BASED QUESTIONS

19. Let $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$, then $\lim_{x\to\infty} f(x)$ is equal to

20. $\lim_{x\to 0}\left(\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}+\sqrt[3]{\dfrac{1+\cos^3 x+3\cos^2 x+3\cos x}{\cos x+63}}\right)$ is equal to

21. If $\lim_{x\to\infty} f(x)$ exists and is finite and nonzero and $\lim_{x\to\infty}\left(f(x)+\dfrac{3f(x)-1}{f^2(x)}\right)=3$, then the value of $\lim_{x\to\infty} f(x)$ is equal to

22. If $f(x)=\begin{cases} x-1, & x\geq 1 \\ 2x^2-2, & x<1 \end{cases}$, $g(x)=\begin{cases} x+1, & x>0 \\ -x^2+1, & x\leq 0 \end{cases}$ and $h(x)=|x|$, then $\lim_{x\to 0} f(g(h(x)))$ is equal to

23. If $f(x)=\begin{cases} \sin x & ,x\neq n\pi,\ n=0,\pm 1,\pm 2,..... \\ 2 & ,\text{otherwise} \end{cases}$

and $g(x)=\begin{cases} x^2+1 & ,x\neq 0,2 \\ 4 & ,x=0 \\ 5 & ,x=2 \end{cases}$, then $\lim_{x\to 0} g(f(x))$ is equal to

24. The value of $\lim_{x\to 0} x^2\left[\dfrac{1}{x^2}\right]$ where [.] denotes G.I.F., is

25. $\lim_{x\to 0}\left(\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}+\dfrac{84x\tan^{-1}\left(\sqrt{2}-1\right)}{\sin\pi x}\right)$ is equal to

26. If $f(x)=\sum_{\lambda=1}^{n}\left(x-\dfrac{5}{\lambda}\right)\left(x-\dfrac{4}{\lambda+1}\right)$, then $\lim_{n\to\infty} f(0)$ is equal to

27. Let $f(x)=\begin{cases} (-1)^{[x^2]} & \text{if } x<0 \\ \lim_{n\to\infty}\dfrac{1}{1+x^n} & \text{if } x\geq 0 \end{cases}$. Then $\lim_{x\to 0^-} 5f(x)+\lim_{x\to 0^+} 7f(x)$ equals (where [.] represents greatest integer function)

28. The value of $\left[\lim_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}\right]$ where [.] denotes GIF is

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

[*JEE Main-2016*]

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) $\dfrac{1}{2}$

2. Let $p=\lim\limits_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then $\log p$ is equal to :

[*JEE Main-2016*]

(*a*) 2 (*b*) 1

(*c*) 1/2 (*d*) 1/4

3. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$ equals **[*JEE Main-2017*]**

(*a*) $\dfrac{1}{4}$ (*b*) $\dfrac{1}{24}$

(*c*) $\dfrac{1}{16}$ (*d*) $\dfrac{1}{8}$

4. For each $t\in R$, let[t] be the greatest integer less than or equal to t. Then $\lim\limits_{x\to 0^+}x\left(\left[\dfrac{1}{x}\right]+\left[\dfrac{2}{x}\right]+....+\left[\dfrac{15}{x}\right]\right)$

[*JEE Main - 2018*]

(*a*) is equal to 15. (*b*) is equal to 120.

(*c*) does not exist (in R). (*d*) is equal to 0.

5. $\lim\limits_{y\to 0}\dfrac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}=$

[*JEE Main - 2019 (January)*]

(*a*) exists and equals $\dfrac{1}{4\sqrt{2}}$

(*b*) exists and equals $\dfrac{1}{2\sqrt{2}\left(\sqrt{2}+1\right)}$

(*c*) exists and equals $\dfrac{1}{2\sqrt{2}}$

(*d*) does not exist

6. For each $x\in R$, let [x] be the greatest integer less than or equal to x. Then

$\lim\limits_{x\to 0^-}\dfrac{x\left([x]+|x|\right)\sin[x]}{|x|}$ is equal to

[*JEE Main - 2019 (January)*]

(*a*) $-\sin 1$ (*b*) 0

(*c*) 1 (*d*) $\sin 1$

7. For each $t\in R$, let [t] be the greatest integer less than or equal to t. then,

$\lim\limits_{x\to 1+}\dfrac{(1-|x|+\sin|1-x|)\sin\left(\dfrac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$

[*JEE Main - 2019 (January)*]

(*a*) Equals 1 (*b*) Equals 0

(*c*) Equals -1 (*d*) Does not exist

8. $\lim\limits_{x\to 0}\dfrac{x\cot(4x)}{\sin^2 x\cot^2(2x)}$ is equal to:

[*JEE Main - 2019 (January)*]

(*a*) 0 (*b*) 2

(*c*) 4 (*d*) 1

9. Let [x] denote the greatest integer less than or equal to x.

Then $\lim\limits_{x\to 0}\dfrac{\tan\left(\pi\sin^2 x\right)+\left(|x|-\sin\left(x[x]\right)\right)^2}{x^2}$:

[*JEE Main - 2019 (January)*]

(*a*) does not exist (*b*) equals π

(*c*) equal $\pi+1$ (*d*) equals 0

10. $\lim\limits_{x\to\pi/4}\dfrac{\cot^3 x-\tan x}{\cos\left(x+\dfrac{\pi}{4}\right)}$ is :

[*JEE Main - 2019 (January)*]

(*a*) 4 (*b*) $4\sqrt{2}$

(*c*) $8\sqrt{2}$ (*d*) 8

11. $\lim\limits_{x\to 1^-}\dfrac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$ is equal to :

[*JEE Main - 2019 (January)*]

(*a*) $\dfrac{1}{\sqrt{2\pi}}$ (*b*) $\sqrt{\dfrac{2}{\pi}}$

(*c*) $\sqrt{\dfrac{\pi}{2}}$ (*d*) $\sqrt{\pi}$

12. Let $f: R \to R$ be a differentiable function satisfying $f'(3) + f'(2) = 0$ **[JEE Main-2019(April)]**

Then $\lim\limits_{x\to 0}\left(\dfrac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}}$ is equal to

(*a*) e^2 (*b*) e

(*c*) e^{-1} (*d*) 1

13. If $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1} = \lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$, then k is :

[JEE Main-2019(April)]

(*a*) $\dfrac{3}{8}$ (*b*) $\dfrac{3}{2}$

(*c*) $\dfrac{4}{3}$ (*d*) $\dfrac{8}{3}$

14. If $\lim\limits_{x\to 1}\dfrac{x^2-ax+b}{x-1} = 5$, then a + b is equal to :-

[JEE Main-2019(April)]

(*a*) -7 (*b*) -4

(*c*) 5 (*d*) 1

15. $\lim\limits_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$ is :

[JEE Main-2019(April)]

(*a*) 3 (*b*) 2

(*c*) 6 (*d*) 1

16. $\lim\limits_{x\to 2}\dfrac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}}$ is equal to

[JEE Main-2020 (January)]

17. $\lim\limits_{x\to 0}\left(\dfrac{3x^2+2}{7x^2+2}\right)^{1/x^2}$ is equal to

[JEE Main-2020 (January)]

(*a*) $\dfrac{1}{e^2}$ (*b*) e

(*c*) e^2 (*d*) $\dfrac{1}{e}$

18. $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$ is equal to :

[JEE Main-2020 (September)]

(*a*) e^2 (*b*) 1

(*c*) e (*d*) 2

19. If $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+....+x^n-n}{x-1} = 820$, $(n \in N)$ then the value of n is equal to _____.

[JEE Main-2020 (September)]

20. $\lim\limits_{x\to a}\dfrac{(a+2x)^{1/3}-(3x)^{1/3}}{(3a+x)^{1/3}-(4x)^{1/3}}(a \neq 0)$ is equal to :

[JEE Main-2020 (September)]

(*a*) $\left(\dfrac{2}{9}\right)\left(\dfrac{2}{3}\right)^{1/3}$ (*b*) $\left(\dfrac{2}{3}\right)\left(\dfrac{2}{9}\right)^{1/3}$

(*c*) $\left(\dfrac{2}{3}\right)^{4/3}$ (*d*) $\left(\dfrac{2}{9}\right)^{4/3}$

21. If $\lim\limits_{x\to 0}\left\{\dfrac{1}{x^8}\left(1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)\right\} = 2^{-k}$, then the value of k is _____.

[JEE Main-2020 (September)]

22. Let [t] denote the greatest integer $\leq$ t. If for some $\lambda \in R - \{0,1\}$, $\lim\limits_{x\to 0}\left|\dfrac{1-x+|x|}{\lambda-x+[x]}\right| = L$, then L is equal to:

[JEE Main-2020 (September)]

(*a*) 2 (*b*) $\dfrac{1}{2}$

(*c*) 0 (*d*) 1

23. $\lim\limits_{x\to 0}\dfrac{x\left(e^{(\sqrt{1+x^2+x^4}-1)/x}-1\right)}{\sqrt{1+x^2+x^4}-1}$

[JEE Main-2020 (September)]

(*a*) is equal to 0. (*b*) is equal to $\sqrt{e}$.

(*c*) is equal to 1. (*d*) does not exist.

24. If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$ then $\lim\limits_{x\to\alpha^+}\dfrac{\sqrt{1-\cos(p(x))}}{x+\alpha-4}$ is equal to :

[JEE Main-2020(September)]

(*a*) $\dfrac{1}{\sqrt{2}}$ (*b*) $\dfrac{1}{2}$

(*c*) $\dfrac{3}{\sqrt{2}}$ (*d*) $\dfrac{3}{2}$

25. $\lim\limits_{x\to 1}\left(\dfrac{\int_0^{(x-1)^2} t\cos(t^2)dt}{(x-1)\sin(x-1)}\right)$

[JEE Main-2020(September)]

(*a*) does not exist (*b*) is equal to $-\dfrac{1}{2}$

(*c*) is equal to 1 (*d*) is equal to $\dfrac{1}{2}$

26. Let $f(x) = x^6 + 2x^4 + x^3 + 2x + 3, x \in R$. Then the natural number n for which $\lim\limits_{x\to 1}\dfrac{x^n f(1) - f(x)}{x-1} = 44$ is

[JEE Main-2021]

27. The value of **[JEE Main-2021]**

$$\lim_{h\to 0}\left\{\frac{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h\left(\sqrt{3}\cosh-\sinh\right)}\right\}\text{is:}$$

(*a*) $\frac{3}{4}$ (*b*) $\frac{2}{\sqrt{3}}$

(*c*) $\frac{4}{3}$ (*d*) $\frac{2}{3}$

28. If $\lim_{x\to 0}\frac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}$ exists and is equal to b, then the value of a – 2b is **[JEE Main-2021]**

JEE-ADVANCED PREVIOUS YEAR'S

1. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then **[IIT JEE-2012]**

(*a*) $a=1, b=4$ (*b*) $a=1, b=-4$

(*c*) $a=2, b=-3$ (*d*) $a=2, b=3$

2. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where a > -1. Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[IIT JEE-2012]**

(*a*) $-\frac{5}{2}$ and 1 (*b*) $-\frac{1}{2}$ and –1

(*c*) $-\frac{7}{2}$ and 2 (*d*) $-\frac{9}{2}$ and 3

3. The largest value of the non-negative integer a for which

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}\text{ is}$$

[JEE Advanced-2014]

4. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x\in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x\in R$. Let (fog) (x) dentoe f(g(x)) and (gof) (x) denote g(f(x)). Then which of the following is(are)true?

(*a*) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$ (*b*) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(*c*) $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$ (*d*) There is an $x\in R$ such that (gof)(x) = 1

5. Let m and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is

[JEE Advanced -2015]

6. Let $\alpha,\beta\in\mathbb{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$. Then $6(\alpha+\beta)$ equals **[JEE Advanced -2016]**

7. Let $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x\neq 1$.

Then **[JEE Advanced -2017]**

(*a*) $\lim_{x\to 1^+}f(x)=0$

(*b*) $\lim_{x\to 1^-}f(x)$ does not exist

(*c*) $\lim_{x\to 1^-}f(x)=0$

(*d*) $\lim_{x\to 1^+}f(x)$ does not exist

8. Let $F: R\to R$ be a function. We say that f has

PROPERTY 1; if $\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 ; if $\lim_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite.

[JEE Advanced -2019]

Then which of the following options is/are correct ?

(*a*) $f(x)=x|x|$ has PROPERTY 2

(*b*) $f(x)=x^{2/3}$ has PROPERTY 1

(*c*) $f(x)=\sin x$ has PROPERTY 2

(*d*) $f(x)=|x|$ has PROPERTY 1

9. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x\to 0^+}\frac{(1-x)^{\frac{1}{x}}-e^{-1}}{x^a}$$

is equal to a nonzero real number, is .....

[JEE(Advanced) - 2020]

10. The value of the limit

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(\sin 3x+\sin x)}{\left(2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{3x}{2}\right)}$$

is .....

[JEE(Advanced) - 2020]

ANSWER KEY

Topicwise Questions

1. (*d*)	**2.** (*a*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)	**6.** (*c*)	**7.** (*c*)	**8.** (*d*)	**9.** (*d*)	**10.** (*a*)
11. (*b*)	**12.** (*a*)	**13.** (*a*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*d*)	**18.** (*a*)	**19.** (*a*)	**20.** (*d*)
21. (*a*)	**22.** (*c*)	**23.** (*a*)	**24.** (*c*)	**25.** (*c*)	**26.** (*a*)	**27.** (*d*)	**28.** (*c*)	**29.** (*b*)	**30.** (*b*)

Learning Plus

1. (*d*)	**2.** (*d*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)	**6.** (*c*)	**7.** (*d*)	**8.** (*c*)	**9.** (*c*)	**10.** (*a*)
11. (*b*)	**12.** (*d*)	**13.** (*c*)	**14.** (*a*)	**15.** (*a*)	**16.** (*b*)	**17.** (*a*)	**18.** (*c*)	**19.** (*a*)	**20.** (*c*)
21. (*a*)	**22 .** (*b*)	**23.** (*c*)	**24.** (*b*)	**25.** (*b*)	**26.** (*a*)	**27.** (*c*)	**28.** (*a*)	**29.** (*c*)	**30.** (*d*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING\NUMERICAL

1. (*a*,*b*,*c*) **2.** (*a*,*b*) **3.** (*a*,*b*,*c*,*d*) **4.** (*b*,*c*) **5.** (*a*,*b*) **6.** (*a*,*d*) **7.** (*a*,*b*,*c*,*d*) **8.** (*a*,*b*,*c*) **9.** (*a*,*d*)

10. (*a*,*b*,*c*,*d*) **11.** (*a*) **12.** (*d*) **13.** (*a*) **14.** (*d*) **15.** (*a*) **16.** (*b*)

17. (*a*) → (s),(*b*) → (r), (*c*) → (p), (*d*) → (q), **18.** (*a*) → (q), (*b*) → (r), (*c*) → (s), (*d*) → (p), **19.** [1] **20.** [2]

21. [1] **22.** [0] **23.** [1] **24.** [1] **25.** [11] **26.** [20] **27.** [12] **28.** [1]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*)	**2.** (*c*)	**3.** (*c*)	**4.** (*b*)	**5.** (*a*)	**6.** (*a*)	**7.** (*b*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*b*)	**12.** (*d*)	**13.** (*d*)	**14.** (*a*)	**15.** (*b*)	**16.** [36]	**17.** (*a*)	**18.** (*a*)	**19.** [40]	**20.** (*b*)
21. [8]	**22.** (*a*)	**23.** (*c*)	**24.** (*c*)	**25.** (No Option is correct)		**26.** [7]	**27.** (c)	**28.** [5]	

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*b*)	**2.** (*b*)	**3.** [0]	**4.** (*a*,*b*,*c*)	**5.** [2]	**6.** (7)	**7.** [*c*, *d*]	**8.** [*b*,*d*]	**9.** [1.00]	**10.** [8]

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

4 Continuity and Derivability Methods of Differentiation

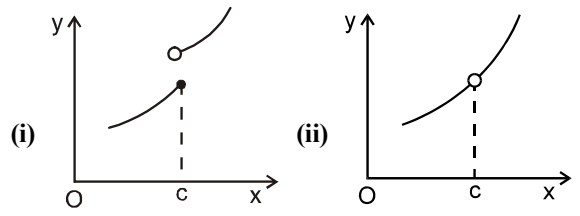
CONTINUITY DEFINITION

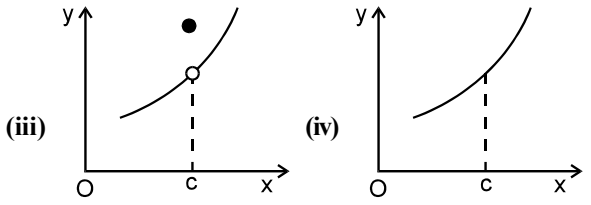
A function f(x) is said to be continuous at x = c, if $\lim_{x\to c} f(x) = f(c)$

i.e. f is continuous at x = c

if $\lim_{h\to 0^+} f(c-h) = \lim_{h\to 0^+} f(c+h) = f(c)$.

If a function f(x) is continuous at x = c, the graph of f(x) at the corresponding point (c, f(c)) will not be broken. But if f(x) is discontinuous at x = c, the graph will be broken when x = c





(i), (ii) and (iii) are discontinuous at x = c)

(iv) is continuous at x = c)

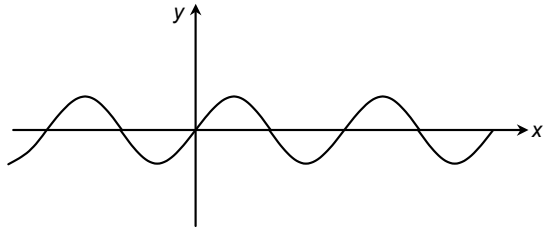
GEOMETRICAL MEANING OF CONTINUITY

Function f(x) will be continuous at x = c if there is no break in the graph of function f(x) at the point (c, f(c)).

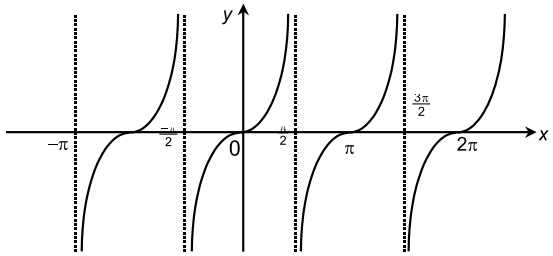
In an interval, function is said to be continuous if there is no break in graph of function in the entire interval.

For example:

- f(x) = sinx is continuous in its entire domain.



f(x) = tanx is discontinuous at $x = (2n+1)\frac{\pi}{2}$ where $n \in \mathbf{I}$.



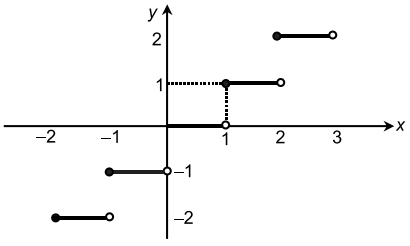
THE REASONS OF DISCONTINUITY

A function f can be discontinuous due to any of the following three reasons:

f(*x*) will be discontinuous at $x = a$, in any of the following cases:

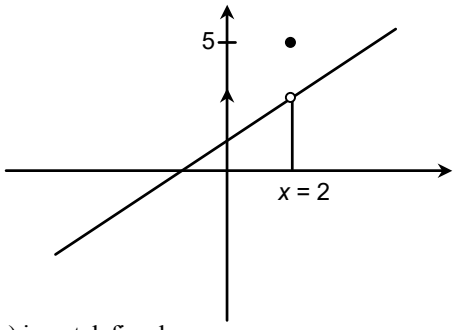
(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.

For example $y = [x]$ at $x \in \mathbf{I}$.



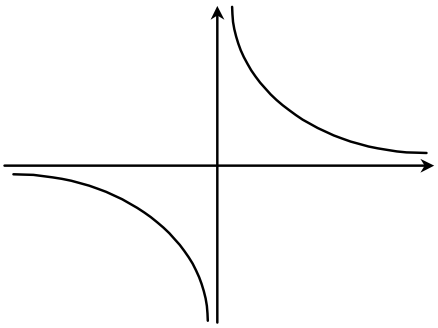
(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but not equal to f(a).

For example $f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases}$ at $x = 2$.



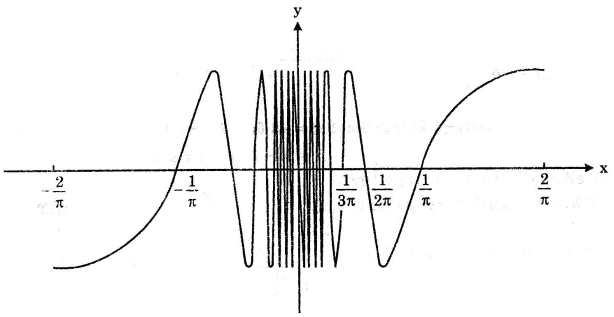
(iii) $f(a)$ is not defined.

For example $y = \frac{1}{x}$ at $x = 0$



(iv) At least one of the limits does not exist.

For example $y = \sin\left(\frac{1}{x}\right)$ at $x = 0$



TRAIN YOUR BRAIN

Example 1: If $f(x) = \begin{cases} \sin\frac{\pi x}{2} & , \quad x < 1 \\ [x] & , \quad x \geq 1 \end{cases}$, then find whether f(x) is continuous or not at x = 1, where [.] is greatest integer function.

Sol. $f(x) = \begin{cases} \sin\frac{\pi x}{2} & , \quad x < 1 \\ [x] & , \quad x \geq 1 \end{cases}$

For continuity at x = 1, we determine f(1), $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$.

Now, $f(1) = [1] = 1$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1$

and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} [x] = 1$

so $f(1) = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1

Example 2: Let $f(x) = \begin{cases} -2\sin x & \text{if} \quad x \leq -\pi/2 \\ A\sin x + B & \text{if} \quad -\frac{\pi}{2} < x < \pi/2 \\ \cos x & \text{if} \quad x \geq \pi/2 \end{cases}$

Find A and B so as to make the function continuous.

Sol. At $x = -\pi/2$

L.H.L. $= \lim\limits_{x \to -\frac{\pi^-}{2}} (-2\sin x)$; R.H.L. $= \lim\limits_{x \to -\frac{\pi^+}{2}} (A\sin x + B)$

Replace x by $-\frac{\pi}{2} - h$; Replace x by $-\frac{\pi}{2} + h$

where $h \to 0$ where $h \to 0$.

Since, L.H.L. = R.H.L.

$\lim\limits_{h \to 0} -2\sin\left(-\frac{\pi}{2} - h\right) = 2 = \lim\limits_{h \to 0} A\sin\left(-\frac{\pi}{2} + h\right) + B = B - A$

So $B - A = 2$ ...(i)

At $x = \pi/2$

L.H.L. $= \lim\limits_{x \to \frac{\pi^-}{2}} (A\sin x + B)$ R.H.L. $= \lim\limits_{x \to \frac{\pi^+}{2}} \cos x$

Replace x by $\frac{\pi}{2} - h$; Replace x by $\frac{\pi}{2} + h$

where $h \to 0$; where $h \to 0$

Since, L.H.L. = R.H.L.

$= \lim\limits_{h \to 0} A\sin\left(\frac{\pi}{2} - h\right) + B = A + B = \lim\limits_{h \to 0} \cos\left(\frac{\pi}{2} + h\right) = 0$

So $A + B = 0$ ...(ii)

Solving (i) & (ii), B = 1, A = –1

Example 3: Test the continuity of f(x) at x = 0 if

$f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

Sol. For $x < 0$,

L.H.L. $= \lim\limits_{x \to 0-} f(x) = \lim\limits_{h \to 0} f(0 - h) = \lim\limits_{h \to 0}$

$(0 - h + 1)^{2-\left(\frac{1}{|0-h|} + \frac{1}{(0-h)}\right)} = \lim\limits_{h \to 0} (1-h)^2 = (1-0)^2 = 1$

$f(0) = 0$. & R.H.L $= \lim\limits_{x \to 0+} f(x) = \lim\limits_{h \to 0} f(0 + h)$

$= \lim\limits_{h \to 0} (h+1)^{2-\left(\frac{1}{|h|} + \frac{1}{h}\right)}$

$\exp\left(\lim\limits_{h \to 0} (h + 1 - 1) \times \frac{2h - 2}{h}\right) = e^{-2}$

$\therefore$ L.H.L. = R.H.L. $\neq f(0)$

Hence f(x) is discontinuous at x = 0.

Example 4: If f(x) be continuous function for all real values of x and satisfies;

$x^2+(f(x)-2).x+2\sqrt{3}-3-\sqrt{3}\ .f(x)=0,\ \forall\ x\in R$. Then find the value of $f(\sqrt{3})$.

Sol. As f(x) is continuous for all $x \in R$.

Thus, $\lim\limits_{x\to\sqrt{3}} f(x)=f(\sqrt{3})$ where

$$f(x)=\frac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x},\ x\neq\sqrt{3}$$

$$\lim_{x\to\sqrt{3}} f(x)=\lim_{x\to\sqrt{3}}\frac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}$$

$$=\lim_{x\to\sqrt{3}}\frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)}=2(1-\sqrt{3})$$

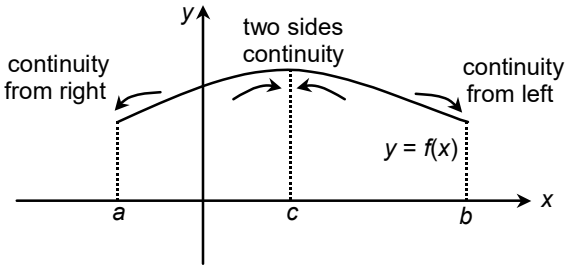
$f(\sqrt{3})=2(1-\sqrt{3})$.

CONTINUITY IN INTERVAL

(a) Continuity in an open interval: *A* function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each point of (a, b).

(b) Continuity in a closed interval: *A* function $f(x)$ is said to be continuous in an closed interval $[a, b]$ if it is

- continuous at each point (a, b)



Continuity at points a, b and c

- $f(x)$ is continuous from right at $x = a$

  i.e. $\lim\limits_{x\to a^+} f(x)=f(a)$

- $f(x)$ is continuous from left at $x = b$

  i.e. $\lim\limits_{x\to b^-} f(x)=f(b)$

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Example 5: If $f(x)=\lim\limits_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$, examine continuity of f(x).

Sol. If $|x|<1$, $f(x)=\lim\limits_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$

$$=\frac{\log(x+2)}{1}=\log(x+2)$$

If $|x|>1$, $f(x)=\lim\limits_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$

$$=\lim_{n\to\infty}\frac{\frac{\log(x+2)}{x^{2n}}-\sin x}{1+\frac{1}{x^{2n}}}=-\sin x$$

If $|x|=1$, $f(x)=\lim\limits_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$

$$=\frac{\log(x+2)-\sin x}{2}$$

$$\text{Thus } f(x)=\begin{cases}-\sin x, & -2<x<-1\\ \log(x+2), & -1<x<1\\ -\sin x, & x>1\\ \frac{\log(x+2)-\sin x}{2}, & x=\pm1\end{cases}$$

Obviously $f(x)$ is discontinuous at $x = \pm 1$.

Example 6: Let $f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$ where [.] denotes greatest integer function. Then find domain of f and the points of discontinuity of f in the domain.

Sol. Let $f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$

Domain of f(x) is $x \in \mathbf{R}$ excluding the points where $[x+1]=0$
($\because$ denominator can not be zero)
$\Rightarrow\ 0\le x+1<1$
$\Rightarrow\ -1\le x<0$
i.e. for all $x \in [-1, 0)$, denominator is zero. So, domain is $x \in \mathbf{R}-[-1, 0)$.

Points of Discontinuity

Check the continuity at $x = a$ (where $a \in \mathbf{I}$)

$$LHL=\lim_{h\to0}[a-h]\sin\left(\frac{\pi}{[a+1-h]}\right)$$

$$LHL=(a-1)\sin\left(\frac{\pi}{a}\right) \quad \ldots(i)$$

$$RHL=\lim_{h\to0}[a+h]\sin\left(\frac{\pi}{[a+1+h]}\right)$$

$$\Rightarrow RHL=a\sin\left(\frac{\pi}{a+1}\right) \quad \ldots(ii)$$

From (i) and (ii), LHL ≠ RHL
$\Rightarrow$ f(x) is discontinuous at $x = a$ (i.e., at integral values of x)
So, points of discontinuity are $x \in I \cap D$
(i.e. integer lying in the set of domain)
$\Rightarrow\ x \in \mathbf{I}-\{-1\}$

TYPES OF DISCONTINUITY

Definition :– Let a function f be defined in the neighbourhood of a point c, except perhaps at c itself. Also let both one–sided limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist, where $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$. Then the point c is called a **discontinuity of the first kind** in the function f(x).

In more complicated case $\lim_{x\to c} f(x)$ may not exist because one or both one-sided limits do not exist. Such condition is called a **discontinuity of the second kind.**

Removable & Irremovable Discontinuity

(a) In case $\lim_{x\to c} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity. In this case we can redefine the function at single point such that $\lim_{x\to c} f(x) = f(c)$ & make it continuous at x=c.

Removable Type Of Discontinuity Can Be Further Classified As:

(i) Missing Point Discontinuity : where $\lim_{x\to a} f(x)$ exists finitely but f(a) is not defined . e.g.

$f(x) = \dfrac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at $x = 1$.

(ii) Isolated Point Discontinuity : where $\lim_{x\to a} f(x)$ exists & f(a) also exists but ; $\lim_{x\to a} f(x) \neq f(a)$.

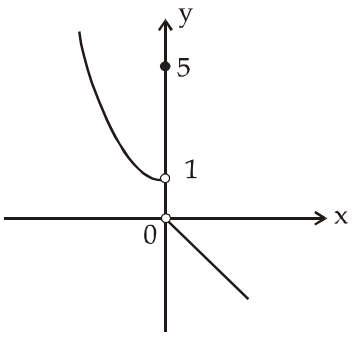
e.g. $f(x) = \dfrac{x^2-16}{x-4}$, $x \neq 4$ & $f(4) = 9$ has a break at $x = 4$.

(b) In case $\lim_{x\to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it . Such discontinuities are known as non - removable discontinuity.

Irremovable Type Of Discontinuity Can Be Further Classified As:

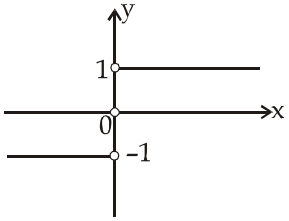
(a) Finite discontinuity: Both the limits (i.e. L.H.L and R.H.L.) are finite but not equal

(i) The function $y = \begin{cases} x^2+1 & \text{for } x < 0, \\ 5 & \text{for } x = 0, \\ -x & \text{for } x > 0, \end{cases}$



has a discontinuity of the first kind at x = 0

(ii) The function $y = |x|/x$ is defined for all $x \in \mathbf{R}$, $x \neq 0$; but at x = 0 it has a discontinuity of the first kind.



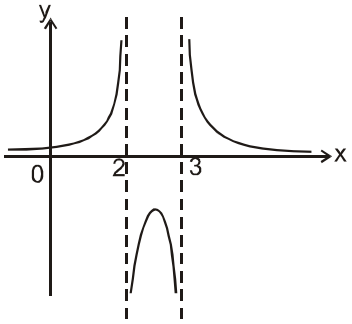
The left-hand limit is $\lim_{x\to 0^-} y = -1$, while the right-hand limit is $\lim_{x\to 0^+} y = 1$

(b) Infinite discontinuity: Atleast one of two limits is infinite

e.g. (i) $f(x) = \dfrac{1}{x-4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at $x = 4$.

(ii) The function $y = \dfrac{1}{(x-2)(x-3)}$ has no limits (neither one-sided nor two-sided) at x = 2 and x = 3. Therefore

x = 2 and x = 3 are discontinuities of the second kind

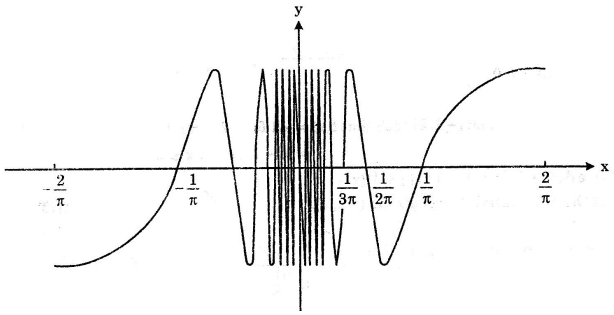


The function $y = \ln|x|$ at the point x = 0 has the limits $\lim_{x\to 0} \ln|x| = -\infty$. Consequently, $\lim_{x\to 0} f(x)$ (and also the one-sided limits) do not exist; x = 0 is a discontinuity of the second kind.

It is not true that discontinuities of the second kind only arise when $\lim_{x\to c} f(x) = \pm\infty$.

(c) Oscillatory discontinuity: Limits oscillate between tow finite quantities

e.g. The function $y = \sin(1/x)$, has no one-sided limits for $x \to 0^-$ and $x \to 0^+$



There is no limit as $x \to 0$ since the values of the function sin (1/x) do not approach a certain number, but repeat an infinite number of times within the interval from –1 to 1 as $x \to 0$.

Remark:

(i) In case of finite discontinuity the non-negative difference between the value of the RHL at x = c & LHL at x = c is called **The Jump Of Discontinuity**. A function having a finite number of jumps in a given interval I is called a **Piece-wise Continuous** or **Sectionally Continuous** function in this interval.

(ii) All Polynomials, Trigonometrical functions, Exponential & Logarithmic functions are continuous in their domains.

(iii) Point functions are to be treated as discontinuous. eg . $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(iv) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg . f(x) $= \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0 , hence the composite g of $(x) = \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0.

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Example 7: If $f(x) = \begin{cases} x & , \ x < 1 \\ x^2 & , \ x > 1 \end{cases}$, then check f(x) is continuous at x = 1 or not if not, then comment on the type of discontinuity.

Sol. $f(x) = \begin{cases} x & \forall \ x < 1 \\ x^2 & \forall \ x > 1 \end{cases}$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$ and

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2 = 1$

$\therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) =$ finite

and f(1) is not defined.

So f(x) is discontinuous at x = 1 and this discontinuity is of 1st type (removable).

Example 8: If $f(x) = \begin{cases} \cos^{-1}\{\cot x\}, & x < \frac{\pi}{2} \\ \pi[x]-1, & x \le \frac{\pi}{2} \end{cases}$. Find jump of discontinuity, where [.] is greatest integer function and {.} fractional part function.

Sol. $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & \text{if} \quad x < \frac{\pi}{2} \\ \pi[x]-1 & \text{if} \quad x \ge \frac{\pi}{2} \end{cases}$

L.H.L. $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \cos^{-1}\{\cot x\}$

$= \cos^{-1}\{0^+\} = \cos^{-1} 0 = \frac{\pi}{2}$

R.H.L. $\lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{x \to \frac{\pi}{2}^+} \pi[x]-1 = \pi - 1$

$\therefore$ Jump of discontinuity $= |\pi - 1 - \frac{\pi}{2}| = \frac{\pi}{2} - 1$

THEOREMS ON CONTINUITY

(i) If f & g are two functions which are continuous at x = c, then the functions defined by:
$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(ii) If f(x) is continuous & g(x) is discontinuous at x = a, then the product function $\phi(x) = f(x).g(x)$ may or may not be continuous but sum or difference function $\phi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at x = a.

e.g. $f(x) = x$ & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

(iii) If f (x) and g(x) both are discontinuous at x = a, then the product function $\phi(x) = f(x).\ g(x)$ is not necessarily be discontinuous at x = a.

e.g. $f(x) = g(x) = \begin{cases} 1 & , \ x \ge 0 \\ -1 & , \ x < 0 \end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

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Example 9: If f(x) = [sin(x–1)] – {sin(x–1)}. Comment on continuity of f(x) at $x = \frac{\pi}{2} + 1$

(where [.] denotes G.I.F. and { . } denotes fractional part function).

Sol. $f(x) = [\sin(x-1)] - \{\sin(x-1)\}$

Let $g(x) = [\sin(x-1)] + \{\sin(x-1)\} = \sin(x-1)$

which is continuous at $x = \frac{\pi}{2} + 1$

as $[\sin(x-1)]$ and $\{\sin(x-1)\}$ both are discontinuous at $x = \frac{\pi}{2} + 1$

$\therefore$ At most one of f(x) or g(x) can be continuous at $x = \frac{\pi}{2} + 1$

As g(x) is continuous at $x = \frac{\pi}{2} + 1$, therefore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluating $\lim\limits_{x \to \frac{\pi}{2}+1} f(x)$ and $f\left(\frac{\pi}{2}+1\right)$.

CONTINUITY OF COMPOSITE FUNCTIONS

If f is continuous at x = c and g is continuous at x = f(c), then the composite g[f(x)] is continuous at x = c.

eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite function $(gof)(x) = \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0.

Note: While checking points of discontinuity of gof(x), always include points of discontinuity of f(x).

Example 10: If $f(x) = \begin{cases} [\sin \pi x] & ; \quad 0 \le x < 1 \\ \left\{x - \frac{2}{3}\right\} \cdot \operatorname{sgn}\left(x - \frac{5}{4}\right) & ; \quad 1 \le x \le 2 \end{cases}$,

where { . } represents fractional part function and [.] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

Sol. (i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2

(ii) For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$

i.e. $x = 0, \frac{1}{2}$

(iii) For $\left\{x - \frac{2}{3}\right\} \cdot \operatorname{sgn}\left(x - \frac{5}{4}\right)$, continuity should be checked when $x - \frac{5}{4} = 0$

(as sgn (x) is discontinuous at x = 0)

i.e. $x = \frac{5}{4}$ and when $x - \frac{2}{3} \in I$

i.e. $x = \frac{5}{3}$ (as {x} is discontinuous when $x \in I$)

$\therefore$ overall discontinuity should be checked at x = 0, $\frac{1}{2}$, 1, $\frac{5}{4}$, $\frac{5}{3}$ and 2 check the discontinuity your self.

discontinuous at $x = \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$

Example 11: If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x).

Sol. $f(x) = \frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain

and f is not defined at x = 1

$\therefore$ f is discontinuous at x = 1

$g(x) = \frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2

$\therefore$ g is discontinuous at x = 2

Now fog (x) will be discontinuous at

(i) x = 2 (point of discontinuity of g(x))

(ii) g(x) = 1(when g(x) = point of discontinuity of f(x)) if g(x) = 1

$\Rightarrow \quad \frac{1}{x-2} = 1 \Rightarrow x = 3$

$\therefore$ discontinuity of fog(x) should be checked at x = 2 and at x = 3

$$\text{fog}(x) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$$

fog (2) is not defined

$$\lim_{x \to 2} \text{fog}(x) = \lim_{x \to 2} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x \to 2} \frac{1+x-2}{1-x+2} = 1$$

$\therefore$ fog (x) is discontinuous at x = 2 and it is removable discontinuity at x = 2

fog (3) = not defined

$$\lim_{x \to 3^+} \text{fog}(x) = \lim_{x \to 3^+} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

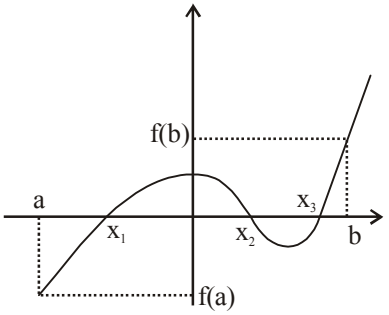
$$\lim_{x \to 3^-} \text{fog}(x) = \lim_{x \to 3^-} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

$\therefore$ fog (x) is discontinuous at x = 3 and it is non removable discontinuity of IInd kind.

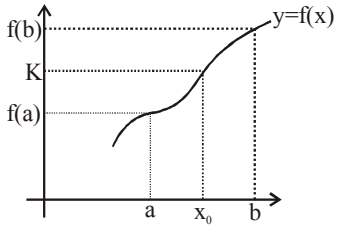
INTERMEDIATE VALUE THEOREM

A function f which is continuous in $[a, b]$ such that f(a) ≠ f(b) then f possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).



(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).



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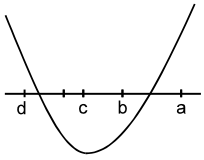
Example 12: Given that a > b > c > d, then prove that the equation (x – a) (x – c) + 2(x – b) (x – d) = 0 will have real and distinct roots.

Sol.



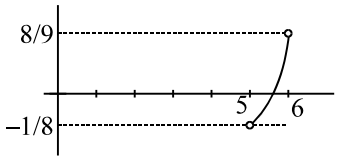
(x – a) (x – c) + 2 (x – b) (x – d) = 0
f(x) = (x – a) (x – c) + 2 (x – b) (x – d)
f(a) = (a – a) (a – c) + 2 (a – b) (a – d) = + ve
f(b) = (b – a) (b – c) + 0 = –ve
f(c) = 0 + 2 (c – b) (c – d) = –ve
f(d) = (d – a) (d – c) + 0 = +ve
hence (x – a) (x – c) + 2(x – b) (x – d) = 0
have real and distinct roots

Example 13: State intermediate value theorem and use it to prove that the equation $\sqrt{x-5} = \frac{1}{x+3}$ has at least one real root.

Sol. Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$ first, $f(x)$ is continuous on [5, 6]

Also $f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0, f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$

Hence by intermediate value theorem ∃ at least one value of c ∈ (5, 6) for which $f(c) = 0$



$\therefore \sqrt{c-5} - \frac{1}{c+3} = 0$

$\therefore$ c is root of the equation $\sqrt{x-5} = \frac{1}{x+3}$ and c ∈ (5, 6)

Example 14: If f(x) be a continuous function in [0, 2π] and f(0) = f(2π) then prove that there exists point c ∈ (0, π) such that f(c) = f(c + π).

Sol. Let g(x) = f(x) – f(x + π) ....(i)
at x = π; g(π) = f(π) – f(2π) ....(ii)
at x = 0, g(0) = f(0) – f(π) ...(iii)
adding (ii) and (iii),
g(0) + g(π) = f(0) – f(2π)
⇒ g(0) + g(π) = 0 [Given f(0) = f(2π) ⇒ g(0) = –g(π)
⇒ g(0) and g(π) are opposite in sign.
⇒ There exists a point c between 0 and π such g(c) = 0 as shown in graph;
From (i) putting x = c g(c) = f(c) – f(c + π) = 0
Hence, f(c) = f(c + π)

Example 15: Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Sol. Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x}$

$f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n.\sin^2\frac{\pi}{4}} = \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$

Now $f(0) = \lim_{n\to\infty} \frac{1}{n.\sin^2(0)+1} = \frac{1}{1+0} = 1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty}\frac{1}{1+n\sin^2 x}\right] = 0$

{here $\sin^2 x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞}
$\therefore$ f(x) is not continuous at x = 0

Important Points to be Remember

(a) If f(x) is continuous at x = a & g(x) is discontinuous at x = a, then f(x) ± g(x) must be discontinuous at x = a
Proof : (Desirable)
Let F(x) = f(x) + g(x) be a continuous function hence g(x) = F(x) – f(x), should also be a continuous function, which contradicts the given fact.

(b) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ will **not necessarily be discontinuous at x = a**, e.g.

$$f(x) = \sin x \ \& \ g(x) = \begin{cases} \cos\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f(x) is continuous at x = 0 & g(x) is discontinuous at x = 0, but f(x).g(x) is continuous at x = 0.

(c) If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ **is not necessarily be discontinuous at x = a**, e.g.

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \ \& \ g(x) \begin{cases} -1/x & x \geq 0 \\ 1/x & x < 0 \end{cases}$$

f(x) & g(x) both are discontinuous at x = a then f(x) ± g(x) is not necessarilly be discontinuous at x = a.

Single Point Continuity

Function which are continuous only at one point are said to exhibit single point continuity

Exmples:

(1) $f(x) = \begin{cases} x \text{ if } & x \in Q \\ x \text{ if } & x \notin Q \end{cases}$ continuous only at x = 0

(2) $f(x) = \begin{cases} x & \text{if } \ x \in Q \\ 1-x & \text{if } \ x \notin Q \end{cases}$ continuous only at $x = \frac{1}{2}$

(3) $f(x) = \begin{cases} x^2 \text{ if } & x \in Q \\ 1 \text{ if } & x \notin Q \end{cases}$ continuous only at x=1 & –1

DIFFERENTIABILITY OF A FUNCTION

Definition of Tangent:

Tangent to the curve y = f(x) at a point A(a, f(a)) is a limiting case of secant through A.

If f is defined on an open interval containing a, and if the limit

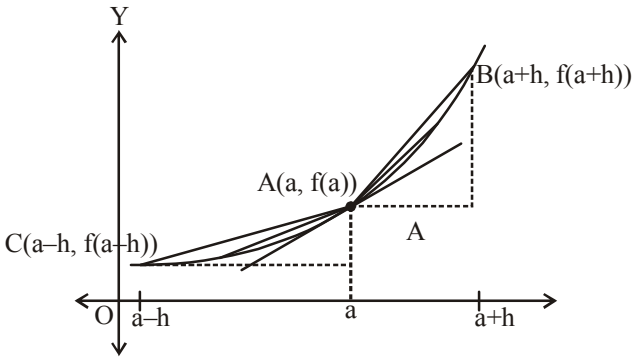
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = m$$

exists, then the line passing through (a, f(a)) with slope m is the tangent line to the graph of f at the point (a, f(a)).
The slope of the tangent line to the graph of f at the point (a, f(a)) is also called the slope of the graph of f at x = a.

⇒ Stope of tangent at point A(if it exists) = Limiting value of slope of secant through A i.e.

$$\begin{cases} AB(\text{through right side}) \\ AC(\text{through left sdie}) \end{cases}$$

DERIVABILITY AT A POINT



⇒ Derivative at x = a is limiting value of slope of secant through A.
Let secant through, A(a, f(a)) intersect the curve at B(a + h, f(a + h)) and C(a – h, f(a – h))

⇒ $m_{AB} = \frac{f(a+h) - f(a)}{h}, h > 0$

and $m_{AC} = \frac{f(a) - f(a-h)}{h}, h > 0$

⇒ $\lim_{B \to A} m_{AB} = \lim_{C \to A} m_{AC}$ = slope of tangent A i.e. derivative at x = a.
as, $B \to A \ \& \ C \to A \Rightarrow h \to 0$

⇒ $\lim_{B \to A} m_{AB} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ = Slope of tangent at A through right side

and $\lim_{C \to A} M_{AC} = \frac{f(a-h) - f(a)}{h}$ = Slope of tangent A through left side

⇒ derivative of f(x) at x = a exist,

If $\lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ = Finite quantity.

Right hand & Left hand Derivatives:

By definition : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

(i) The right hand derivative of f′ at x = a denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists & is finite.

(ii) The left hand derivative of f at x = a denoted by $f'(a^-)$ is defined by :

$$f'(a^-) = \lim_{h \to 0^+} \frac{f(a-h) - f(a)}{-h}$$, Provided the limit exists & is finite.

f′(a) exists if and only if these one-sided derivatives exist and are equal.

CONCEPT OF VERTICAL TANGENT

The above definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \left| \frac{f(c+\Delta x) - f(c)}{\Delta x} \right| = \infty$$

then the vertical line, x = c, passing through (c, f(c)) is a vertical tangent line to the graph of f. For example, the function shown in Figure has a vertical tangent line at (c, f(c)). If the domain of f is the closed interval [a, b], then you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for x = a) and from the left (for x = b).

$$\lim_{\Delta x \to 0^+} \left| \frac{f(a+\Delta x) - f(a)}{\Delta x} \right| = \infty$$

$$\lim_{\Delta x \to 0^-} \left| \frac{f(b+\Delta x) - f(b)}{\Delta x} \right| = \infty$$

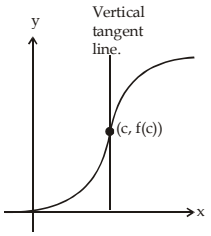


Figure
The graph of f has a vertical tanent line at (c, f(c)).

AVERAGE AND INSTANTANEOUS RATE OF CHANGE

Suppose y is a function of x, say y = f(x). Corresponding to a change from x to x + Δx, the variable y changes from f(x) to f(x + Δx). The change in y is Δy = f(x + Δx) – f(x), and the **average rate of change of y with respect to x** is

Average rate of change

$$= \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

As the interval over which we are averaging becomes shorter (that is, as $\Delta x \to 0$), the average rate of change approaches what we would intuitively call the **instantaneous rate of change of y with respect to x,** and the difference quotient approaches the derivative $\frac{dy}{dx}$. Thus, we have

Instantaneous Rate of Change

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

To summarize:

Instantaneous Rate of Change

Suppose f(x) is differentiable at $x = x_0$. Then the **instantaneous rate of cange** of y = f(x) with respect to x at x_0 is the value of the derivative of f at x_0. That is

$$\text{Instantaneous Rate of Change} = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$$

TRAIN YOUR BRAIN

Example 16: Find the rate at which the function $y = x^2 \sin x$ is changing with respect to x when $x = \pi$.

For any x, the instantaneous rate of change in the derivative,

Sol. $\frac{dy}{dx} = 2x \sin x + x^2 \cos x$

Thus, the rate when $x = \pi$ is $\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi \sin \pi + \pi^2 \cos \pi$

$= 2\pi(0) + \pi^2(-1) = -\pi^2$

The negative sign indicates that when $x = \pi$, the function is decreasing at the rate of $\pi^2 \approx 9.9$ units of y for each one-unit increase in x.

Let us consider an example comparing the average rate of change and the instantaneous rate of change.

Example 17: Comment on the differentiability of $f(x) = \begin{cases} x & , \quad x < 1 \\ x^2 & , \quad x \geq 1 \end{cases}$ at x = 1.

Sol. $\text{R.H.D.} = f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} \frac{1 + h^2 + 2h - 1}{h}$$

$$= \lim_{h \to 0^+} (h + 2) = 2$$

$$\text{L.H.D.} = f'(1^-) = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0^+} \frac{1 - h - 1}{-h} = 1$$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

Example 18: If $f(x) = \begin{cases} A + Bx^2 & , \quad x < 1 \\ 3Ax - b + 2 & , \quad x \geq 1 \end{cases}$, then find A and B so that f(x) become differentiable at x = 1.

Sol. $f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0^+} \frac{3A(1+h) - B + 2 - 3A + B - 2}{h}$$

$$= \lim_{h \to 0^+} \frac{3Ah}{h} = 3A$$

$$f'(1^-) = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0^+} \frac{A + B(1-h)^2 - 3A + B - 2}{-h}$$

$$= \lim_{h \to 0^+} \frac{(-2A + 2B - 2) + Bh^2 - 2Bh}{-h}$$

hence for this limit to be defined $-2A+2B-2=0$

$B=A+1$

$\therefore \quad f'(1^-)=\lim_{h\to 0} -(Bh-2B)=2B$

For f(x) to be differentiable at x = 1

$\therefore \quad f'(1^-)=f'(1^+)$

$\Rightarrow \quad 3A=2B=2(A+1) \quad \because B=A+1$

$A=2, B=3$

RELATION BETWEEN CONTINUITY & DIFFERENTIABILITY

If a function f is derivable at x then f is continuous at x.

For : $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x)=\frac{f(x+h)-f(x)}{h}.h \quad [h\neq 0]$

Therefore $\lim_{h\to 0}[f(x+h)-f(x)]$

$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.h=f'(x).0=0$

Therefore $\lim_{h\to 0}[f(x+h)-f(x)]=0$

$\Rightarrow \lim_{h\to 0} f(x+h)=f(x) \Rightarrow$ f is continuous at x.

If f(x) is derivable for every point of its domain, then it is continuous in that domain .

The converse of the above result is not true: "If f is continuous at x, then f may or maynot be derivable at x"

The functions $f(x)=|x|$ & $g(x)=x\sin\frac{1}{x}$; $x\neq 0$ & $g(0)=0$ are continuous at x = 0 but not derivable at x = 0.

Remark:

(a) Let $f'_+(a)=p$ & $f'_-(a)=q$ where p & q are finite then :

(i) $p=q \Rightarrow$ f is derivable at $x=a \Rightarrow$ f is continuous at x = a.

(ii) $p\neq q \Rightarrow$ f is not derivable at x = a but f is continuous at x = a

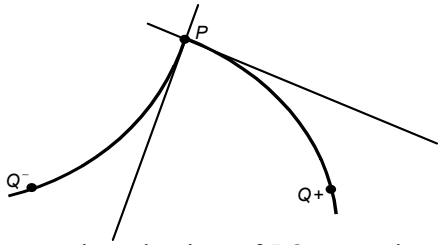
Differentiable $\Rightarrow$ Continuous ;

Non-differentiable $\not\Rightarrow$ Discontinuous

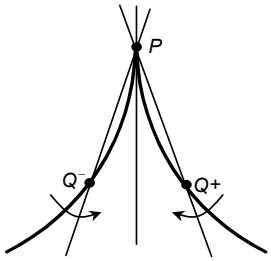
But Discontinuous $\Rightarrow$ Non-differentiable .

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

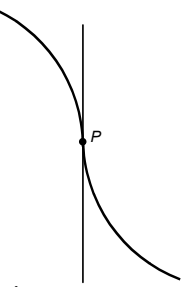
(i) a corner, where the one-sided derivatives differ



(ii) a cusp, where the slope of *PQ* approaches ∞ from one side and $-\infty$ from the other



(iii) a vertical tangent, where the slope of *PQ* approaches ∞ from both sides or approaches $-\infty$ from both sides.



(iv) a discontinuity



A function is continuous at every point where it has a derivative.

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Example 19: If f(x) is differentiable at x = a, prove that it will be continuous at x = a.

Sol. $f'(a^+)=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}=\ell$

$\lim_{h\to 0^+}[f(a+h)-f(a)]=h\ell$

as $h\to 0$ and ℓ is finite, then $\lim_{h\to 0^+} f(a+h)-f(a)=0$

$\Rightarrow \quad \lim_{h\to 0^+} f(a+h)=f(a).$

Similarly $\lim_{h\to 0^+}[f(a-h)-f(a)]=-h\ell$

$\Rightarrow \lim_{h\to 0^+} f(a-h)=f(a)$

$\therefore \quad \lim_{h\to 0^+} f(a+h)=f(a)=\lim_{h\to 0^+} f(a-h)$

Hence, f(x) is continuous.

Example 20: Find the equation of tangent to $y=(x)^{1/3}$ at $x=1$ and $x=0$.

Sol. At $x=1$ Here $f(x)=(x)^{1/3}$

$$\text{L.H.D} = f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0^+} \frac{(1-h)^{1/3}-1}{-h} = \frac{1}{3}$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^{1/3}-1}{h} = \frac{1}{3}$$

$$\text{As R.H.D.} = \text{L.H.D.} = \frac{1}{3}$$

$$\therefore \text{ slope of tangent} = \frac{1}{3}$$

$$\therefore y - f(1) = \frac{1}{3}(x-1)$$

$$y - 1 = \frac{1}{3}(x-1)$$

$\Rightarrow$ $3y - x = 2$ is tangent to $y = x^{1/3}$ at $(1, 1)$

At $x=0$

$$\text{L.H.D.} = f'(0^-) = \lim_{h\to 0^+} \frac{(0-h)^{1/3}-0}{-h} = +\infty$$

$$\text{R.H.D.} = f'(0^+) = \lim_{h\to 0^+} \frac{(0+h)^{1/3}-0}{h} = +\infty$$

As L.H.D. and R.H.D are infinite. $y = f(x)$ will have a vertical tangent at origin.

$\therefore$ $x = 0$ is the tangent to $y = x^{1/3}$ at origin.

DIFFERENTIABILITY OF SUM, PRODUCT & COMPOSITION OF FUNCTIONS

(i) If $f(x)$ & $g(x)$ are differentiable at $x=a$, then the functions $f(x)\pm g(x)$, $f(x).g(x)$ will also be differentiable at $x=a$ & if $g(a) \neq 0$, then the function $f(x)/g(x)$ will also be differentiable at $x=a$.

(ii) If $f(x)$ is not differentiable at $x=a$ & $g(x)$ is differentiable at $x=a$, then $f(x)\pm g(x)$ is non differentiable at $x=a$.

(iii) If $f(x)$ is not differentiable at $x=a$ & $g(x)$ is differentiable at $x=a$, then the product function $F(x)=f(x).g(x)$ can still be differentiable at $x=a$
e.g. $f(x)=|x|$ and $g(x)=x^2$.

(iv) If $f(x)$ & $g(x)$ both are not differentiable at $x=a$, then the product function $F(x)=f(x).g(x)$ can still be differentiable at $x=a$ e.g. $f(x)=|x|$ & $g(x)=|x|$.

(v) If $f(x)$ & $g(x)$ both are non-differentiable at $x=a$, then the sum function $F(x)=f(x)+g(x)$ may be a differentiable function. e.g. $f(x)=|x|$ & $g(x)=|x|$.

(vi) If f is differentiable at $x=a$, then

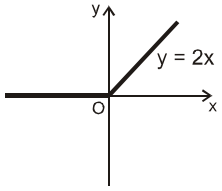
$$\lim_{h\to 0} \frac{f(a+g(h))-f(a+p(h))}{g(h)-p(h)} = f'(a), \text{ where}$$

$$\lim_{h\to 0} p(h) = \lim_{h\to 0} g(h) = 0$$

TRAIN YOUR BRAIN

Example 21: Discuss the differentiability of $f(x)=x+|x|$.

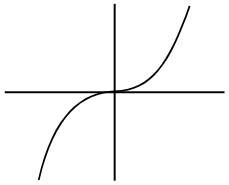
Sol.



Non-differentiable at $x=0$.

Example 22: Discuss the differentiability of $f(x)=x|x|$

Sol. $\because f(x) = \begin{cases} x^2 & , \ x \geq 0 \\ -x^2 & , \ x < 0 \end{cases}$



Differentiable at $x=0$

Example 23: If $f(x)$ is differentiable and $g(x)$ is differentiable, then prove that $f(x)\,.\,g(x)$ will be differentiable.

Sol. Given, $f(x)$ is differentiable

$$\text{i.e. } \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = f'(a)$$

$g(x)$ is differentiable

$$\text{i.e. } \lim_{h\to 0^+} \frac{g(a+h)-g(a)}{h} = g'(a)$$

let $p(x)=f(x)\,.\,g(x)$

$$\text{Now, } \lim_{h\to 0^+} \frac{p(a+h)-p(a)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(a+h).g(a+h)-f(a).g(a)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(a+h)g(a+h)+f(a+h).g(a)-f(a+h).g(a)-f(a).g(a)}{h}$$

$$= \lim_{h\to 0^+} \left[\frac{f(a+h)\,(g(a+h)-g(a))}{h} + \frac{g(a)(f(a+h)-f(a))}{h}\right]$$

$$= \lim_{h\to 0^+} \left[f(a+h).\frac{g(a+h)-g(a)}{h} + g(a).\frac{f(a+h)-f(a)}{h}\right]$$

$= f(a)\,.\,g'(a) + g(a)\,f'(a) = p'(a)$

Hence $p(x)$ is differentiable.

DIFFERENTIABILITY OVER AN INTERVAL

f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each point of the interval (a, b) and f(x) is said to be differentiable over a closed interval [a, b] if:

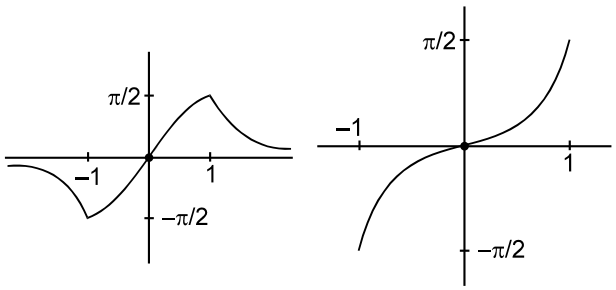
(i) for the points a and b, $f'(a^+)$ and $f'(b^-)$ exist finitely

(ii) for any point c such that $a < c < b$, $f'(c^+)$ & $f'(c^-)$ exist finitely and are equal.

Note :

(i) All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin^{-1}\frac{2x}{1+x^2}$ Graph of $y = \sin^{-1} x$.



Non differentiable at x=1 & x=–1 Non differentiable at x=1 & x=– 1

(ii) Derivability should be checked at following points

(a) At all points where continuity is required to be checked.

(b) At the critical points of modulus and inverse trigonometric function.

TRAIN YOUR BRAIN

Example 24: Suppose $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \geq 1 \end{cases}$. If f" (1) exist then find the value of $a^2 + b^2 + c^2$.

Sol. For continuity at x = 1 we leave $f(1^-) = 1$ and $f(1^+) = a + b + c$

$\therefore \quad a + b + c = 1 \quad ....(1)$

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 2ax + b & \text{if } x \geq 1 \end{cases}$$ for continuity of f' (x) at x = 1

$f'(1^-) = 3$; $f'(1^+) = 2a + b$

hence $2a + b = 3 \quad ....(2)$

$$f''(x) = \begin{cases} 6x & \text{if } x < 1 \\ 2a & \text{if } x \geq 1 \end{cases}$$ $f''(1^-) = 6$; $f''(1^+) = 2a$ for

continuity of f" (x) $2a = 6 \Rightarrow a = 3$

from (2), $b = -3$; $c = 1$. Hence $a = 3, b = -3$; $c = 1$

$\therefore \sum a^2 = 19$

Example 25: If a function f is defined by $f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ show that f is continuous but not derivable at x = 0

Sol. We have $f(0^+) = \lim\limits_{x\to 0^+} \frac{xe^{1/x}}{1+e^{1/x}} = \lim\limits_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$

$$= \lim_{x\to 0^+} \frac{x}{e^{1/x}+1} = 0$$

$$f(0^-) = \lim_{x\to 0^-} \frac{xe^{1/x}}{1+e^{1/x}} = 0$$

Also f(0) = 0

$\therefore f(0^+) = f(0^-) = f(0) \Rightarrow$ f is continuous at x = 0

Again

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{xe^{1/x}}{1+e^{1/x}} - 0}{x}$$

$$= \lim_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+} \frac{1}{e^{-1/x}+1} = 1$$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\frac{xe^{1/x}}{1+e^{1/x}} - 0}{x}$$

$$= \lim_{x\to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = 0$$

Since $f'(0^+) \neq f'(0^-)$, the derivative of f(x) at x = 0 does not exist.

Example 26: A function f(x) is such that $f\left(x+\frac{\pi}{2}\right) = \frac{\pi}{2} - |x| \ \forall\ x$.

Find $f'\left(\frac{\pi}{2}\right)$, if it exists.

Sol. Given that $f\left(x+\frac{\pi}{2}\right) = \frac{\pi}{2} - |x|$

$$\Rightarrow f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h} = \frac{\frac{\pi}{2} - |h| - \frac{\pi}{2}}{h} = -1$$

$$\Rightarrow f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \frac{\frac{\pi}{2} - |-h| - \frac{\pi}{2}}{-h} = 1$$

$\Rightarrow f'\left(\frac{\pi}{2}\right)$ doesn't exist.

PROBLEMS OF FINDING FUNCTIONS SATISFYING GIVEN CONDITIONS:

TRAIN YOUR BRAIN

Example 27: If f(x) is a function satisfies the relation for all x, y ∈ R, f(x + y) = f(x) + f(y) and if f'(0) = 2 and function is differentiable every where, then find f(x).

Sol. $f'(x)=\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0^+}\frac{f(x)+f(h)-f(x)-f(0)}{h}$ $(\because f(0)=0)$

$=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=f'(0)$

$\Rightarrow \quad f'(x)=2$

$\Rightarrow \quad \int f'(x)\,dx=\int 2\,dx$

$f(x)=2x+c$

$\because \quad f(0)=2.0+c$

as $f(0)=0$

$\therefore \quad c=0$

$\therefore \quad f(x)=2x$

Second Method:

Since $f(x+y)=f(x)+f(y)$ is true for all values of x and y is independent of differentiating both sides w.r.t x (here y is constant with respect to x).

$f'(x+y)=f'(x)$

Put $x=0$

$f'(y)=f'(0)\Rightarrow \int f'(y)\,dy=\int 2\,dy$

$f(y)=2y+c\Rightarrow f(0)=0+c=0$

$\therefore \; c=0$

$\therefore \; f(y)=2y$

$\Rightarrow f(x)=2x.$

Example 28: Let $f(xy)=xf(y)+yf(x)$ for all $x, y\in R^+$ and $f(x)$ be differentiable in $(0,\infty)$ then determine $f(x)$.

Given $f(xy)=xf(y)+yf(x)$

Sol. Replacing x by 1 and y by x then we get $x\,f(1)=0$

$\therefore \quad f(1)=0, x\neq 0 \qquad (\because x, y, \in R^+)$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$

$=\lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$

$=\lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h}$

$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)}+\lim_{h\to 0}\frac{f(x)}{x}=f'(x)+\frac{f(x)}{x}$

$\Rightarrow f'(x)-\frac{f(x)}{x}=f'(1)\Rightarrow \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$

$\Rightarrow \frac{d}{dx}\left\{\frac{f(x)}{x}\right\}=\frac{f'(1)}{x}$

On integrating w.r.t.x and taking limit 1 to x then

$\frac{f(x)}{x}-\frac{f(1)}{1}=f'(1)(\ln x-\ln 1)$

$\Rightarrow \quad \frac{f(x)}{x}-0=f'(1)\ln x \quad \because \quad f(1)=0)$

$\therefore \; f(x)=f'(1)(x\ln x)$

Alternative Method:

Given $f(xy)=xf(y)+yf(x)$

Differentiating both sides w.r.t.x treating y as constant,

$f'(xy)\,.\,y=f(y)+yf'(x)$

Putting $y=x$ and $x=1$, then

$f'(xy).\,x=f(x)+xf'(x)\Rightarrow \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$

$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x}\right)=\frac{f'(1)}{x}$

Integrating both sides w.r.t.x taking limit 1 to x,

$\frac{f(x)}{x}-\frac{f(1)}{1}=f'(1)\{\ln x-\ln 1\}$

$\Rightarrow \frac{f(x)}{x}-0=f'(1)\ln x \qquad (\because f(1)=0)$

Hence, $f(x)=-f'(1)(x\ln x)$.

Example 29: $f(x+y)=f(x)\,.\,f(y)\;\forall\; x, y\in R$ and $f(x)$ is a differentiable function and $f'(0)=1$, $f(x)\neq 0$ for any x. Find $f(x)$

Sol. $f(x)$ is a differentiable function

$\therefore \; f'(x)=\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0^+}\frac{f(x).f(h)-f(x).f(0)}{h} \qquad (\because f(0)=1)$

$=\lim_{h\to 0^+}\frac{f(x).(f(h)-f(0))}{h}=f(x)\,.\,f'(0)=f(x)$

$\therefore \quad f'(x)=f(x)$

$\therefore \quad \int\frac{f'(x)}{f(x)}\,dx=\int 1\,dx$

$\Rightarrow \quad \ell n\, f(x)=x+c$

$\therefore \quad \ell n\, 1=0+c$

$\Rightarrow \quad c=0$

$\therefore \quad \ell n\, f(x)=x$

$\Rightarrow \quad f(x)=e^x$

Example 30: $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}\;\forall\, x, y\in R$ and $f(0)=1$ and $f'(0)=-1$ and function is differentiable for ll x, then find $f(x)$.

Sol. $f'(x)=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f\left(\frac{2x+0}{2}\right)}{h}$

$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-\frac{f(2x)+f(0)}{2}}{h}$

$=\lim_{h\to 0}\frac{f(2h)-f(0)}{2h}=f'(0)=-1$

$f'(x) \qquad =-1$

Integrating both sides, we get

$f(x)=-x+c$

$\because c=+1 \quad (\text{as } f(0)=1)$

$\therefore f(x)=-x+1=1-x$

METHODS OF DIFFERENTIATION

INTRODUCTION

The essence of calculus is the derivative. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the tangent line to the function at a point. Let us use the view of derivatives as tangents to motivate a geometric definition of the derivative.

FIRST PRINCIPLE OF DIFFERENTIATION

1. The derivative of a given function f at a point x = a in its domain represent the slope of the tangent at that point, and it is defined as:

$\underset{h\to 0}{\text{Limit}}\ \frac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by f′(a).

i.e. $f'(a)=\underset{x\to a}{\text{Limit}}\ \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

2. If x and x + h belong to the domain of a function f defined by y = f(x), then

$\underset{h\to 0}{\text{Limit}}\ \frac{f(x+h)-f(x)}{h}$ if it exists, is called the Derivative of f at x & is denoted by f′(x) or $\frac{dy}{dx}$.

i.e., $f'(x)=\underset{h\to 0}{\text{Limit}}\ \frac{f(x+h)-f(x)}{h}$

This method of differentiation is also called ab-initio method or first principle method.

Note that if y = f (x) then the symbols

$\frac{dy}{dx}$ = Dy = f ' (x) = y_1 or y' have the same meaning.

However a dot, denotes the time derivative.

e.g. $\dot{S}=\frac{dS}{dt}$; $\dot{\theta}=\frac{d\theta}{dt}$ etc.

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Example 31: Find the derivative of the following functions with respect to x using first principle.

(i) $y=\frac{x}{x^2+1}$; (ii) $y=\cos^2 x$; (iii) $y=\sin 3x$; (iv) $y=x^3-3^x$; (v) $y=\ln^2 x$.

Sol. (i) $y=\frac{x}{x^2+1}=f(x)$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+h}{(x+h)^2+1}-\frac{x}{x^2+1}}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)(x^2+1)-x\left[(x^2+1)+2hx+h^2\right]}{h(x^2+1)\left[(x+h)^2+1\right]}$$

$$=\lim_{h\to 0}\frac{(x^2+1)-x(2x+h)}{(x^2+1)\left[(x+h)^2+1\right]}$$

$$=\frac{(x^2+1)-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}.$$

(ii) $y=\cos^2 x=f(x)$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{-h}{2}\right)}{h}(\cos(x+h)+\cos x))$$

$$=2\sin x\left(\frac{-1}{2}\right)(2\cos x)=-2\sin x\cos x=-\sin 2x.$$

(iii) $y=\sin 3x=f(x)$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(3x+3h)-\sin 3x}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\left(\frac{3h}{2}\right)\cos\left(3x+\frac{3h}{2}\right)}{h}$$

$$=2\cdot\left(\frac{3}{2}\right)\cdot\cos(3x+0)=3\cos 3x.$$

(iv) $y=x^3-3^x=f(x)$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3-3^{x+h}-(x^3-3^x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} - \lim_{h\to 0}\frac{(3^{x+h} - 3^x)}{h}$$

$$= \lim_{h\to 0}\frac{3xh\,(x+h) + h^3}{h} - 3^x \lim_{h\to 0}\frac{(3^h - 1)}{h}$$

$$= 3x\,(x+0) - 3^x \ln 3 = 3x^2 - 3^x \ln 3.$$

(v) $y = f(x) = \ln^2 x$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\ln^2(x+h) - \ln^2 x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\ln(x+h) - \ln x}{h}\right)\left(\ln(x+h) + \ln x\right)$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}\left(\ln(x+h) + \ln x\right) = \frac{2\ln x}{x}.$$

STANDARD DERIVATIVES

(i) $\frac{d}{dx}x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \ln a$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

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Example 32: If $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right)$, then find $\frac{dy}{dx}$.

Sol. $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right)$

$$= \left(1+x^{\frac{1}{4}}\right)\left(1-x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)$$

$$= \left(1-x^{\frac{1}{2}}\right)\left(1+x^{\frac{1}{2}}\right)$$

$$= 1-x \Rightarrow \frac{dy}{dx} = -1.$$

Example 33:

(i) $D\left(\frac{1}{\sin x}\right)$;

(ii) $D(\tan(\tan^{-1}x))$;

(iii) $D\left(\cos^4\frac{x}{2} - \sin^4\frac{x}{2}\right)$ = ____

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n$ = ____

(v) $D(\cos x \operatorname{cosec} x)$ = ____

(vi) $D\left(\frac{1}{\log_x e}\right)$ when $x = \frac{1}{e}$ = ____

(vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right)$ = ____

Sol. (i) $D\left(\frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x$

(ii) $D(\tan(\tan^{-1}x)) = 1$ $(\because \tan\tan^{-1}x = x\ \forall x \in R)$

(iii) $D\left(\cos^4\frac{x}{2} - \sin^4\frac{x}{2}\right)$

$$= \frac{d}{dx}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)$$

$$= \frac{d}{dx}(\cos x) = -\sin x$$

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n = \frac{d}{dx}\left(\frac{\pi}{2}\right)^n = 0$

(v) $D(\cos x \operatorname{cosec} x) = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $D\left(\frac{1}{\log_x e}\right) = \frac{1}{x} \quad \therefore \quad D\left(\frac{1}{\log_x e}\right)_{\text{at } x = \frac{1}{e}} = e$

(vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right) = D\left(\frac{2\sin^2 x}{2\sin x \cos x}\right)$

$= D(\tan x) = \sec^2 x$

THEOREM ON DERIVATIVES

1. $\frac{d}{dx}\left(f_1(x) \pm f_2(x)\right) = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$.

2. $\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx} f(x)$, where k is any constant.

3. **Product rule :**

$$\frac{d}{dx}\left\{f_1(x)\, f_2(x)\right\} = f_1(x)\frac{d}{dx} f_2(x) + f_2(x)\frac{d}{dx} f_1(x).$$

4. **Quotient rule:**

$$y = \frac{f(x)}{g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)},$$

to be remembered as

$$\frac{d}{dx}\left(\frac{N^r}{D^r}\right) = \frac{D^r \frac{d}{dx}(N^r) - N^r \frac{d}{dx}(D^r)}{(D^r)^2};$$

Note: (i) If $y = \frac{1}{f(x)}$ then $\frac{dy}{dx} = -\frac{f'(x)}{f^2(x)}$

(ii) If 3 functions are involved then remember

$$\frac{d}{dx}\left((f(x)\cdot g(x)\cdot h(x))\right)$$

$= f(x)\cdot g(x)\cdot h'(x) + g(x)\cdot h(x)\cdot f'(x) + h(x)\cdot f(x)\cdot g'(x)$

This result can be generalised to the product of n functions

DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

If f(x) and g(x) are differentiable functions, then fog is also differentiable and (fog)' (x) = f'(g(x)).g'(x).

$$\text{or, } \frac{d}{dx}\{fog(x)\} = \frac{d}{d\,g(x)}\{(fog)(x)\}\frac{d}{dx}(g(x))$$

or

If y is a function of t and t is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Thus, if $y = f(t)$ and $t = \phi(x)$

$$\frac{dt}{dx} = \phi'(x)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = f'(t)\,\phi'(x).$$

This rule is called Chain Rule.

This chain rule can be extended as follows

Let $y = f(t)$, $t = \phi(z)$, $z = \psi(x)$

then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dz}{dx} = f'(t)\,\phi'(z)\,\psi'(x)$

Let $y = \log \sin x^3 = \log t$

Putting $t = \sin x^3 = \sin z$, $z = x^3$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dz}{dx} = (1/t)\cos z\, 3x^2$$

$$= \left(\frac{1}{\sin x^3}\right)(\cos x^3) \times 3x^2 = 3x^2 \cot x^3$$

Note: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$.

It can be extended for any number of chain. In general if $y = (E)^C$ where E = function of x and C = constant then

$$\frac{dy}{dx} = C(E)^{C-1}\cdot\frac{d}{dx}(E)$$

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Example 34: Find the derivative of the following functions

(i) $x \sin x$; (ii) $e^x \cdot \tan x$

Sol. (i) $y = x \sin x$

$$y = x \sin x \text{ so } \frac{dy}{dx} = \sin x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \sin x + x\cos x$$

(ii) $y = e^x \tan x$

$$\text{so } \frac{dy}{dx} = \tan x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\tan x) = e^x(\tan x + \sec^2 x)$$

Example 35: Find the derivative of the following functions with respect to x.

(i) $f(x) = \sqrt{\sin(2x+3)}$ (ii) $f(x) = \frac{x}{1+x^2}$

Sol. (i) $f(x) = \sqrt{\sin(2x+3)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\sqrt{\sin(2x+3)}\right)$$

$$= \frac{1}{2\sqrt{\sin(2x+3)}}\cdot\frac{d}{dx}(\sin(2x+3)) \quad \text{(chain rule)}$$

$$= \frac{\cos(2x+3)}{\sqrt{\sin(2x+3)}}$$

(ii) $f(x) = \frac{x}{1+x^2}$

$\Rightarrow f'(x) = \frac{(1+x^2)-x(2x)}{(1+x^2)^2}$ (Quotient rule)

$= \frac{1-x^2}{(1+x^2)^2}$

Example 36: If $f(x) = \sin(x + \tan x)$, then find value of $f'(0)$.

Sol. $\because f(x) = \sin(x + \tan x)$

$\Rightarrow f'(x) = \cos(x + \tan x)(1 + \sec^2 x)$ (chain rule)

Hence, $f'(0) = 2$

Example 37: Find the derivative of the following functions

(i) $y = \frac{1-\ln x}{1+\ln x}$; (ii) $y = \frac{x^3+2^x}{e^x}$;

(iii) $y = \frac{x \sin x}{1+\tan x}$

Sol. (i) $y = \frac{1-\ln x}{1+\ln x}$

$\frac{dy}{dx} = \frac{(1+\ln x)\left(\frac{-1}{x}\right) - \frac{(1-\ln x)}{x}}{(1+\ln x)^2}$

$= \frac{-(1+\ln x+1-\ln x)}{x(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2}$

(ii) $y = \frac{x^3+2^x}{e^x} = x^3 e^{-x} + 2^x e^{-x}$

$\frac{dy}{dx} = 3x^2e^{-x} - x^3e^{-x} - 2^xe^{-x} + 2^xe^{-x}(\ln 2)$

$= (3x^2 - x^3 - 2^x + 2^x \ln 2)\, e^{-x}$

(iii) $y = \frac{x \sin x}{(1+\tan x)}$

$\frac{dy}{dx} = \frac{(1+\tan x)(\sin x + x\cos x) - (x \sin x)\sec^2 x}{(1+\tan x)^2}$

$= \frac{\sin x + x\cos x + \sin x \tan x + x \sin x - x \tan x \sec x}{(1+\tan x)^2}$

$= \frac{(1+x)\sin x + x\cos x + (\sin x - x \sec x)\tan x}{(1+\tan x)^2}$

Example 38: Find derivative of following functions?

(i) $y = \sin^3 \sqrt{x}$ (ii) $y = \ln(\sec x)$

(iii) $y = \cos(\ln x)$ (iv) $y = e^{ax} \sin bx$.

(v) $y = e^{ax} \cos bx$

Sol. (i) $y = \sin^3 \sqrt{x}$

Let $v = \sqrt{x}$; $u = \sin \sqrt{x} = \sin v$

$y = u^3$

$\frac{dy}{dx} = \frac{d}{dx}(u^3) = \frac{d}{dx}(u^3) \cdot \left(\frac{du}{dx}\right)$

$= 3u^2 \cdot \frac{d}{dv}(\sin v)\left(\frac{dv}{dx}\right) = 3u^2 \cos v \frac{d}{dx}\left(\sqrt{x}\right)$

$= 3 \sin^2 v \cos v \left(\frac{1}{2\sqrt{x}}\right) = \frac{3\sin^2 \sqrt{x} \cdot \cos \sqrt{x}}{2\sqrt{x}}$

(ii) $y = \ln(\sec x)$

Let $u = \sec x$

$\therefore y = \ln u$

$\frac{dy}{dx} = \frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u} \cdot \frac{d}{dx}(\sec x)$

$= \frac{1}{\sec x}(\sec x \tan x) = \tan x$

(iii) $y = \cos(\ln x)$

Let $u = \ln x$

$\therefore y = \cos u$

$\frac{dy}{dx} = \frac{d}{dx}(\cos u) = \frac{d}{du}(\cos u)\frac{du}{dx}$

$= -\sin u \cdot \frac{d}{dx}(\ln x) = \frac{-\sin(\ln x)}{x}$

(iv) $y = e^{ax} \sin bx$

$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\sin bx) + \sin bx \frac{d}{dx} e^{ax}$

$= e^{ax} \cos bx \frac{d}{dx}(bx) + \sin bx \cdot e^{ax} \frac{d}{dx}(ax)$

$= be^{ax} \cos bx + a \sin bx \cdot e^{ax}$

$= (b \cos bx + a \sin bx)\, e^{ax}$

(v) $y = e^{ax} \cos bx$

$\frac{dy}{dx} = e^{ax} \frac{d}{dx}(\cos bx) + \cos bx \frac{d}{dx}(e^{ax})$

$= e^{ax}(-\sin bx) \frac{d}{dx}(bx) + \cos bx\, e^{ax} \cdot a$

$= (a \cos bx - b \sin bx)\, e^{ax}$

Example 39: Find derivative of following functions ?

(i) $y = (fog)(x)$ (ii) $y = (gof)(x)$

(iii) $y = \dfrac{1}{(f(x))^n}$ (iv) $y = \sec^2\left(f^3(x)\right)$

(v) $y = \sqrt{f(x)}$ (vi) $y = f(1/x)$

Sol. (i) $y = fog(x) = f(g(x))$

$$\frac{dy}{dx} = \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \frac{d}{dx} g(x)$$

$$= f'(g(x)) \cdot g'(x)$$

(ii) $y = gof(x) = g(f(x))$

$$\frac{dy}{dx} = \frac{d}{dx} g(f(x)) = g'(f(x)) \frac{d}{dx} f(x) = g'(f(x)) \cdot f'(x)$$

(iii) $y = \dfrac{1}{(f(x))^n} = (f(x))^{-n}$

$$\frac{dy}{dx} = -n (f(x))^{-n-1} \frac{d}{dx} f(x) = \frac{-n f'(x)}{(f(x))^{n+1}}$$

(iv) $y = \sec^2 (f^3(x))$

$$\frac{dy}{dx} = 2\sec\left(f^3(x)\right) \frac{d}{dx} \sec\left(f^3(x)\right)$$

$$= 2 \sec (f^3(x)) \cdot \sec (f^3(x)) \tan (f^3(x)) \frac{d}{dx}\left(f^3(x)\right)$$

$$= 2 \sec^2 (f^3(x)) \tan (f^3(x)) \; 3f^2(x) \frac{d}{dx} f(x) .$$

$$= 6 \sec^2 (f^3(x)) \tan (f^3(x)) f^2(x) \cdot f'(x)$$

(v) $y = \sqrt{f(x)}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(f(x)\right)^{\frac{-1}{2}} \frac{d}{dx} f(x) = \frac{f'(x)}{2\sqrt{f(x)}} .$$

(vi) $y = f\left(\dfrac{1}{x}\right)$

$$\therefore \frac{dy}{dx} = f'\left(\frac{1}{x}\right) \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= f'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{-f'\left(\frac{1}{x}\right)}{x^2} .$$

Example 40: If $f(x) = (1 + x)(3 + x^2)^{1/2}(9 + x^3)^{1/3}$ then $f'(-1)$ is equal to

(*a*) 0 (*b*) $2\sqrt{2}$

(*c*) 4 (*d*) 6

Sol. $f(x) = (1 + x)(3 + x^2)^{1/2}(9 + x^3)^{1/3}$

$$f'(x) = (3 + x^2)^{1/2}(9 + x^3)^{1/3} + (1 + x)\frac{1}{2}(3 + x^2)^{-1/2} \cdot 2x (9 + x^3)^{1/3}$$

$$+ (1 + x)(3 + x^2)^{1/2}\left(\frac{1}{3}(9 + x^3)^{-2/3} \cdot 3x^2\right)$$

$$f'(-1) = (3 + 1)^{1/2}(9 - 1)^{1/3} + 0 + 0$$

$$f'(-1) = 2 \cdot 2 = 4$$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithm of the function first and then differentiate is called **Logarithmic differentiation**. It is often useful in situations when

(i) a function which is the product or quotient of a number of functions

$$y = f_1(x) f_2(x) f_3(x) .....$$

or $$y = \frac{f_1(x) f_2(x) f_3(x).....}{g_1(x) g_2(x) g_3(x).....} .$$

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate *i.e.*,

$y = \left(f(x)\right)^{g(x)} = e^{g(x).\ln(f(x))}$ and then differentiate.

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Example 41: Find the derivative of the following functions

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)(x^x)$

(ii) $y = (x^{\ln x})(\sec x)^{3x}$;

(iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

Sol.

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)\left(x^x\right)$ take logarithm on both sides

$$\ln y = \ln (\sin x) + \sqrt{\sin x} + \ln (\ln x) + x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{(\cos x)}{2\sqrt{\sin x}} + \frac{1}{(\ln x)} \cdot \frac{1}{x} + \ln x + 1$$

$$\frac{dy}{dx} = y\left(\cot x + \frac{1}{2}\frac{\cos x}{\sqrt{\sin x}} + \frac{1}{x \ln x} + \ln x + 1\right)$$

(ii) $y = x^{\ln x}\left(\sec x\right)^{3x}$

$$\ln y = (\ln x)^2 + 3x \ln (\sec x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2 \ln x}{x} + 3 \ln\left(\sec x\right) + \frac{3x}{(\sec x)} (\sec x \tan x)$$

$$\frac{dy}{dx} = y\left(\frac{2 \ln x}{x} + 3 \ln (\sec x) + 3x \tan x\right) .$$

(iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

$\ln y = x \ln(\ln x) - (3x+1)\ln 2$

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln(\ln x) + \dfrac{x}{(\ln x)}\cdot\dfrac{1}{x} - 3\ln 2$

$\dfrac{dy}{dx} = y\left(\ln(\ln x) + \dfrac{1}{(\ln x)} - 3\ln 2\right).$

Example 42: If $y = (\sin x)^{\ln x}\,\text{cosec}\,(e^x(a+bx))$ and $a + b = \dfrac{\pi}{2e}$

then the value of $\dfrac{dy}{dx}$ at $x = 1$ is

(*a*) $(\sin 1)\ln\sin(1)$ (*b*) 0

(*c*) $\ln\sin(1)$ (*d*) $1 + \ln(\sin 1)$

Sol. $y = (\sin x)^{\ln x}\,\text{cosec}\,(e^x(a+bx))$

$\ln y = (\ln x)\ln(\sin x) - \ln\left(\sin\left(e^x(a+bx)\right)\right)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}\ln(\sin x) + (\ln x)\cot x$

$$-\frac{\cos\left(e^x(a+bx)\right)\left(ae^x + bxe^x + be^x\right)}{\sin\left(e^x(a+bx)\right)}$$

$$\left.\frac{1}{y}\frac{dy}{dx}\right|_{\text{at } x=1} = \ln(\sin 1) + 0 - \frac{\cos\left(\frac{\pi}{2}\right)(a+2b)e}{\sin\left(\frac{\pi}{2}\right)}$$

$\left(\because \text{using } a + b = \dfrac{\pi}{2e}\right)$

$= \ln(\sin 1) + 0 - 0$

Also $y = 1$ at $x = 1$

so $\left.\dfrac{dy}{dx}\right|_{\text{at } x=1} = \ln(\sin 1)\; y\,|_{\text{at } x=1}$

$= \ln(\sin 1)\cdot 1 = \ln(\sin 1)$

Hence, option (C) is correct

Example 43: Find the derivative of $\dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$ w.r.t. x,

Sol. Let $y = \dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$\log y = \dfrac{1}{2}\log x + \dfrac{3}{2}\log(x+4) - \dfrac{4}{3}\log(4x-3)$

Diff. both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2(x+4)}\times 1 - \frac{4}{3}\times\frac{1}{4x-3}\times 4$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

DERIVATIVE OF IMPLICIT FUNCTIONS

Implicit functions: If the variable x and y are connected by a relation of the form f(x, y) = 0 and it is not possible to express y as a function of x in the form $y = \phi(x)$, then such functions are said to be implicit functions.

For example,

(i) $x^2 + xy + y^3 = 1$ (ii) $x + y + \sin(xy) = 1$

(iii) $x^y + y^x = 1$ (iv) $16^{x^2+y} + 16^{x+y^2} = 1$

Derivative of Implicit Functions:

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) Corresponding to every curve represented by an implicit equation, there exist one or more explicit functions representing that equation. It can be shown that dy/dx at any point on the curve remains the same whether the process of differentiation is done explictly or implicitly.

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\cdot\frac{dy}{dx} = f'(y)\cdot\frac{dy}{dx}$$

For example :

(i) $\dfrac{d}{dx}(\sin y) = \dfrac{d}{dy}(\sin y)\dfrac{dy}{dx} = \cos y\left(\dfrac{dy}{dx}\right)$

(ii) $\dfrac{d}{dx}(y^3) = \dfrac{d}{dy}(y^3)\cdot\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx}$

A direct formula for implicit functions: Let f(x, y) = 0. Take all the terms of left side and put left side equal to f(x, y).

$$\text{Then } \frac{dy}{dx} = -\frac{\text{diff. of f w.r.t. x keeping y as constant}}{\text{diff. of f w.r.t. y keeping x as constant}}$$

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Example 44: If $x^2 + 2xy + y^3 = 4$, find $\dfrac{dy}{dx}$.

Sol. We have $x^2 + 2xy + y^3 = 4$

Diff. both sides w.r.t. x,

$$\Rightarrow \frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

$$\Rightarrow\ 2x+2\left(x\frac{dy}{dx}+y.1\right)+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow\ \frac{dy}{dx}=-\frac{2(x+y)}{(2x+3y^2)}$$

2nd Method :

$$\frac{dy}{dx}=-\frac{\text{diff. of f w.r.t. x keeping y as constant}}{\text{diff. of f w.r.t. y keeping x as constant}}$$

$$=-\frac{2x+2y}{2x+3y^2}$$

Example 45: If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+...}}}$, prove that $\frac{dy}{dx}=\frac{y}{2y-x}$.

Sol. We have, $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+...}}}$,

$$\Rightarrow y=x+\frac{1}{y}\Rightarrow y^2=xy+1$$

$$\Rightarrow\ 2y\frac{dy}{dx}=y+x\frac{dy}{dx}+0$$

[Diff. both sides with respect to x] $\Rightarrow \frac{dy}{dx}(2y-x)=y$

$$\Rightarrow\ \frac{dy}{dx}=\frac{y}{2y-x}.$$

Example 46: If $\sqrt{x}+\sqrt{y}=4$, then find $\frac{dx}{dy}$ at $y=1$.

Sol. Diff. both sides of the given equation w.r.t. y, we get

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy}+\frac{1}{2\sqrt{y}}=0\Rightarrow\frac{dx}{dy}=-\frac{\sqrt{x}}{\sqrt{y}}=\frac{\sqrt{y}-4}{\sqrt{y}}$$

$$\Rightarrow\left[\frac{dx}{dy}\right]_{y=1}=\frac{1-4}{1}=-3.$$

Example 47: If $x^y=e^{x-y}$ then prove that $\frac{dy}{dx}=\frac{\ln x}{(1+\ln x)^2}$.

Sol. $x^y=e^{x-y}$

$y\ln x=(x-y)$ ...(i)

now differentiating with respect to x.

$$\frac{y}{x}+\ln x\cdot\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{1-\frac{y}{x}}{1+\ln x}=\frac{x-y}{x(1+\ln x)}=\frac{y\ln x}{x(1+\ln x)}$$

$$\left(\text{From (i)},\ y=\frac{x}{1+\ln x}\right)$$

$$\therefore\ \frac{dy}{dx}=\frac{\ln x}{(1+\ln x)^2}.$$

Example 48: If $\sin y = x\sin(a+y)$ then prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

Sol. $\sin y = x\sin(a+y)$ differentiating both sides w.r.t. x

$$\cos y\cdot\frac{dy}{dx}=\sin(a+y)+x\cos(a+y)\cdot\left(0+\frac{dy}{dx}\right).$$

$$\Rightarrow\ \frac{dy}{dx}\left(\cos y-x\cos(a+y)\right)=\sin(a+y)$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{\sin(a+y)}{\cos y-x\cos(a+y)}$$

$$=\frac{\sin(a+y)}{\cos y-\frac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$=\frac{\sin^2(a+y)}{\sin(a+y)\cos y-\sin y\cos(a+y)}=\frac{\sin^2(a+y)}{\sin a}$$

Example 49: If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Sol. $x\sqrt{1+y}+y\sqrt{1+x}=0$

$\Rightarrow\ x^2(1+y)=y^2(1+x)$

$\Rightarrow\ (x^2-y^2)+x^2y-y^2x=0$

$\Rightarrow\ (x-y)(x+y+xy)=0$

$\Rightarrow\ x-y=0$ or $x+y+xy=0$

$\therefore\ y=x$ or $\frac{-x}{x+1}$.

$$\Rightarrow\ \frac{dy}{dx}=1\ \text{or}\ \frac{-1}{(x+1)^2}$$

Example 50: If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+.....\infty}}}$

find $\frac{dy}{dx}$ $(\sin x>0)$.

Sol. $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+.....\infty}}}$

$$\Rightarrow\ y=\sqrt{\sin x+y}$$

$$\Rightarrow\ y^2-y=\sin x\Rightarrow 2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x$$

$$\Rightarrow\ \frac{dy}{dx}=\left(\frac{\cos x}{2y-1}\right).$$

Example 51:

(i) $y = x^{x^{x^{\cdot^{\cdot^{\cdot^{\infty}}}}}}$ (ii) $y = (\ln x)^{(\ln x)^{(\ln x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}}$

Sol.

(i) $y = x^{x^{x^{\cdots\infty}}}$

$\Rightarrow \quad y = x^y \qquad \Rightarrow \qquad \ln y = y \ln x$

$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x} + \ln x \frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}\times\left(\frac{1}{y} - \ln x\right) = \frac{y}{x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{(1-y\ln x)}{y}} = \frac{y^2}{x(1-y\ln x)}$

(ii) $y = (\ln x)^y$

$\Rightarrow \quad \ln y = y\ln(\ln x)$

$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x\ln x} + \ln\ln x \frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - \ln\ln x\right) = \frac{y}{x\ln x}$

$\Rightarrow \quad \frac{dy}{dx}\frac{\frac{y}{x\ln x}}{\frac{(1-y\ln\ln x)}{y}} = \frac{y^2}{x\ln x(1-y\ln\ln x)}$

Example 52: If $y = \sqrt{x\log_e x}$, then find $\frac{dy}{dx}$ at $x = e$.

Sol. $\frac{dy}{dx} = \frac{1}{2\sqrt{x\log_e x}}\frac{d}{dx}[x\log_e x]$

$= \frac{1}{2\sqrt{x\log_e x}}\left[x\cdot\frac{1}{x} + 1\cdot\log_e x\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=e} = \frac{1}{2\sqrt{e\cdot 1}}(1+1) = \frac{1}{\sqrt{e}} \quad (\because \log_e e = 1)$

DERIVATIVE OF INVERSE FUNCTION

Let the functions f(x) and g(x) be inverse of each other then

$f(g(x)) = g(f(x)) = x$

$\therefore \; f'(g(x))\; g'(x) = g'(f(x))\, f'(x) = 1$

If $\frac{dy}{dx}$ exists and $\frac{dy}{dx} \neq 0$, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ or $\frac{dy}{dx}\cdot\frac{dx}{dy} = 1$

or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad [\frac{dx}{dy} \neq 0]$

Note : **Some Standard Substitutions**

Expressions	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos 2\theta$

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Example 53: If $f(x) = \ell n(\sin^{-1}x^2)$, then find $f'(x)$.

Sol. $f'(x) = \frac{1}{(\sin^{-1}x^2)}\cdot\frac{1}{\sqrt{1-(x^2)^2}}\cdot 2x$

$= \frac{2x}{(\sin^{-1}x^2)\sqrt{1-x^4}}$

Example 54: If $f(x) = 2x\sec^{-1}x - \text{cosec}^{-1}(x)$, then find $f'(-2)$.

Sol. $f'(x) = 2\sec^{-1}(x) + \frac{2x}{|x|\sqrt{x^2-1}} + \frac{1}{|x|\sqrt{x^2-1}}$

Hence, $f'(-2) = 2.\sec^{-1}(-2) - \frac{2}{\sqrt{3}} + \frac{1}{2\sqrt{3}}$

$f'(-2) = \frac{4\pi}{3} - \frac{\sqrt{3}}{2}$

Example 55: Find $\frac{dy}{dx}$ for $y = \tan^{-1}\left\{\frac{1-\cos x}{\sin x}\right\}$, where $-\pi < x < \pi$.

Sol. $y = \tan^{-1}\left\{\frac{1-\cos x}{\sin x}\right\} = \tan^{-1}\left\{\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

$\left(\because -\pi < x < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}.$

Example 56: $D\left(\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)\right)$ is equal to

Sol. $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases}\frac{\pi}{2}, & x > 0\\ \frac{-\pi}{2}, & x < 0\end{cases}$

$D\left(\tan^{-1} + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 \;\forall\; x \in R - \{0\}.$

Example 57: If $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}$, find $\left.\dfrac{dy}{dx}\right]_{x=-1}$;

(*a*) 0 (*b*) 1

(*c*) $2/\pi$ (*d*) -1

Sol. $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}=\dfrac{2}{\pi}\left(\tan^{-1}x-\cot^{-1}x\right)$

$$\therefore \frac{dy}{dx}=\frac{2}{\pi}\left(\frac{1}{1+x^2}+\frac{1}{1+x^2}\right)=\frac{4}{\pi(1+x^2)}$$

$$\left.\frac{dy}{dx}\right|_{x=-1}\frac{4}{\pi(1+(-1)^2)}=\frac{2}{\pi}$$

Example 58:

(i) $y=\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$, $x\in(0,1)$

(ii) $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, $x\in(0,1)$

(iii) If $f(x)=\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$, find $f'(0)$.

Sol.

(i) $y=\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Let $x=\cos 2\theta \Rightarrow \theta=\dfrac{1}{2}\cos^{-1}(x)$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$y=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)=\frac{\pi}{4}-\theta$$

$$y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}.$$

(ii) $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Put $x=\tan\theta$ $\quad\therefore \theta=\tan^{-1}x$

$$y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow y=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x \qquad \Rightarrow \frac{dy}{dx}=\frac{1}{2(1+x^2)}.$$

(iii) $f(x)=\sin^{-1}\left(\dfrac{2^{x+1}}{1+2^{2x}}\right)$

Let $2^x=\tan\theta$

$$f(x)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$$

$$f(x)=2\theta=2\tan^{-1}(2^x) \quad\Rightarrow\quad f'(x)=\frac{2}{1+2^{2x}}(2^x\ln 2)$$

$$f'(0)=\frac{2}{1+1}(2^0\ln 2)=\ln 2.$$

Example 59: If $y=\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$, and $0<x<1$, then find $\dfrac{dy}{dx}$

Sol. $y=\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$, where $0<x<1$

$$=\sin^{-1}\left[x\sqrt{1-\left(\sqrt{x}\right)^2}-\sqrt{x}\sqrt{1-x^2}\right]$$

$$=\sin^{-1}x-\sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$$

PARAMETRIC DIFFERENTIATION

In some situation curves are represented by the equations e.g. $x=\sin t$ & $y=\cos t$

If $x=f(t)$ and $y=g(t)$ then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{g'(t)}{f'(t)}$$

Example 60: If $x = a\cos^3 t$ and $y = a\sin^3 t$, then find the value of $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$

Example 61: If $y = a\cos t$ and $x = a(t - \sin t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

Sol. $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)}$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1.$

Example 62: If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Sol. We have $x = a\sec^3\theta$ and $y = a\tan^3\theta$

$\frac{dx}{d\theta} = 3a\sec^2\theta\cdot(\sec\theta\tan\theta) = 3a\tan^2\theta\sec^3\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

DIFFERENTIATION OF ONE FUNCTION W.R.T. OTHER FUNCTION

If $y = f(x)$ and $z = g(x)$ then derivative of $f(x)$ w.r.t. $g(x)$ is given by

$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$

$\therefore$ Differential coefficient of $f(x)$ w.r.t. $g(x) = \frac{\text{derivative of } f(x) \text{ w.r.t. } x}{\text{derivative of } g(x) \text{ w.r.t. } x} = \frac{f'(x)}{g'(x)}$

Example 63: Find derivative of $y = \ell n\, x$ with respect to $z = e^x$.

Sol. $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1}{xe^x}$

Example 64: Find derivative of $(\ln x)^{\tan x}$ w.r.t. x^x.

Sol. $u = (\ln x)^{\tan x}$; $v = x^x$ $\ln(u) = \tan x\ln(\ln x)$

$\Rightarrow \frac{1}{u}\left(\frac{du}{dx}\right) = (\sec^2 x)\ln(\ln x) + \tan x\left(\frac{1}{\ln x}\cdot\frac{1}{x}\right)$

$\frac{du}{dx} = \frac{u\left((x\ln x)\ln(\ln x)\sec^2 x + \tan x\right)}{(x\ln x)}$

$\ln v = x\ln x$

$\Rightarrow \frac{1}{v}\left(\frac{dv}{dx}\right) = (\ln x + 1)$

$\frac{du}{dx} = \frac{(\ln x)^{\tan x}}{x^x}\left(\frac{x\ln x\ln(\ln x)\sec^2 x + \tan x}{(x\ln x)(\ln x + 1)}\right).$

Example 65: Derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\sqrt{1-x^2}$ when $x = \frac{1}{2}$.

Sol. $u = \cos^{-1}(2x^2 - 1)$; $v = \sqrt{(1-x^2)}$

put $x = \cos\theta$ as $x = \frac{1}{2}$ so $\theta = \frac{\pi}{3}$

$u = \cos^{-1}(\cos 2\theta) = 2\theta$

put $v = \sin\theta$

$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{\cos\theta} = 4.$

Example 66:

(i) Differential coefficient of $e^{\sin^{-1}x}$ w.r.t. $e^{-\cos^{-1}x}$ is independent of x. (T or F)

(ii) Find the derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$.

Sol. (i) Let $u = e^{\sin^{-1}x}$ and $v = e^{-\cos^{-1}x}$

$u = e^{\frac{\pi}{2} - \cos^{-1}x}$ $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$u = e^{\frac{\pi}{2}}e^{-\cos^{-1}x} = e^{\frac{\pi}{2}}v$

So, $\frac{du}{dv} = e^{\frac{\pi}{2}}$. (True)

(ii) Let $u = f(\tan x)$ and $v = g(\sec x)$

$\Rightarrow \frac{du}{dx} = f'(\tan x)\sec^2 x$ and

$\frac{dv}{dx} = g'(\sec x)\sec x\tan x$

$\Rightarrow \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$

$\Rightarrow \left[\frac{du}{dv}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}} = \frac{f'(1)\sqrt{2}}{g'\left(\sqrt{2}\right)}$

$= \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}.$

HIGHER ORDER DERIVATIVES

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists in (a, b) is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w. r. t. x.

If it happens that the first derivative has a derivative in (a, b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''. While the first derivative denotes slope of the graph, the second derivative denotes it's concavity.

Similarly, the 3rd order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$, it is also denoted by $f'''(x)$ or y'''.

It must be carefully noted that in case of parametric functions

although $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ but $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{dx^2/dt^2}$ rather

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

which on applying chain rule can be resolved as

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{dy/dt}{dx/dt}\right\}\cdot\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2}\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\left[\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right]}{\left(\frac{dx}{dt}\right)^3}$$

Note: A homogeneous equation of degree n represents 'n' straight lines passing through the origin, hence

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 0$$

e.g. If $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$, then $\left.\frac{d^2y}{dx^2}\right]_{(1,1)} = 0$

Example 67:

(i) If $y = a\cos(\ln x) + b\sin(\ln x)$ then prove that $x^2y_3 + 3xy_2 + 2y_1 = 0$

(ii) If $y = e^{a\cos^{-1}x}$, then prove that $(1-x^2)y_2 - xy_1 - a^2y = 0$.

(iii) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ then prove that $(x^2-1)y_3 + 3xy_2 + (1-m^2)y_1 = 0$

Sol. (i) $y = a\cos(\ln x) + b\sin(\ln x)$

$$y_1 = \frac{-a\sin(\ln x)}{x} + \frac{b\cos(\ln x)}{x}$$

$$\Rightarrow x\,y_1 = -a\sin(\ln x) + b\cos(\ln x)$$

$$\Rightarrow x\,y_2 + y_1 = \frac{-a\cos(\ln x)}{x} - \frac{b\sin(\ln x)}{x}$$

$$\Rightarrow x^2y_2 + x\,y_1 = -a\cos(\ln x) - b\sin(\ln x) = -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

$$\Rightarrow (x^2y_3 + 2xy_2) + (xy_2 + y_1) + y_1 = 0$$

$$\Rightarrow x^2y_3 + 3xy_2 + 2y_1 = 0.$$

(ii) $y = e^{a\cos^{-1}x}$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = y_1$$

$$\Rightarrow y_1 = \frac{-ay}{\sqrt{1-x^2}} \Rightarrow (1-x^2)(y_1)^2 = a^2y^2$$

now differentiating w.r.t. to x

$$(1-x^2)\cdot 2y_1y_2 - 2x(y_1)^2 = a^2(2yy_1)$$

$$\Rightarrow (1-x^2)y_2 - x\,y_1 = a^2y.$$

(iii) $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$

$$\Rightarrow y^{2/m} - 2x\,y^{1/m} + 1 = 0 \Rightarrow y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$= \left(x \pm \sqrt{x^2-1}\right)$$

$$\Rightarrow y = \left(x \pm \sqrt{x^2-1}\right)^m$$

$$\Rightarrow y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left(1 \pm \frac{x}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow y_1 = \frac{m}{\sqrt{x^2-1}}\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left(\sqrt{x^2-1} \pm x\right)$$

$$\Rightarrow y_1 = \pm\frac{my}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)(y_1)^2 = m^2y^2$$

$$\Rightarrow (x^2-1)2y_1y_2 + (y_1)^2 2x = m^2y2y_1$$

$$\Rightarrow (x^2-1)y_2 + x\,y_1 = m^2y$$

$$\Rightarrow (x^2-1)y_3 + 2x\,y_2 + x\,y_2 + y_1 = m^2y_1$$

$$\Rightarrow (x^2-1)y_3 + 3xy_2 + (1-m^2)y_1 = 0.$$

Example 68: Use the substitution x = tanθ to show that the equation,

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0$$

changes to $\frac{d^2y}{d\theta^2}+y=0.$

Sol. $$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\left(\frac{dy}{dx}\right)+\frac{y}{\left(1+x^2\right)^2}=0 \quad .....(1)$$

$$x=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta \Rightarrow \frac{d\theta}{dx}=\frac{1}{\sec^2\theta}=\cos^2\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\left(\frac{d\theta}{dx}\right)=\frac{1}{\sec^2\theta}\left(\frac{dy}{d\theta}\right)=\cos^2\theta\left(\frac{dy}{d\theta}\right)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\sec^2\theta}\cdot\frac{dy}{d\theta}\right)$$

$$=\frac{d}{d\theta}\left(\cos^2\theta\frac{dy}{d\theta}\right)\cdot\frac{d\theta}{dx}$$

$$=\left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)\cos^2\theta$$

From equation (1) putting x = tan θ

$$\cos^2\theta\left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)$$

$$+\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\left(\cos^2\theta\frac{dy}{d\theta}\right)+\left(\frac{y}{\left(1+\tan^2\theta\right)^2}\right)=0$$

$$\Rightarrow \left(\frac{d^2y}{d\theta^2}-2\tan\theta\frac{dy}{d\theta}\right)+\left(2\tan\theta\frac{dy}{d\theta}\right)+y=0$$

$$\Rightarrow \frac{d^2y}{d\theta^2}+y=0.$$

Example 69: If $e^{x+y}=y^2$ then prove that $y''=\frac{2y}{(2-y)^3}$.

Sol. $e^{x+4}=y^2 \Rightarrow x+y=2\ln y$

$$1+\frac{dy}{dx}=\frac{2}{y}\left(\frac{dy}{dx}\right)\Rightarrow 1=\left(\frac{2-y}{y}\right)\frac{dy}{dx}\Rightarrow \frac{dy}{dx}=\frac{y}{2-y}$$

$$\frac{d^2y}{dx^2}=\frac{(2-y)\frac{dy}{dx}-y\left(\frac{-dy}{dx}\right)}{(2-y)^2}$$

$$=\frac{2}{(2-y)^2}\left(\frac{dy}{dx}\right)=\frac{2y}{(2-y)^3}.$$

DEDUCTION OF NEW IDENTITIES BY DIFFERENTIATING A GIVEN IDENTITY

Example 70: If

$$\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.....\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(x/2^n\right)}$$, then prove that

$$\sum_{r=1}^{n}\frac{1}{2^r}\tan\frac{x}{2^r}=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x$$

Sol. $$\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}....\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(x/2^n\right)} \quad ...(1)$$

Taking logarithm of both sides

$$\log\left(\cos\frac{x}{2}\right)+\log\left(\cos\frac{x}{2^2}\right)+\log\left(\cos\frac{x}{2^3}\right)+.....+\log\left(\cos\frac{x}{2^n}\right)$$

$$=\log(\sin x)-\log\left(2^n\sin\left(\frac{x}{2^n}\right)\right)$$

Differentiating w.r.t. to x

$$-\left(\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{1}{2^2}\frac{\sin\left(\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)}+....+\frac{1}{2^n}\frac{\sin\left(\frac{x}{2^n}\right)}{\cos\left(\frac{x}{2^n}\right)}\right)$$

$$=\cot x-\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\cdot\cos\left(\frac{x}{2^n}\right)$$

so $$\sum_{r=1}^{n}\frac{1}{2^r}\tan\left(\frac{x}{2^r}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^r}\right)-\cot x.$$

DERIVATIVE OF A DETERMINANT

$$\text{If } F(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ l(x) & m(x) & n(x)\\ u(x) & v(x) & w(x)\end{vmatrix},$$

where f, g, h, l, m, n, u, v, w are differentiable functions of x, then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x)\\ l(x) & m(x) & n(x)\\ u(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x)\\ l'(x) & m'(x) & n'(x)\\ u(x) & v(x) & w(x)\end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x)\\ l(x) & m(x) & n(x)\\ u'(x) & v'(x) & w'(x)\end{vmatrix}$$

Example 71: If

$$f(x)=\begin{vmatrix}\cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2)\\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2)\\ \sin 2x & 0 & \sin 2x^2\end{vmatrix}$$

then find $f'(x)$.

Sol. $$f(x)=\begin{vmatrix}\cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2)\\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2)\\ \sin 2x & 0 & \sin 2x^2\end{vmatrix}$$

Expanding the determinant

$f(x) = \cos(x+x^2)\cos(x-x^2)\sin 2x^2 - \sin(x+x^2)$

$\left(\sin(x-x^2)\right)\left(\sin 2x^2 - \sin 2x\right)$

$+ \cos(x+x^2)\cos(x-x^2)\sin 2x$

$= \sin 2x^2 \cos 2x + \sin 2x \cos(2x^2)$

$= \sin(2x^2+2x)$

$f'(x) = \cos(2x^2+2x)\,2(2x+1)$

L' HOSPITAL'S RULE:

If $f(x)$ & $g(x)$ are functions of x such that:

(i) $\underset{x\to a}{\text{Limit}}\ f(x) = \infty = \underset{x\to a}{\text{Limit}}\ g(x)$

OR

(ii) $\underset{x\to a}{\text{Limit}}\ f(x) = 0 = \underset{x\to a}{\text{Limit}}\ g(x)$, both $f(x)$ and $g(x)$ are continuous at $x = a$, both $f(x)$ and $g(x)$ are differentiable at $x = a$ and both $f'(x)$ and $g'(x)$ are continuous at $x = a$,

then $\underset{x\to a}{\text{Limit}}\ \dfrac{f(x)}{g(x)} = \underset{x\to a}{\text{Limit}}\ \dfrac{f'(x)}{g'(x)}$

Example 72: Evaluate $\underset{x\to 0}{\text{Lim}}\ \dfrac{x\cos x - \ln(1+x)}{x^2}$.

Sol. $\underset{x\to 0}{\text{Lim}}\ \dfrac{x\cos x - \ln(1+x)}{x^2}$ $\left(\dfrac{0}{0}\right)$ form

$= \underset{x\to 0}{\text{Lim}}\ \dfrac{(\cos x - x\sin x) - \dfrac{1}{1+x}}{2x}$ $\left(\dfrac{0}{0}\right)$ form

$= \underset{x\to 0}{\text{Lim}}\ \dfrac{-\sin x - (\sin x + x\cos x) + \dfrac{1}{(1+x)^2}}{2}$

$= \dfrac{-0-(0+0)+1}{2} = \dfrac{1}{2}$.

Example 73: $\underset{x\to 0}{\text{Lim}}\ \dfrac{e^x \sin x - x - x^2}{x^2 + x - \ln(1-x)}$

Sol. $L = \underset{x\to 0}{\text{Lim}}\ \dfrac{e^x \sin x - x - x^2}{x^2 + x - \ln(1-x)} \Rightarrow \left(\dfrac{0}{0}\right)$ form

$= \underset{x\to 0}{\text{Lim}}\ \dfrac{e^x(\cos x + \sin x) - 1 - 2x}{2x + 1 + \dfrac{1}{(1-x)}}$ $\left(\dfrac{0}{0}\right)$ form

$= \dfrac{1-1-0}{0+1+1} = 0$.

Example 74: Evaluate $\underset{x\to 1}{\text{Lim}}\ \dfrac{x^x - x}{x - 1 - \ln x}$

Sol. $\underset{x\to 1}{\text{Lim}}\ \dfrac{x^x - x}{x - 1 - \ln x}$

$= \underset{x\to 1}{\text{Lim}}\ \dfrac{e^{x\ln x} - x}{x - 1 - \ln x}$ $\left(\dfrac{0}{0}\right)$ form

$= \underset{x\to 1}{\text{Lim}}\ \dfrac{e^{x\ln x}(1+\ln x) - 1}{\left(1 - \dfrac{1}{x}\right)}$ $\left(\dfrac{0}{0}\right)$ form

$= \underset{x\to 1}{\text{Lim}}\ \dfrac{e^{x\ln x}(1+\ln x)^2 + e^{x\ln x}\cdot\dfrac{1}{x} - 0}{\left(\dfrac{1}{x^2}\right)}$ $\left(\dfrac{0}{0}\right)$ form

$= 1 + 1 - 0 = 2.$

Example 75: Evaluate $\underset{x\to 0}{\text{Lim}}\ \dfrac{1 - \cos x\cdot\cos 2x\cdot\cos 3x}{x^2}$

Sol. $\underset{x\to 0}{\text{Lim}}\ \dfrac{1 - \cos x\cos 2x\cos 3x}{x^2}$

$= \underset{x\to 0}{\text{Lim}}\ \dfrac{1 - e^{(\ln\cos x + \ln\cos 2x + \ln\cos 3x)}}{x^2}$

$= \underset{x\to 0}{\text{Lim}}\ \dfrac{-e^{\ln(\cos x) + \ln\cos 2x + \ln\cos 3x}\cdot(-\tan x - 2\tan 2x - 3\tan 3x)}{2x}$

$= \dfrac{e^0}{2}(1+4+9) = 7.$ **Ans.**

Topicwise Questions

CONTINUITY AND DIFFERENTIABILITY

1. Let $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & \text{if } x\neq 2\\ k, & \text{if } x=2\end{cases}$.

If $f(x)$ be continuous for all x, then k =

(*a*) 7 (*b*) –7

(*c*) ±7 (*d*) 14

2. The points at which the function $f(x)=\dfrac{x+1}{x^2+x-12}$ is discontinuous, are

(*a*) –3, 4 (*b*) 3, –4

(*c*) –1, –3, 4 (*d*) –1, 3, 4

3. If function $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & \text{when } x\neq 1\\ k, & \text{when } x=1\end{cases}$ is continuous at x = 1, then the value of k will be

(*a*) –1 (*b*) 2

(*c*) –3 (*d*) –2

4. If $f(x)=\begin{cases}\dfrac{\sin 2x}{5x}, & \text{when } x\neq 0\\ k, & \text{when } x=0\end{cases}$ is continuous at x = 0, then the value of k will be

(*a*) 1 (*b*) $\dfrac{2}{5}$

(*c*) $-\dfrac{2}{5}$ (*d*) 2/3

5. If $f(x)=\begin{cases}1+x, & \text{when } x\leq 2\\ 5-x, & \text{when } x\leq 3\end{cases}$, then

(*a*) $f(x)$ is continuous at x = 2

(*b*) $f(x)$ is discontinuous at x = 2

(*c*) $f(x)$ is continuous at x = 3

(*d*) $f(x)$ is discontinuous at x= 3

6. If $f(x)=\begin{cases}1, & \text{when } 0<x\leq\dfrac{3\pi}{4}\\ 2\sin\dfrac{2}{9}x, & \text{when } \dfrac{3\pi}{4}<x<\pi\end{cases}$, then

(*a*) $f(x)$ is continuous at $x=0$

(*b*) $f(x)$ is continuous at $x=\pi$

(*c*) $f(x)$ is continuous at $x=\dfrac{3\pi}{4}$

(*d*) $f(x)$ is discontinuous at $x=\dfrac{3\pi}{4}$

7. If $f(x)=\begin{cases}x^2, & \text{when } x\leq 1\\ x+5, & \text{when } x>1\end{cases}$, then

(*a*) $f(x)$ is continuous at $x=1$

(*b*) $f(x)$ is discontinuous at $x=1$

(*c*) $\lim\limits_{x\to 1} f(x)=1$

(*d*) $\lim\limits_{x\to 1} f(x)=2$

8. If $f(x)=\begin{cases}x+\lambda, & x<3\\ 4, & x=3\\ 3x-5, & x>3\end{cases}$ is continuous at x = 3, then λ

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) 1

9. If $f(x)=\begin{cases}\dfrac{1-\sin x}{\pi-2x}, & x\neq\dfrac{\pi}{2}\\ \lambda, & x=\dfrac{\pi}{2}\end{cases}$, be continuous at x = π/2, then value of λ is

(*a*) –1 (*b*) 1

(*c*) 0 (*d*) 2

10. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}$, $(x\neq 0)$, is continuous function at $x=0$, then $f(0)$ equals

(*a*) $\dfrac{1}{4}$ (*b*) $-\dfrac{1}{4}$

(*c*) $\dfrac{1}{8}$ (*d*) $-\dfrac{1}{8}$

11. If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{for } -1\leq x<0\\ 2x^2+3x-2, & \text{for } 0\leq x\leq 1\end{cases}$, is continuous at x = 0, then k =

(*a*) –4 (*b*) –3

(*c*) –2 (*d*) –1

12. The function $f(x)=|x|+\dfrac{|x|}{x}$ is

(*a*) Continuous at the origin

(*b*) Discontinuous at the origin because |x| is discontinuous there

(*c*) Discontinuous at the origin because $\dfrac{|x|}{x}$ is discontinuous there

(*d*) Discontinuous at the origin because both |x| and $\dfrac{|x|}{x}$

TYPES OF DISCONTINUITY

13. If $f(x)=\frac{x^2-10x+25}{x^2-7x+10}$ for $x \neq 5$ and f is continuous at $x=5$, then $f(5)=$

(*a*) 0 (*b*) 5
(*c*) 10 (*d*) 25

DIFFERENTIABILITY AT A POINT AND OVER AN INTERVAL

14. If $f(x)=\begin{cases} x+1, \text{when } x<2 \\ 2x-1, \text{when } x \geq 2 \end{cases}$, then $f'(2)$ equals

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) Does not exist

15. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x)=[x \sin \pi x]$, then $f(x)$ is

(*a*) Continuous at $x=0$ (*b*) Continuous in $(-1,0)$
(*c*) Differentiable in $(-1,1)$ (*d*) All the above

16. The function $y=|\sin x|$ is continuous for any x but it is not differentiable at

(*a*) $x=0$ only
(*b*) $x=\pi$ only
(*c*) $x=k\pi$ (k is an integer) only
(*d*) $x=0$ and $x=k\pi$ (k is an integer)

17. The function $f(x)=x^2 \sin\frac{1}{x}, x \neq 0, f(0)=0$ at $x=0$

(*a*) Is continuous but not differentiable
(*b*) Is discontinuous
(*c*) Is having continuous derivative
(*d*) Is continuous and differentiable

18. If $f(x)=\begin{cases} \frac{x-1}{2x^2-7x+5} & \text{for } x \neq 1 \\ -\frac{1}{3} & \text{for } x=1 \end{cases}$, then $f'(1)=$

(*a*) –1/9 (*b*) –2/9
(*c*) –1/3 (*d*) 1/3

THEOREMS ON CONTINUITY AND NON-DIFFERENTIABILITY

19. Which of the following is not true

(*a*) Every differentiable function is continuous
(*b*) If derivative of a function is zero at all points, then the function is constant
(*c*) If a function has maximum or minima at a point, then the function is differentiable at that point and its derivative is zero
(*d*) If a function is constant, then its derivative is zero at all points

20. If f is a real-valued differentiable function satisfying $|f(x)-f(y)| \leq (x-y)^2$, x, $y \in R$ and $f(0)=0$, then $f(1)$ equals

(*a*) 1 (*b*) 2
(*c*) 0 (*d*) –1

METHOD OF DIFFERENTIATION

BASIC DIFFERENTIATION

21. If $y=x+\frac{1}{x}$, then

(*a*) $x^2\frac{dy}{dx}+xy=0$

(*b*) $x^2\frac{dy}{dx}+xy+2=0$

(*c*) $x^2\frac{dy}{dx}-xy+2=0$

(*d*) $x\frac{dy}{dx}-xy=0$

22. $\frac{d}{dx}\log|x|=.....,(x \neq 0)$

(*a*) $\frac{1}{x}$ (*b*) $-\frac{1}{x}$
(*c*) x (*d*) –x

23. If $f(x)=\sqrt{1+\cos^2(x^2)}$, then $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is

(*a*) $\sqrt{\pi}/6$ (*b*) $-\sqrt{\pi}/6$
(*c*) $1/\sqrt{6}$ (*d*) $\pi/\sqrt{6}$

24. $\frac{d}{dx}(e^x \log\sin 2x)=$

(*a*) $e^x(\log\sin 2x+2\cot 2x)$

(*b*) $e^x(\log\cos 2x+2\cot 2x)$

(*c*) $e^x(\log\cos 2x+\cot 2x)$

(*d*) $e^x(\log\sin 2x-\cot 2x)$

25. $\frac{d}{dx}\sqrt{\sec^2 x+\operatorname{cosec}^2 x}=$

(*a*) 4 cosec 2x.cot 2x (*b*) –4 cosec 2x. cot 2x
(*c*) –4 cosec x.cot 2x (*d*) 4 cosec x . cot 2x

26. If $y = \log x.e^{(\tan x + x^2)}$, then $\frac{dy}{dx} =$

(*a*) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x + x)\log x\right]$

(*b*) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x - x)\log x\right]$

(*c*) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x + 2x)\log x\right]$

(*d*) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x - 2x)\log x\right]$

PARAMETRIC DIFFERENTIATION

27. If $x = a\cos^4\theta, y = a\sin^4\theta$, then $\frac{dy}{dx}$, at $\theta = \frac{3\pi}{4}$, is

(*a*) -1 (*b*) 1
(*c*) $-a^2$ (*d*) a^2

LOGARITHMIC DIFFERENTIATION

28. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}}$, then $\frac{dy}{dx} =$

(*a*) $\frac{y^2\cot x}{1-y\log\sin x}$ (*b*) $\frac{y^2\cot x}{1+y\log\sin x}$

(*c*) $\frac{y\cot x}{1-y\log\sin x}$ (*d*) $\frac{y\cot x}{1+y\log\sin x}$

DIFF. OF ONE FUNCTION W.R.T OTHER FUNCTION

29. Differential coefficient of $\cos^{-1}(\sqrt{x})$ with respect to $\sqrt{(1-x)}$ is

(*a*) $\sqrt{x}$ (*b*) $-\sqrt{x}$

(*c*) $\frac{1}{\sqrt{x}}$ (*d*) $-\frac{1}{\sqrt{x}}$

30. Differential coefficient of $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ *w.r.t.* $\tan^{-1}x$ is

(*a*) $\frac{1}{1+\tan^{-1}x}$ (*b*) $\frac{-1}{1+\tan^{-1}x}$

(*c*) $\frac{1}{(1+\tan^{-1}x)^2}$ (*d*) $\frac{-1}{2(1+\tan^{-1}x)^2}$

31. The differential of e^{x^3} with respect to $\log x$ is

(*a*) e^{x^3} (*b*) $3x^2e^{x^3}$

(*c*) $3x^3e^{x^3}$ (*d*) $3x^2e^{x^3}+3x^2$

INVERSE FUNCTIONS DIFFERENTIATION

32. If $y = \sin^{-1}(x\sqrt{1-x}+\sqrt{x}\sqrt{1-x^2})$, then $\frac{dy}{dx} =$

(*a*) $\frac{-2x}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$

(*b*) $\frac{-1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$

(*c*) $\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$

(*d*) $\frac{-2}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$

33. If $y = \tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$, then $\frac{dy}{dx} =$

(*a*) $\frac{1}{1+25x^2}+\frac{2}{1+x^2}$ (*b*) $\frac{5}{1+25x^2}+\frac{2}{1+x^2}$

(*c*) $\frac{5}{1+25x^2}$ (*d*) $\frac{1}{1+25x^2}$

34. The first derivative of the function $\left[\cos^{-1}\left(\sin\sqrt{\frac{1+x}{2}}\right)+x^x\right]$ with respect to x at $x = 1$ is

(*a*) $\frac{3}{4}$ (*b*) 0

(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{2}$

35. If $y = \cos^{-1}\cos(|x| - f(x))$, where

$f(x)\begin{cases}=1 & ,\text{if } x>0\\ =-1, & \text{if } x<0\\ =0, & \text{if } x=0\end{cases}$, then $\left.\frac{dy}{dx}\right|_{x=\frac{5\pi}{4}}$ is

(*a*) -1 (*b*) 1
(*c*) 0 (*d*) Indeterminate

36. If $x = \sin^{-1}(3t-4t^3)$ and $y = \cos^{-1}\sqrt{(1-t^2)}$, then $\frac{dy}{dx}$ is equal to

(*a*) 1/2 (*b*) 2/5
(*c*) 3/2 (*d*) 1/3

37. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is equal to

(*a*) -1 (*b*) 1
(*c*) $\log 2$ (*d*) $-\log 2$

38. Let $g(x)$ be the inverse of the function $f(x)$ and $f'(x) = \frac{1}{1+x^3}$. Then $g'(x)$ is equal to

(*a*) $\frac{1}{1+(g(x))^3}$ (*b*) $\frac{1}{1+(f(x))^3}$

(*c*) $1+(g(x))^3$ (*d*) $1+(f(x))^3$

DETERMINANTS DIFFERENTIATION HIGHER ORDER DERIVATIVES

39. If $x = a\sin\theta$ and $y = b\cos\theta$, then $\frac{d^2y}{dx^2}$ is

(*a*) $\frac{a}{b^2}\sec^2\theta$ (*b*) $\frac{-b}{a}\sec^2\theta$

(*c*) $\frac{-b}{a^2}\sec^3\theta$ (*d*) $\frac{-b}{a^2}\sec^3\theta$

40. If $y = e^{\tan^{-1}x}$, then $(1+x^2)\frac{d^2y}{dx^2} =$

(*a*) $(1-2x)\frac{dy}{dx}$

(*b*) $-2x\frac{dy}{dx}$

(*c*) $-x\frac{dy}{dx}$

(*d*) 0

Learning Plus

CONTINUITY AND DIFFERENTIABILITY

1. If $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x^2-1}$ then find the points of discontinuous of fog(x)

(*a*) $x = \pm1, \pm\sqrt{2}$ (*b*) $x = \pm2, \pm\sqrt{2}$

(*c*) $x = \pm2, \pm\sqrt{3}$ (*d*) $x = \pm1, \pm\sqrt{3}$

2. $f(x) = \begin{cases} \frac{\sqrt{(1+px)}-\sqrt{(1-px)}}{x} & ,-1\le x<0 \\ \frac{2x+1}{x-2} & ,0\le x\le 1 \end{cases}$ is continuous in the interval [–1, 1], then 'p' is equal to:

(*a*) –1 (*b*) –1/2

(*c*) 1/2 (*d*) 1

3. For the function $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & x\neq 0 \\ 0 & , x=0 \end{cases}$, which of the following is correct

(*a*) $\lim_{x\to0} f(x)$ does not exist

(*b*) $f(x)$ is continuous at $x = 0$

(*c*) exists but is not continuous at

(*d*) none of these

4. Function $f(x) = (|x-1| + |x-2| + \cos x)$ where $x\in[0,4]$ is not continuous at number of points

(*a*) 3 (*b*) 2

(*c*) 1 (*d*) 0

5. The set of all point for which $f(x) = \frac{|x-3|}{|x-2|} + \frac{1}{[1+x]}$ is continuous is

(where [$*$] represents greatest integer function)

(*a*) R

(*b*) R – [–1, 0]

(*c*) R – ({2} $\cup$ [–1, 0])

(*d*) R – {(–1, 0) $\cup$ n, n $\in$ I}

6. Let f(x) be a continuous function defined for $1 \le x \le 3$. If f(x) takes rational values of for all x and f(2) = 10 then the value of f(1.5) is

(*a*) 7.5 (*b*) 10

(*c*) 8 (*d*) None of these

7. If $f(x) = \begin{cases} \tan^{-1}(\tan x); & x\le\frac{\pi}{4} \\ \pi[x]+1 \quad ; & x>\frac{\pi}{4} \end{cases}$, then jump of discontinuity is

(where [.] denotes greatest integer function)

(*a*) $\frac{\pi}{4} - 1$ (*b*) $\frac{\pi}{4} + 1$

(*c*) $1 - \frac{\pi}{4}$ (*d*) $-1 - \frac{\pi}{4}$

8. The function $f(x) = \begin{cases} \sin(\ln|x|) & ; x\neq0 \\ 1 & ; x=0 \end{cases}$

(*a*) is continuous at $x = 0$

(*b*) has removable discontinuity at $x = 0$

(*c*) has jump discontinuity at $x = 0$

(*d*) has discontinuity of IInd type at $x = 0$

9. Consider $f(x) = \underset{n\to\infty}{\text{Limit}} \frac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0$, $x \neq 1$, $f(1) = 0$ then

(*a*) f is continuous at x = 1

(*b*) f has a finite discontinuity at x = 1

(*c*) f has an infinite or oscillatory discontinuity at x = 1.

(*d*) f has a removable type of discontinuity at x = 1.

10. Let $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$, then for all values of x

(*a*) *f* is continuous but not differentiable

(*b*) *f* is differentiable but not continuous

(*c*) *f'* is continuous but not differentiable

(*d*) *f* is continuous and differentiable

11. If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then indicate the correct alternative(s)

(*a*) f(x) is continuous but not differentiable at x = 0

(*b*) f(x) is differentiable at x = 0

(*c*) f(x) is not differentiable at x = 0

(*d*) None of these

12. If $f(x) = \begin{cases} \frac{x(3e^{1/x} + 4)}{2 - e^{1/x}} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$ then f(x) is

(*a*) continuous as well differentiable at x = 0

(*b*) continuous but not differentiable at x = 0

(*c*) neither differentiable at x = 0 not continuous at x = 0

(*d*) None of these

13. If $f(x) = \begin{cases} \sqrt{x}\left(1 + \sin\frac{1}{x}\right) & , \ x > 0 \\ -\sqrt{-x}\left(1 + \sin\frac{1}{x}\right) & , \ x < 0 \\ 0 & , \ x = 0 \end{cases}$, then f(x) is

(*a*) continuous as well diff. at x = 0

(*b*) continuous at x = 0, but not diff. at = 0

(*c*) neither continuous at x=0 nor diff. at x=0

(*d*) None of these

14. No. of points is $f(x) = \text{Max.}\ \{|\tan x|, \cos|x|\}$ where function is not differentiable in $[-\pi, \pi]$?

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 8

15. If $f(x) = \begin{cases} \frac{x^2 - 1}{x^2 + 1} & , \ 0 < x \leq 2 \\ \frac{1}{4}\left(x^3 - x^2\right) & , \ 2 < x \leq 3 \\ \frac{9}{4}\left(|x - 4| + |2 - x|\right) & , \ 3 < x < 4 \end{cases}$,

then.

(*a*) f (x) is differentiable at x = 2 & x = 3

(*b*) f (x) is non-differentiable at x = 2 & x = 3

(*c*) f (x) is differentiable at x = 3 but not at x = 2

(*d*) f (x) is differentiable at x = 2 but not at x = 3

16. If $f(x) = p|\sin x| + q \cdot e^{|x|} + r|x|^3$ and f(x) is differentiable at x = 0, then

(*a*) p = q = r = 0 (*b*) p = 0, q = 0, r ∈ R

(*c*) q = 0, r = 0, p ∈ R (*d*) p + q = 0, r ∈ R

17. If f(x) is differentiable everywhere, then

(*a*) | f | is differentiable everywhere

(*b*) $|f|^2$ is differentiable everywhere

(*c*) f | f | is not differentiable at some point

(*d*) f + | f | is differentiable everywhere

18. If *f*(x) is a differentiable function such that $f : R \to r$ and $f\left(\frac{1}{n}\right) = 0 \ \forall\ n \geq 1, n \in I$ then

(*a*) $f(x) = 0 \ \forall\ x \in (0,1)$

(*b*) $f(x) = 0 \forall x \in (0,1)$

(*c*) $f(0) = 0$ but $f'(0)$ may or may not be 0

(*d*) $|f(x)| \leq 1 \ \forall\ x \in (0,1)$

19. Let $f : R \to R$ be a function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{3}$, f(0) = 0 and f'(0) = 3, then

(*a*) $\frac{f(x)}{x}$ is differentiable in R

(*b*) f(x) is continuous but not differentiable in R

(*c*) f(x) is continuous in R

(*d*) f(x) is bounded in R

20. If for all values of x & y ; f(x + y) = f(x).f(y) and f(5) = 2, f '(0) = 3, then f '(5) is-

(*a*) 3 (*b*) 4

(*c*) 5 (*d*) 6

METHOD OF DIFFERENTIATION

21. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then $\frac{dy}{dx}$ at x = 1 is equal to

(*a*) 2 (*b*) 1

(*c*) – 2 (*d*) –1

22. Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then $\frac{dx}{dt}$ at $t = 0$ is given by

(*a*) 1 (*b*) $\frac{19}{2}$

(*c*) $\frac{2}{19}$ (*d*) none of these

23. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(*a*) 4 (*b*) 2

(*c*) -2 (*d*) 0

24. If $\frac{d}{dx}\left(\frac{1+x^2+x^4}{1+x+x^2}\right) = ax + b$ then the value of a and b are respectively

(*a*) 2 and 1 (*b*) –2 and 1

(*c*) 2 and –1 (*d*) None of these

25. If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1+t^2}$, then $\frac{dy}{dx}$ =

(*a*) $\frac{2}{1+t^2}$ (*b*) $\frac{1}{1+t^2}$

(*c*) 1 (*d*) 2

26. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to

(*a*) $(\sin x)^{\tan x}.(1+\sec^2 x.\log\sin x)$

(*b*) $\tan x.(\sin x)^{\tan x-1}.\cos x$

(*c*) $(\sin x)^{\tan x}.\sec^2 x.\log\sin x$

(*d*) $\tan x.(\sin x)^{\tan x-1}$

27. If $f(x) = |x|^{|\sin x|}$, then $f'(\pi/4)$ is equal to

(*a*) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ell n\frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right)$

(*b*) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ell n\frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}\right)$

(*c*) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ell n\frac{\pi}{4} - \frac{2\sqrt{2}}{\pi}\right)$

(*d*) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ell n\frac{\pi}{4} + \frac{2\sqrt{2}}{\pi}\right)$

28. The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is

(*a*) 4 (*b*) 1/4

(*c*) 1 (*d*) None of these

29. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ =

(*a*) $\frac{\sin x}{2y-1}$ (*b*) $\frac{\sin x}{1-2y}$

(*c*) $\frac{\cos x}{1-2y}$ (*d*) $\frac{\cos x}{2y-1}$

30. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$, then $\frac{dy}{dx}$ =

(*a*) $\frac{x}{\sqrt{1-x^2}}$ (*b*) $\frac{1-2x}{\sqrt{1-x^2}}$

(*c*) $\frac{1-2x}{2\sqrt{1-x^2}}$ (*d*) $\frac{1}{1+x^2}$

31. If $y = \sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}\frac{1+x^2}{1-x^2}$, then $\frac{dy}{dx}$ =

(*a*) $\frac{4}{1-x^2}$ (*b*) $\frac{1}{1+x^2}$

(*c*) $\frac{4}{1-x^2}$ (*d*) $\frac{-4}{1+x^2}$

32. If $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$ and $\frac{dy}{dx} = \frac{1}{2\sqrt{x(1-x)}} + p$, then p =

(*a*) 0 (*b*) $\frac{1}{\sqrt{1-x}}$

(*c*) $\sin^{-1}\sqrt{x}$ (*d*) $\frac{1}{\sqrt{1-x^2}}$

33. If g is the inverse function of f an $f'(x) = \frac{x^5}{1+x^4}$. If $g(2) = a$, then $f'(2)$ is equal to

(*a*) $\frac{a^5}{1+a^4}$ (*b*) $\frac{1+a^4}{a^5}$

(*c*) $\frac{1+a^5}{a^4}$ (*d*) $\frac{a^4}{1+a^5}$

34. If f(x), g(x), h(x) are polynomials in x of degree 2 and $F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$, then $F'(x)$ is equal to

(*a*) 1 (*b*) 0

(*c*) –1 (*d*) $f(x).g(x).h(x)$

35. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right) =$

(*a*) 0 (*b*) 1
(*c*) 4 (*d*) None of these

36. If $y = x \log\left(\frac{x}{a+bx}\right)$, then $x^3 \frac{d^2y}{dx^2} =$

(*a*) $x\frac{dy}{dx} - y$ (*b*) $\left(x\frac{dy}{dx} - y\right)^2$

(*c*) $y\frac{dy}{dx} - x$ (*d*) $\left(y\frac{dy}{dx} - x\right)^2$

37. Let f(x) be a polynomial in x. Then the second derivative of $f(e^x)$, is
(*a*) $f''(e^x) \cdot e^x + f'(e^x)$
(*b*) $f''(e^x) \cdot e^{2x} + f'(e^x).e^{2x}$
(*c*) $f''(e^x)\, e^{2x}$
(*d*) $f''(e^x) \cdot e^{2x} + f'(e^x) \cdot e^x$

38. If $y^2 = ax^2 + bx + c$, then $y^3 \frac{d^2y}{dx^2}$ is
(*a*) a constant (*b*) a function of *x* only
(*c*) a function of *y* only (*d*) a function of *x* and *y*

39. Let $y = e^{2x}$. Then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is
(*a*) 1 (*b*) e^{-2x}
(*c*) $2e^{-2x}$ (*d*) $-2e^{-2x}$

40. The value of $\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} + \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$ is equal to

(*a*) –1 (*b*) $\frac{y}{x}$
(*c*) $-\frac{x}{y}$ (*d*) 0

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. Let f(x) and g(x) be defined by f(x) = [x] and $g(x) = \begin{cases} 0, & x \in I \\ x^2, & x \in R - I \end{cases}$ (where [.] denotes the greatest integer function), then

(*a*) $\lim_{x\to 1}$ g(x) exists, but g is not continuous at x = 1

(*b*) $\lim_{x\to 1}$ f(x) does not exist and f is not continuous at x = 1

(*c*) gof is continuous for all x

(*d*) fog is continuous for all x

2. Which of the following function(s) defined below has/ have single point continuity.

(*a*) $f(x) = \begin{bmatrix} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$ (*b*) $g(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{bmatrix}$

(*c*) $h(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$ (*d*) $k(x) = \begin{bmatrix} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{bmatrix}$

3. $f(x) = (\sin^{-1}x)^2 \cdot \cos(1/x)$ if $x \neq 0$; $f(0) = 0$, f(x) is:
(*a*) continuous no where in $-1 \le x \le 1$
(*b*) continuous everywhere in $-1 \le x \le 1$
(*c*) differentiable no where in $-1 \le x \le 1$
(*d*) differentiable everywhere in $-1 < x < 1$

4. $f(x) = (\sin^{-1} x)^2 \cdot \cos(1/x)$ if $x \neq 0$; $f(0) = 0$, f(x) is
(*a*) cont. no where in $-1 \le x \le 1$
(*b*) cont. every where in $-1 \le x \le 1$
(*c*) differentiable no where in $-1 \le x \le 1$
(*d*) differentiable everywhere in $-1 < x < 1$

5. If $f(x) = x^2 \cdot \sin(1/x)$, $x \neq 0$ and $f(0) = 0$ then,
(*a*) f(x) is continuous at x = 0
(*b*) f(x) is derivable at x = 0
(*c*) f'(x) is continuous at x = 0
(*d*) f'(x) is not derivable at x = 0

6. A function which is continuous & not differentiable at x = 0 is
(*a*) $f(x) = x$ for $x < 0$ & $f(x) = x^2$ for $x \ge 0$
(*b*) $g(x) = x$ for $x < 0$ & $g(x) = 2x$ for $x \ge 0$
(*c*) $h(x) = x|x|$ $x \in R$
(*d*) $K(x) = 1 + |x|$, $x \in R$

7. If $f(x) = |(x - 4)(x - 5)|$, then f'(x) is equal to
(*a*) $-2x + 9$, for all $x \in R$
(*b*) $2x - 9$ if $x > 5$
(*c*) $-2x + 9$ if $4 < x < 5$
(*d*) not defined for x = 4, 5

8. If $x = \cos t$, $y = \log_e t$ then

(*a*) $\frac{dy}{dx} = -\frac{2}{\pi}$ at $t = \frac{\pi}{2}$

(*b*) $\frac{dy}{dx} = \frac{4}{\pi^2}$ at $t = \frac{\pi}{2}$

(*c*) $\frac{dy}{dx} = \frac{144}{\pi^2}$ at $t = \frac{\pi}{6}$

(*d*) $\frac{dy}{dx} = -\frac{12}{\pi}$ at $t = \frac{\pi}{6}$

9. The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation

(*a*) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$

(*b*) $\frac{d^2u}{dx^2} = 2v$

(*c*) $\frac{d^2v}{dx^2} = -2u$

(*d*) $\frac{du}{dx} + \frac{dv}{dx} = 2v$

10. If $x^p \cdot y^q = (x+y)^{p+q}$ then $\frac{dy}{dx}$ is

(*a*) independent of p
(*b*) independent of q
(*c*) dependent on both p and q
(*d*) $\frac{y}{x}$

11. The differential coefficient of $\sin^{-1}\frac{t}{\sqrt{1+t^2}}$ w.r.t $\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ is

(*a*) $1 \ \forall \ t > 0$ (*b*) $-1 \ \forall \ t < 0$
(*c*) $1 \ \forall \ t \in R$ (*d*) None of these

12. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to

(*a*) $-\frac{2^y}{2^x}$ (*b*) $\frac{1}{1-2^x}$

(*c*) $1 - 2^y$ (*d*) $\frac{2^x\left(1-2^y\right)}{2^y\left(2^x-1\right)}$

13. If $\sqrt{x^2+y^2} = e^t$ where $t = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$, then $\frac{dy}{dx}$ is equal to

(*a*) $\frac{x-y}{x+y}$ (*b*) $\frac{x+y}{x-y}$

(*c*) $\frac{y-x}{y+x}$ (*d*) $\frac{x-y}{2x+y}$

14. If $f(x) = \begin{vmatrix} \cos\left(x+x^2\right) & \sin\left(x+x^2\right) & -\cos\left(x+x^2\right) \\ \sin\left(x-x^2\right) & \cos\left(x-x^2\right) & \sin\left(x-x^2\right) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$,

then

(*a*) $f(-2) = 0$ (*b*) $f'(-1/2) = 0$
(*c*) $f'(-1) = -2$ (*d*) $f''(0) = 4$

15. If f is twice differentiable such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If h(x) is a twice differentiable function such that $h'(x) = (f(x))^2 + (g(x))^2$. If $h(0) = 2$, $h(1) = 4$, then the equation $y = h(x)$ represents

(*a*) a curve of degree 2
(*b*) a curve passing through the origin
(*c*) a straight line with slope 2
(*d*) a straight line with y intercept equal to 2.

Comprehension–1 (Q.16 to 18)

Let $f(x) = \begin{cases} x\,g(x) & , \ x \le 0 \\ x + ax^2 - x^3 & , \ x > 0 \end{cases}$, where $g(t) = \lim_{x\to 0} (1 + a\tan x)^{t/x}$, a is positive constant, then

16. If a is even prime number, then g(2) =

(*a*) e^2 (*b*) e^3
(*c*) e^4 (*d*) None of these

17. Set of all values of a for which function f(x) is continuous at x = 0

(*a*) $(-1, 10)$ (*b*) $(-\infty, \infty)$
(*c*) $(0, \infty)$ (*d*) none of these

18. If f(x) is differentiable at x = 0, then a $\in$

(*a*) $(-5, -1)$ (*b*) $(-10, 3)$
(*c*) $(0, \infty)$ (*d*) none of these

Comprehension–2 (Q.19 to 21)

Consider the implicit equation $x^2 + 5xy + y^2 - 2x + y - 6 = 0$

19. The value of $\frac{dy}{dx}$ at (1, 1) is

(*a*) $\frac{5}{8}$ (*b*) $-\frac{5}{8}$

(*c*) $\frac{8}{5}$ (*d*) $-\frac{8}{5}$

20. The value of $\frac{d^2y}{dx^2}$ at (1, 1) is

(*a*) $\frac{111}{256}$ (*b*) $-\frac{111}{256}$

(*c*) $\frac{256}{111}$ (*d*) $-\frac{256}{111}$

21. The equation of normal to the conic (i) at (1, 1) is

(*a*) $5x-8y-3=0$

(*b*) $8y-5x-3=0$

(*c*) $8x-5y-3=0$

(*d*) $8x-5y+3=0$

22. Match the Column

Column–I	**Column – II**
(*a*) $f(x)=\lvert x^3\rvert$ is	(p) continuous in (–1, 1)
(*b*) $f(x)=\sqrt{\lvert x\rvert}$ is	(q) differentiable in (–1, 1)
(*c*) $f(x)=\lvert\sin^{-1}x\rvert$ is	(r) differentiable in (0, 1)
(*d*) $f(x)=\cos^{-1}\lvert x\rvert$ is	(s) not differentiable atleast at one point in (–1, 1)

23. Match the Column

Column – I	**Column – II**
(*a*) If $f'(x)=\sqrt{3x^2+6}$ and $y=f(x^3)$, then at $x=1$, $\frac{dy}{dx}=$	(p) -2
(*b*) If f be a differentiable function such that $f(xy)=f(x)+f(y)$; $x, y\in R$, then $f(e)+f(1/e)=$	(q) -1
(*c*) If f be a twice differentiable function such that $f''(x)=-f(x)$ and $f'(x)=g(x)$, If $h(x)=(f(x))^2+(g(x))^2$ & $h(5)=9$, then $h(10)=$	(r) 0
(*d*) $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$, $\frac{\pi}{2}<x<\pi$, then $\frac{dy}{dx}=$	(s) 9

NUMERICAL VALUE BASED

24. Let $f(x+y)=f(x)+f(y)-2xy-1$ for all x and y. If $f'(0)$ exists and $f'(0)=-\sin\alpha$, then the value of $f\{f'(0)\}$ is

25. Let f(x) be a function satisfying the condition $f(-x)=f(x)$ for all real x. If $f'(0)$ exists, its value is

26. If $f(x)=\begin{cases}\frac{\sin[x^2]\pi}{x^2-3x+8}+ax^3+b & ,0\le x\le 1\\ 2\cos\pi x+\tan^{-1}x & ,1<x\le 2\end{cases}$ is

differentiable in [0, 2], then the value of [a + b + 6] is
(Here [.] stands for the greatest integer function)

27. If $f(x)=\begin{cases}x^2e^{2(x-1)} & \text{for } 0\le x\le 1\\ a\ \text{sgn}\ (x+1)\cos(2x-2)+bx^2 & \text{for } 1<x\le 2\end{cases}$ is differentiable at x = 1 then $a^3+b^3=$

28. Find number of points of non-differentiability of

$f(x)=\lim\limits_{n\to\infty}\frac{\{e^x\}^n-1}{\{e^x\}^n+1}$ in interval [0, 1]

where {.} represents fractional part function

29. $f(x)=x+\cfrac{1}{2x+\cfrac{1}{2x+-----}}$ then $\frac{f(50)f'(50)}{50}$ is

30. If $x^2+y^2=t-\frac{1}{t}$ and $x^4+y^4=t^2+\frac{1}{t^2}$ then $x^3y\frac{dy}{dx}=$

31. If y is a function of x and $\log(x+y)=2xy$ then $y'(0)=$

32. $xy=(x+y)^n$ and $\frac{dy}{dx}=\frac{y}{x}$ then $n=$

33. $y=\frac{2x^2}{(x-2)(x-3)(x-4)}+\frac{3x}{(x-3)(X-4)}+\frac{x}{x-4}$.

If $\frac{x}{y}\frac{dy}{dx}=\frac{a}{2-x}+\frac{b}{3-x}+\frac{c}{4-x}$

then a + b + c is

JEE-MAIN
PREVIOUS YEAR'S
CONTINUITY AND DIFFERENTIABILITY

1. Let $S = \{t \in R : f(x) = |x-\pi| \cdot \left(e^{|x|}-1\right)\sin|x| \text{ is not differentiable at } t\}$. Then the set S is equal to :

[JEE Main - 2018]

(*a*) $\{0\}$ (*b*) $\{\pi\}$

(*c*) $\{0, \pi\}$ (*d*) ϕ (an empty set)

2. Let $f : R \to R$ be a function defined as $f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a+bx, & \text{if } 1 < x < 3 \\ b+5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$ Then f is :

[JEE Main - 2019 (January)]

(*a*) continuous if a=5 and b=5

(*b*) continuous if a=5 and b=10

(*c*) continuous if a=0 and b=5

(*d*) not continuous for any values of a and b

3. Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8-2|x|, & 2 < |x| \le 4 \end{cases}$. Let S the set of points in the interval $(-4, 4)$ at which f is not differentiable. Then S:

[JEE Main - 2019 (January)]

(*a*) Is an empty set

(*b*) Equals $\{-2, -1, 0, 1, 2\}$

(*c*) Equals $\{-2, -1, 1, 2\}$

(*d*) Equals $\{-2, 2\}$

4. Let K be the set of all real values of x where the function. $f(x) = \sin|x| - |x| + 2(x-\pi)\cos|x|$ is not differentiable. Then the set K is equal to:

[JEE Main - 2019 (January)]

(*a*) ϕ (en empty set) (*b*) $\{\pi\}$

(*c*) $\{0\}$ (*d*) $\{0, \pi\}$

5. Let $f(x) = \begin{cases} -1, -2 \le x < 0 \\ x^2 - 1, 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$, Then in the interval $(-2, 2)$, g(x) is

[JEE Main - 2019 (January)]

(*a*) differentiable at all points

(*b*) not continuous

(*c*) not differentiable at two points

(*d*) not differentiable at one point

6. Let $f : [-1, 3] \to R$ be defined as

[JEE Main - 2019(April)]

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases},$$

where [t] denotes the greatest integer less than or equal to t. Then *f* is discontinuous at :

(*a*) four or more points

(*b*) only one point

(*c*) only two points

(*d*) only three points

7. Let $f(x) = 15 - |x-10|$; $x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is:

[JEE Main - 2019(April)]

(*a*) $\{5, 10, 15, 20\}$ (*b*) $\{10, 15\}$

(*c*) $\{5, 10, 15\}$ (*d*) $\{10\}$

8. If the function *f* defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \ne \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$ is continuous, then k is equal to

[JEE Main - 2019(April)]

(*a*) $\frac{1}{2}$ (*b*) 1

(*c*) $\frac{1}{\sqrt{2}}$ (*d*) 2

9. If the function $f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x-\pi| + 3, & x > 5 \end{cases}$ is continuous at $x = 5$, then the value of $a - b$ is :-

[JEE Main - 2019(April)]

(*a*) $\frac{2}{5-\pi}$ (*b*) $\frac{2}{\pi-5}$

(*c*) $\frac{2}{\pi+5}$ (*d*) $\frac{-2}{\pi+5}$

10. If $f(x) = [x] - \left[\frac{x}{4}\right]$, $x \in R$, where [x] denotes the greatest integer function, then :

[JEE Main - 2019(April)]

(*a*) Both $\lim\limits_{x\to4^-} f(x)$ and $\lim\limits_{x\to4^+} f(x)$ exist but are not equal

(*b*) $\lim\limits_{x\to4^-} f(x)$ exists but $\lim\limits_{x\to4^+} f(x)$ does not exist

(*c*) $\lim\limits_{x\to4^+} f(x)$ exists but $\lim\limits_{x\to4^-} f(x)$ does not exist

(*d*) f is continuous at x = 4

11. If $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , \ x < 0 \\ q & , \ x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}} & , \ x > 0 \end{cases}$ is continuous at x = 0 , then the ordered pair (p, q) is equal to :

[JEE Main - 2019(April)]

(*a*) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (*b*) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$

(*c*) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (*d*) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

12. Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at x = c, g is : **[JEE Main - 2019(April)]**

(*a*) differentiable if f'(c) = 0 (*b*) not differentiable

(*c*) differentiable if $f' \neq (c)$

(*d*) not differentiable if f'(c) = 0

13. Let S be the set of points where the function, $f(x) = |2 - |x - 3||$, $x \in R$ is not differentiable. Then $\sum\limits_{x\in S} f(f(x))$ is equal to **[JEE Main-2020 (January)]**

14. If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by $f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+3x}{1-2x}\right), & \text{when } x \neq 0 \\ k \quad , & \text{when } x = 0 \end{cases}$ is continuous, then k is equal to_____. **[JEE Main-2020 (January)]**

15. Let S be the set of all function $f : [0, 1] \to R$, which are continuous on [0, 1] and differentiable on (0, 1). Then for every f in S, there exists a $c \in (0, 1)$, depending on f, such that

[JEE Main-2020 (January)]

(*a*) $|f(c) - f(1)| < (1 - c)|f'(c)|$

(*b*) $|f(c) - f(1)| < |f'(c)|$

(*c*) $\frac{f(1) - f(c)}{1 - c} = f'(c)$

(*d*) $|f(c) + f(1)| < (1 + c)|f'(c)|$

16. If $f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & ; \ x < 0 \\ b & ; \ x = 0 \\ \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & ; \ x > 0 \end{cases}$

[JEE Main-2020 (January)]

is continuous at x = 0, then a + 2b is equal to :

(*a*) 1 (*b*) 0

(*c*) –2 (*d*) –1

17. Let [t] denote the greatest integer $\le t$ and $\lim\limits_{x\to0} x\left[\frac{4}{x}\right] = A$.

Then the function, $f(x) = [x^2] \sin(\pi x)$ is discontinuous, when x is equal to:

[JEE Main-2020 (January)]

(*a*) $\sqrt{A}$ (*b*) $\sqrt{A+1}$

(*b*) $\sqrt{A+5}$ (*d*) $\sqrt{A+21}$

18. If a function f(x) defined by

[JEE Main-2020 (September)]

$f(x) = \begin{cases} ae^x + be^{-x} & , \ -1 \le x < 1 \\ cx^2 & , \ 1 \le x < 3 \\ ax^2 + 2cx & , \ 3 \le x \le 4 \end{cases}$

be continuous for some a, b, $c \in R$ and f'(0) + f'(2) = e, then the value of a is :

(*a*) $\frac{e}{e^2 + 3e + 13}$ (*b*) $\frac{e}{e^2 - 3e + 13}$

(*c*) $\frac{e}{e^2 - 3e - 13}$ (*d*) $\frac{e}{e^2 - 3e + 13}$

19. Contrapositive of the statement :

[JEE Main-2020 (September)]

'If a function f is differentiable at a, then it is also continuous at a', is :

(*a*) If a function f is continuous at a, then it is differentiable at a.

(*b*) If a function f is not continuous at a, then it is not differentiable at a.

(*c*) If a function f is not continuous at a, then it is differentiable at a.

(*d*) If a function f is continuous at a, then it is not differentiable at a.

20. The function $f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x, & |x| \le 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$ is

[JEE Main-2020 (September)]

(*a*) both conti nuous and differentiable on R – {–1}.

(*b*) both conti nuous and differentiable on R – {1}.

(*c*) continuous on R – {–1} and differentiable on R – {–1, 1}.

(*d*) continuous on R – {1} and differentiable on R – {–1, 1}.

21. Let $f:(0,\infty)\to(0,\infty)$ be a differentiable function such that $f(1)=e$ and $\lim\limits_{t\to x}\dfrac{t^2f^2(x)-x^2f^2(t)}{t-x}=0$

If $f(x)=1$, then x is equal to :

[*JEE Main-2020 (September)*]

(*a*) $2e$ (*b*) e

(*c*) $\dfrac{1}{2e}$ (*d*) $\dfrac{1}{e}$

22. If the function [*JEE Main-2020 (September)*]

$f(x)=\begin{cases}k_1(x-\pi)^2-1 & x\le\pi\\ k_2\cos x, & x>\pi\end{cases}$ is twice

differentiable, then the ordered pair (k_1, k_2) is equal to :

(*a*) $(1,0)$ (*b*) $\left(\dfrac{1}{2},1\right)$

(*c*) $(1,1)$ (*d*) $\left(\dfrac{1}{2},-1\right)$

23. Let $f(x)=x\cdot\left[\dfrac{x}{2}\right]$, for $-10<x<10$, where [t] denotes the greatest integer function. Then the number of points of discontinuity of f is equal to _____.

[*JEE Main-2020 (September)*]

24. Let $f:R\to R$ be defined as

[*JEE Main-2020 (September)*]

$$f(x)=\begin{cases}x^5\sin\left(\dfrac{1}{x}\right)+5x^2, & x<0\\ 0, & x=0\\ x^5\cos\left(\dfrac{1}{x}\right)+\lambda x^2, & x>0\end{cases}$$

The value of λ for which $f''(0)$ exists, is ——.

METHOD OF DIFFERENTIATION

25. For $x\in R, f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then: [*JEE Mains- 2016*]

(*a*) g is not differentiable at $x=0$

(*b*) $g'(0)=\cos(\log 2)$

(*c*) $g'(0)=-\cos(\log 2)$

(*d*) g is differentiable at $x=0$ and $g'(0)=-\sin(\log 2)$

26. If for $x\in\left(0,\dfrac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\dfrac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}.g(x)$, then g(x) equals : [*JEE Main - 2017*]

(*a*) $\dfrac{3}{1+9x^3}$ (*b*) $\dfrac{9}{1+9x^3}$

(*c*) $\dfrac{3x\sqrt{x}}{1-9x^3}$ (*d*) $\dfrac{3x}{1-9x^3}$

27. If $x=3\tan t$ and $y=3\sec t$, then the value of $\dfrac{d^2y}{dx^2}$ at $t=\dfrac{\pi}{4}$, is; [*JEE Main - 2019 (January)*]

(*a*) $\dfrac{3}{2\sqrt{2}}$ (*b*) $\dfrac{1}{3\sqrt{2}}$

(*c*) $\dfrac{1}{6}$ (*d*) $\dfrac{1}{6\sqrt{2}}$

28. Let $f:R\to R$ be a function such that $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$, $x\in R$. Then f(2) equals:

[*JEE Main - 2019 (January)*]

(*a*) -4 (*b*) 30

(*c*) -2 (*d*) 8

29. Let x, y be positive real numbers and m, n positive integers, The maximum value of the expression $\dfrac{x^my^n}{(1+x^{2m})(1+y^{2n})}$ is [*JEE Main - 2019 (January)*]

(*a*) 1 (*b*) $\dfrac{1}{2}$

(*c*) $\dfrac{1}{4}$ (*d*) $\dfrac{m+n}{6mn}$

30. For $x>1$, if $(2x)^{2y}=4e^{2x-2y}$, then $(1+\log_e 2x)^2\dfrac{dy}{dx}$ equal to:

[*JEE Main - 2019 (January)*]

(*a*) $\dfrac{x\log_e 2x-\log_e 2}{x}$ (*b*) $\log_e 2x$

(*c*) $\dfrac{x\log_e 2x+\log_e 2}{x}$ (*d*) $x\log_e 2x$

31. If $x\log_e(\log_e x)-x^2+y^2=4(y>0)$, then $\dfrac{dy}{dx}$ at $x=e$ is equal to : [*JEE Main - 2019 (January)*]

(*a*) $\dfrac{(1+2e)}{2\sqrt{4+e^2}}$ (*b*) $\dfrac{(2e-1)}{2\sqrt{4+e^2}}$

(*c*) $\dfrac{(1+2e)}{\sqrt{4+e^2}}$ (*d*) $\dfrac{e}{\sqrt{4+e^2}}$

32. If $2y=\left(\cot^{-1}\left(\dfrac{\sqrt{3}\cos x+\sin x}{\cos x-\sqrt{3}\sin x}\right)\right)^2$, $x\in\left(0,\dfrac{\pi}{2}\right)$, then $\dfrac{dy}{dx}$ is equal to [*JEE Main - 2019(April)*]

(*a*) $2x-\dfrac{\pi}{3}$ (*b*) $\dfrac{\pi}{3}-x$

(*c*) $\dfrac{\pi}{6}-x$ (*d*) $x-\dfrac{\pi}{6}$

33. If $f(1)=1, f'(1)=3$, then the derivative of $f(f(f(x)))+(f(x))^2$ at $x=1$ is :

[*JEE Main - 2019(April)*]

(*a*) 12 (*b*) 33
(*c*) 9 (*d*) 15

34. If $^{20}C_1+(2^2)^{20}C_2+(3^2)^{20}C_3+....+(20^2)^{20}C_{20}=A(2^\beta)$, then the ordered pair (A, β) is equal to :

[*JEE Main - 2019(April)*]

(*a*) (420, 18) (*b*) (380, 19)
(*c*) (380, 18) (*d*) (420, 19)

35. The derivation of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x\in\left(0,\frac{\pi}{2}\right)\right)$ is :

[*JEE Main - 2019(April)*]

(*a*) $\frac{1}{2}$ (*b*) $\frac{2}{3}$
(*c*) 1 (*d*) 2

36. If $y(\alpha)=\sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}$, $\alpha\in\left(\frac{3\pi}{4},\pi\right)$; then $\frac{dy}{d\alpha}$ at $\alpha=\frac{5\pi}{6}$ is [*JEE Main-2020 (January)*]

(*a*) 4 (*b*) $\frac{4}{3}$
(*c*) $\frac{-1}{4}$ (*d*) -4

37. Let $x^k+y^k=a^k$ $(a, k>0)$ and $\frac{dy}{dx}+\left(\frac{y}{x}\right)^{\frac{1}{3}}=0$, then k is

[*JEE Main-2020 (January)*]

(*a*) $\frac{4}{3}$ (*b*) $\frac{2}{3}$
(*c*) $\frac{1}{3}$ (*d*) $\frac{3}{2}$

38. Let $y=y(x)$ be a function of x satisfying $y\sqrt{1-x^2}=k-x\sqrt{1-y^2}$ where k is a contant and $y\left(\frac{1}{2}\right)=-\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x=\frac{1}{2}$, is equal to :

[*JEE Main-2020 (January)*]

(*a*) $\frac{\sqrt{5}}{2}$ (*b*) $-\frac{\sqrt{5}}{2}$
(*c*) $\frac{2}{\sqrt{5}}$ (*d*) $\frac{-\sqrt{5}}{4}$

39. If $f'(x)=\tan^{-1}(\sec x+\tan x)$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$, and $f(0)=0$, then $f(1)$ is equal to :

[*JEE Main-2020 (January)*]

(*a*) $\frac{1}{4}$ (*b*) $\frac{\pi+2}{4}$
(*c*) $\frac{\pi-1}{4}$ (*d*) $\frac{\pi+1}{4}$

40. If $x=2\sin\theta-\sin2\theta$ and $y=2\cos\theta-\cos2\theta$, $\theta\in[0,2\pi]$, then $\frac{d^2y}{dx^2}$ at $\theta=\pi$ is :

[*JEE Main-2020 (January)*]

(*a*) $-\frac{3}{8}$ (*b*) $\frac{3}{8}$
(*c*) $\frac{3}{4}$ (*d*) $-\frac{3}{4}$

41. Let f and g be differentiable functions on R such that fog is the identity function. If for some $a, b\in R$, $g'(a)=5$ and $g(a)=b$, then $f'(b)$ is equal to :

[*JEE Main-2020 (January)*]

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$
(*c*) 1 (*d*) 5

42. If $y=\sum_{k=1}^{6}k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at $x=0$ is _____.

[*JEE Main-2020 (September)*]

43. If $y^2+\log_e(\cos^2x)=y$, $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then "

[*JEE Main-2020 (September)*]

(*a*) $|y'(0)|+|y''(0)|=3$
(*b*) $|y'(0)|+|y''(0)|=1$
(*c*) $y''(0)=0$
(*d*) $|y''(0)|=2$

44. Suppose a dif ferentiable function f(x) satisfies the identity $f(x+y)=f(x)+f(y)+xy^2+x^2y$, for all real x and y. If $\lim_{x\to0}\frac{f(x)}{x}=1$, then $f'(3)$ is equal to _____.

[*JEE Main-2020 (September)*]

45. If $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y)=a^2-b^2$, where $a>b>0$, then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{2a+b}{2a-b}$ (*b*) $\frac{a-b}{a+b}$
(*c*) $\frac{a+b}{a-b}$ (*d*) $\frac{a-2b}{a+2b}$

46. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=\frac{1}{2}$ is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{2\sqrt{3}}{3}$ (*b*) $\frac{2\sqrt{3}}{5}$

(*c*) $\frac{\sqrt{3}}{10}$ (*d*) $\frac{\sqrt{3}}{12}$

NUMERICAL BASED QUESTIONS

47. Let $a,b\in R, b\neq 0$, Define a function

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x-1), & \text{for } x\le 0 \\ \frac{\tan 2x-\sin 2x}{bx^3}, & \text{for } x>0 \end{cases}.$$

If f is continuous at $x=0$, then $10-ab$ is equal to

[*JEE Main-2021*]

48. Let $f:R\to R$ satisfy the equation $f(x+y)=f(x).f(y)$ for all $x,y\in R$ and $f(0)\neq 0$ for any $x\in R$. If the function f is differentiable at $x=0$ and $f'(0)=2$, then $\lim\limits_{h\to 0}\frac{1}{2}(f(h)-1)$ is equal to

[*JEE Main-2021*]

49. If $\lim\limits_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$ then $a+b+c$ is equal to

[*JEE Main-2021*]

JEE-ADVANCED PREVIOUS YEAR'S CONTINUITY & DIFFERENTIABILITY

1. Let $f(x)=\begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x\neq 0 \\ 0, & x=0 \end{cases}$, $x\in IR$, then f is

[*IIT JEE - 2012*]

(*a*) differentiable both at x = 0 and at x = 2
(*b*) differentiable at x = 0 but not differentiable at x = 2
(*c*) not differentiable at x = 0 but differentiable at x = 2
(*d*) differentiable neither at x = 0 nor at x = 2

2. For every integer n, let a_n and b_n be real numbers. Let function $f: IR\to IR$ be given by

$$f(x)=\begin{cases} a_n+\sin\pi x, & \text{for } x\in[2n,2n+1] \\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n) \end{cases}, \text{ for all integers n.}$$

If f is continuous, then which of the following hold(s) for all n ?

[*IIT JEE-2012*]

(*a*) $a_{n-1}-b_{n-1}=0$ (*b*) $a_n-b_n=1$

(*c*) $a_n-b_{n+1}=1$ (*d*) $a_{n-1}-b_n=-1$

3. For every pair of continuous functions f, $g:[0,1]\to R$ such that max $\{f(x): x\in[0,1]\}$ = max $\{g(x): x\in[0,1]\}$, the correct statement(s) is (are):

[*JEE Advanced-2014*]

(*a*) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$
(*b*) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$
(*c*) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c\in[0,1]$
(*d*) $(f(c))^2=(g(c))^2$ for some $c\in[0,1]$

4. Let $f:R\to R$ and $g:R\to R$ be respectively given by $f(x)=|x|+1$ and $g(x)=x^2+1$. Define $h:R\to R$ by

$$h(x)=\begin{cases} \max\{f(x),g(x)\} & \text{if } x\le 0, \\ \min\{f(x),g(x)\} & \text{if } x>0. \end{cases}$$

The number of points at which h(x) is not differentiable is

[*JEE Advanced -2014*]

5. Let $f_1: R\to R$, $f_2:[0,\infty)\to R$, $f_3: R\to R$ and $f_4: R\to[0,\infty)$ be defined by

[*JEE Advanced -2014*]

$$f_1(x)=\begin{cases} |x| & \text{if } x<0, \\ e^x & \text{if } x\ge 0; \end{cases} \quad f_2(x)=x^2;$$

$$f_3(x)=\begin{cases} \sin x & \text{if } x<0, \\ x & \text{if } x\ge 0 \end{cases} \text{ and}$$

$$f_4(x)=\begin{cases} f_2(f_1(x)) & \text{if } x<0, \\ f_2(f_1(x))-1 & \text{if } x\ge 0 \end{cases}$$

List I	List II
P. f_4 is	**1.** onto but not one-one
Q. f_3 is	**2.** neither continuous nor one-one
R. $f_2 o f_1$ is	**3.** differentiable but not one-one
S. f_2 is	**4.** continuous and one-one

	P	Q	R	S
(*a*)	3	1	4	2
(*b*)	1	3	4	2
(*c*)	3	1	2	4
(*d*)	1	3	2	4

6. Let g: $R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$. Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let (foh)(x) denote f(h(x)) and (hof)(x) denote h(f(x)). Then which of the following is(are) true? **[*JEE Advanced -2015*]**

(*a*) f is differentiable at x = 0
(*b*) h is differentiable at x = 0
(*c*) foh is differentiable at x = 0
(*d*) hof is differentiable at x = 0

7. Let [x] be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous? **[*JEE Advanced -2017*]**

(*a*) $x = -1$ (*b*) $x = 1$
(*c*) $x = 0$ (*d*) $x = 2$

8. Let $f : R \to R$ be a differentiable function with $f(0) = 1$ and satisfying the equation
$f(x + y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in \mathbb{R}$
Then, then value of $\log_e (f(4))$ is ....
[*JEE Advanced - 2018*]

9. Let $f_1 : R \to R, f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to R$

and $f_4 : R \to R$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan_x^{-1}$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = \left[\sin\left(\log_e (x + 2)\right)\right]$, where for $t \in \mathbb{R}$, [t] denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

[*JEE Advanced - 2018*]

List - I	List - II
(P) The function f_1 is	(*a*) NOT continuous at x = 0
(Q) The function f_2 is	(*b*) continuous at x = 0 and NOT differentiable at x = 0
(R) The function f_3 is	(*c*) differentiable at x = 0 and its derivative is NOT continuous at x = 0
(S) The function f_4 is	(*d*) differentiable at x = 0 and its derivative is continuous at x = 0

The correct option is :
(*a*) P → 2; Q → 3; R → 1; S → 4
(*b*) P → 4; Q → 1; R → 2; S → 3
(*c*) P → 4; Q → 2; R → 1; S → 3
(*d*) P → 2; Q → 1; R → 4; S → 3

10. If the function $f : R \to R$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE?** **[*JEE(Advanced) - 2020*]**

(*a*) *f* is one-one, but **NOT** onto
(*b*) *f* is onto, but **NOT** one-one
(*c*) *f* is **BOTH** one-one and onto
(*d*) *f* is **NEITHER** one-one **NOR** onto

11. Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g : R \to R$ be an arbitrary function. Let fg: $R \to R$ be the product function defined by $(fg)(x) = f(x)\,g(x)$. Then which of the following statements is/are TRUE ? **[*JEE(Advanced) - 2020*]**

(*a*) If g is continuous at x = 1, then fg is differentiable at x = 1
(*b*) If fg is differentiable at x = 1, then g is continuous at x = 1
(*c*) If g is differentiable at x = 1, then fg is differentiable at x = 1
(*d*) If fg is differentiable at x = 1, then g is differentiable at x = 1

12. Let the functions $f : (-1, 1) \to \mathbb{R}$ and $g : (-1, 1) \to (-1, 1)$ be defined by
$f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,
where [x] denotes the greatest integer less than or equal to x. Let *fog*: $(-1, 1) \to \mathbb{R}$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval (–1,1) at which *fog* is **NOT** continuous, and suppose d is the number of points in the interval (–1, 1) at which *fog* is **NOT** differentiable. Then the value of c + d is _____ **[*JEE(Advanced) - 2020*]**

13. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in \mathbb{R}$. If $\lim_{x \to 0} g(x) = 1$, then which of the following statements is/are TRUE? **[*JEE(Advanced) - 2020*]**

(*a*) *f* is differentiable at every $x \in \mathbb{R}$
(*b*) If g(0)=1, then g is differentiable at every $x \in \mathbb{R}$
(*c*) The derivative $f'(1)$ is equal to 1
(*d*) The derivative $f'(0)$ is equal to 1

METHOD OF DIFFERENTIATIION

14. Let $f(x) = x \sin \pi x, x > 0$. Then for all natural numbers n, f' (x) vanishes at **[*JEE Advanced -2013*]**

(*a*) a unique point in the interval $\left(n, n + \frac{1}{2}\right)$
(*b*) a unique point in the interval $\left(n + \frac{1}{2}, n + 1\right)$
(*c*) a unique point in the interval (n, n +1)
(*d*) two points in the interval (n, n +1)

15. Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x)=x^3+3x+2$, $g(f(x))=x$ and $h(g(g(x)))=x$ for all $x\in\mathbb{R}$. Then **[*JEE Advanced -2016*]**

(*a*) $g'(2)=\frac{1}{15}$ (*b*) $h'(1)=666$

(*c*) $h(0)=16$ (*d*) $h(g(3))=36$

16. Let f: $(0,\pi)\to\mathbb{R}$ be a twice differentiable function such that $\lim\limits_{t\to x}\frac{f(t)\sin t-f(t)\sin x}{t-x}=\sin^2 x$ for all $x\in(0,\pi)$

If $f\left(\frac{\pi}{6}\right)=-\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE?

[*JEE Advanced - 2018*]

(*a*) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4\sqrt{2}}$ then which of the following statement(s) is (are) TRUE?

(*b*) $f(x)<\frac{x^4}{6}-x^2$ for all $x\in(0,\pi)$

(*c*) There exists $\alpha\in(0,\pi)$ such that $f'(\alpha)=0$

(*d*) $f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$

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ANSWER KEY

Topicwise Questions

1. (*a*) **2.** (*b*) **3.** (*b*) **4.** (*b*) **5.** (*a*) **6.** (*c*) **7.** (*b*) **8.** (*d*) **9.** (*c*) **10.** (*d*)
11. (*c*) **12.** (*c*) **13.** (*a*) **14.** (*d*) **15.** (*d*) **16.** (*d*) **17.** (*d*) **18.** (*b*) **19.** (*c*) **20.** (*c*)
21. (*c*) **22.** (*a*) **23.** (*b*) **24.** (*a*) **25.** (*b*) **26.** (*c*) **27.** (*a*) **28.** (*a*) **29.** (*c*) **30.** (*c*)
31. (*c*) **32.** (*c*) **33.** (*c*) **34.** (*a*) **35.** (*b*) **36.** (*d*) **37.** (*a*) **38.** (*c*) **39.** (*c*) **40.** (*a*)

Learning Plus

1. (*a*) **2.** (*b*) **3.** (*c*) **4.** (*d*) **5.** (*d*) **6.** (*b*) **7.** (*c*) **8.** (*d*) **9.** (*b*) **10.** (*c*)
11. (*b*) **12.** (*b*) **13.** (*b*) **14.** (*b*) **15.** (*b*) **16.** (*d*) **17.** (*b*) **18.** (*b*) **19.** (*c*) **20.** (*d*)
21. (*a*) **22.** (*c*) **23.** (*c*) **24.** (*c*) **25.** (*c*) **26.** (*a*) **27.** (*d*) **28.** (*a*) **29.** (*d*) **30.** (*c*)
31. (*c*) **32.** (*d*) **33.** (*b*) **34.** (*b*) **35.** (*c*) **36.** (*b*) **37.** (*d*) **38.** (*a*) **39.** (*d*) **40.** (*d*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*a,b,c*) **2.** (*b,c,d*) **3.** (*b,d*) **4.** (*b,d*) **5.** (*a,b,d*) **6.** (*a,b,d*) **7.** (*b,c,d*) **8.** (*a,d*) **9.** (*a,b,c,d*)
10. (*a,b,d*) **11.** (*a,b*) **12.** (*a,b,c,d*) **13.** (*b,c*) **14.** (*b,c,d*) **15.** (*c, d*) **16.** (*c*) **17.** (*c*) **18.** (*c*) **19.** (*b*)
20. (*a*) **21.** (*c*) **22.** (*a*) $\rightarrow$ (p, q, r), (*b*) $\rightarrow$ (p, r, s), (*c*) $\rightarrow$ (p, r, s), (*d*) $\rightarrow$ (p, r, s)
23. (*a*) $\rightarrow$ (s); (*b*) $\rightarrow$ (r) ; (*c*) $\rightarrow$ (s); (*d*) $\rightarrow$ (p) **24.** (*a*) **25.** (0) **26.** (*d*) **27.** (7) **28.** (0) **29.** (*a*)
30. (*a*) **31.** (*a*) **32.** (*b*) **33.** (9)

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*d*) **2.** (*d*) **3.** (*b*) **4.** (*a*) **5.** (*d*) **6.** (*d*) **7.** (*c*) **8.** (*a*) **9.** (*a*) **10.** (*d*)
11. (*d*) **12.** (*a*) **13.** (*c*) **14.** (5) **15.** (Bonus) **16.** (*b*) **17.** (*b*) **18.** (*b*) **19.** (*b*) **20.** (*d*)
21. (*d*) **22.** (*b*) **23.** (8) **24.** (5) **25.** (*b*) **26.** (*b*) **27.** (*d*) **28.** (*c*) **29.** (*c*) **30.** (*a*)
31. (*b*) **32.** (*d*) **33.** (*b*) **34.** (*a*) **35.** (*d*) **36.** (*a*) **37.** (*b*) **38.** (*b*) **39.** (*d*) **40.** (*b*)
41. (*a*) **42.** (91) **43.** (*d*) **44.** (10) **45.** (*c*) **46.** (*c*) **47.** [14] **48.** [0] **49.** [4]

JEE-ADVANCED PREVIOUS YEAR'S

1. (*b*) **2.** (*b,d*) **3.** (*a,d*) **4.** (*c*) **5.** (*d*) **6.** (*a,d*) **7.** (*a,b,d*) **8.** (*b*) **9.** (*d*) **10.** (*c*)
11. (*a,c*) **12.** (*d*) **13.** (*a,b,d*) **14.** (*b,c*) **15.** (*b,c*) **16.** (*b,c,d*)



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2. Inverse Trigonometric Functions **19-33**

3. Limit of Function **34-48**

4. Continuity and Derivability Methods of Differentiation **49-72**

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1 Relations and Function

Topicwise Questions

1. (*a*) For any $a \in N$, we find that $a|a$, therefore R is reflexive but R is not symmetric, because aRb does not imply that bRa.

2. (*a*) $A = \{2,4,6\}$; $B = \{2,3,5\}$

$\therefore A \times B$ contains $3 \times 3 = 9$ elements.

Hence, number of relations from A to $B = 2^9$.

3. (*b*) Reflexive relation : a R a

but identity relation is $y = x : x \in A \,\&\, y \in A$ so $I \subset R$

4. (*b*) The relation is not symmetric, because $A \subset B$ does not imply that $B \subset A$. But it is anti-symmetric because and $\Rightarrow A=B$.

5. (*d*) We have $(a,b)R(a,b)$ for all $(a,b) \in N \times N$

Since $a+b=b+a$. Hence, R is reflexive.

R is symmetric for we have $(a,b)R(c,d) \Rightarrow a+d=b+c$

$\Rightarrow d+a=c+b$

$\Rightarrow c+b=d+a \Rightarrow (c,d)R(e,f)$.

Then by definition of R, we have

$a+d=b+c$ and $c+f=d+e$,

whence by addition, we get

$a+d+c+f=b+c+d+e$ or $a+f=b+e$

Hence, $(a,b)\ R(e,f)$

Thus, $(a,b)\ R(c,d)$ and $(c,d)R(e,f) \Rightarrow (a,b)R(e,f)$.

6. (*b*) Total number of reflexive relations in a set with n elements $= 2^{n^2-n}$.

Therefore, total number of reflexive relation set with 4 elements $= 2^{12}$.

7. (*d*) Since $n \mid n$ for all $n \in N$, therefore R is reflexive. Since $2 \mid 6$ but $6 \nmid 2$, therefore R is not symmetric.

Let $n\,R\,m$ and $m\,R\,p \Rightarrow n|m$ and $m|p \Rightarrow n|p \Rightarrow nRp$. So, R is transitive.

8. (*d*) $-1 \le \log_3 x \le 1$; $3^{-1} \le x \le 3 \Rightarrow \frac{1}{3} \le x \le 3$

$\therefore$ Domain of function $= \left[\frac{1}{3}, 3\right]$.

9. (*c*) (i) $x \le 2$

(ii) $\sqrt{9-x^2} > 0 \Rightarrow |x| < 3$

or $-3 < x < 3$

Hence domain is $(-3, 2]$.

10. (*c*) $f(x) = 1 + \dfrac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \Rightarrow$ Range $= (1, 7/3]$.

11. (*a*) Let $x_1, x_2 \in R$, then $f(x_1) = \cos x_1$, $f(x_2) = \cos x_2$, so $f(x_1) = f(x_2)$

$\Rightarrow \cos x_1 = \cos x_2 \Rightarrow x_1 = 2n\pi \pm x_2$

$\Rightarrow x_1 \neq x_2$, so it is not one-one.

Again the value of f-image of x lies in between –1 to 1

$\Rightarrow f[R] = \{f(x) : -1 \le f(x) \le 1)\}$

So other numbers of co-domain (besides –1 and 1) is notf-image. $f[R] \in R$, so it is also not onto. So this mapping is neither one-one nor onto.

12. (*b*) For any $x, y \in R$, we have

$$f(x) = f(y) \Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n} \Rightarrow x = y$$

$\therefore$ f is one-one.

Let $\alpha \in R$ such that $f(x) = \alpha \Rightarrow \dfrac{x-m}{x-n} = \alpha \Rightarrow$

$$x = \frac{m - n\alpha}{1-\alpha}$$

Clearly $x \notin R$ for $\alpha = 1$. So, f is not onto.

13. (*c*) Let $f(x) = f(y) \Rightarrow \dfrac{x^2-4}{x^2+4} = \dfrac{y^2-4}{y^2+4}$

$\Rightarrow \dfrac{x^2-4}{x^2+4} - 1 = \dfrac{y^2-4}{y^2+4} - 1 \Rightarrow x^2 + 4 = y^2 + 4$

$\Rightarrow x = \pm y$, $\therefore f(x)$ is many-one.

Now for each $y \in (-1,1)$, there does not exist $x \in X$ such that $f(x) = y$. Hence f is into.

14. (*a*) $-\sqrt{1+(-\sqrt{3})^2} \le (\sin x - \sqrt{3}\cos x) \le \sqrt{1+(-\sqrt{3})^2}$

$-2 \le (\sin x - \sqrt{3}\cos x) \le 2$

$-2+1 \le (\sin x - \sqrt{3}\cos x + 1) \le 2+1$

$-1 \le (\sin x - \sqrt{3}\cos x + 1) \le 3$ i.e., range $= [-1, 3]$

$\therefore$ For f to be onto $S = [-1, 3]$

15. (*d*) Let $f(x_1) = f(x_2) \Rightarrow [x_1] = [x_2] \not\Rightarrow x_1 = x_2$
{For example, if x_1=1.4, x_2=1.5, then [1.4]=[1.5]=1}
$\therefore$ f is not one-one.
Also, f is not onto as its range I (set of integers) is a proper subset of its co-domain R.

16. (*c*) $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$
$f(\theta) = \sin^2\theta + \sin\theta\,(3\sin\theta - 4\sin^3\theta)$
$f(\theta) = \sin^2\theta + 3\sin^2\theta - 4\sin^4\theta$
$f(\theta) = 4\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta\,(1 - \sin^2\theta)$
$= 4\sin^2\theta\cos^2\theta$
$f(\theta) = (\sin 2\theta)^2$
$-1 \le \sin 2\theta \le 1 \Rightarrow \sin^2 2\theta \le 1$
$f(\theta) = \sin^2 2\theta \le 1$ for all real θ
$\Rightarrow f(\theta) \ge 0$ for all real θ
$\therefore$ Option (3) is correct Answer.

17. (*c*) $\dfrac{f(a)}{f(a+1)} = \dfrac{a/(a-1)}{(a+1)/a} = \dfrac{a^2}{a^2-1} = f(a^2)$.

18. (*d*) The given expression is

$$\cos(\log x^2)\cos(\log y^2) - \frac{1}{2}\left[\cos\log\frac{x^2}{2} + \cos\log\frac{x^2}{y^2}\right]$$

$$= \frac{1}{2}\left[\cos(\log x^2 + \log y^2) + \cos(\log x^2 - \log y^2)\right]$$

$$-\frac{1}{2}\left[\cos\log\frac{x^2}{2} + \cos(\log x^2 - \log y^2)\right]$$

$$= \frac{1}{2}\left[\cos\log x^2y^2 - \cos\log\frac{x^2}{2}\right].$$

19. (*a*) $\phi(x) = a^x \Rightarrow \phi(p) = a^p$

$\therefore [\phi(p)]^3 = [a^p]^3 = a^{3p} = \phi(3p)$

20. (*d*) $f(f(x)) = \dfrac{\alpha f(x)}{f(x)+1} = \dfrac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\left(\frac{\alpha x}{x+1}+1\right)} = \dfrac{\alpha^2.x}{\alpha x + x + 1}$

$\therefore x = \dfrac{\alpha^2.x}{(\alpha+1)x+1}$ or $x((\alpha+1)x + 1 - \alpha^2) = 0$

or $(\alpha+1)x^2 + (1-\alpha^2)x = 0$.
This should hold for all x.
$\Rightarrow \alpha + 1 = 0, 1 - \alpha^2 = 0$, $\therefore \alpha = -1$.

21. (*b*) $g(x) = 1 + \{x\}$; $f\{g(x)\} = f\{1 + \{x\}\} = f(k) = 1$
where, $k = 1 + \{x\}, 1 \le k < 2$

22. (*a*) $f(x) = 3x^2 - 4x + 8\log(1 + |x|)$; $x \in [0, 1]$
$f(-x) = 3x^2 + 4x + 8\log(1 + |x|)$; $x \in [-1, 0]$
$\therefore$ Option (1) is correct Answer.

23. (*b*) $f(x) = 2x^6 + 3x^4 + 4x^2$

$f(-x) = 2(-x)^6 + 3(-x)^4 + 4(-x)^2 = f(x)$

$\Rightarrow f(x)$ is an even function and derivative of an even function is always odd.

24. (*b*) In (a), $f(-x) = \dfrac{a^{-x}+1}{a^{-x}-1} = \dfrac{1+a^x}{1-a^x} = -\dfrac{a^x+1}{a^x-1} = -f(x)$
So, it is an odd function.

In (b),

$$f(-x) = (-x)\frac{a^{-x}-1}{a^{-x}+1} = -x\frac{1-a^x}{1+a^x} = x\frac{a^x-1}{a^x+1} = f(x)$$

So, it is an even function.

In (c), $f(-x) = \dfrac{a^{-x}-a^x}{a^{-x}+a^x} = -f(x)$
So, it is an odd function.
In (d), $f(-x) = \sin(-x) = -\sin x = -f(x)$
So, it is an odd function.

25. (*d*) Here, $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$

and $f(-x) = \log\left(\dfrac{1-x}{1+x}\right) = \log\left(\dfrac{1+x}{1-x}\right)^{-1}$

$= -\log\left(\dfrac{1+x}{1-x}\right) = -f(x) \Rightarrow f(x)$ is an odd function.

26. (*b*) $f(x) = \log(x + \sqrt{x^2+1})$

and $f(-x) = -\log(x + \sqrt{x^2+1}) = -f(x)$
f(x) is odd function.

27. (*b*) $x \longrightarrow x + 2$
(ii) $f(x+3) + f(x+5) = 2$
(i) $f(x+1) + f(x+3) = 2$
(ii) – (i) $\Rightarrow f(x+1) = f(x+5)$
$f(x+1) = f((x+1)+4)$
Hence T = 4

28. (*d*) $f(x) = \cos(ax) + \sin bx$
cosax is periodic with period = $2\pi/a$
sinbx is periodic with period = $2\pi/b$

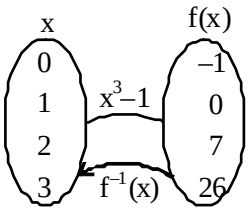
$\dfrac{1}{2}$ L.C.M of $\left[\dfrac{2\pi}{a}, \dfrac{2\pi}{b}\right]$

Option (1), (2), (3) are correct
$\therefore$ Option (4) is correct Answer.

29. (*a*) $f(x) = \sin x + \tan x + \text{sgn}(x^2-6x+10)$
$\downarrow \qquad\qquad\qquad \downarrow$
L.C.M.: T = $2\pi \qquad 2\pi$,

30. (*c*) Let $f(x)$ be periodic with period T.
Then, $f(x+T) = f(x)$ for all $x \in R$
$\Rightarrow x + T - [x+T] = x - [x]$, for all $x \in R$
$\Rightarrow x + T - x = [x+T] - [x]$
$\Rightarrow [x+T] - [x] = T$ for all $x \in R$
$\Rightarrow$ T= 1, 2, 3, 4
The smallest value of T satisfying
$f(x+T) = f(x)$ for all $x \in R$ is 1.
Hence $f(x) = x - [x]$ has period 1.

31. (*c*)



$\therefore$ Domain of $f^{-1}(x)$
= range of f(x)
= {–1, 0, 7, 26}

32. (*a*) $y = \dfrac{e^{2x}-1}{e^{2x}+1}$; Let $e^{2x} = t$

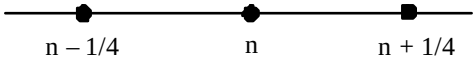
$y = \dfrac{t-1}{t+1} \Rightarrow ty + y = t - 1$

$\Rightarrow \dfrac{y+1}{1-y} = t$

$\Rightarrow e^{2x} = \dfrac{y+1}{1-y} \Rightarrow 2x = \log\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow f^{-1}(x) = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

33. (*d*) If $f(x) = \sqrt{\log_{10}\cos 2\pi x}$ exists, then
$\log_{10}\cos 2\pi x \geq 0$ and $\cos 2\pi x > 0$
or $\cos 2\pi x \geq 1$ and $\cos 2\pi x > \cos \pi/2$
or $\cos 2\pi x \geq \cos 0°$ and $\cos 2\pi x > \cos \pi/2$
or $2\pi x \geq 2n\pi$ and $2\pi x > 2n\pi \pm \pi/2$
or $x \geq n$ and $x > n \pm \pi/4$
where, $n \in$ Integer



If we take $x > n + 1/4$, then $x \geq n$ will be excluded which is not possible
If we take $x > n - 1/4$, then $x \geq n$ will be included which is possible
Therefore,
Option (4) is correct answer.

34. (*d*) A = N × N Binary operation * is defined as
(a, b) * (c, d) = (a + c,b + d)
(a) Now (c,d) * (a, b) = (c + a, d + b) = (a + c, b + d)
= (a, b) * (c, d)
$\Rightarrow$ (a, b) * (c, d) = (c, d) * (a, b)
$\therefore$ This operation * is commutative
(b) Next (a, b) * [(c, d) * (e, f)]
= (a, b) * (c + e, d + f) = ((a + c + e), (b + d + f)) and
[(a, b) * (c, d)] * (e, f)
=(a + c, b + d) * (e, f) = ((a + c +e,b + d + f))
$\Rightarrow$ (a, b) * [(c, d)* (e, f)] = [(a, b) * (c, d)] * (e, f)
$\therefore$ The binary operation given is associative
(c) An element $e = (c_1, e_2) \in A$ will be an identify element for the operation it
$a * e = a = e * a \; \forall \; a = (a_1, a_2) \in A$
i.e. $(a_1 + e_1, a_2 + e_2) = (a_1, a_2) = (e_1 + a_1, e_2 + a_2)$
Which is not true for any element therefore e.
Identity element does not exists.

35. (*d*) A binary operation * on {a, b} is a function form {a,b} × {a, b} → {a, b}
i.e, * is a function form
{(a,a), (a,b), (b,a), (b,b)} → {a, b}
Hence, the total number of bianry oporations on the set {a, b} is 2^4 i.e 16

Learning Plus

1. (*b*) Reflexive relation : a R a
but identity relation is y = x : x ∈ A & y ∈ A
so $I \subset R$

2. (*b*) $n^2 \geq 0$ reflexine,
if $nm \geq 0 \Rightarrow mn \geq 0$ symetric
if $nm \geq 0$
$mp \geq 0$
$\Rightarrow np \geq 0$ transitine

3. (*a*) Since $1 + a.a = 1 + a^2 > 0, \forall a \in S$, $\therefore (a, a) \in R$
$\therefore R$ is reflexive.
Also $(a, b) \in R \Rightarrow 1 + ab > 0$
$\Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$,
$\therefore R$ is symmetric.
$\because (a, b) \in R$ and $(b, c) \in R$ need not imply
$(a, c) \in R$. Hence, R is not transitive.

4. (*a*) A = {3, 6, 9, 12}

(i) $\because$ 3R3, 6R6, 9R9, 12R12 so R is reflexive.

(ii) $\because$ 6R12 but 12 $\not R$ 6 so R is not symmetric.

(iii)Also there are not pairs for whom transitivity fails.

3R6 & 6R12 $\Rightarrow$ 3R12

$\therefore$ R is transitive.

5. (*a*) xRx : As word x has all letters common with itself, so reflexive.

(i) xRy $\Rightarrow$ yRx so symmetric

(ii) xRy, yRz $\not\Rightarrow$ xRz

Ex. (OX. Box) $\in$ R & (Box, Bird) $\in$ R

but (OX, Bird) $\notin$ R, so not transitive.

6. (*a*) Let $A = \{1, 2, 3\}$ and $R = \{(1,1),(1,2)\}, S = \{(2,2)(2,3)\}$ be transitive relations on A.

Then $R \cup S = \{(1, 1); (1, 2); (2, 2); (2, 3)\}$

Obviously, $R \cup S$ is not transitive. Since $(1, 2) \in R \cup S$ and $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

7. (*d*) On the set N of natural numbers, $R = \{(x, y) : x, y \in N, 2x + y = 41\}$.

Since $(1,1) \notin R$ as $2.1 + 1 = 3 \neq 41$. So, R is not reflexive.

$(1, 39) \in R$ but $(39,1) \notin R$. So R is not symmetric (20, 1)

$(1, 39 \in R$. But $(20,39) \notin R$, So R is not transitive.

8. (*b*) For any $a \in R$, we have $a \geq a$, Therefore the relation R is reflexive but it is not symmetric as $(2, 1) \in R$ but $(1, 2) \notin R$. The relation R is transitive also, because $(a, b) \in R, (b, c) \in R$ imply that $a \geq b$ and $b \geq c$ which is turn imply that $a \geq c$.

9. (*b*) To define f(x), $9 - x^2 > 3 \Rightarrow -3 < x < 3$ .....(i)

$-1 \leq (x - 3) \leq 1 \Rightarrow 2 \leq x \leq 4$ .....(ii)

From (i) and (ii), $2 \leq x < 3$ i.e., [2, 3).

10. (*a*) $f(x) = \log_e (3x^2 - 4x + 5)$

$$3x^2 - 4x + 5 \geq \frac{11}{3}$$

$$\Rightarrow \ell n\,(3x^2 - 4x + 5) \geq \ell n\,\frac{11}{3}$$

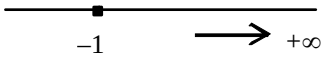
[$\because \ell$n is an increasing function]

$\therefore$ Range is $\left[\ell n \frac{11}{3}, \infty\right)$

11. (*b*) We have to check for every option

For (1) : $f(x) = |x + 1|$; $x \in [-1, \infty)$

$|x + 1| = x + 1$



For one-one

$f(x) = f(y) \Rightarrow x + 1 = y + 1$

$\Rightarrow x = y$ for $x \in [-1, \infty)$

option (1) is Injective

For (2) : $g(x) = x + \frac{1}{x}$; $x \in (0, \infty)$

If g(x) = g(y)

$$\Rightarrow x + \frac{1}{x} = y + \frac{1}{y}$$

$$\Rightarrow (x - y) + \frac{1}{x} - \frac{1}{y} = 0$$

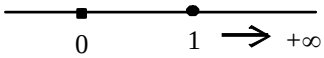
$$\Rightarrow (x - y) + \frac{y - x}{xy} = 0$$

$$\Rightarrow (x - y)\,\frac{[xy - 1]}{xy} = 0$$

$\Rightarrow x = y$ and $xy - 1 \neq 0$

$\Rightarrow x^2 - 1 \neq 0 \Rightarrow x \neq \pm 1$

But $x \in (0, \infty)$



Hence option (2) must-not be Injective

For (3) : $h(x) = x^2 + 4x - 5$; $x \in (0, \infty)$

If $h(x) = h(y) \Rightarrow x^2 + 4x - 5 = y^2 + 4y - 5$

$\Rightarrow (x - y)(x + y) + (x - y).4 = 0$

$\Rightarrow (x - y)(x + y + 4) = 0$

$\Rightarrow x = y$ and $x + y + 4 \neq 0$

$\Rightarrow x \neq -2$

But $x \in (0, \infty)$

$\therefore$ Option (3) must be Injective

For (4) : $k(x) = e^{-x}$; $x \in [0, \infty)$

If $k(x) = k(y) \Rightarrow e^{-x} = e^{-y}$

$\Rightarrow -x = -y$

$\Rightarrow x = y$ for $x \in [0, \infty)$

$\therefore$ Option (4) is Injective

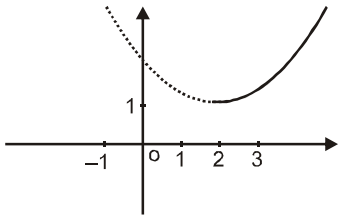
$\therefore$ Correct option is (2)

12. (*b*) $f : [2, \infty) \to Y$

$f(x) = x^2 - 4x + 5$

$f(x) = (x - 2)^2 + 1$

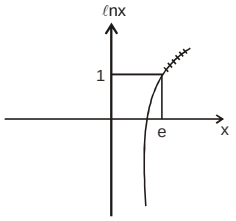
For given domain by graph range is $[1, \infty)$



For function to be onto codomain y = $[1, \infty)$

13. (*c*) $f : (e, \infty) \to R$

$f(x) = \ell n\,(\ell n\,(\ell n\,x))$

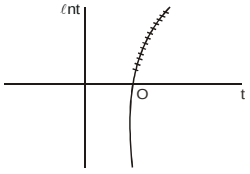


$e < x < \infty$

$1 < \ell nx < \infty$

$0 < \ell n\,(\ell n\,x) < \infty$

$-\infty < \ell n\,(\ell n\,(\ell n\,x) < \infty$

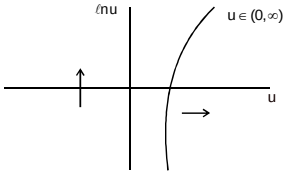


Range = Co-domain onto

Let $\ell n\,t = u$

Let $\ell nx = t$; $t \in (1, \infty)$

$\ell n\,t \in (0, \infty)$

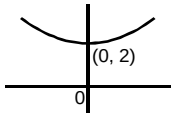


one-one

Hence, one-one onto

14. (*d*) $f : R \to R$; $f(x) = 6^x + 6^{|x|}$

$x \geq 0 \quad f(x) = 2.6^x$



$x < 0 \quad f(x) = 6^x + 6^{-x} = 6^x + \dfrac{1}{6^x} \geq 2$

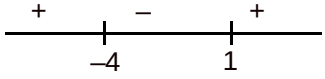
Many-one into

15. (*c*) $f : R \to R$; $f(x) = \dfrac{x^2-4}{x^2+1} \Rightarrow f(-x) = \dfrac{x^2-4}{x^2+1} = f(x)$

f(x) is even that's why many-one.

$y = \dfrac{x^2-4}{x^2+1} \Rightarrow yx^2 + y = x^2 - 4$

$\Rightarrow \quad x^2 = \dfrac{y+4}{1-y} \geq 0$



$\Rightarrow \quad \dfrac{y+4}{y-1} \leq 0$

$\therefore \quad y \in [-4, 1)$

Range ≠ Co-Domain $\Rightarrow$ into

16. (*d*) $f(x) = \cos\left[\dfrac{\pi^2}{2}\right] x + \sin\left[-\dfrac{\pi^2}{2}\right] x$; $\dfrac{\pi^2}{2} \cong 4.9$

$f(x) = \cos 4x - \sin 5x$

$f(0) = 1$

$f\left(\dfrac{\pi}{3}\right) = \cos\dfrac{4\pi}{3} - \sin\dfrac{5\pi}{3} = \dfrac{\sqrt{3}-1}{2} = \dfrac{1}{\sqrt{3}+1}$

$f\left(\dfrac{\pi}{2}\right) = 0$

$f(\pi) = 1$

17. (*b*) (*a*) $f(x) = \sin^{-1}x + \cos^{-1}x, x \in [-1, 1]$ and $g(x) = \dfrac{\pi}{2}, x \in R$

$f(x) = \dfrac{\pi}{2}, \quad x \in [-1, 1]$ and $g(x) = \dfrac{\pi}{2}, x \in R$ Non-identical functions

(*b*) $f(x) = \tan^{-1}x + \cot^{-1}x$ and $g(x) = \dfrac{\pi}{2}, x \in R$

$f(x) = \dfrac{\pi}{2}, x \in R$ and $g(x) = \dfrac{\pi}{2}, x \in R$ Identical functions

(*c*) $f(x) = \sec^{-1}x + \text{cosec}^{-1}x$ and $g(x) = \dfrac{\pi}{2}, x \in R$

$f(x) = \dfrac{\pi}{2}, |x| \in [1, \infty)$ and $g(x) = \dfrac{\pi}{2}, x \in R$ Non-identical functions

18. (*b*) $f(x) = 2\sin^2\theta + 4\cos(x+\theta)\sin x.\sin\theta + \cos(2x+2\theta)$

$f(x) = \cos 2x$

$\Rightarrow \quad f\left(\dfrac{\pi}{4} - x\right) = \cos\left(2\left(\dfrac{\pi}{4} - x\right)\right) = \sin 2x$

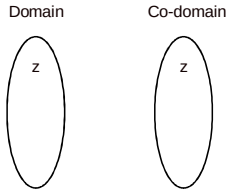
$\Rightarrow \quad f^2(x) + f^2\left(\dfrac{\pi}{4} - x\right) = 1$

19. (*b*) $z \to$ integer

$f(x) = ax^2 + bx + c$

$f(0) = c \in I$

$f(1) = a + b + c = I$



$f(1) - f(0) = a + b \in I$

20. (*c*) (*a*) $e^{(\ell nx)/2}$ and $\sqrt{x} \Rightarrow D_1 \in (0, \infty)$; $D_2 \in [0, \infty)$

Domain are not same so not identical

(*b*) $\tan^{-1}(\tan x)$ and $\cot^{-1}(\cot x)$

Domain are not same so non identical

(*c*) $\cos^2x + \sin^4x$ and $\sin^2x + \cos^4x$

$\Rightarrow \quad x \in R \;\&\; x \in R$ Identical

21. (*d*) $f(x) = \cos(\ell n\, x)$

$f(x) \cdot f(y) = \cos(\ell n\, x) \cdot \cos(\ell n\, y)$

$$f\left(\frac{x}{y}\right) + f(xy) = \cos(\ell n\, x - \ell n y) + \cos(\ell n\, x + \ell n\, y)$$

$$\therefore \quad f(x) \cdot f(y) - \frac{1}{2}\left(f\left(\frac{x}{y}\right) + f(xy)\right) = \cos(\ell n\, x)\cos(\ell n\, y)$$

$$-\frac{1}{2}[2\cos(\ell n x)\cos(\ell n\, y) = 0$$

22. (*d*) $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

Replace $x + \frac{1}{x} = t$, where $|t| \geq 2$

$\therefore \quad f(t) = t^2 - 2,\ |t| \geq 2$

23. (*c*) $f(1) = 1 = 2 - 1$

$f(n + 1) = 2f(n) + 1$

$\therefore \quad f(2) = 2f(1) + 1 = 2.1 + 1 = 3 = 2^2 - 1$

$f(3) = 7 = 2^3 - 1$

$f(4) = 15 = 2^4 - 1$

Similarly $f(n) = 2^n - 1$

24. (*b*) **Method 1 :**

(usual but lengthy)

$x^2 f(x) + f(1 - x) = 2x - x^4$.....(1)

replace x by $(1 - x)$ in equation (1)

$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4$.....(2)

eliminate $f(1 - x)$ by equation (1) and (2)

we get

$f(x) = 1 - x^2$

Method 2 : Since R.H.S. is polynomial of 4th degree and also by options consider $f(x) = ax^2 + bx + c$

$x^2 f(x) + f(1 - x) = 2x - x^4$

$\Rightarrow \quad x^2(ax^2 + bx + c) + a(1 - x)^2 + b(1 - x) + c = 2x - x^4$

by comparing coefficients

$a = -1$

$b = 0$

$c = 1$

$\therefore \quad f(x) = -x^2 + 1$

25. (*c*) $y = 2[x] + 3$ and $y = 3[x - 2]$

$2[x] + 3 = 3[x] - 6$

$\Rightarrow \quad [x] = 9 \Rightarrow x \in [9, 10)$

$\therefore \quad y = 21$

$\therefore \quad [x + y] = 30$

26. (*a*) $f(x)$ where $x \in [0, 1]$

$f(\ell n(1 - x^2))$ where $0 \leq \ell n(1 - x^2) \leq 1$

$1 \leq 1 - x^2 \leq e$

$0 \leq -x^2 \leq e - 1$

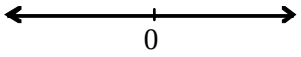
$0 \geq x^2 \geq 1 - e$

(i) $x = 0$

(ii) $x \in (-\infty, -\sqrt{1-e}\,] \cup [\sqrt{1-e}\,, \infty)$

$\downarrow \qquad \downarrow$

Imaginary numbers as $1 < e$



$\Rightarrow \quad x \in \{0\}$

27. (*a*) $f(g(x_1)) = f(g(x_2))$

$\Rightarrow \quad g(x_1) = g(x_2)$

as f is one - one function

$\Rightarrow \quad x_1 = x_2$

as g is one - one function hence $f(g(x_1)) = f(g(x_2))$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad f(g(x))$ is one - one function

28. (*b*) $f(x) = [x] + \frac{1}{2},\ x \notin I$

$$f(-x) = [-x] + \frac{1}{2} = -[x] - 1 + \frac{1}{2} = -\left([x] + \frac{1}{2}\right) = -f(x)$$

odd function

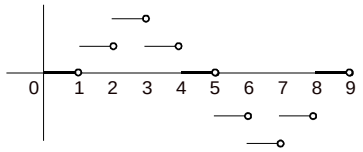
29. (*d*) $f(x) = \frac{x f(x^2)}{2 + \tan^2 x . f(x^2)}$ given that $f(-x) = f(x)$ ....(1)

$$f(-x) = \frac{-x\, f(x^2)}{\tan^2 x \cdot f(x^2)} \Rightarrow f(-x) = -f(x) \quad ...(2)$$

When both conditions are there only one possibility is there when $f(x) = 0 \Rightarrow f(10) = 0$

30. (*c*) $f(x) = \sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cos\frac{\pi}{3}[x]$

$\frac{2\pi}{\left(\frac{\pi}{4}\right)} = 8$



If $0 \leq x < 1$ then $y = 0$

If $1 \leq x < 2$ then $y = \frac{1}{\sqrt{2}}$

If $2 \leq x < 3$ then $y = 1$

If $3 \leq x < 4$ then $y = \frac{1}{\sqrt{2}}$

If $4 \leq x < 5$ then $y = 0$

Period $8/4/6 \Rightarrow$ Lcm = 24

31. (*c*) $f(x) = x(2-x)$; $f(x+2) = f(x) \Rightarrow$ period $= 2$

32. (*a*) $f(x) = x + (-1)^{x-1}$

$$f(x) = \begin{cases} x+1 \ ; & x \in \text{odd natural number} \\ x-1 \ ; & x \in \text{even natural number} \end{cases}$$

$y = f(x)$

$$x = \begin{cases} y-1 \ ; & y \in \text{even natural number} \\ y+1 \ ; & y \in \text{odd natural number} \end{cases}$$

$$f^{-1}(x) = \begin{cases} x-1 \ ; & x \in \text{even natural number} \\ x+1 \ ; & x \in \text{odd natural number} \end{cases}$$

$\therefore$ $f(x)$ is inverse to itself

33. (*c*) Greatest integer function defined as $[x] \le x$ for all $x \in R$.

then $\left[\frac{1}{2}+\frac{1}{1000}\right]+\left[\frac{1}{2}+\frac{2}{1000}\right]+.....$

$$+\left[\frac{1}{2}+\frac{999}{1000}\right] (999 \text{ term})$$

i.e. $\left[\frac{1}{2}+\frac{1}{1000}\right]+\left[\frac{1}{2}+\frac{2}{1000}\right]+.....$

$$+\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+.....+\left[\frac{1}{2}+\frac{999}{1000}\right]$$

First 499 terms, each will be zero and remaining 500 terms will be as follows

$$\left[\frac{1}{2}+\frac{500}{1000}\right]+\left[\frac{1}{2}+\frac{501}{1000}\right]+.....+\left[\frac{1}{2}+\frac{999}{1000}\right]$$

$= +1+1 ..... +1(500 \text{ terms}) = 500$

$\therefore$ Option (3) is correct Answer.

Advanced Level Multiconcept Questions

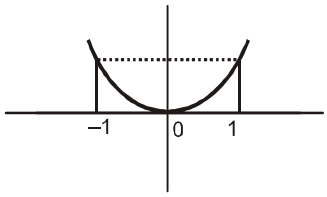
1. (*b,c*) $f(x) = \ell n(\sin^{-1}(\log_2 x))$

Domain $0 < \log_2 x \le 1$, $x \in (1, 2]$

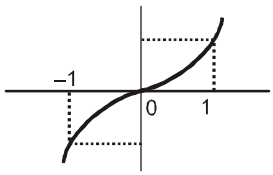
Range $\left(-\infty, \ell n\frac{\pi}{2}\right]$

2. (*b,d*) Domain $D \equiv [-1, 1]$

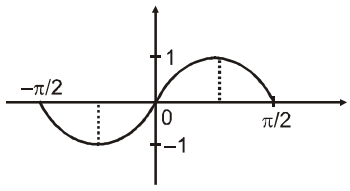
(*a*) $f(x) = x^2$ many -one



(*b*) $g(x) = x^3$ one - one

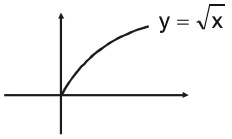


(*c*) $h(x) = \sin 2x$ many - one



(*d*) $k(x) = \sin\left(\frac{\pi x}{2}\right)$ one-one function

3. (*a,d*) $f : I^+ \to R$, $f(x) = \sqrt{x}$



Range $= \{y : y = \sqrt{n}, n \in I^+\}$

$\therefore$ function is not onto but one-one

4. (*b,c,d*) $f : R \to [-1, 1]$

$$f(x) = \sin\left(\frac{\pi}{2}[x]\right) = \begin{cases} -1 & , \quad -1 \le x < 0 \\ 0 & , \quad 0 \le x < 1 \\ 1 & , \quad 1 \le x \le 2 \end{cases}$$

Many - one function
into function

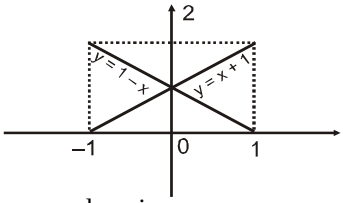
Also $f(x+4) = \sin\left(\frac{\pi}{2}[x+4]\right)$

$= \sin\left(2\pi + \frac{\pi}{2}[x]\right)$

$= \sin\left(\frac{\pi}{2}[x]\right)$

$= f(x)$ and hence periodic

5. (*a,b*) $f ; [-1, 1] \to [0, 2]$
for onto function



Range = codomain
only two linear functions possible as show in graph.

6. (*a,b,c*) By definition

7. (*abcd*) $f(x)=\dfrac{x-2\sqrt{5}}{5\sqrt{2}x-1}$

$\Rightarrow f^{-1}(x)=\dfrac{x-2\sqrt{5}}{5\sqrt{2}x-1}$

$\Rightarrow f(x)=f^{-1}(x)\ \forall\ x\neq\dfrac{1}{5\sqrt{2}}$

$\Rightarrow f\left(f(x)\right)=x$

8. (*a,c*) $f(x)=\dfrac{1-x}{1+x}, 0\le x\le 1$

$g(x)=4x(1-x), 0\le x\le 1$

$fog(x)=\dfrac{1-g(x)}{1+g(x)}=\dfrac{1-4x(1-x)}{1+4x(1-x)}=\dfrac{1-4x+4x^2}{1+4x-4x^2}$

$gof(x)=4f(x).(1-f(x))$

$=4\left(\dfrac{1-x}{1+x}\right)\left(1-\left(\dfrac{1-x}{1+x}\right)\right)$

$=\dfrac{8x(1-x)}{(1+x)^2}$

9. (*b,c,d*)

(*a*) $f(x)=e^{\ell n(\sec^{-1}x)}=\sec^{-1}x$; $\sec^{-1}x>0$ but not equal to zero, so $x\neq 1$
$x\in(-\infty,-1]\cup(1,\infty)$
$g(x)=\sec^{-1}x, x\in(-\infty,-1]\cup[1,\infty)$
non-identical functions

(*b*) $f(x)=\tan(\tan^{-1}x)=x, x\in R$
$g(x)=\cot(\cot^{-1}x)=x, x\in R$
identical functions

(*c*) $f(x)=\operatorname{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$

$g(x)=\operatorname{sgn}(\operatorname{sgn}x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$

Identical functions

(*d*) $f(x)=\cot^2x.\cos^2x$,
$x\in R-\{n\pi\}$, $n\in I$
$g(x)=\cot^2x-\cos^2x$
$=\cot^2x(1-\sin^2x)$
$=\cot^2x.\cos^2x$
$x\in R-\{n\pi\}, n\in I$
Identical functions

10. (*a*) $f'(x)=x^2+x+a$
$f'(x)\ge 0$

$\Rightarrow\left(x+\dfrac{1}{2}\right)^2+\left(a-\dfrac{1}{4}\right)\ge 0$

Hence $a\ge\dfrac{1}{4}$

11. (*b*) If $a=-1$
$f'(x)=x^2+x-1$
$D=1+4>0$
$\Rightarrow$ many one function and cubic polynomial have range $(-\infty,\infty)$ hence onto function

12. (*a*) For f(x) to be invertible f(x) should be one-one that is $f'(x)\ge 0$,

$\Rightarrow a\in\left[\dfrac{1}{4},\infty\right)$

13. (*a*) $f_2(x)=\dfrac{1}{2}\Rightarrow\left|f(|x|)\right|=\dfrac{1}{2}$

$\Rightarrow f(|x|)=\pm\dfrac{1}{2}$

$\Rightarrow |x|=\sqrt{e}, \sqrt{\dfrac{1}{2}}$

14. (*d*) $f(|x|)=\left|f(|x|)\right|$
$\Rightarrow f(|x|)\ge 0$

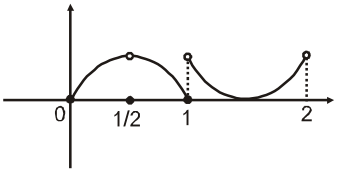
15. (*c*) Range of $f_3(x)=[-1,1]$
$\Rightarrow\log_{27}(f_3(x)+2)=[\log_{27}1,\log_{27}3]$

16. (*a*) $\rightarrow$ (q, r), (*b*) $\rightarrow$ (q, r), (*c*) $\rightarrow$ (q), (*d*) $\rightarrow$ (s)

(*a*) $y=\sin\left(2\pi t+\dfrac{\pi}{3}\right)+2\sin\left(3\pi+\dfrac{\pi}{4}\right)+3\sin(5\pi t)$

$\text{LCM}\left(\dfrac{2\pi}{2\pi},\dfrac{3\pi}{2\pi},\dfrac{5\pi}{2\pi}\right)=\text{LCM}=\left(1,\dfrac{2}{3},\dfrac{2}{5}\right)=2$

(*b*) $y=\{\sin\pi x\}$ period $=2$



For $x\in\left(0,\dfrac{1}{2}\right)$ function is one-one
For $x\in(0,2)$ and $x\in(0,8)$
function is money - one

(*c*) $y=\dfrac{1}{2}\left(\dfrac{\left|\sin\dfrac{\pi}{4}x\right|}{\cos\dfrac{\pi}{4}x}+\dfrac{\sin\dfrac{\pi}{4}x}{\left|\cos\dfrac{\pi}{4}x\right|}\right)$

(*d*) Since f(x) is bijective,
$\therefore$ $f(0)=0$ or 2 but $f(0)=0\Rightarrow c=0$
(which is not true)
$\therefore$ $f(0)=2$ & $f(2)=0$

17. $(a) \to (q), (b) \to (s), (c) \to (p), (d) \to (r)$,

$(a)\ \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2} \Rightarrow x \in [0, 1]$

$(b)\ \sin^{-1}\sqrt{x} + \cos^{-1}\left(1-\sqrt{x^2}\right) = 0$

$\Rightarrow \cos^{-1}\sqrt{1-x^2} = -\sin^{-1}(x) \Rightarrow x \in [-1, 0]$

$(c)\ g\left(\dfrac{1-x^2}{1+x^2}\right) = 2h(x)$

$\Rightarrow \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

$\Rightarrow x \in [0, \infty)$

$(d)\ h(x) + h(1) = h\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow \tan^{-1}x + \tan^{-1} = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow x \in (-\infty, 1)$

18. (1.75) $2f(x) - 3f(1/x) = x^2$ ... (i)

replacing x by 1/x

$2f\left(\dfrac{1}{x}\right) - 3f(x) = 1/x^2$ ... (ii)

solving (i) & (ii) we get

$-5f(x) = 2x^2 + 3/x^2$

$f(x) = -1/5\,(2x^2 + 3/x^2)$

$\therefore\ f(2) = -1/5\,(8 + 3/4) = -7/4$

19. (1000)

$f(x) + f(x+4) = f(x+2) + f(x+6)$... (i)

Replacing x by x+2 in equation (i), we get

$f(x+2) + f(x+6) = f(x+4) + f(x+8)$ ... (ii)

Adding (i) & (ii) $f(x) = f(x+8)$

$f(5) = f(13) = 10$

$f(13) = f(21) = 10$

$f(805) = 10$

$\therefore\ \sum_{r=1}^{100} f(5+8r) = f(13) + f(21) + ... + f(805)$

$= 10 + 10 + ...$ 100 times. $= 10 \times 100 = 1000.$

20. (0001) $y = 1 + a\,x \Rightarrow x = \dfrac{y-1}{\alpha}$

$f^{1}(x) = \dfrac{x-1}{\alpha} = f(x) = 1 + \alpha x$

$\Rightarrow \dfrac{x-1}{\alpha} = 1 + \alpha x$

$\Rightarrow x - 1 = \alpha + \alpha^2 x$

Equating the coefficient of x

$\alpha^2 = 1$ & $\alpha = -1$

$\alpha = \pm 1$

$\alpha = -1.$

21. (0005)

$f'(x) = -f(x)$ and $f'(x) = g(x)$

$\Rightarrow f''(x) \cdot f'(x) + f(x) \cdot f'(x) = 0$

$\Rightarrow f(x)^2 + (f'(x))^2 = c$

$\Rightarrow (f(x)^2 + (g(x))^2 = c$

$\Rightarrow F(x) = c$

$\Rightarrow F(10) = 5.$

22. (0000)

$f(x) = 0\ \forall\ x \in R$

$\Rightarrow f(3) - f(2) = 0.$

23. (0013)

$x = 0 \Rightarrow f(2) = 2f(0) - f(1) = 2 \times 2 - 3 = 1$

$x = 1 \Rightarrow f(3) = 6 - 1 = 5$

$x = 2 \Rightarrow f(4) = 2f(2) - f(3) = 2 \times 1 - 5 = -3$

$x = 3 \Rightarrow f(5) = 2f(3) - f(4) = 2(5) - (-3) = 13$.

24. (0.875) $f(2a - x) = f(x) \Rightarrow f(2a + x) = -f(x)$ $\because$ f is odd

$\Rightarrow f(x + 4a) = f(x)$

$\Rightarrow$ f is periodic with period 4a $\Rightarrow f(1 + 4r) = f(1)$

Now $\sum_{r=0}^{\infty} [f(1)]^r = 8$

$\Rightarrow \dfrac{1}{1-f(1)} = 8 \qquad \Rightarrow f(1) = 7/8$

25. (0002) Putting $x = f(y) = 0$ then $f(0) = f(0) + 0 + f(0) - 1$

$\Rightarrow f(0) = 1$

Putting $x = f(y)$

We get $f(0) = f(x) + x^2 + f(x) - 1 \Rightarrow f(x) = 1 - x^2/2$

$f(16) = 1 - \dfrac{256}{2} = 1 - 128 = -127$

$|f(16)| = 127.$

26. (0004) $3f(x) + 2f\left(\dfrac{x+59}{x-1}\right) = 10x + 30$

For $x = 7$, $3f(7) + 2f(11) = 70 + 30 = 100$

For $x = 11$, $3f(11) + 2f(7) = 140.$

$\dfrac{f(7)}{-20} = \dfrac{f(11)}{-220} = \dfrac{-1}{9-4}$ or $f(7) = 4$

27. (0004)

$[x]^2 = x + 2\{x\}$

or $[x]^2 = [x] + 3\{x\}$

or $\{x\} = \dfrac{[x]^2 - [x]}{3}$

or $0 \le \dfrac{[x]^2 - [x]}{3} < 1$

or $0 \le [x]^2 - [x] < 3$

or $[x] \in \left(\dfrac{1-\sqrt{3}}{2}, 0\right] \cup \left[1, \dfrac{1+\sqrt{3}}{2}\right)$

or $[x] = -1, 0, 1, 2$

or $\{x\} = \dfrac{2}{3}, 0, 0, \dfrac{2}{3}$ (respectively)

or $x = -\dfrac{1}{3}, 0, 1, \dfrac{8}{3}$

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*c*) Given $R = \{(x, y) : x, y \in Z, x^2 + 3y^2 \le 8\}$
So $R = \{(1, 1), (2, 1), (1, -1), (0, 1), (1, 0)\}$

So $D_{R^{-1}} = \{-1, 0, 1\}$

2. (*a*) (i) If $(a, b) \in R_1$ and $(b,c) \in R_1$
$\Rightarrow a^2 + b^2 \in Q$ and $b^2 + c^2 \in Q$
then $a^2 + 2b^2 + c^2 \in Q$ but we cannot say anything about $a^2 + c^2$, that it is rational or not. So R_1 is not transitive.

(ii) If $(a, b) \in R_2$ and $(b,c) \in R_2$
$\Rightarrow a^2 + b^2 \notin Q$ and $b^2 + c^2 \notin Q$
but we can't say anything about $a^2 + c^2$
that it is rational or irrational.
So R_2 is not transitive.

3. (*c*) $f(x) + 2f\left(\frac{1}{x}\right) = 3x$ ……..(1)

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \text{ ……..(2)}$$

Multiply eq (2) by 2 & subtract from eq (1)

$$f(x) = \frac{2}{x} - x$$

Now, $f(x) = f(-x)$

$$\frac{2}{x} - x = \frac{2}{-x} + x$$

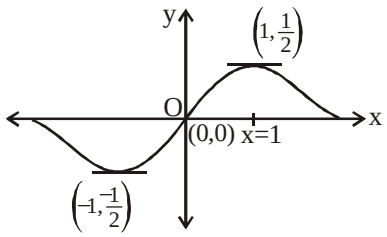
$$\Rightarrow 2x = \frac{4}{x} \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$

So, S contains exactly two elements.

4. (*d*) $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$f(x) = \frac{x}{1+x^2} \forall\ x \in R$$

$$\Rightarrow f'(x) = \frac{(1+x^2).1 - x.2x}{(1+x^2)^2} = \frac{-(x+1)(x-1)}{(1+x^2)^2}$$



singn of f'(x)

$\therefore$ From above diagram of f(x), f(x) is surjective but not injective.

5. (*b*) Case - I : $x \in [0,9]$

$$2(3 - \sqrt{x}) + x - 6\sqrt{x} + 6 = 0$$

$$\Rightarrow x - 8\sqrt{x} + 12 = 0 \Rightarrow \sqrt{x} = 4, 2$$

$x = 16, 4 \Rightarrow x = 4$ ↑

rejected

Case - II : $x \in [9, \infty]$

$$2(\sqrt{x} - 3) + x - 6\sqrt{x} + 6 = 0$$

$$x - 4\sqrt{x} = 0 \Rightarrow x = 16, 0$$

↑

rejected

So, x = 4, 16

6. (*a*) $x \in R - \{0,1\}$

$$f_1(x) = \frac{1}{x}, f_2(x) = 1 - x, f_3(x) = \frac{1}{1-x}$$

Given $f_2\left(J\left(f_1(x)\right)\right) = f_3(x)$

$$1 - J\left(f_1(x)\right) = f_3(x)$$

$$J\left(f_1(x)\right) = 1 - f_3(x) = 1 - \frac{1}{1-x}$$

$$J\left(f_1(x)\right) = \frac{x}{x-1}$$

$$J\left(\frac{1}{x}\right) = \frac{x}{x-1} = \frac{1}{1 - \frac{1}{x}}$$

$$J(x) = \frac{1}{1-x} = f_3(x)$$

7. (*a*) $f(x) = 2\left(1 + \frac{1}{x-1}\right)$

$$f'(x) = -\frac{2}{(x-1)^2}$$

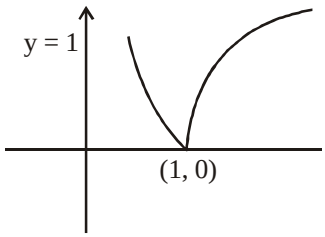
$\Rightarrow$ f is one – one but not onto

8. (*a*) $\left.\begin{matrix} f(g(1)) = 1 \\ f(g(2)) = 1 \end{matrix}\right\}$ Many one

$f(g(2)) = k$
$f(g(2k + 1)) = k + 1$

$\therefore$ Onto

9. (*c*) $y = \left|1-\frac{1}{x}\right|$



Neither one-one nor Onto

10. (*a*) $F_4(x) = \frac{\sin^4 x + \cos^4 x}{4} = \frac{1-2\sin^2 x.\cos^2 x}{4}$

$= \frac{1}{4} - \frac{1}{2}\sin^2 x.\cos^2 x$

$F_6(x) = \frac{\sin^6 x + \cos^6 x}{6}$

$= \frac{1-3\sin^2 x.\cos^2 x(\sin^2 + \cos^2 x)}{6}$

$= \frac{1}{6} - \frac{1}{2}\sin^2 x.\cos^2 x$

$F_4(x) - f_6(x) = \frac{1}{4} - \frac{1}{6} = \frac{6-4}{24} = \frac{2}{24} = \frac{1}{12}$

11. (*a*) $f(x) = a^x, a > 0$

$f(x) = \frac{a^x + a^{-x} + a^x - a^{-x}}{2}$

$\Rightarrow f_1(x) = \frac{a^x + a^{-x}}{2}$

$\Rightarrow f_2(x) = \frac{a^x - a^{-x}}{2}$

$\Rightarrow f_1(x+y) + f_1(x-y)$

$= \frac{a^{x+y} + a^{-x-y}}{2} + \frac{a^{x-y} + a^{-x+y}}{2}$

$= \frac{\left(a^x + a^{-x}\right)}{2}\left(a^y + a^{-y}\right)$

$= f_1(x) \times 2 f_1(y)$

$= 2 f_1(x)\ f_1(y)$

12. (*b*) From the given functional equation :

$f(x) = 2x \qquad \forall\ x \in N$

$2^{a+1} + 2^{a+2} + ... + 2^{a+10} = 16(2^{10}-1)$

$2^a (2 + 2^2 + ... + 2^{10})\ 16(2^{10}-1)$

$2^a.\frac{2.\left(2^{10}-1\right)}{1} = 16\left(2^{10}-1\right)$

$2^{a+1} = 16 = 2^4$

$a = 3$

13. (*a*) $y = \frac{x^2}{1-x^2}$

Range of y : R–[–1,0)

for surjective function, A must be same as above range.

14. (*d*) g(S) = [–2, 2]

So, $f(g(S)) = [0, 4] = S$

And $f(S) = [0, 16]$

$\Rightarrow f(g(S)) \neq f(S)$

Also, $g(f(S)) = [-4, 4] \neq g(S)$

So, $g(f(S) \neq S$

15. (*d*) $fog(x) = (-x) \Rightarrow (fg(\alpha)) = -\alpha = b$

$(fg(x))' = -1 \Rightarrow (fg(\alpha))' = -1 = a$

16. (*c*) $f(x) = \sqrt{x}$, $g(x) = \tan x$, $h(x) = \frac{1-x^2}{1+x^2}$

$fog(x) = \sqrt{\tan x}$

$hofog(x) = h(\sqrt{\tan x}) = \frac{1-\tan x}{1+\tan x}$

$= -\tan\left(\frac{\pi}{4} - x\right)$

$\phi(x) = \tan\left(\frac{\pi}{4} - x\right)$

$\phi\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{12}\right) = -\tan\frac{\pi}{12}$

$= \tan\left(\pi - \frac{\pi}{12}\right) = \tan\frac{11\pi}{12}$

17. (*c*) $g((x)) = f^2(x) + f(x) - 1$

$g\left(f\left(\frac{5}{4}\right)\right) = 4\left(\frac{5}{4}\right)^2 - 10.\frac{5}{4} + 5 = -\frac{5}{4}$

$g\left(f\left(\frac{5}{4}\right)\right) = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$

$\frac{-5}{4} = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$

$f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) + \frac{1}{4} = 0$

$\left(f\left(\frac{5}{4}\right) + \frac{1}{2}\right)^2 = 0$

$f\left(\frac{5}{4}\right) = \frac{-1}{2}.$

18. (*c*) $y = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$

$$\frac{1+y}{1-y} = \frac{8^{2x}}{8^{-2x}} \Rightarrow 8^{4x} = \frac{1+y}{1-y}$$

$$4x = \log_8\left(\frac{1+y}{1-y}\right) \Rightarrow x = \frac{1}{4}\log_8\left(\frac{1+y}{1-y}\right)$$

$$f^{-1}(x) = \frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right) = \frac{\log_8 e}{4}\log_e\left(\frac{1+x}{1-x}\right).$$

19. (*c*) $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$. If $f(1) = 2$
$\Rightarrow f(x) = 2x$

Now, $g(n) = \sum_{k-1}^{(n-1)} f(k)$

$= f(1) + f(2) + f(3) + ...f(n-1)$
$= 2 + 4 + 6 + ... + 2(n-1)$
$= 2[1 + 2 + 3 + ... + (n-1)]$

$= 2 \times \dfrac{(n-1)(n)}{2} = n^2 - n$

So, $n^2 - n = 20$ (given)
$\Rightarrow n^2 - n - 20 = 0$
$(n-5)(n+4) = 0$
$\Rightarrow n = 5$

22. (*a*) $\because f(x) = \sin^{-1}\left(\dfrac{|x|+5}{x^2+1}\right)$

$$\therefore\ -1 \le \frac{|x|+5}{x^2+1} \le 1$$

$$\therefore\ x^2 - |x| - 4 \ge 0$$

$$\left(|x| - \frac{1-\sqrt{17}}{2}\right)\left(|x| - \frac{1+\sqrt{17}}{2}\right) \ge 0$$

$$\therefore\ |x| \ge \frac{1-\sqrt{17}}{2}$$

$$\therefore x \in \left(-\infty, -\frac{1+\sqrt{17}}{2}\right] \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$$

21. (19)The desired functions will contain either one element or two elements in its codomain of which '2' always belongs to f(A).
$\therefore$ The set B can be
$\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}$
Total number of functions
$= 1 + (2^3 - 2)3$
$= 19$

22. (*d*) $f(f(x)) = \dfrac{a - \left(\dfrac{a-x}{a+x}\right)}{a + \left(\dfrac{a-x}{a+x}\right)} = x$

$$\Rightarrow \frac{a^2 + ax - a + x}{a^2 + ax + a - x} = x$$
$\Rightarrow a^2 + (a+1)x - a = a^2x + (a-1)x^2 + ax$
$\Rightarrow (a-1)x^2 + (a^2-1)x + (a - a^2) = 0$
$\forall x \in R - \{-a\}$
Hence $a = 1$

$$\therefore\ f(x) = \frac{1-x}{1+x} \Rightarrow f\left(-\frac{1}{2}\right) = 3$$

23. (*b*) $\dfrac{3^{200}}{8} = \dfrac{1}{8}(9^{100})$

$$= \frac{1}{8}(1+8)^{100}$$

$$= \frac{1}{8} + \text{Integer} \therefore \left\{\frac{3^{200}}{8}\right\} = \left\{\frac{1}{8} + \text{intger}\right\}$$

$$= \frac{1}{8}$$

24. (*b*) Let $f(1) = a$
then $f(1+1) = a^2$
$f(2+1) = a^2$
and so on.

$$\sum_{x=1}^{\infty} f(x) = 2 \Rightarrow a + a^2 + a^3 + ... \infty = 2$$

$$\Rightarrow \frac{a}{1-a} = 2$$

$$\Rightarrow a = \frac{2}{3}$$

Now, $\dfrac{f(4)}{f(2)} = \dfrac{a^4}{a^2} = a^2 = \dfrac{4}{9}$

25. (*b*) $f(m+n) = f(m) + f(n)$

Put $m = 1, n = 1$

$f(2) = 2f(1)$

Put $m = 2, n = 1$

$f(3) = f(2) + f(1) = 3f(1)$

Put $m = 3, n = 3$

$f(6) = 2f(3) \Rightarrow f(3) = 9$

$\Rightarrow f(1) = 3, f(2) = 6$

$f(2) \cdot f(3) = 6 \times 9 = 54$

26. (*c*) $f(x)=\sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right)+\cos^{-1}\left(\frac{x-1}{x+1}\right)$ ……..(1)

$$-1\le\frac{x-1}{x+1}\le 1\Rightarrow 0\le x<\infty$$

$$-1\le\frac{3x^2+x-1}{(x-1)^2}\le 1\Rightarrow x\in\left[\frac{-1}{4},\frac{1}{2}\right]\cup\{0\}\ldots\ldots\ldots(2)$$

(1) & (2)

$\Rightarrow$ Domain $=\left[\frac{1}{4},\frac{1}{2}\right]\cup\{0\}$

27. (*d*) $\frac{1+x}{x}\in(-\infty,-1]\cup[1,\infty)$

$$\Rightarrow\frac{1}{x}\in(-\infty,-2]\cup[0,\infty)$$

$$\Rightarrow x\in\left[-\frac{1}{2},0\right)\cup(0,\infty)\Rightarrow x\in\left[-\frac{1}{2},0\right)\cup\{0\}$$

28. (*c*) $\therefore$ (gof) $^{-1}$ exist $\Rightarrow$ gof is bijective

$\Rightarrow$ 'f' must be one-one and 'g' must be ONTO.

29. (*b*) $x^3-3x^2y-xy^2+3y^3=0$

$$\Rightarrow x\left(x^2-y^2\right)-3y\left(x^2-y^2\right)=0$$

$$\Rightarrow(x-3y)(x-y)(x+y)=0$$

30. (*a*) $O\le x^2-x+1\le 1$

$\Rightarrow x^2-x\le 0\ \Rightarrow x\in[0,1]$

Also, $0<\sin^{-1}\left(\frac{2x-1}{2}\right)\le\frac{\pi}{2}$

$$\Rightarrow 0<\frac{2x-1}{2}\le 1$$

$$\Rightarrow 0<2x-1\le 2$$

$1<2x\le 3$

$\frac{1}{2}<x\le\frac{3}{2}$

Taking intersection

$x\in\left(\frac{1}{2},1\right]$

$$\Rightarrow\alpha=\frac{1}{2},\beta=1$$

$$\Rightarrow\alpha+\beta=\frac{3}{2}$$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*a*) $f(x)=x^2$; $g(x)=\sin x\Rightarrow gof(x)=\sin x^2$

$\Rightarrow$ gogof $(x)=\sin(\sin x^2)$

$\Rightarrow$ (fogogof) $(x)=(\sigma in(\sin x^2))^2=\sin^2(\sin x^2)$

Now$\sin^2(\sin x^2)=\sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2)=0,1$

$\Rightarrow \sin x^2=n\pi,(4n+1)\frac{\pi}{2}$; $\eta\in I$

$\Rightarrow \sin x^2=0$

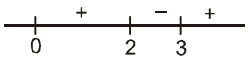
$\Rightarrow x^2=n\pi$

$\Rightarrow x=\pm\sqrt{n\pi}$; $n\in W$

2. (*b*) $F:[0,3]\to[1,29]$

$f(x)=2x^3-15x^2+36x+1$

$f'(x)=6x^2-30x+36=6(x^2-5x+6)=6(x-2)(x-3)$



in given domain function has local maxima, it is many-one

Now at $x=0$ $f(0)=1$

$x=2f(2)=16-60+72+1=29$

$x=3f(3)=54-135+108+1=163-135=28$

Has range = [1, 29]

Hence given funχτιon is onto

3. (*ab*)**NOTE :** Since a functional mapping can't have two images for pre-image 1/3, so this is ambiguity in this question perhaps the answer can be A or B or AB or marks to all.

$$\cos4\theta=\frac{1}{3}\Rightarrow 2\cos^2 2\theta-1=\frac{1}{3}\Rightarrow\cos^2 2\theta=\frac{2}{3}$$

$$\Rightarrow\cos2\theta=\pm\sqrt{\frac{2}{3}}$$

$$\text{Now f}(\cos4\theta)=\frac{2}{2-\sec^2\theta}=\frac{1+\cos2\theta}{\cos2\theta}=1+\frac{1}{\cos2\theta}$$

$$\Rightarrow f\left(\frac{1}{3}\right)=1\pm\sqrt{\frac{3}{2}}$$

4. (*a,b,c*) (i) $f(-x) = -f(x)$ so it is odd function

(ii) $f'(x) = 3(\log(\sec x + \tan x))^2 \dfrac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x) > 0$

(iii) Range of f(x) is R as $f\left(-\dfrac{\pi}{2}\right) \Rightarrow -\infty$

5. (*b,c,d*) $\alpha = 3\sin^{-1}\dfrac{6}{11} > 3\sin^{-1}\dfrac{6}{12}$ and

$\beta = 3\cos^{-1}\dfrac{4}{9} > 3\cos^{-1}\dfrac{4}{8}$

$\Rightarrow \alpha > \dfrac{\pi}{2}$ & $\beta > \pi$

$\Rightarrow \alpha + \beta > \dfrac{3\pi}{2}$

6. (19.00) $f(x) + f(1-x) = \dfrac{4^x}{4^x+2} + \dfrac{4^{1-x}}{4^{1-x}+2}$

$= \dfrac{4^x}{4^x+2} + \dfrac{4/4^x}{\frac{4}{4^x}+2}$

$= \dfrac{4^x}{4^x+2} + \dfrac{4}{4+2.4^x}$

$= \dfrac{4^x}{4^x+2} + \dfrac{2}{2+4^x}$

$= 1$

so, $f\left(\dfrac{1}{40}\right) + f\left(\dfrac{2}{40}\right) + ... + f\left(\dfrac{39}{40}\right) - f\left(\dfrac{1}{2}\right)$

$= 19 + f\left(\dfrac{1}{2}\right) - f\left(\dfrac{1}{2}\right) = 19$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

2 Inverse Trigonometric Functions

Topicwise Questions

1. (*b*) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

2. (*b*) $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$

$(4\pi - 12) - (14 - 4\pi)$ $8\pi - 26$.

3. (*c*) Given that $\tan^{-1} x - \tan^{-1} y = \tan^{-1} A$

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} A$$

Hence $A = \frac{x-y}{1+xy}$.

4. (*b*) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(\pi + \frac{\pi}{6}\right)\right\}$

$$= \cos^{-1}\left(-\cos\frac{\pi}{6}\right) = \pi - \cos^{-1}\cos\frac{\pi}{6} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

5. (*b*) Obviously $x = \sin(\theta + \beta)$ and $y = \sin(\theta - \beta)$

$$\therefore 1 + xy = 1 + \sin(\theta+\beta)\sin(\theta-\beta)$$

$$= 1 + \sin^2\theta - \sin^2\beta = \sin^2\theta + \cos^2\beta$$

6. (*a*) $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right] = \tan^{-1}\left[\frac{\sin(\pi/2 - x)}{1+\cos(\pi/2 - x)}\right]$

$$= \tan^{-1}\left[\frac{2\sin(\pi/4 - x/2)\cos(\pi/4 - x/2)}{2\cos^2(\pi/4 - x/2)}\right]$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{\pi}{4} - \frac{x}{2}$$

7. (*c*) Let $\tan^{-1} 2 = \alpha \Rightarrow \tan\alpha = 2$

and $\cot^{-1} 3 = \beta \Rightarrow \cot\beta = 3$

$\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3)$

$= \sec^2\alpha + \text{cosec}^2\alpha = 1 + \tan^2\alpha + 1 + \cot^2\alpha$

$= 2 + (2)^2 + (3)^2 = 15$

8. (*a*) Given, $\tan^{-1} x = \sin^{-1}\left[\frac{3}{\sqrt{10}}\right]$

$\Rightarrow x = \tan\left\{\sin^{-1}\left[\frac{3}{\sqrt{10}}\right]\right\} = \tan\{\tan^{-1} 3\}$

$\Rightarrow x = 3.$

9. (*c*) $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \cos\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$$= \cos\cos^{-1}\sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

10. (*b*) $\cot^{-1} 3 + \text{cosec}^{-1}\sqrt{5} = \cot^{-1} 3 + \cot^{-1} 2$

$$= \cot^{-1}\left(\frac{3\times 2 - 1}{3+2}\right) = \cot^{-1}(1) = \frac{\pi}{4}.$$

11. (*a*) $\cos^{-1}\left[\cos\frac{5\pi}{3}\right] + \sin^{-1}\left[\frac{\cos 5\pi}{3}\right] = \frac{\pi}{2}$

$(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2})$.

12. (*d*) $x \in [-1, 1]$.

13. (*d*) $\tan\left[\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}\right]=\tan\left[\tan^{-1}\frac{\frac{1}{2}-\frac{1}{3}}{1+\frac{1}{6}}\right]$

$=\tan\tan^{-1}\left(\frac{1}{6}\times\frac{6}{7}\right)=\frac{1}{7}$.

14. (*b*) $\cos\left(\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}\right)=\cos\left(\frac{\pi}{2}+\cos^{-1}\frac{1}{5}\right)$

$=-\sin\left(\cos^{-1}\frac{1}{5}\right)=-\sin\left(\sin^{-1}\sqrt{\frac{24}{25}}\right)=-\frac{2\sqrt{6}}{5}$

15. (*a*) $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}$

$=\{\sin^{-1}(x)+\cos^{-1}(x)\}+\left\{\sin^{-1}\left(\frac{1}{x}\right)+\cos^{-1}\left(\frac{1}{x}\right)\right\}$

$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$

16. (*a*) $\sin^{-1}\frac{4}{5}=\tan^{-1}\frac{4}{3}, 2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}=\cot^{-1}\frac{4}{3}$ and

$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

17. (*a*) Let $\cos^{-1}x=\theta$. Then $x=\cos\theta$

$\tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{x^2}-1}=\sqrt{\frac{1-x^2}{x}}$

$\therefore\tan(\cos^{-1}x)=\tan\theta=\frac{\sqrt{1-x^2}}{x}$.

18. (*d*) $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]$

$=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]=\tan\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)=-\frac{7}{17}$.

19. (*b*) Let $x=\tan^2\theta\Rightarrow\theta=\tan^{-1}\sqrt{x}$

Now, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

$=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\frac{1}{2}\cos^{-1}\cos 2\theta=\frac{2\theta}{2}=\theta=\tan^{-1}\sqrt{x}$.

20. (*b*) $\sin\left(4\tan^{-1}\frac{1}{3}\right)=\sin\left[2\tan^{-1}\left(\frac{2/3}{1-(1/9)}\right)\right]$

$=\sin\left[2\tan^{-1}\frac{3}{4}\right]=\sin\sin^{-1}\left(\frac{2\times(3/4)}{1+(9/16)}\right)$

$=\frac{3}{2}\times\frac{16}{25}=\frac{24}{25}$

$\left(\because 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\right)$

21. (*b*) $\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]+\cos[\tan^{-1}(2\sqrt{2})]$

$=\sin\left[\tan^{-1}\frac{2/3}{1-1/9}\right]+\cos[\tan^{-1}(2\sqrt{2})]$

$=\sin[\tan^{-1}3/4]+\cos[\tan^{-1}2\sqrt{2}]$

$=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}$.

22. (*d*) $\sin^{-1}2x=\sin^{-1}x-\sin^{-1}\frac{\sqrt{3}}{2}$

$\sin^{-1}2x=\sin^{-1}\left(x\sqrt{\left(1-\frac{3}{4}\right)}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)$

$2x=\left(\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)$

$\frac{\sqrt{3}}{2}\sqrt{1-x^2}=\frac{x}{2}-2x=\frac{-3x}{2}$

$\frac{3(1-x^2)}{4}=\frac{9x^2}{4}$

$\Rightarrow\ 3-3x^2=9x^2\Rightarrow x^2=\frac{1}{4}\Rightarrow x=\pm\frac{1}{2}$.

23. (*a*) $\sin\left[3\sin^{-1}\frac{1}{5}\right]=\sin\left[\sin^{-1}\left\{3\left(\frac{1}{5}\right)-4\left(\frac{1}{5}\right)^3\right\}\right]$

$=\sin\left[\sin^{-1}\left\{\frac{3}{5}-\frac{4}{125}\right\}\right]=\sin\left[\sin^{-1}\left(\frac{75-4}{125}\right)\right]$

$=\sin\left[\sin^{-1}\frac{71}{125}\right]=\frac{71}{125}$

24. (*c*) $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}(1+x)=\frac{\pi}{2}-\tan^{-1}(1-x)$

$\Rightarrow \tan^{-1}(1+x)=\cot^{-1}(1-x)$

$\Rightarrow \tan^{-1}(1+x)=\tan^{-1}\left(\frac{1}{1-x}\right)$

$\Rightarrow 1+x=\frac{1}{1-x}\Rightarrow 1-x^2=1\Rightarrow x=0$

25. (*c*) $x+y=\tan^{-1}33$

$\Rightarrow y=\tan^{-1}33-\tan^{-1}3$

$=\tan^{-1}\frac{33-3}{1+99}=\tan^{-1}\frac{30}{100}\Rightarrow y=\tan^{-1}(0.3)$

26. (*c*) The given equation may be written as

$\tan^{-1}x+\cot^{-1}x+\cot^{-1}x=\frac{2\pi}{3}\Rightarrow$

$\cot^{-1}x=\frac{2\pi}{3}-\frac{\pi}{2}=\frac{\pi}{6}\Rightarrow x=\sqrt{3}.$

27. (*b*) $\because \cot^{-1}\frac{1}{2}=\cos^{-1}\frac{1}{\sqrt{5}}$

Hence given equation can be written as

$\sin^{-1}x+\cos^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{2}\Rightarrow x=\frac{1}{\sqrt{5}}$

28. (*b*) $\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)=\pi$

$\Rightarrow \tan^{-1}x+\tan^{-1}y=\pi-\tan^{-1}z$

$\Rightarrow \frac{x+y}{1-xy}=-z\Rightarrow x+y=-z+xyz$

$\Rightarrow x+y+z=xyz$

Dividing by *xyz*, we get

$\frac{1}{yz}+\frac{1}{xz}+\frac{1}{xy}=1.$

Note: Students should remember this question as a formula.

29. (*c*) $\cos^{-1}x+\cos^{-1}(2x)=-\pi$

$\Rightarrow \cos^{-1}2x=-\pi-\cos^{-1}x$

$\Rightarrow 2x=\cos(\pi+\cos^{-1}x)$

$\Rightarrow 2x=\cos\pi(\cos\cos^{-1}x)-\sin\pi\sin(\cos^{-1}x)$

$2x=-x\Rightarrow x=0$

But x = 0 does not satisfy the given equation.

No solution will exist.

30. (*a*) Given equation is $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x+(\cos^{-1}x+\sin^{-1}x)=\frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x+\frac{\pi}{2}=\frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x=4\pi/3$

which is not possible as $\cos^{-1}x\in[0,\pi]$.

Learning Plus

1. (*b*) Given $=\frac{\pi}{3}+\frac{\pi}{4}=\frac{7\pi}{12}$

2. (*d*) $\therefore -1\le x\le 1$ .....(1)

$x\in R$ .....(2)

$x\le -1$ or $x\ge 1$ .....(3)

By $(1)\cap(2)\cap(3$

$\Rightarrow x\in\{-1,1\}$

3. (*c*) Domain of f(x) is $x\in\{-1,\ 1\}$

$f(-1)=-\frac{\pi}{2}-\frac{\pi}{4}+\pi=\frac{\pi}{4}$

$f(1)=\frac{\pi}{2}+\frac{\pi}{4}+0=\frac{3\pi}{4}$

4. (*b*) $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}]=y$

$=\cos[\tan^{-1}(\sin\frac{\pi}{6})]=\cos\left[\tan^{-1}\frac{1}{2}\right]$

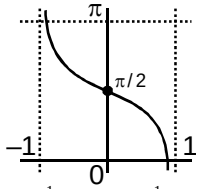
$=\cos\left[\cos^{-1}\frac{2}{\sqrt{5}}\right]=\frac{2}{\sqrt{5}}$

5. (*a*) $\sum_{i=1}^{n}\cos^{-1}\alpha_i=0$

$\cos^{-1}\alpha_1+\cos^{-1}\alpha_2+.....+\cos^{-1}\alpha_n$

so,$\cos^{-1}\alpha_1$ is always positive

So, in order to have their sum 0 all should be equal to 0



$\cos^{-1}\alpha_1=\cos^{-1}\alpha_2=.....=\cos^{-1}\alpha_n=0$

$\Rightarrow \cos^{-1}\alpha_1=0\Rightarrow \alpha_1=1$

$\cos^{-1}\alpha_2=0\Rightarrow \alpha_2=1$

$\vdots$

$\cos^{-1}\alpha_n=0\Rightarrow \alpha_n=1$

$\therefore \sum_{i=1}^{m}\alpha_i=n$

6. (*b*) $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6},\ n \in N$

$\frac{n}{\pi} < \cot\frac{\pi}{6} \Rightarrow n < \sqrt{3}\,-\pi$

$\Rightarrow n < \sqrt{3} \times 3.14 \quad \Rightarrow n = 5$

7. (*d*) $\text{cosec}^{-1}(\cos x)$ is define if

$\cos x \geq 1$ or $\cos x \leq -1$

$\Rightarrow \cos x = \pm 1 \Rightarrow \quad x = n\pi$

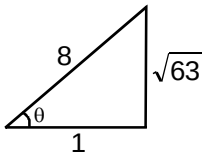
8. (*d*) $\pi \leq x \leq 2\pi\ \cos^{-1}\cos x = 2\pi - x$

9. (*a*) $y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$

Let, $\cos^{-1}\frac{1}{8} = \theta \Rightarrow \cos\theta = \frac{1}{8}$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{1}{8}}{2}}$



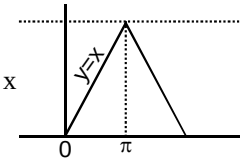
$\Rightarrow \sqrt{\frac{9}{16}} = \frac{3}{4}$

&$\cos\frac{\theta}{2} = \cos\left(\frac{\cos^{-1}\left(\frac{1}{8}\right)}{2}\right) = \frac{3}{4}$

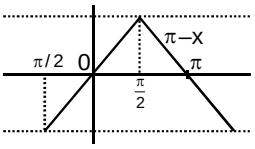
10. (*d*) $y = \sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$

given $x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{\pi}{2} < x < \pi$

Now $\cos^{-1}(\cos x) = x$



$\sin^{-1}(\sin x) = \pi - x$



so, $y = \sin^{-1}[\cos\{x + \pi - x\}]$

$y = \sin^{-1}(\cos\pi) \Rightarrow \sin^{-1}(-1) \Rightarrow -\frac{\pi}{2}$

11. (*d*) $x \geq 0,\ \theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$

$\downarrow \qquad\qquad \downarrow$

$[-1,1] \quad [-1,1] \quad x \in R$

but $x \geq 0$ so, $x \in [0, 1]$

$\theta = \frac{\pi}{2} - \tan^{-1}x$

$R_\theta : \left[\theta\big|_{x=1}, \theta\big|_{x=0}\right] = \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

12. (*b*) $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2} - \cos^{-1}x - \cos^{-1}y + \frac{\pi}{2}$

$= \pi - (\cos^{-1}x + \cos^{-1}y) = \frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$

13. (*b*) $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2}$

$\frac{\pi}{2} - 2\cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\Rightarrow \cos^{-1}x = \frac{\pi}{6}$

$x = \frac{\sqrt{3}}{2}$

14. (*c*) $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) - \frac{\pi}{6} = 0$

$\sin^{-1}(1) - \sin^{-1}\sqrt{\frac{3}{x}} = \frac{\pi}{6} \Rightarrow \sin^{-1}\sqrt{\frac{3}{x}} = \frac{\pi}{2} - \frac{\pi}{6} =$

$\frac{\pi}{3}$

$\Rightarrow \sqrt{\frac{3}{x}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\Rightarrow x = 4$

15. (*b*) By property if $x < 0,\ \tan^{-1}\frac{1}{x} = \cot^{-1}x - \pi$

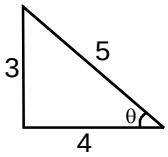
$\therefore \quad \tan^{-1}x + \tan^{-1}\frac{1}{x} = \tan^{-1}x + \cot^{-1}x - \pi = \frac{\pi}{2} - \pi$

$\Rightarrow \tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$

16. (*d*) $y = \tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

Let $\theta = \sin^{-1}\frac{3}{5} \Rightarrow \sin\theta = \frac{3}{5}$

$\tan\theta = \frac{3}{4}$



$so\ \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$so\ y = \tan\left[\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$$y = \frac{\tan\left\{\tan^{-1}\left(\frac{3}{4}\right)\right\}+\tan\left\{\tan^{-1}\left(\frac{2}{3}\right)\right\}}{1-\tan\left(\tan^{-1}\left(\frac{3}{4}\right)\right)\ .\ \tan\left(\tan^{-1}\left(\frac{2}{3}\right)\right)}$$

$$y = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}} = \frac{9+8}{6} = \frac{17}{6}$$

17. (*c*) $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{\cos^{-1}x}{2}\right)$

Let $\cos^{-1}x = \theta \Rightarrow x = \cos\theta$

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{\left(1+\tan\frac{\theta}{2}\right)^2+\left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \frac{2\left(\sec^2\frac{\theta}{2}\right)}{\left(1-\tan^2\frac{\theta}{2}\right)} = \frac{2}{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{2}{\cos\theta} = \frac{2}{x}$$

18. (*c*) $\tan^{-1}a+\tan^{-1}b = \pi+\tan^{-1}\left(\frac{a+b}{1-ab}\right)$

if $ab > 1, a > 0, b > 0$

19. (*a*) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3} = \tan^{-1}1 = \frac{\pi}{4}$

20. (*b*) $y = \cot^{-1}\left(\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right)$

given $\frac{\pi}{2} < x < \pi$

$$y = \cot^{-1}\left(\frac{\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|}\right)$$

$\Rightarrow$ Now if $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

Now $\sin\frac{x}{2} > \cos\frac{x}{2}$ so, modulus will open directly

$$y = \cot^{-1}\left(-\frac{2\sin\frac{x}{2}}{2\cos\frac{x}{2}}\right) = \cot^{-1}\left(-\tan\frac{x}{2}\right)$$

$$y = \pi-\cot^{-1}(\tan\frac{x}{2}) \Rightarrow \pi-\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

$$y = \pi-\frac{\pi}{2}+\frac{x}{2} = \frac{\pi}{2}+\frac{x}{2}$$

21. (*b*) $y = \tan^{-1}\left(\frac{1-x}{1+x}\right), 0 \le x \le 1$

put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x\ \theta \in [0, \pi]$

$$y = \tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

here given, $0 \le x \le 1$

$0 \le \cos\theta \le 1$

$$0 < \theta \le \frac{\pi}{2}$$

comes in PVR of $\theta = \cos^{-1}x$

$$0 < \frac{\theta}{2} \le \frac{\pi}{4}$$

$$so,\ y = \frac{\theta}{2} = \frac{\cos^{-1}x}{2}$$

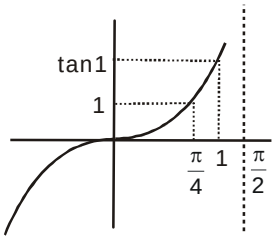
$$Now\ y_{min} = \left.\frac{\theta}{2}\right|_{\theta=0} = 0$$

$$y_{max} = \left.\frac{\theta}{2}\right|_{\theta=\frac{\pi}{2}} = \frac{\pi}{4}$$

$$so,\ \left(0, \frac{\pi}{4}\right)$$

22. (*a*) $\tan 1 > 1$

$\tan 1 > \frac{\pi}{4}$



$\tan 1 > \tan^{-1} 1$

23. (*c*) $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$

$$\tan^{-1}\left\{\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{(2x+1)(4x+1)}}\right\} = \tan^{-1}\frac{2}{x^2}$$

$$\frac{4x+1+2x+1}{(2x+1)(4x+1)-1} = \frac{2}{x^2}$$

$$\frac{6x+2}{(8x^2+6x)} = \frac{2}{x^2} \Rightarrow 6x^3+2x^2 = 16x^2+12x$$

$\Rightarrow 6x^3 - 14x^2 - 12x = 0$

$\Rightarrow x(6x^2 - 14x - 12) = 0$

$x = 0, 6x^2 - 14x - 12 = 0$

$x = 0, x = 3, \frac{-3}{2}$

check : $x = 0$

$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$ (Accepted)

$x = 3$

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

$$\tan^{-1}\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}$$

$\Rightarrow \tan^{-1}\left(\frac{13+7}{90}\right)$ (Accepted)

$x = -3/2$

$$\tan^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\left(\frac{-1}{5}\right)$$

$\Rightarrow \tan^{-1}\left(\frac{-6}{9}\right)$ (Rejected)

So, 2 solutions

24. (*b*) $\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{5}{13}$

$$= \cos^{-1}\left(\frac{3}{5}\times\frac{5}{13} - \frac{4}{5}\times\frac{12}{13}\right)$$

$$= \cos^{-1}\left(-\frac{33}{65}\right)$$

25. (*b*) $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{4}$

$$\Rightarrow \frac{3\sin 2\theta}{5+4\cos 2\theta} = 1$$

$$\Rightarrow \frac{6\tan\theta}{\tan^2\theta+9} = 1$$

$\Rightarrow \tan^2\theta - 6\tan\theta + 9 = 0$

$\Rightarrow (\tan\theta - 3)^2 = 0$

$\therefore \tan\theta = 3$

26. (*d*) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$= \begin{cases} 2\sin^{-1}x & \text{if} \quad |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if} \quad \frac{1}{\sqrt{2}} < x \le 1 \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if} \quad -1 \le x < -\frac{1}{\sqrt{2}} \end{cases}$$

$\therefore 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ is true for $|x| \le \frac{1}{\sqrt{2}}$

27. (*c*) $\cos^{-1}\left\{\frac{x^2}{2} + \sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right\} = \cos^{-1}\frac{x}{2} - \cos^{-1}x$

The above holds iff

$1 \ge x \ge 0 \;\&\; 1 \ge \frac{x}{2} \ge 0$

$0 \le x \le 1 \;\&\; 0 \le x \le 2$

$\downarrow \cap$

$0 \le x \le 1$

28. (*c*) $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$

$$\cos^{-1}(\sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p}) + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$$

$$\cos^{-1}0 + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$$

$$\cos^{-1}\sqrt{1-q} = \frac{\pi}{4} \Rightarrow 1-q = \frac{1}{2}$$

$\Rightarrow q = \frac{1}{2}$ so, $0 \le \sqrt{p} \le 1 \;\&\; 0 \le \sqrt{1-p} \le 1$

$0 \le p \le 1 \;\&\; 0 \le 1-p \le 1$

$-1 \le -p \le 0 \Rightarrow 0 \le p \le 1$

29. (*b*) $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\sin^{-1}\left\{\frac{x}{5}\times\frac{3}{5}+\frac{4}{5}\times\frac{\sqrt{25-x^2}}{5}\right\} = \frac{\pi}{2}$

$\frac{3x}{25} + \frac{4\sqrt{25-x^2}}{25} = 1$

$x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0$

$\Rightarrow\ x = 3$

30. (*a*) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

$(\sin^{-1}x + \sin^{-1}(1-x)) = \cos^{-1}x$

$\sin^{-1}(x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2}) = \sin^{-1}\sqrt{1-x^2}$

$x\sqrt{2x-x^2} = \sqrt{1-x^2}\ (1-1+x)$

$x^2(2x-x^2) - (1-x^2)x^2 = 0$

$2x^3 - x^2 = 0$

$x^2(2x-1) = 0$

$\Rightarrow\ x = 0, \frac{1}{2}$

Both accepted.

31. (*b*) $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$-2\sin^{-1}x = \cos^{-1}(1-x)$

$1 - 2x^2 = 1 - x$

$2x^2 - x = 0 \Rightarrow x = 0, \frac{1}{2}$

check : $x = 0$

L.H.S. $\Rightarrow \frac{\pi}{2} - 0$

$= \frac{\pi}{2} = \text{RHS}$

$x = \frac{1}{2}$

$\frac{\pi}{6} - \frac{\pi}{3} \Rightarrow -\frac{\pi}{2} \Rightarrow$ so rejected

Advanced Level Multiconcept Questions

1. (*cd*) $x^2 - x - 2 > 0$

α satisfies it

$\Rightarrow\ \alpha^2 - \alpha - 2 > 0$

$(\alpha - 2)(\alpha + 1) > 0$

$\alpha < -1 \text{ or } \alpha > 2$

2. (*b,c*) Given : $\alpha = 2\tan^{-1}(\sqrt{2} - 1)$

$\beta = 3\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$\gamma = \cos^{-1}\left(\frac{1}{3}\right)$

$\alpha = 2 \times 22.5^\circ = 45^\circ$

$\beta = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$

$\gamma = \cos^{-1}\left(\frac{1}{3}\right) = \cos^{-1}(0.33)$

$\alpha < \gamma < \beta$

so, $\gamma > \alpha$ $\qquad$ $\beta > \gamma$

3. (*ab*) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

$\Rightarrow\ x = y = z = 1$

$\Rightarrow\ x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$

4. (*bcd*) $\cos\left[\frac{1}{2}\cos^{-1}\cos\left(-\frac{14\pi}{5}\right)\right]$

$= \cos\left[\frac{1}{2}\cos^{-1}\cos\left(\frac{14\pi}{5}\right)\right] \because \cos\theta = \cos(-\theta)$

$= \cos\left[\frac{1}{2}\left(\frac{14\pi}{5} - 2\pi\right)\right]$ since $\frac{14\pi}{5} \in (2\pi, 3\pi)$

$= \cos\left[\frac{2\pi}{5}\right]$

$= -\cos\left(\pi - \frac{2\pi}{5}\right) = -\cos\left(\frac{3\pi}{5}\right)$

$= \sin\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = \sin\left(\frac{\pi}{10}\right)$

5. (*bd*) $6\sin^{-1}\left(x^2-6x+\frac{17}{2}\right)=\pi$

$\Rightarrow \sin^{-1}\left(x^2-6x+\frac{17}{2}\right)=\frac{\pi}{6}$

$\Rightarrow x^2-6x+\frac{17}{2}=\frac{1}{2}$

$\Rightarrow x^2-6x+8=0$

$\Rightarrow x=2,4$

6. (*cd*) $\sin^{-1}x>\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$

$\Rightarrow \sin^{-1}>\frac{\pi}{4}\Rightarrow x>\frac{1}{\sqrt{2}}$ also $|x|\le 1$

$\Rightarrow x\in\left(\frac{1}{\sqrt{2}},1\right]$

7. (*ac*) $f(x)=\sin^{-1}x+\cos^{-1}x$

Now f(x) will be equal to $\frac{\pi}{2}$ iff arg lie b/w –1 to 1

8. (*acd*) $\text{cosec}^{-1}x=\sin^{-1}\frac{1}{x}\ |x|\ge 1$

9. (*abc*) $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ since $0<x<1$

$=\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$ (let$\cos^{-1}x=\theta\ 0<\theta<\frac{\pi}{2}$)

$=\tan^{-1}\tan\frac{\theta}{2}\because\frac{\theta}{2}\in\left(0,\frac{\pi}{4}\right)$

$=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}x$ ....(1)

also $\cos\theta=2\cos^2\frac{\theta}{2}-1$

$\cos\frac{\theta}{2}=\sqrt{\frac{1+x}{2}}$ (taking $\cos^{-1}$ on both side)

$\cos^{-1}\cos=\frac{\theta}{2}=\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ since $\frac{\theta}{2}\in\left(0,\frac{\pi}{4}\right)$

$\Rightarrow \frac{\theta}{2}=\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ ...(2)

similarly $\sin\frac{\theta}{2}=\sqrt{\frac{1-x}{2}}$

$\sin^{-1}\sin\frac{\theta}{2}=\frac{\theta}{2}=\sin^{-1}\sqrt{\frac{1-x}{2}}$ ...(3)

also $\frac{\theta}{2}=\tan^{-1}\sqrt{\frac{1+x}{1-x}}$ ...(4)

10. (*abc*) $\tan\left(\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)=\frac{a}{b}$

$\tan\left(\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)=\frac{a}{b}$

$\tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)=\frac{a}{b}$

$\frac{a}{b}=\frac{17}{6}\Rightarrow a-b=11$

$\Rightarrow a+b=23$

$\Rightarrow 3b=3.6=18=a+11$

11. (*ad*) $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4-2n^2+2}$

$=\lim_{k-\infty}\sum_{n=1}^{k}\left\{\tan^{-1}(n+1)^2-\tan^{-1}(n-1)^2\right\}$

$=\lim_{k-\infty}\left\{\tan^{-1}(k+1)^2+\tan^{-1}k^2-\tan^{-1}1-\tan^{-1}0\right\}$

$=\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}-0=\frac{3\pi}{4}$

Also $\tan^{-1}2+\tan^{-1}3=\pi+\tan^{-1}\left(\frac{3+2}{1-3.2}\right)$

Since $xy=6>1$

$=\frac{3\pi}{4}$

and $\sec^{-1}\left(-\sqrt{2}\right)=\frac{3\pi}{4}$

12. (*bc*) $2x=\tan(2\tan^{-1}a)+2\tan(\tan^{-1}a+\tan^{-1}a^3)$

$2x=\tan(\tan^{-1}a+\tan^{-1}a)+2\tan(\tan^{-1}a+\tan^{-1}a^3)$

$2x=\frac{2a}{1-a^2}+2\frac{a+a^3}{(1-a^2)(1+a^2)}$

$x=\frac{a}{1-a^2}+\frac{a}{(1-a^2)}$

$a^2x-x+2a=0$ (A) is valid

& $a\ne -1$ & 1 (D) is valid

13. (*ac*) Given $a=\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)+\cos^{-1}\left(\frac{-1}{2}\right)$

$b=\tan^{-1}\left(-\sqrt{3}\right)-\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$a=-\frac{\pi}{4}+\pi-\frac{\pi}{3}=\frac{5\pi}{12}$

$b=-\frac{\pi}{3}-\pi+\frac{\pi}{3}=-\pi$

$a-b=\frac{17\pi}{12}$ & $a+b=-\frac{7\pi}{12}$

14. (*ac*) $\cos^{-1} x = \tan^{-1} x \Rightarrow x \in (0, 1]$

$\tan^{-1} \frac{\sqrt{1-x^2}}{x} = \tan^{-1} x$

taking tan on both side $\sqrt{1-x^2} = x^2$

$\Rightarrow 1 - x^2 = x^4 \Rightarrow x^4 + x^2 - 1 = 0$

$x^2 = \frac{-1+\sqrt{5}}{2}$ since x^2 is +ve avoid negative result

$\sin(\cos^{-1} x) = \sin(\sin^{-1} \sqrt{1-x^2}) = \sqrt{1-x^2} = x^2 = \frac{\sqrt{5}-1}{2}$

$\tan(\cos^{-1} x) = \tan(\tan^{-1} x) = x \neq \frac{\sqrt{5}-1}{2}$

15. (*d*) Let $\cos^{-1} x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$\therefore \sin^{-1} \sqrt{1-x^2} = \sin^{-1}(\sin\theta)$

$$= \begin{cases} \theta & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ \pi - \theta & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

$$= \begin{cases} \cos^{-1} x & \text{if } 0 \le x \le 1 \\ \pi - \cos^{-1} x & \text{if } -1 \le x < 0 \end{cases}$$

$\therefore \cos^{-1} x = \sin^{-1} \sqrt{1-x^2}$ if $0 < x < 1$ is true.

16. (*c*) Let $\sin^{-1} x = \theta$, then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\therefore \cos^{-1} \sqrt{1-x^2} = \cos^{-1}(\cos\theta)$

$$= \begin{cases} -\theta & , -\frac{\pi}{2} \le \theta \le 0 \\ \theta & , 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} -\sin^{-1} x & , -1 \le x \le 0 \\ \sin^{-1} x & , 0 \le x \le 1 \end{cases}$$

$\therefore \sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$ if $0 < x < 1$ is true

17. (*d*) Let $\cos^{-1} x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$\therefore \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \tan^{-1}(\tan\theta)$

$$= \begin{cases} \theta & , 0 \le \theta < \frac{\pi}{2} \\ \theta - \pi & , \frac{\pi}{2} < \theta \le \pi \end{cases}$$

$$= \begin{cases} \cos^{-1} x & , 0 < x \le 1 \\ -\pi + \cos^{-1} x & , -1 \le x < 0 \end{cases}$$

i.e $\cos^{-1} x = \pi + \tan^{-1} \frac{\sqrt{1-x^2}}{x}$, $-1 < x < 0$ is correct.

18. (*d*) $\sin^{-1} x < \frac{3\pi}{4}$

$x \in [-1, 1]$

19. (*a*) $\sin^{-1}(-1) + \text{cosec}^{-1}(-1) = \frac{-\pi/2}{2} + \frac{3\pi}{2} = \pi$

20. (b) If $-1 \le x \le 1$

then $-1 \le -x \le 1$

So $\frac{3\pi}{4} \le \tan^{-1}(-x) \le \frac{5\pi}{4}$

21. (*a*) $\to$ (t), (*b*) $\to$ (s), (*c*) $\to$ (t), (*d*) $\to$ (p)

(*a*) Given expression = $2\sin^{-1} x - \frac{\pi}{2}$ maximum at

(*b*) Given expression = $t^2 + \left(\frac{\pi}{2} - t\right)^2$, $t = \sin^{-1} x$ minimum

at $t = \frac{\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}}$

(*c*) Similar to (*b*)

(*d*) Given expression = $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \left(\frac{\pi}{2}\right)^3$

$- 3\sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right).\frac{\pi}{2}$

$= \frac{3\pi}{2}(\sin^{-1} x)^2 - \frac{3\pi^2}{4}\sin^{-1} x + \frac{\pi^3}{8}$

This is quadratic in $\sin^{-1} x$. Therefore it will give maximum value when $\sin^{-1} x = -\frac{\pi}{2} \Rightarrow x = -1$

Let the domain and range of inverse circular functions are defined as follows Domain Range

$\sin^{-1} x$ $[-1, 1]$ $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ $\cos^{-1} x$ $[-1, 1]$ $[0, \pi]$ $\tan^{-1} x$ R

$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\cot^{-1} x$ R $(0, \pi)$ $\text{cosec}^{-1} x$ $(-\infty, -1] \cup [1, \infty)$ $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$

$\sec^{-1} x$ $(-\infty, -1] \cup [1, \infty)$ $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

22. (a) $\to (p)$, $(b) \to$ (q), $(c) \to$ (r) $(d) \to$ (s)

$$(a)\ \text{G.E.} = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\frac{2}{3}} = \frac{17}{6}$$

$$(b)\ \text{G.E.} \ \frac{\frac{2.\frac{1}{5}}{1-\frac{1}{25}}-1}{1+\frac{2.\frac{1}{5}}{1-\frac{1}{25}}.1} = \frac{-7}{17}$$

$$(c)\ \text{G.E.} = \sqrt{\frac{1+\frac{1}{8}}{2}} =$$

$$(d)\ \text{G.E.} = \cos\tan^{-1}\ \frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{3}$$

23. (A) $\to$ (p, q, r, s); (B) $\to$ (p); (C) $\to$ (P,R,S); (D) $\to$ (p)

$$\text{(A) } f(x) = \sin^{-1}\left(\frac{2}{|\sin x - 1| + |\sin x + 1|}\right)$$

For all values of x
$|\sin x - 1|$ will open negative
$|\sin x + 1|$ is positive

$f(x) = \sin^{-1}(1)$ $\qquad \frac{\pi}{2}$

so P, Q, R. S

(B) $f(x) = \cos^{-1}(|x-1| - |x-2|)$
In $(-\infty, 1]$ $\ -x + 1 + x - 2 = -1$
$\cos^{-1}(-1) = \pi$
(1, 2) not in domain.
$[2, \infty)$ $\cos^{-1}[1] = 0$So, Ans. P

$$\text{(C) } f(x) = \sin^{-1}\left[\frac{\pi}{\left|\sin^{-1}x - \frac{\pi}{2}\right| + \left|\sin^{-1}x + \frac{\pi}{2}\right|}\right]$$

Domain $|x| \le 1$
Ans. P,R,S

(D) $f(x) = \cos(\cos^{-1}|x|)$
$+ \sin^{-1}(\sin x) - \text{cosec}^{-1}(\text{cosec } x) + \text{cosec}^{-1}|x|$
domain of f(x)
$|x| = 1$
$x = \pm 1$

24. (0013) Put $\sin^{-1}\frac{5}{x} = A$ or $\frac{5}{x} = \sin A$

$$\sin^{-1}\frac{12}{x} = B \quad \text{or} \quad \frac{12}{x} = \sin B$$

$$\Rightarrow A + B = \frac{\pi}{2}$$

$$\Rightarrow \sin A = \sin\left(\frac{\pi}{2} - B\right) = \cos B = \sqrt{1-\sin^2 B}$$

$$\Rightarrow \frac{5}{x} = \sqrt{1-\frac{144}{x^2}} \text{ or } \frac{169}{x^2} = 1$$

or $x^2 = 169$ or $x = 13$
$[\because x = -13$ does not satisfy the given equation]

25. (0003)

$$\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x} = \tan^{-1}\frac{2}{x^2}$$

$$\text{or } \tan^{-1}\left[\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\frac{1}{1+4x}}\right] = \tan^{-1}\frac{2}{x^2}$$

$$\text{or } \frac{2+6x}{6x+8x^2} = \frac{2}{x^2}$$

or $6x^3 - 14x^2 - 12x = 0$
or $x(x-3)(3x+2) = 0$
or $x = 3$ or $x = -2/3$
But for $x = -2/3$, L.H.S < 0 and R.H.S. > 0
Hence, the only solution is $x = 3$.

26. (0003) We have $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1/\sqrt{5}}{\sqrt{1-\frac{1}{5}}} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2} = \tan^{-1}1$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}1 - \tan^{-1}\frac{1}{2}$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1-\frac{1}{2}}{1+1.\frac{1}{2}}\right)$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow x = 3$$

27. (0001) Given $\sin^{-1}A+\cos^{-1}B=\frac{\pi}{2}$ ..... (1)

But $\sin^{-1}A+\cos^{-1}A=\frac{\pi}{2}$ ..... (2)

$\therefore$ from (1) & (2). $A=B$

$$\Rightarrow x-\frac{x^2}{2}+\frac{x^3}{4}-....\text{to}\ \infty=x^2-\frac{x^4}{2}+\frac{x^6}{4}-....\text{to}\ \infty$$

$$\Rightarrow \frac{x}{1-\left(-\frac{x}{2}\right)}=\frac{x^2}{1-\left(-\frac{x^2}{2}\right)}\Rightarrow x(2+x^2)=x^2(2+x)$$

$$\Rightarrow 2x(x-1)=0$$

$\therefore$ $x=0$ or 1

but $x\neq 0$

$\therefore$ $x=1$

28. (0010) $\tan\left\{\pi-\cos^{-1}\left(\frac{2}{7}\right)-\frac{\pi}{2}\right\}=\tan\left\{\frac{\pi}{2}-\cos^{-1}\left(\frac{2}{7}\right)\right\}$

$$=\tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}=\tan\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)=\frac{2}{3\sqrt{5}}$$

29. (0002) $\cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right)+\sin^{-1}\left(\sqrt{x^2+x+1}\right)=\frac{\pi}{2}$

$$\Rightarrow \frac{1}{\sqrt{x^2+x+1}}=\sqrt{x^2+x+1}$$

$$\Rightarrow x^2+x+1=1\Rightarrow x^2+x=0\Rightarrow x=-1,\ 0$$

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEAR'S

1. (*a*) $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}\left(x>\frac{3}{4}\right)$

$$\cos^{-1}\left(\frac{2}{3x}\times\frac{3}{4x}-\sqrt{1-\frac{4}{9x^2}}\sqrt{1-\frac{9}{16x^2}}\right)=\frac{\pi}{2}$$

$$\Rightarrow \frac{1}{2x^2}=\frac{\sqrt{9x^2-4}\sqrt{16x^2-9}}{12x^2}$$

$$\Rightarrow 6=\sqrt{9x^2-4}\sqrt{16x^2-9}$$

Square both side

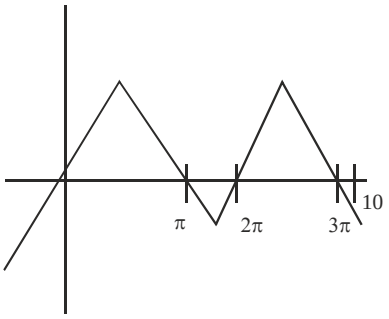
$36=144x^4-81x^2-64x^2+36$

$$\Rightarrow 144x^4=145x^2$$

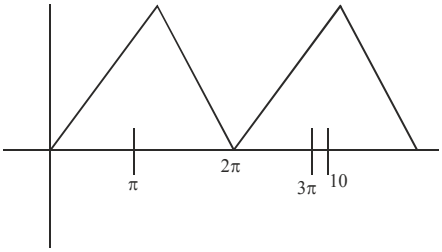
$$\Rightarrow x^4=\frac{145x^2}{144}\Rightarrow x=\pm\frac{\sqrt{145}}{12},0$$

$\therefore x>\frac{3}{4}$ hence $x=\frac{\sqrt{145}}{12}$

2. (*a*)



$x=\sin^{-1}(\sin 10)=3\pi-10$



$y=\cos^{-1}(\cos 10)=4\pi-10$

$y-x=\pi$

3. (*a*) $\cot\left[\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right]$

$$=\cot\left[\sum_{n=1}^{19}\cot^{-1}\left(1+n^2+n\right)\right]$$

$$=\cot\left[\sum_{n=1}^{19}\tan^{-1}\left(\frac{1}{1+n^2+n}\right)\right]$$

$$=\cot\left[\sum_{n=1}^{19}\tan^{-1}(n+1)-\tan^{-1}1\right]$$

$$=\cot\left[\tan^{-1}20-\tan^{-1}1\right]$$

$$=\cot\left(\tan^{-1}\frac{19}{21}\right)$$

$$=\frac{21}{19}$$

4. (*b*) $(\cot^{-1}(x)-5)(\cot^{-1}(x)-2)>0$

$\Rightarrow \cot^{-1}(x)\in(-\infty,2)\cup(5,\infty)$

Put $0<\cot^{-1}(x)<\pi$

$\Rightarrow \cot^{-1}(x) \Rightarrow (0,2)$

$\Rightarrow x\in(\cot 2,\infty)$

5. (*c*) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

Taking tangent on both side, we get $\frac{2x+3x}{1-6x^2}=1$

$\Rightarrow 6x^2+5x-1=0\Rightarrow (x+1)(6x-1)=0$

$\Rightarrow x=\frac{1}{6}$ {–1 is rejected as it does not satisfies the given equation }

Hence number of element is S is one.

6. (*a*) $\cos\alpha=\frac{3}{5},\tan\beta=\frac{1}{3}$

$\Rightarrow \tan\alpha=\frac{4}{3}$

$\Rightarrow \tan(\alpha-\beta)=\frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}.\frac{1}{3}}=\frac{9}{13}$

$\Rightarrow \sin(\alpha-\beta)=\frac{9}{5\sqrt{10}}$

$\Rightarrow \alpha-\beta=\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

7. (*c*) $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$

$\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

$=\sin^{-1}\left(\frac{33}{65}\right)=\cos^{-1}\left(\frac{56}{65}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$

8. (*c*) $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$

$\cos(\cos^{-1}x-\cos^{-1}\frac{y}{2})=\cos\alpha$

$\Rightarrow x\times\frac{y}{2}+\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}=\cos\alpha$

$\Rightarrow \left(\text{coa}\,\alpha-\frac{xy}{2}\right)^2=(1-x^2)\left(1-\frac{y^2}{4}\right)$

$x^2+\frac{y^2}{4}-xy\cos\alpha=1-\cos^2\alpha=\sin^2\alpha$

9. (*c*) $2\pi-\left(\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}\right)$

$=2\pi-\left(\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{16}{63}\right)$

$=2\pi-\left(\tan^{-1}\left(\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}\right)+\tan^{-1}\frac{16}{63}\right)$

$=2\pi-\left(\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{16}{63}\right)$

$=2\pi-\left(\tan^{-1}\frac{63}{16}+\cot^{-1}\frac{63}{16}\right)$

$=2\pi-\frac{\pi}{2}$

$=\frac{3\pi}{2}$

10. (*d*) $S=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+.....$

$=\tan^{-1}\left(\frac{1}{1+1.2}\right)+\tan^{-1}\left(\frac{1}{1+2.3}\right)+\tan^{-1}\left(\frac{1}{1+3.4}\right)+...$

$=\tan^{-1}\left(\frac{2-1}{1+2.1}\right)+\tan^{-1}\left(\frac{3-2}{1+3.2}\right)+$

$\tan^{-1}\left(\frac{4-3}{1+3.4}\right)+.....+\tan^{-1}\left(\frac{11-10}{1+11.10}\right)$

$=(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+(\tan^{-1}4-\tan^{-1}3)$
$+.....+(\tan^{-1}11-\tan^{-1}10)$
$=\tan^{-1}11-\tan^{-1}1$

$=\tan^{-1}\left(\frac{11-1}{1+11.1}\right)=\tan^{-1}\left(\frac{5}{6}\right)$

$\tan(S)=\frac{5}{6}$

11. (*c*) $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$

$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right)=\sin^{-1}x$

$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$

$x=0$ or $3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$

$4\sqrt{25-9x^2} = 25-3\sqrt{25-16x^2}$

Squaring we get

$16\left(25-9x^2\right) = 625-9\left(25-16x^2\right)-150\sqrt{25-16x^2}$

$400 = 625+225-150\sqrt{25-16x^2}$

$\sqrt{25-16x^2} = 3 \Rightarrow 25-16x^2 = 9$

$\Rightarrow x^2 = 1$

Put $x = 0, 1, -1$ in the original equation

We see that all values satisfy the original equation.

Number of solution $= 3$

12. (*a*) $\tan^{-1}(x+1)+\cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}(x+1)+\tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \frac{(1+x)+(x-1)}{1-(1+x)(x-1)} = \frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 4x^2+31x-8 = 0$

$\Rightarrow x = -8, \frac{1}{4}$

but at $x = \frac{1}{4}$

$LHS > \frac{\pi}{2}$ and $RHS < \frac{\pi}{2}$

So, only solution is $x = -8 = -\frac{32}{4}$

13. (*c*) $\cot^{-1}(\alpha) = \cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+\ldots 100$ terms

$= \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}+\tan^{-1}\frac{1}{32}+\ldots 100$ term

$= \sum_{k=1}^{100}\tan^{-1}\frac{1}{2k^2} = \sum_{k=1}^{100}\left(\tan^{-1}(2k+1)-\tan^{-1}(2k-1)\right)$

$= \tan^{-1}201-\tan^{-1}1$

$= \tan^{-1}\frac{200}{202}$

$= \cot^{-1}(1.01)$

14. (*a*) There are three cases possible for $x \in [-1,1]$

Case I: $x \in \left[-1, -\sqrt{\frac{2}{3}}\right)$

$\therefore \sin^{-1}(1)+\cos^{-1}(0) = x^2$

$\Rightarrow x^2 = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

$\Rightarrow x = \pm\sqrt{\pi} \rightarrow$ (Reject)

Case II : $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

$\therefore \sin^{-1}(0)+\cos^{-1}(-1) = x^2$

$\Rightarrow 0+\pi = x^2 \Rightarrow x = \pm\sqrt{x} \rightarrow$ (Reject)

Case III : $x \in \left(\sqrt{\frac{2}{3}}, 1\right)$

$\therefore \sin^{-1}(0)+\cos^{-1}(0) = x^2$

$\Rightarrow x^2-\pi \Rightarrow x-\pm\sqrt{x}$ (Reject)

$\therefore$ No solution. There, the correct answer is (1).

JEE-ADVANCED
PREVIOUS YEAR'S

1. (*b*) $\cot \sum_{n=1}^{23} \cot^{-1}(1+2+4+6+.....+2n) \Rightarrow \cot \sum\cot^{-1}(1+n(n+1))$

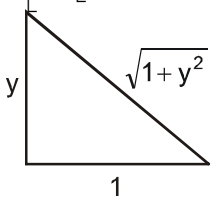
$\cot \sum \tan^{-1} \frac{(n+1)-n}{1+n(n+1)} \Rightarrow \chi ot$

$\sum_{n=1}^{23}\left(\tan^{-1}(n+1)-\tan^{-1}n\right)$

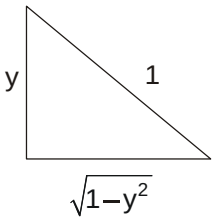
$\cot(\tan^{-1}24-\tan^{-1}1) \Rightarrow \cot\left(\tan^{-1}\frac{24-1}{1+24}\right) \Rightarrow \cot\left(\cot^{-1}\frac{25}{23}\right)=\frac{25}{23}$

2. (*b*) (P) $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$

$$=\left[\frac{1}{y^2}\left[\frac{\left(\frac{1}{\sqrt{1+y^2}}+\frac{y.y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}+\frac{y}{\sqrt{1-y^2}}\right)}\right]^2+y^4\right]^{1/2}$$



$$=\left(\frac{1}{y^2}.y^2(1-y^4)+y^4\right)^{1/2}=1$$



Ans. 4

(Q) $\cos x+\cos y=-\cos z$ $\quad \sin x+\sin\psi=-\sin z$ square and add

$2+2\chi os(x-\psi)=1$

$\Rightarrow \cos(x-y)=-1/2$

$\Rightarrow 2\cos^2\left(\frac{x-y}{2}\right)-1=-1/2, \Rightarrow \cos\left(\frac{x-y}{2}\right)=1/2$

Ans. 3

(R) $\cos 2x\left(\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{\pi}{4}+x\right)\right)+2\sin^2 x=2\sin x\cos x$

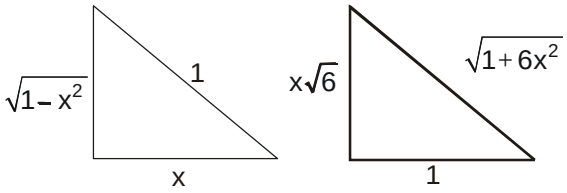
$\cos 2x\,(\sqrt{2}\sin x)+2\sin^2 x=2\sin x\cos x \Rightarrow \sqrt{2}\sin x[\cos 2x+\sqrt{2}\sin x-\sqrt{2}\cos x]=0$

Either $\sin x=0$ OR $\cos^2x-\sin^2x=\sqrt{2}(\cos x-\sin x)$

$\boxed{\sec x=1}$ OR $\cos x=\sin x \Rightarrow \boxed{\sec x=\sqrt{2}}$

Ans. 2 OR 4

(S) $\cot(\sin^{-1}\sqrt{1-x^2})=\sin(\tan^{-1}(x\sqrt{6}))$

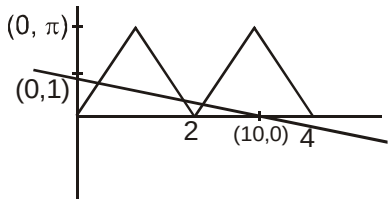


$\frac{x}{\sqrt{1-x^2}}=\frac{x\sqrt{6}}{\sqrt{1+6x^2}} \Rightarrow 1+6x^2=6-6x^2 \Rightarrow 12\xi^2=5$

$x=\sqrt{\frac{5}{12}}==\frac{1}{2}\sqrt{\frac{5}{3}}$

Ans. 1

3. (*c*) $f(x)=(siv^{\square})\,x\in[0,4\pi]$ & $f(x)=\frac{10-x}{10}=1-\frac{x}{10}$



so, 3 solution.

4. (*b,c,d*) $\alpha=3\sin^{-1}\frac{6}{11}>3\sin^{-1}\frac{6}{12}$

and $\beta=3\cos^{-1}\frac{4}{9}>3\cos^{-1}\frac{4}{8}$

$\Rightarrow \alpha>\frac{\pi}{2}$ & $\beta>\pi$

$\Rightarrow \alpha+\beta>\frac{3\pi}{2}$

5. (*b*) $\sum_{i=1}^{\infty}x^{i+1}=\frac{x^2}{1-x}$

$\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i=\frac{x}{2-x}$

$\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i=\frac{-x}{2-x}$

$$\sum_{i=1}^{\infty}(-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty} x^{i+1} - \sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$x(x^3 + 2x^2 + 5x - 2) = 0$

$\therefore$ $x = 0$ and let $f(x) = x^3 + 2x^2 + 5x - 2$

$$f\left(\frac{1}{2}\right) . f\left(-\frac{1}{2}\right) < 0$$

Hence two solutions exist

6. (*d*) $f_n(x)\sum_{j=1}^{n}\tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$$f_n(x) = \sum_{j=1}^{n}\left[\tan^{-1}(x+j) - \tan^{-1}(x+j-1)\right]$$

$$f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$$

$$\therefore \tan\left(f_n(x)\right) = \tan\left[\tan^{-1}(x+n) - \tan^{-1}x\right]$$

$$\tan\left(f_n(x)\right) = \frac{(x+n)-x}{1+x(x+n)}$$

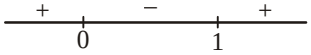
$$\tan\left(f_n(x)\right) = \frac{n}{1+x^2+nx}$$

$$\therefore \sec^2\left(f_n(x)\right) = 1 + \tan^2\left(f_n(x)\right)$$

$$\sec^2\left(f_n(x)\right) = 1 + \left(\frac{n}{1+x^2+nx}\right)^2$$

$$\lim_{x\to\infty}\sec^2\left(f_n(x)\right) = \lim_{x\to\infty} 1 + \left(\frac{n}{1+x^2+nx}\right)^2 = 1$$

7. (*a*) $E_1 : \frac{x}{x-1} > 0$



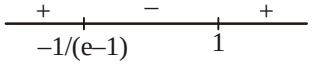
$\Rightarrow$ $E_1 : x \in (-\infty, 0) \cup (1, \infty)$

$$E_2 : -1 \le \ell n\left(\frac{x}{x+1}\right) \le 1$$

$$\frac{1}{e} \le \frac{x}{x-1} \le e$$

Now $\frac{x}{x-1} - \frac{1}{e} \ge 0$

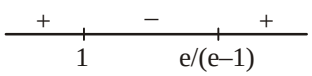
$\Rightarrow$ $\frac{(e-1)x+1}{e(x-1)} \ge 0$



$\Rightarrow$ $x \in \left(-\infty, \frac{1}{1-e}\right] \cup (1, \infty)$

also $\frac{x}{x-1} - e \le 0$

$\frac{(e-1)x-e}{x-1} \ge 0$



$\Rightarrow$ $x \in (-\infty, 1) \cup \left[\frac{e}{e-1}, \infty\right]$

So $E_2 : \left(-\infty, \frac{1}{1-e}\right) \cup \left[\frac{e}{e-1}, \infty\right]$

as Range of $\frac{x}{x-1}$ is $R^+ - \{1\}$

$\Rightarrow$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Rane of g is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]/\{0\}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Now $P \to 4$, $Q \to 2$, $R \to 1$, $S \to 1$

Hence A is correct

8. (0.01)

$$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left(\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)-\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\sum_{k=0}^{10}\tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{11\pi}{2}+\frac{7\pi}{12}\right)-\tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\cot\frac{7\pi}{12}-\tan\frac{7\pi}{12}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\frac{1}{\sin\frac{7\pi}{12}\cos\frac{7\pi}{12}}\right)\right)$$

$$= \sec^{-1}\left(-\frac{1}{2}\times\frac{1}{\sin\frac{7\pi}{6}}\right) = \sec^{-1}(1) = 0.00$$

3 Limit of Function

Topicwise Questions

1. (*d*) Since $\lim_{x\to 0-}\frac{|x|}{x}=-1$ and $\lim_{x\to 0+}\frac{|x|}{x}=1$, hence limit does not exist.

2. (*a*) $\lim_{x\to n^+}(x-[n])=\lim_{x\to n^+}x-\lim_{x\to n^+}[n]=n-n=0$.

3. (*a*) $\lim_{x\to 1}\frac{\log[(x-1)+1]}{x-1}=1.$

Aliter : Apply L-Hospital's rule,

$$\lim_{x\to 1}\frac{\log x}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1$$

4. (*b*) $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=n.2^{n-1}$

$\Rightarrow n.2^{n-1}=80$

$\Rightarrow n=5$

5. (*a*) $\lim_{x\to 0}\frac{x.2\sin^2 x}{x^2}=2.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}x=0$.

6. (*c*) $\lim_{x\to 0}kx\,\text{cosec}\,x=\lim_{x\to 0}x\,\text{cosec}\,kx$

$\Rightarrow k.\lim_{x\to 0}\frac{x}{\sin x}=\frac{1}{k}\lim_{x\to 0}\frac{kx}{\sin kx}$

$\Rightarrow k=\frac{1}{k}\Rightarrow k=\pm 1$.

7. (*c*) $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}=\lim_{x\to 0}\left(\frac{x^3\cot x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$

$=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3\times\lim_{x\to 0}\cos x\times\lim_{x\to 0}(1+\cos x)=2$

8. (*d*) $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}=\lim_{x\to 0}\frac{2x(e^x-1)}{4.\sin^2\frac{x}{2}}$

$=2\lim_{x\to 0}\left[\frac{(x/2)^2}{\sin^2\frac{x}{2}}\right]\left(\frac{e^x-1}{x}\right)=2.$

9. (*d*) $\lim_{x\to 0}\frac{2\sin 4x\cos 2x}{2\sin x\cos 4x}=\lim_{x\to 0}4\left(\frac{\sin 4x}{4x}\right)\left(\frac{x}{\sin x}\right)\frac{\cos 2x}{\cos 4x}=4$

Aliter : $\lim_{x\to 0}\frac{\frac{2\sin 2x}{2x}+\frac{6\sin 6x}{6x}}{\frac{5\sin 5x}{5x}-\frac{3\sin 3x}{3x}}=\frac{2+6}{5-3}=4.$

10. (*a*) Expand log(1 + x) and then solve.

Aliter : Apply L-Hospital's rule,

$$\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]$$

$$=\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}=\lim_{x\to 0}\frac{1}{2}\left(\frac{1}{1+x}\right)^2=\frac{1}{2}.$$

11. (*b*) Apply L-Hospital's rule,

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$$

$$=\lim_{x\to 0}\frac{\cos x}{2\sqrt{1+\sin x}}+\frac{\cos x}{2\sqrt{1-\sin x}}=\frac{1}{2}+\frac{1}{2}=1.$$

12. (*a*)

$$\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{\cos\theta}=\lim_{\theta\to\pi/2}\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}=0$$

13. (*a*) Apply the L-Hospital's rule, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

14. (*c*) $\frac{d}{da}[a^2\sin a]=2a\sin a+a^2\cos a.$

Aliter : Apply L-Hospital's rule,

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{2(a+h)\sin(a+h)+(a+h)^2\cos(a+h)}{1}$$

$$=2a\sin a+a^2\cos a.$$

15. (*a*)

$$\lim_{x\to 3}\left\{\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\right\}=\lim_{x\to 3}\frac{(x-3)\left\{\sqrt{x-2}+\sqrt{4-x}\right\}}{2(x-3)}=1$$

Aliter : Apply *L*-Hospital's rule.

16. (*b*) $\lim\limits_{x\to\pi/4}\dfrac{(\sqrt{2}-\sec x)\cos x(1+\cot x)}{\cot^2 x[2-\sec^2 x]}$

$$=\lim_{x\to\pi/4}\frac{\tan x\sin x(1+\cot x)}{(\sqrt{2}+\sec x)}=\frac{\frac{1}{\sqrt{2}}(2)}{\sqrt{2}+\sqrt{2}}=\frac{1}{2}.$$

Aliter : Apply L-Hospital's rule.

17. (*d*) $\lim\limits_{x\to 0}\dfrac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=\lim\limits_{x\to 0}\dfrac{|\sin x|}{x}$

So, $\lim\limits_{x\to 0+}\dfrac{|\sin x|}{x}=1$ and $\lim\limits_{x\to 0-}\dfrac{|\sin x|}{x}=-1$

Hence limit does not exist.

18. (*a*) Let $y=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{mx}\right)^x=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{mx}\right)^{mx\cdot\frac{1}{m}}$

$$\Rightarrow\quad y=e^{1/m},\ \left(\because\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e\right).$$

19. (*a*) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4},\quad\left(\dfrac{0}{0}\text{form}\right)$

Applying L-Hospital's rule, we get

$$\lim_{x\to 2}\frac{3x^2}{2x}=\lim_{x\to 2}\frac{3\times 2\times 2}{2\times 2}=3$$

20. (*d*) Let $y=\lim\limits_{x\to 3}\dfrac{x^3-x^2-18}{x-3},\left(\dfrac{0}{0}\text{form}\right)$

Applying L-Hospital's rule, we get

$$y=\lim_{x\to 3}3x^2-2x=(27-6)=21.$$

21. (*a*) $\lim\limits_{x\to\infty}\dfrac{2x^2+3x+4}{3x^2+3x+4}=\lim\limits_{x\to\infty}\dfrac{2+\frac{3}{x}+\frac{4}{x^2}}{3+\frac{3}{x}+\frac{4}{x^2}}=\dfrac{2}{3}$

22. (*c*) $\lim\limits_{x\to 2}\left(\dfrac{3^{x/2}-3}{3^x-9}\right)=\lim\limits_{x\to 2}\left(\dfrac{3^{x/2}-3}{(3^{x/2})^2-3^2}\right)$

$$=\lim_{x\to 2}\frac{1}{3^{x/2}+3}=\frac{1}{6}.$$

23. (*a*)

$$\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}=\lim_{x\to\infty}\frac{\log x^n}{[x]}-\lim_{x\to\infty}\frac{[x]}{[x]}=0-1=-1.$$

24. (*c*) $\lim\limits_{x\to 0}\dfrac{\log(3+x)-\log(3-x)}{x}=k$

By L-Hospital's rule, $\lim\limits_{x\to 0}\dfrac{\frac{1}{3+x}+\frac{1}{3-x}}{1}=k$

$$\Rightarrow\frac{2}{3}=k$$

25. (*c*) Using L-Hospital's rule, $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{-1}{-\text{cosec}^2\theta}=1$.

26. (*a*) $\lim\limits_{n\to\infty}\dfrac{1-n^2}{\Sigma n}$

$$=\lim_{n\to\infty}\frac{(1-n)(1+n)}{\frac{1}{2}n(n+1)}=\lim_{n\to\infty}\frac{2(1-n)}{n}$$

$$=\lim_{n\to\infty}2\left(\frac{1}{n}-1\right)=2(0-1)=-2.$$

27. (*d*) $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{3^x-1},\left(\dfrac{0}{0}\text{ form}\right)$

Using L-Hospital's rule,

$$\lim_{x\to 0}\frac{\frac{1}{1+x}}{3^x\log_e 3}=\frac{1}{\log_e 3}=\log_3 e.$$

28 (*c*) $\lim\limits_{x\to\infty}\left(\dfrac{x+a}{x+b}\right)^{x+b}=\lim\limits_{x\to\infty}\left(1+\dfrac{a-b}{x+b}\right)^{x+b}$

$$=\lim_{x\to\infty}\left\{\left(1+\frac{a-b}{x+b}\right)^{\frac{x+b}{a-b}}\right\}^{a-b}=e^{a-b}$$

29 (*b*) $\lim\limits_{x\to a}\dfrac{f(a)[g(x)-g(a)]-g(a)[f(x)-f(a)]}{[x-a]}$

$$=f(a)g'(a)-g(a)f'(a)=2\times 2-(-1)(1)=5.$$

30 (*b*) $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+.....+\dfrac{n}{1-n^2}\right]$

$$=\lim_{n\to\infty}\frac{\Sigma n}{1-n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{1-n^2}=-\frac{1}{2}.$$

1. (*d*) $\lim_{x\to 0^+} \sec x > 1$

So limit not exist

2. (*d*) $\lim_{x\to\infty} \sec^{-1}\left(\frac{x}{x+1}\right)$ Put $x = \frac{1}{y}$

$= \lim_{y\to 0} \sec^{-1}\left(\frac{1}{y+1}\right) = \lim_{y\to 0} \cos^{-1}(y+1)$

and $\lim_{y\to 0^+} \cos^{-1}(y+1) = \lim_{h\to 0} \cos^{-1}(1+h) \Rightarrow$ Not possible

so, Limit does not exists.

3. (*c*) $\lim_{x\to 0} \left\{\cot\left(\frac{\pi}{4}+x\right)\right\}^{(\operatorname{cosec} x)}$

$\ell = e^{\lim_{x\to 0} \operatorname{cosec} x\left(\cot\left(\frac{\pi}{4}+x\right)-1\right)}$

$= e^{\lim_{x\to 0} \operatorname{cosec} x\left(\frac{1}{\tan(\frac{\pi}{4}+x)}-1\right)}$

$= e^{\lim_{x\to 0} \operatorname{cosec} x\left(\frac{1-\tan x-1-\tan x}{1+\tan x}\right)}$

$= e^{\lim_{x\to 0} -\operatorname{cosec} x\left(\frac{2\tan x}{1+\tan x}\right)} = e^{-\lim_{x\to 0}\frac{2}{\cos x+\sin x}} = e^{-2}$

4. (*b*) $\lim_{x\to 1} \frac{\left(\sum_{k=1}^{100} x^k\right)-100}{x-1}$

$= \lim_{x\to 1}\left[\frac{x-1}{x-1}+\frac{x^2-1}{x-1}+.....+\frac{x^{100}-1}{x-1}\right]$

$= 1 + 2 + 3 + ...+ 100 = \frac{(100).(101)}{2}$

$= 5050$

5. (*d*) $\lim_{n\to\infty} [(1+x)(1+x^2).....(1+x^{2^n})]$

Multipling $(1-x)$ in N^r & D^r

$= \lim_{n\to\infty}\left[\frac{(1-x)(1+x).....(1+x^{2^n})}{(1-x)}\right] = \lim_{n\to\infty}\left(\frac{1-x^{2^{n+1}}}{1-x}\right)$

$= \frac{1}{(1-x)} \lim_{n\to\infty}\left(1-x^{2^{n+1}}\right) = \frac{1}{(1-x)}$

$\because |x \le 1|$

6. (*c*) $\lim_{x\to 3} \frac{(x^3+27)\ell n(x-2)}{(x^2-9)}$

$= \lim_{x\to 3} \frac{(x+3)(x^2-3x+9)\ell n[1+(x-3)]}{(x-3)(x+3)}$

$= \lim_{x\to 3} (x^2-3x+9).\frac{\ell n[1+(x-3)]}{(x-3)}$

$= (9-9+9)(1) = 9$

7. (*d*) $\operatorname{Lim}_{x\to 0} \frac{\sin(\ell n(1+x))}{\ell n(1+\sin x)}$

$= \operatorname{Lim}_{x\to 0} \frac{\sin(\ell n(1+x))}{\ell n(1+x)} \times \frac{\ell n(1+x)}{x} \times \frac{\sin x}{\ell n(1+\sin x)} \times \frac{x}{\sin x}$

$= 1 . 1 . 1 = 1$

8. (*c*) $\operatorname{Lim}_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right]$ Let $x = \left(\frac{\pi}{2}+h\right)$

$= \operatorname{Lim}_{h\to 0}\left[\frac{\frac{\pi}{2}+h-\frac{\pi}{2}}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \operatorname{Lim}_{h\to 0}\left[\frac{h}{-\sin h}\right] = -2$

9. (*c*) $\operatorname{Lim}_{x\to\infty} \frac{x^3\sin\left(\frac{1}{x}\right)+x+1}{x^2+x+1}$

$\operatorname{Lim}_{x\to\infty} \frac{x^2\left\{\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}+\frac{1}{x}+\frac{1}{x^2}\right\}}{x^2\left\{1+\frac{1}{x}+\frac{1}{x^2}\right\}} = \frac{1}{1} = 1$

10. (*a*) $\operatorname{Lim}_{n\to\infty} \frac{-3n+(-1)^n}{4n-(-1)^n} = \operatorname{Lim}_{n\to\infty} \frac{-3+\frac{(-1)^n}{n}}{4-\frac{(-1)^n}{n}}$

$= \operatorname{Lim}_{n\to\infty} \frac{-3+\frac{(-1\text{ or }1)}{\infty}}{4-\frac{(-1\text{ or }1)}{\infty}} = -\frac{3}{4}$

11. (*b*) $\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$

$= \lim_{n\to\infty}\cos\left(\frac{\pi}{4n}\right)\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\times\frac{\pi}{4}$

$= 1\,.\,1\,.\,\frac{\pi}{4} = \frac{\pi}{4}$

12. (*d*) $\lim_{n\to\infty}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$

$= \lim_{n\to\infty}\frac{5.5^n+3^n-4^n}{5^n+2^n+27.9^n}$

$= \lim_{n\to\infty}\frac{5.\left(\frac{5}{9}\right)^n+\left(\frac{3}{9}\right)^n-\left(\frac{4}{9}\right)^n}{\left(\frac{5}{9}\right)^n+\left(\frac{2}{9}\right)^n+27}$

$= \frac{0+0-0}{0+0+27} = 0$

13. (*c*) $\lim_{x\to 0}\left[\frac{\sin[x-3]}{[x-3]}\right]$

LHL at $x = 0$

$\lim_{h\to 0}\left[\frac{\sin[-3-h]}{[-3-h]}\right] = \lim_{h\to 0}\left[\frac{\sin(-4)}{(-4)}\right] = -1$

RHL at $x = 0$

$\lim_{h\to 0}\left[\frac{\sin[-3+h]}{[-3+h]}\right] = \lim_{h\to 0}\left[\frac{\sin 3}{3}\right] = 0$

since LHL $\neq$ RHL hence limit does not exist.

14. (*a*) $\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^{(x+1)}$ It is of the form 1^∞, so

$\ell = e^{\lim_{x\to\infty}(x+1)\left[\frac{x+2}{x-2}-1\right]} = e^{\lim_{x\to\infty}\left(\frac{x+1}{x-2}\right)[x+2-x+2]}$

$= e^{\lim_{x\to\infty}\left(\frac{1+1/x}{1-2/x}\right).4} = e^4$

15. (*a*) $\lim_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{5}{x}}$ It is of the form 1^∞, so

$\ell = e^{\lim_{x\to 0^+}\frac{5}{x}\left(1+\tan^2\sqrt{x}-1\right)} = e^{\lim_{x\to 0^+}\frac{5}{x}\left(\tan^2\sqrt{x}\right)}$

$= e^{\lim_{h\to 0}\frac{5}{h}\left(\tan^2\sqrt{h}\right)} = e^{5\lim_{h\to 0}\left(\frac{\tan^2\sqrt{h}}{\left(\sqrt{h}\right)^2}\right)} = e^5$

16. (*b*) $\lim_{x\to 0}\frac{\sin(6x^2)}{\ell n\cos(2x^2-x)}$

$= \lim_{x\to 0}\frac{\sin 6x^2}{6x^2}.\frac{6x^2}{\ell n(\cos(2x^2-x))}$

$= 1.\lim_{x\to 0}\frac{6x^2}{\ell n(\cos(2x^2-x))}\left(\frac{0}{0}\text{ form}\right)$ [Using L' Hospital rule]

$= \lim_{x\to 0}\frac{12x}{\frac{(-\sin(2x^2-x))(4x-1)}{\cos(2x^2-x)}}$

$= \lim_{x\to 0}\frac{-12\cos(2x^2-x)}{(4x-1)}.\frac{x(2x-1)}{\sin(2x^2-x)}\frac{1}{(2x-1)}$

$= = \frac{1(-12)}{1.1} = -12$

17. (*a*) $\lim_{x\to\infty}\left(x-x^2\,\ell n\left(1+\frac{1}{x}\right)\right)$

$= \lim_{x\to\infty} x-x^2\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-.....\right)$

$= \lim_{x\to\infty} x-x+\frac{1}{2}-\frac{1}{3x}+.....= \frac{1}{2}$

18. (*c*) $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$ It is of the form 1^∞, so,

$\Rightarrow \ell = e^{\lim_{x\to\infty} x\left[\frac{x^2-2x+1}{x^2-4x+2}-1\right]} = e^{\lim_{x\to\infty}\left(\frac{2x^2-x}{x^2-4x+2}\right)}$

$= e^{\lim_{x\to\infty}\left(\frac{2-1/x}{1-4/x+2/x^2}\right)} = e^2$

19. (*a*) $\lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$ (1^∞ form)

$= e^{\lim_{x\to 0}\left(\frac{\cos x-1}{\sin x}\right)}$

$= e^{\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}}$

$= e^{\lim_{x\to 0}-\tan\left(\frac{x}{2}\right)}$

$= e^0 = 1$

20. (*c*) $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right)$

Put $x=\frac{1}{y} \Rightarrow \lim_{y\to 0}\left(\frac{\sin y}{y}\right)+\lim_{y\to 0}\sin y^2 = 1+0=1$

21 (*a*) $\lim_{x\to\frac{\pi}{2}} \frac{1-2^{\cos x}}{x\left(x-\frac{\pi}{2}\right)}\frac{1}{2\cos x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{2^{\sin\left(\frac{\pi}{2}-x\right)}-1}{\left(\frac{\pi}{2}-x\right)x.2^{\cos x}} = \frac{2\ln 2}{\pi}$

22. (*b*) $\lim_{x\to 0}(\cos\ mx)^{\frac{n}{x^2}}$ since it is of the form 1^∞, so,

$\ell = e^{\lim_{x\to 0}\frac{n}{x^2}(\cos mx-1)} = e^{\lim_{x\to 0}\frac{-2n\sin^2\left(\frac{mx}{2}\right)}{x^2\times\frac{m^2}{4}\times\frac{4}{m^2}}}$

$= e^{\left(-\frac{m^2 n}{2}\right)}$

23. (*c*) $\ell = \lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$\ell = \lim_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})(1+\cos x\sqrt{\cos 2x})}{x^2(1+\cos x\sqrt{\cos 2x})}$

$\ell = \lim_{x\to 0}\frac{1-\cos^2 x.\cos 2x}{x^2(1+1\sqrt{1})}$

$\ell = \lim_{x\to 0}\frac{1-\cos^2 x(1-2\sin^2 x)}{2x^2}$

$\ell = \lim_{x\to 0}\frac{1-\cos^2 x+2\sin^2 x\cos^2 x}{2x^2}$

$\ell = \lim_{x\to 0}\frac{\sin^2 x}{2x^2}+\lim_{x\to 0}\frac{\sin^2 x}{x^2}.\cos^2 x$

$\ell = \frac{1}{2}+1=\frac{3}{2}$

24. (*b*) $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$

$= \lim_{x\to 0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$= \lim_{x\to 0}\frac{2\sin\left\{\frac{\sin x+x}{2}\right\}}{\left\{\frac{\sin x+x}{2}\right\}}\times\frac{\left(\frac{\sin x+x}{2}\right)}{x^4}\times\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\Big/\left(\frac{x-\sin x}{2}\right)$

$= \lim_{x\to 0} 2\left(\frac{x+\sin x}{2}\right)\times\left(\frac{x-\sin x}{2}\right)\times\frac{1}{x^4}$

$= \lim_{x\to 0}\frac{1}{2}\left(\frac{x+\sin x}{x}\right)\left(\frac{x-\sin x}{x^3}\right)$

$= \lim_{x\to 0}\frac{1}{2}\left(1+\frac{\sin x}{x}\right)\cdot\left(\frac{x-\sin x}{x^3}\right)$

$= \frac{1}{2}(1+1)\cdot\left(\frac{1}{6}\right)=\frac{1}{6}$

25. (*b*) $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$ It is of the form 1^∞

$\ell = e^{\lim_{x\to\infty}2x\left(1+\frac{a}{x}+\frac{b}{x^2}-1\right)} = e^{\lim_{x\to\infty}\left(2a+\frac{2b}{x}\right)} = e^{2a}$

but $\ell = e^2$, so $e^{2a}=e^2 \Rightarrow a=1$ & $b\in R$

26. (*a*) $\ell = \lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$ It is of form 1^∞

$\ell = e^{\lim_{x\to\infty}x\left\{\frac{x^2+5x+3}{x^2+x+3}-1\right\}} = e^{\lim_{x\to\infty}x\left(\frac{4x}{x^2+x+3}\right)}$

$= e^{\lim_{x\to\infty}\left(\frac{4}{1+1/x+3/x^2}\right)} = e^4$

27. (*c*) $\ell = \lim_{x\to 0}\frac{\sin\cdot[\cos x]}{1+[\cos x]}$

LHL : $\lim_{h\to 0}\frac{\sin[\cos(0-h)]}{1+[\cos(0-h)]} \Rightarrow \frac{\sin 0}{1+0}=0$ $(\because [\cos(0-h)]=0)$

RHL : $\lim_{h\to 0}\frac{\sin[\cos(0+h)]}{1+[\cos(0+h)]} \Rightarrow \frac{\sin 0}{1+0}=0$

so, $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}=0$

28 (*a*) 1^∞ From

$\ell = \lim_{x\to\infty} x\left[\sin\frac{1}{x}+\cos\frac{1}{x}-1\right]$

put $x=1/t$

$$= \lim_{t\to 0} \frac{1}{t}\left[\sin t + \cos t - 1\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[2\sin\frac{t}{2}\cos\frac{t}{2} - 2\sin^2\frac{t}{2}\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\, 2\sin\frac{t}{2}\left[\cos\frac{t}{2} - \sin\frac{t}{2}\right]$$

$$= 1 \Rightarrow \lim_{x\to\infty}\left[\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)\right]^x = e$$

29 (*c*) $\ell\text{im}_{x\to 0} \dfrac{e^{-\frac{x^2}{2}} - \cos x}{x^3 \sin x}$

$$= \ell\text{im}_{x\to 0} \frac{\left[1-\frac{x^2}{2}+\frac{x^4}{4.2!}-.....\right]-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-.....\right]}{x^3\left[x-\frac{x^3}{3!}+.....\right]}$$

$$= \ell\text{im}_{x\to 0} \frac{x^4\left[\frac{1}{8}-\frac{1}{4!}\right]-x^6\left[\frac{1}{8.3!}-\frac{1}{6!}\right]+.....}{x^4\left[1-\frac{x^2}{3!}+.....\right]}$$

$$= \left[\frac{1}{8}-\frac{1}{24}\right] = \frac{1}{12}$$

30. (*d*) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+.....+\frac{1}{\sqrt{n^2+2n}}\right)$

$$\Rightarrow \lim_{n\to\infty}\frac{(2n+1)}{\sqrt{n^2+2n}} \le \ell \le \frac{(2n+1)}{\sqrt{n^2}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{(2n+1)}{n\sqrt{1+\frac{2n}{n^2}}} \le \ell \le \lim_{n\to\infty}\left(\frac{2n+1}{n}\right)$$

$$\Rightarrow 2 \le \ell \le 2 \Rightarrow \ell = 2$$

Advanced Level Multiconcept Questions

1. (*a*,*b*,*c*)

$$f(x) = \frac{x^2-9x+20}{x-[x]}$$

$$\ell\text{im}_{x\to 5^-} \frac{x^2-9x+20}{x-[x]} = \frac{25-45+20}{1} = 0$$

$$\ell\text{im}_{x\to 5^+} \frac{x^2-9x+20}{x-[x]} = \ell\text{im}_{h\to 0}\frac{(5+h)^2-9(5+h)+20}{5+h-[5+h]}$$

$$= \ell\text{im}_{h\to 0}\frac{25+10h+h^2-45-9h+20}{h} = \ell\text{im}_{h\to 0}\frac{h^2+h}{h} =$$

$$\ell\text{im}_{h\to 0}\frac{h(h+1)}{h} = 1$$

$\because \ \ell\text{im}_{x\to 5^-} f(x) \ne \ell\text{im}_{x\to 5^+} f(x)$

so $\ell\text{im}_{x\to 5} f(x)$ does not exist

2. (*a*,*b*)

(*a*) $\ell\text{im}_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6} = \ell\text{im}_{x\to\infty}\frac{\sqrt{x^2+2}}{-3x-6}$

$$= \ell\text{im}_{x\to\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{-3-\frac{6}{x}} = -\frac{1}{3}$$

(*b*) $\ell\text{im}_{x\to\infty}\frac{\sqrt{x^2+2}}{3x-6}$

$$= \ell\text{im}_{x\to\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}} = \frac{1}{3}$$

3. (*a*,*b*,*c*,*d*)

$$\ell\text{im}_{x\to\infty}\frac{(ax+1)^n}{x^n+A}$$

(*a*) If $n \in N$

$$\ell\text{im}_{x\to\infty}\frac{\left(a+\frac{1}{x}\right)^n}{1+\frac{A}{x^n}} = \frac{(a+0)^n}{1+0} = a^n$$

(*b*) If $n \in Z^-$ & $a = A = 0$

then $\ell\text{im}_{x\to\infty}\frac{(ax+1)^n}{x^n+A}$

$$= \ell\text{im}_{x\to\infty}\frac{1}{x^n} = \infty \ n \in Z^-$$

(*c*) If $n = 0$

then $\ell\text{im}_{x\to\infty}\frac{(ax+1)^n}{x^n+A}$

$= \lim_{x\to\infty} \frac{1}{1+A}$

$= \frac{1}{1+A}$

(*d*) If $n \in Z^-$, $A = 0$ & $a \neq 0$

then $\lim_{x\to\infty} \frac{(ax+1)^n}{x^n + A}$

$= \lim_{x\to\infty} \frac{(ax+1)^n}{x^n}$

$= \lim_{x\to\infty} \left(a + \frac{1}{x}\right)^n$

$= (a + 0)^n$

$= a^n$

4. (*b*,*c*)

$$f(x) = \begin{cases} 1 + \frac{2x}{a}, & 0 \le x < 1 \\ ax, & 1 \le x < 2 \end{cases}$$

L.H.L. $= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} 1 + \frac{2(1-h)}{a} = 1 + \frac{2}{a}$

R.H.L. $= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} a(1 + h) = a$

$\because$ $\lim_{x\to 1} f(x)$ exists

L.H.L.= R.H.L.

$\Rightarrow 1 + \frac{2}{a} = a$

$\Rightarrow a^2 - a - 2 = 0$

$\Rightarrow (a - 2)(a + 1) = 0$

$\Rightarrow a = 2, -1$

5. (*a*,*b*)

Let $f(x) = \frac{\cos 2 - \cos 2x}{x^2 - |x|}$

(*a*) $\lim_{x\to -1} f(x)$

for $x = -1$, $|x| = -x$

$f(x) = \frac{\cos 2 - \cos 2x}{x^2 + x}$

Now $\lim_{x\to -1} \frac{\cos 2 - \cos 2x}{x^2 + x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to -1} \frac{2\sin 2x}{2x + 1} = 2\sin 2$

(*b*) $\lim_{x\to 1} \frac{\cos 2 - \cos 2x}{x^2 - x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 1} \frac{2\sin 2x}{2x - 1} = 2\sin 2$

6. (*a*,*d*)

For $\lim_{x\to 0} \frac{1 + a\cos x}{x^2}$

for $\left(\frac{0}{0}\right)$ form

$1 + a = 0 \Rightarrow a = -1$

for $\lim_{x\to 0} \frac{b\sin x}{x^3} = \lim_{x\to 0} \frac{b}{x^2}$

so $b = 0$

Now $\ell = \lim_{x\to 0} \frac{1 + a\cos x}{x^2} - \lim_{x\to 0} \frac{b\sin x}{x^3}$

$= \lim_{x\to 0} \frac{1 - \cos x}{x^2}$

$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}$

$= \lim_{x\to 0} \frac{2}{4} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$

$= \frac{1}{2} \cdot (1)^2 = \frac{1}{2}$

$\therefore$ $(a, b) = (-1, 0)$ and $\ell = \frac{1}{2}$

7. (*a*,*b*,*c*,*d*)

$\lim_{x\to 0} (\cos x + a \sin bx)^{\frac{1}{x}} = e^2$ (1^∞ form)

$\Rightarrow e^{\lim_{x\to 0} \left(\frac{\cos x + a\sin bx - 1}{x}\right)} = e^2$

$\Rightarrow e^{\lim_{x\to 0} \left(\frac{-\sin x + ab\cos bx}{1}\right)} = e^2$

$\Rightarrow \lim_{x\to 0} (-\sin x + ab\cos bx) = 2$

$\Rightarrow ab = 2$ $\therefore$ $a = 1, b = 2$; $a = 2, b = 1$; $a = 3, b = \frac{2}{3}$;

$a = \frac{2}{3}, b = 3$

8. (*a*,*b*,*c*)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} |0 - h|^{\sin(0-h)}$

$= \lim_{h\to 0} h^{\sin(-h)}$

$= e^{\lim_{h\to 0} (-\sin h)(\ell n h)}$

$= e^0 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{h\to \infty} |0 + h|^{\sin(0+h)}$

$= \lim_{h\to \infty} h^{\sin h}$

$= e^{\lim\limits_{h\to\infty}(\sinh)(\ell nh)}$

$= e^0 = 1$

$\because \lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x) = 1$

$\therefore \lim\limits_{x\to0} f(x) = 1$

9. (*a*,*d*)

$$\lim_{x\to0} \frac{2\sin x\cos x + a\sin x}{x^3} = p \text{ (finite)}$$

$$\Rightarrow \lim_{x\to0} \frac{\sin x}{x}\cdot\left(\frac{2\cos x + a}{x^2}\right) = p$$

$$\Rightarrow \lim_{x\to0} \frac{\sin x}{x}\cdot \lim_{x\to0} \frac{2\cos x + a}{x^2} = p$$

$$\Rightarrow \lim_{x\to0} \frac{2\cos x + a}{x^2} = p$$

For $\left(\frac{0}{0}\right)$ form

$2 + a = 0$

$a = -2$

$$\Rightarrow p = \lim_{x\to0} \frac{2(\cos x - 1)}{x^2}$$

$$\Rightarrow p = \lim_{x\to0} \frac{-2\left(2\sin^2\frac{x}{2}\right)}{x^2}$$

$\Rightarrow p = -1$

10. (*a*,*b*,*c*,*d*)

$$f(x) = \frac{|x+\pi|}{\sin x}$$

(*a*) $f(-\pi^+) = \lim\limits_{h\to0} \frac{|-\pi+h+\pi|}{\sin(-\pi+h)}$

$= \lim\limits_{h\to0} \frac{|h|}{-\sinh} = -1$

(*b*) $f(-\pi^-) = \lim\limits_{h\to0} \frac{|-\pi-h+\pi|}{\sin(-\pi-h)}$

$= \lim\limits_{h\to0} \frac{|h|}{\sin h} = 1$

(*c*) $f(-\pi^+) \neq f(-\pi^-)$

So $\lim\limits_{x\to-\pi} f(x)$ does not exist

(*d*) for $\lim\limits_{x\to\pi} f(x)$

$$\text{LHL} = \lim_{x\to\pi^-} \frac{|x+\pi|}{\sin x}$$

$$= \lim_{h\to0} \frac{2\pi - h}{\sinh} = \frac{2\pi}{0} = \infty$$

$$\text{RHL} = \lim_{x\to\pi^+} \frac{|x+\pi|}{\sin x}$$

$$= \lim_{h\to0} \frac{2\pi + h}{-\sinh} = -\frac{2\pi}{0} = -\infty$$

LHL $\neq$ RHL

so $\lim\limits_{x\to\pi} f(x)$ does not exist.

11. (*a*)

12. (*d*)

13. (*a*)

(11 to 13)

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{\sin x + ae^x + be^{-x} + c\,\ell n(1+x)}{x^3}$$

$$= \lim_{x\to0^+} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) + a\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\right) + b\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots.\right) + c\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.\right)}{x^3}$$

$$= \lim_{x\to0^+} \frac{(a+b) + x(1+a-b+c) + x^2\left(\frac{a}{2!} + \frac{b}{2!} - \frac{c}{2}\right) + x^3\left(-\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}\right) + x^4\left(\ \right) + \ldots.}{x^3}$$

$\because$ limit is finite $\Rightarrow \left.\begin{matrix} a+b=0 \\ 1+a-b+c=0 \\ a+b-c=0 \end{matrix}\right\} \Rightarrow a = -\frac{1}{2}, b = \frac{1}{2}$ and

$c = 0$

Also $\lim\limits_{x\to0^+} f(x) = -\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}$

$= -\frac{1}{6}(1 - a + b - 2c) = -\frac{1}{3}$

14. (*d*)

15. (*a*)

16. (*b*)

(14 to 16)

$$f(x) = e^{\lim\limits_{n\to\infty}\left(\frac{\sin x\left(e^{1/n}-1\right)}{1/n}\right)} = e^{\sin x}$$

$$\therefore a = \frac{2}{11} \times [0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10]$$

$= 10$

$$b = \lim_{x\to0}\left(\frac{x^2}{\left[\frac{\tan x}{x}\right] - \cos x}\right)$$

$$= \lim_{x\to0}\left(\frac{x^2}{1-\cos x}\right) = 2$$

(i) $\therefore\ (a+b)=12$

(ii) $bx^2-b^2x+\lambda=0$

$2x^2-4x+\lambda=0$

$f(1)\cdot f(2)<0$

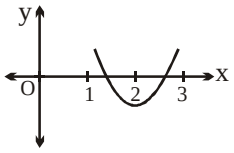
$(\lambda-2)(\lambda)<0$

$\Rightarrow \lambda\in(0,2)$ and

$f(2)\cdot f(3)<0$

$(\lambda)(6+\lambda)<0 \quad \Rightarrow \lambda\in(-6,0)$

Hence, no possible values of λ.



(iii) $\lim\limits_{u\to0}[1+u\ln(1+k^2)]^{\frac{1}{u}}=2k\ ln^2 f(x), k>0$

$\lim\limits_{u\to0}[1+u\,ln(1+k^2)]^{\frac{1}{u}}=e^{ln(1+k^2)}=1+k^2$

$\therefore\ 1+k^2=2k\sin^2x$

Hence, $k+\frac{1}{k}=2\sin^2x$ which is possible only when k is 1 and $x=\frac{\pi}{2}$.

$\therefore (x+k)=1+\frac{\pi}{2}$.

17. $(a)\to(s), (b)\to(r), (c)\to(p), (d)\to(q),$

(a) If $m>n \Rightarrow \quad m-n>0 \ \ \therefore \ \lim\limits_{x\to0}\phi(x)=0$

(b) If $m=n$, then $\lim\limits_{x\to0}\phi(x)=\lim\limits_{x\to0}$

$$\frac{a_0x^m+a_1^{m+1}+.....+a_kx^{m+k}}{b_0x^m+b_1^{m+1}+.....+b_\ell x^{m+\ell}}=\frac{a_0}{b_0}$$

(c) If $n-m$ is even positive, then

$\lim\limits_{x\to0}\phi(x)=\infty \quad$ as $\quad \frac{a_0}{b_0}>0$

(d) If $n-m$ is even positive and $\frac{a_0}{b_0}<0$, then

$\lim\limits_{x\to0}\phi(x)=-\infty$

18. $(a)\to(q), (b)\to(r), (c)\to(s), (d)\to(p),$

(a) $\lim\limits_{x\to0}\dfrac{\tan[e^2]x^2-\tan[-e^2]x^2}{\sin^2x}$

$$=\lim_{x\to0}\frac{[e^2]x^2\frac{\tan[e^2]x^2}{[e^2]x^2}-[-e^2]x^2\frac{\tan[-e^2]x^2}{[-e^2]x^2}}{x^2\frac{\sin^2x}{x^2}}$$

$=[e^2]-[-e^2]$

$=15$

(b) $\lim\limits_{x\to0}\left[\left(\min(t^2+4t+6)\right)\frac{\sin x}{x}\right]=\lim\limits_{x\to0}\left[\frac{2\sin x}{x}\right]$

$\because\ \sin x<x$

$\Rightarrow \frac{2\sin x}{x}<2\Rightarrow\left[\frac{2\sin x}{x}\right]=1$

So $\lim\limits_{x\to0}\left[\frac{2\sin x}{x}\right]=1$

(c) $\lim\limits_{x\to0}\dfrac{(1+x^2)^{1/3}-(1-2x)^{\frac{1}{4}}}{x+x^2}$

$$=\lim_{x\to0}\frac{\left(1+\frac{x^2}{3}+....\right)-\left(1-\frac{x}{2}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}4x^2+....\right)}{x+x^2}$$

$=\frac{1}{2}$

(d) $\lim\limits_{x\to0}=\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2x}=\lim\limits_{x\to0}\dfrac{2\sqrt{2}\sin^2\frac{x}{4}}{\sin^2x}$

$$=\lim_{x\to0}\frac{2\sqrt{2}\sin^2\frac{x}{4}}{16.\frac{x^2}{16}.\frac{\sin^2x}{x^2}}=\frac{\sqrt{2}}{8}$$

19. [1] $\lim\limits_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right)\right)$

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}+\sin\left(\frac{1}{x^2}\right)\right)=1+0=1$$

20. [2] $\lim\limits_{x\to0}\dfrac{1-\cos x\sqrt{\cos2x}}{x^2}$

$$=\lim_{x\to0}\frac{1-\cos^2x\cos2x}{(1+\cos x\sqrt{\cos2x})x^2}$$

$$=\lim_{x\to0}\frac{1}{(1+\cos x\sqrt{\cos2x})}\cdot\lim_{x\to0}\frac{1-\cos^2x\cos2x}{x^2}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{1-\left(\frac{1+\cos2x}{2}\right)\cos2x}{x^2}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{2-\cos2x-(\cos2x)^2}{2x^2}$$

$= \frac{1}{4} \lim_{x\to 0} - \frac{1}{x^2}(\cos 2x + 2)(\cos 2x - 1)$

$= \frac{1}{4} \lim_{x\to 0} (\cos 2x + 2).\ \lim_{x\to 0} \frac{1-\cos 2x}{x^2}$

$= \frac{(1+2)}{4} \cdot \lim_{x\to 0} \frac{2\sin^2 x}{x^2} = \frac{3}{4}.2 \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = \frac{3}{2}$

$\Rightarrow$ limit $= \frac{3}{2} + \frac{1}{2} = 2$

21. [1] $\lim_{x\to\infty} f(x)$ exist and is finite & non zero

$\lim_{x\to\infty}\left[f(x) + \frac{3f(x)-1}{f^2(x)}\right] = 3$

$\Rightarrow \lim_{x\to\infty} f(x) + \frac{3\lim_{x\to\infty} f(x) - 1}{\left[\lim_{x\to\infty} f(x)\right]^2} = 3$

Let $\lim_{x\to\infty} f(x) = A$

$A + \frac{3A-1}{A^2} = 3 \qquad \Rightarrow \quad A = 1$

so $\lim_{x\to\infty} f(x) = 1$

22. [0] $\lim_{x\to 0} f(g(h(x)))$

L.H.L. $\qquad x \to 0^-$

$\lim_{x\to 0^-} h(x) = 0^+$

$\lim_{x\to 0^+} f(g(x))$

then $\lim_{x\to 0^+} g(x) = 1^+$

$\lim_{x\to 1^+} f(x) = 1 - 1 = 0$

R.H.L. $x \to 0^+$

$\lim_{x\to 0^+} h(x) = 0^+$

so $\lim_{x\to 0^+} f(g(x)) = 0$

L.H.L. = R.H.L. = 0

23. [1] $\lim_{x\to 0} g(f(x))$

L.H.L. $= \lim_{x\to 0^-} g[f(x)]$

$\lim_{x\to 0^-} g(\sin x) = \lim_{h\to 0} g(\sinh) = \lim_{h\to 0} (\sin^2 h + 1) = 1$

R.H.L. $= \lim_{x\to 0^+} g[f(x)] = \lim_{x\to 0^+} g(\sin x) = \lim_{h\to 0} g(\sinh) =$

$\lim_{h\to 0} (\sin^2 h + 1) = 1$

L.H.L. = R.H.L. = 1

so $\lim_{x\to 0} g[f(x)] = 1$

24. [1] $\lim_{x\to 0^+} x^2\left[\frac{1}{x^2}\right] = \lim_{x\to 0^+} x^2\left(\frac{1}{x^2} - \left\{\frac{1}{x^2}\right\}\right)$

$\Rightarrow \lim_{x\to 0^+}\left(1 - x^2\left\{\frac{1}{x^2}\right\}\right) = 1$

similarly $\lim_{x\to 0^-} f(x) = 1$

25. [11]

$\lim_{x\to 0}\left(\frac{\sin^{-1} x - \tan^{-1} x}{x^3} + \frac{84x\frac{\pi}{8}}{\sin \pi x}\right)$

$= \lim_{x\to 0} \frac{\left[x + \frac{1^2}{3!}x^3 + \ldots.\right] - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.\right]}{x^3} + \frac{84}{8}$

$= \lim_{x\to 0} \frac{x^3\left(\frac{1}{3!} + \frac{1}{3}\right) + x^5(\ \ ) + \ldots..}{x^3} + \frac{21}{2} = 11$

26. [20]

$f(x) = \sum_{\lambda=1}^{n}\left(x - \frac{5}{\lambda}\right)\left(x - \frac{4}{\lambda+1}\right)$

$f(0) = \sum_{\lambda=1}^{n}\left(-\frac{5}{\lambda}\right)\left(-\frac{4}{\lambda+1}\right)$

$\Rightarrow f(0) = \sum_{\lambda=1}^{n}\left(\frac{20}{(\lambda)(\lambda+1)}\right)$

$\Rightarrow f(0) = 20\sum_{\lambda=1}^{n}\left(\frac{1}{\lambda} - \frac{1}{\lambda+1}\right)$

$\Rightarrow f(0) = 20\left(1 - \frac{1}{n+1}\right)$

$\Rightarrow f(0) = \frac{20n}{n+1}$

Now $\lim_{n\to\infty} f(0) = \lim_{n\to\infty} \frac{20n}{n+1}$

$= \lim_{n\to\infty} \frac{20}{1+\frac{1}{n}} = \frac{20}{1+0} = 20$

27. [12]

LHL $= \lim_{h\to 0^+} f(0-h) = \lim_{h\to 0^+} f(-h) = \lim_{h\to 0^+} (-1)^{[h^2]} = 1$

RHL $= \lim_{h\to 0^+} f(0+h) = \lim_{h\to 0^+} f(h) = \lim_{h\to 0^+}\left(\lim_{n\to\infty} \frac{1}{1+h^n}\right) = 1$

28 [1]

$\because\ (1+x)^{\frac{1}{x}} = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots..\right]$

Now $\ell = \lim_{x\to 0} \frac{e - (1+x)^{\frac{1}{x}}}{\tan x} = \lim_{x\to 0}$

$\frac{e - e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots..\right]}{x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots..} = \frac{e}{2}$

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*c*) $\lim\limits_{x\to0}\dfrac{(1-\cos2x)(3+\cos x)}{x\tan4x}$

$= \lim\limits_{x\to0}\dfrac{2\sin^2x}{x^2}\ \dfrac{(3+\cos x)}{\frac{\sin4x}{4x}}\ \dfrac{\cos4x}{4} = 2$

2. (*c*) $P=\lim\limits_{x\to0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}} = \lim\limits_{x\to0^+}e^{\tan^2\sqrt{x}\times\frac{1}{2x}}$

$P = e^{\frac{1}{2}} \qquad \Rightarrow \log P = \dfrac{1}{2}$

3. (*c*) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{-8\left(x-\frac{\pi}{2}\right)^3}$

$= \lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)}\dfrac{\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{\left(\frac{\pi}{2}-x\right)^2}$

$= \dfrac{1}{8}\cdot1\cdot\dfrac{1}{2} = \dfrac{1}{16}$

4. (*b*) $\lim\limits_{x\to0^+}x\left(\left[\dfrac{1}{x}\right]+\left[\dfrac{2}{x}\right]+.....+\left[\dfrac{15}{x}\right]\right)$

$\lim\limits_{x\to0^+}x\left[\left(\dfrac{1+2+3+....+15}{x}\right)-\left(\left\{\dfrac{1}{x}\right\}+\left\{\dfrac{2}{x}\right\}+...+\left\{\dfrac{15}{x}\right\}\right)\right]$

$\because 0\le\left\{\dfrac{r}{x}\right\}<1$

$0\le x\left\{\dfrac{r}{x}\right\}<x$

$\lim\limits_{x\to0^+}x\left(\dfrac{1+2+3+...+15}{x}\right)=\dfrac{15.16}{2}=120$

5. (*a*) $(1+x)^n\cong1+nx$ (when $x\to0$)

So, $\sqrt{1+y^4}=1+\dfrac{y^4}{2}$

$\lim\limits_{y\to0}\dfrac{\sqrt{2+\frac{y^4}{2}}-\sqrt{2}}{y^4}$

$= \dfrac{\sqrt{2}\left(1+\frac{y^4}{8}-1\right)}{y^4} = \dfrac{\sqrt{2}}{8} = \dfrac{1}{4\sqrt{2}}$

6. (*a*) $\lim\limits_{x\to0^-}\dfrac{x([x]+|x|)\sin[x]}{|x|}$

$x\to0^-$

$\left.\begin{matrix}[x]=-1\\|x|=-x\end{matrix}\right\}\Rightarrow\lim\limits_{x\to0^-}\dfrac{x(-x-1)\sin(-1)}{-x}=-\sin1$

7. (*b*) $\lim\limits_{x\to1+}\dfrac{(1-|x|+\sin|1-x|)\sin\left([1-x]\frac{\pi}{2}\right)}{|1-x|[1-x]}$

$\lim\limits_{x\to1^+}\dfrac{(1-x+\sin(x-1)\sin\left(-\frac{\pi}{2}\right)}{(x-1)(-1)}$

$\text{L}\lim\limits_{x\to1^+}\dfrac{-(x-1)+\sin(x-1)}{(x-1)}=-1+1=0$

8. (*d*) $\lim\limits_{x\to0}\dfrac{x\cos4x\sin^2 2x}{\sin^2x.\cos^2 2x.\sin4x}$

$= \lim\limits_{x\to0}\dfrac{x}{\tan4x}\cdot\dfrac{1}{\sin^2x}\dfrac{\tan^2 2x}{1}$

$= \lim\limits_{x\to0}\dfrac{1}{4}\dfrac{4x}{\tan4x}\dfrac{x^2}{\sin^2x}\cdot\dfrac{\tan^2 2x}{x^2}$

$= \lim\limits_{x\to0}\dfrac{1}{4}\dfrac{4x}{\tan4x}\left(\dfrac{x}{\sin x}\right)^2\cdot\left(\dfrac{\tan2x}{2x}\right)^2\dfrac{4}{1}$

$= \dfrac{1}{4}\cdot1\cdot1\cdot1\cdot\dfrac{4}{1}=1$

9. (*a*) $\text{Lt}\limits_{x\to0}\dfrac{\tan\left(\pi\sin^2x\right)}{\pi\sin^2x}\cdot\dfrac{\pi\sin^2x}{x^2}+\left(\dfrac{|x|-\sin\left(x[x]\right)}{|x|}\right)^2$

$1(\pi).(1)^2+\text{Lt}\limits_{x\to0}\left(1-\dfrac{\sin x[x]}{x[x]}\cdot\dfrac{x[x]}{|x|}\right)^2$ ....(i)

$\text{Lt}\limits_{x\to0^-}\dfrac{x[x]}{|x|}=\dfrac{x}{-x}(-1)=1$

$\text{Lt}\limits_{x\to0^+}\dfrac{x[x]}{|x|}=\dfrac{x(0)}{x}=0$

Put in equation (i)

$\therefore$ Limit does not exist.

10. (*d*) Using LH rule

$\ell\text{im}\limits_{x\to\frac{\pi}{4}}\dfrac{3\cot^2x\left(-\text{cosec}^2x\right)-\sec^2x}{-\sin\left(x+\frac{\pi}{4}\right)}=8$

11. (*b*) $\lim_{x\to1^-}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}}\times\frac{\sqrt{\pi}+\sqrt{2\sin^{-1}x}}{\sqrt{\pi}+\sqrt{2\sin^{-1}x}}$

$$\lim_{x\to1^-}\frac{2\left(\frac{\pi}{2}-\sin^{-1}x\right)}{\sqrt{1-x}\left(\sqrt{\pi}+\sqrt{2\sin^{-1}x}\right)}$$

$$\lim_{x\to1^-}\frac{2\cos^{-1}x}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{\pi}}$$

Assuming $x=\cos\theta$

$$\lim_{\theta\to0^+}\frac{2\theta}{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}\cdot\frac{1}{2\sqrt{\pi}}=\sqrt{\frac{2}{\pi}}$$

12. (*d*) $\lim_{x\to0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}}$ $\left(1^\infty\text{ form}\right)$

$$\Rightarrow e^{\lim_{x\to0}\frac{f(3+x)-f(2-x)-f(3)+(2)}{x(1+f(2-x)-f(2))}}$$

using L 'Hopital

$$\Rightarrow e^{\lim_{x\to0}\frac{f'(3+x)-f'(2-x)}{-xf'(2-x)+(1+f(2-x)-f(2))}}\Rightarrow e^{\frac{f'(3)+f'(2)}{1}}=1$$

13. (*d*) $\lim_{x\to1}\frac{x^4-1}{x-1}=\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$

$$\Rightarrow \lim_{x\to1}(x+1)(x^2+1)=\frac{k^2+k^2+k^2}{2k}$$

$$\Rightarrow k=8/3$$

14. (*a*) $\lim_{x\to1}\frac{x^2-ax+b}{x-1}=5$

$1-a+b=0$ ...(i)

$2-a=5$ ....(ii)

$\Rightarrow a+b=-7.$

15. (*b*) Rationalize

$$\lim_{x\to0}\frac{(x+2\sin x)\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2x-x+1}\right)}{x^2+2\sin x+1-\sin^2x+x-1}$$

$$\lim_{x\to0}\frac{(x+2\sin x)(2)}{x^2+2\sin x-\sin^2x+x}$$

$\frac{0}{0}$ form using L′ hospital

$$\Rightarrow \lim_{x\to0}\frac{(1+2\cos x)\times2}{2x+2\cos x-2\sin x\cos x+1}$$

16. [36]

Let $3^{x/2}=t$

$$\lim_{t\to3}\frac{t^2+\frac{27}{t^2}-12}{\frac{1}{t}-\frac{3}{t^2}}=\lim_{t\to3}\frac{t^4+27-12t^2}{t-3}$$

$$=\lim_{t\to3}\frac{(t^2-3)(t+3)(t-3)}{(t-3)}=6\times6=36.$$

17. (*a*) Let $L=\lim_{x\to0}\left(\frac{3x^2+2}{7x^2+2}\right)^{1/x^2}=e^{\lim_{x\to0}\frac{1}{x^2}\left(\frac{3x^2+2}{7x^2+2}-1\right)}=e^{\frac{-4}{2}}=e^{-2}.$

18. (*a*) $\lim_{x\to0}\left(\frac{1+\tan x}{1-\tan x}\right)^{1/x}$

$$\Rightarrow e^{\lim_{x\to0}\left(\frac{1+\tan x-(1-\tan x)}{1-\tan x}\right)\frac{1}{x}}$$

$$\Rightarrow e^{\lim_{x\to0}\left(\frac{2\tan x}{1-\tan x}\right)\frac{1}{x}}=e^2$$

19. [40]

$$\lim_{x\to1}\frac{x+x^2+x^3+....+x^n-n}{x-1}=820$$

As it is $\left(\frac{0}{0}\right)$ form, Apply L 'Hospital's Rule.

$$\lim_{x\to1}\left(\frac{1+2x+3x^2+....+nx^{n-1}}{1}\right)=820$$

$$\Rightarrow 1+2+3+....n=820$$

$$\Rightarrow \frac{n(n+1)}{2}=820$$

$$\Rightarrow n^2+n-1640=0$$

$$\Rightarrow (n-40)(n+41)=0$$

$$\therefore\ n=40$$

20. (*b*) The indeterminate form is $\frac{0}{0}$

by L'Hospital rule

$$\lim_{x\to a}\frac{\frac{1}{3}(a+2x)^{-2/3}.2-\frac{1}{3}(3x)^{-2/3}.3}{\frac{1}{3}(3a+x)^{-2/3}-\frac{1}{3}(4x)^{-2/3}.4}$$

Put $x=a$

$$\Rightarrow \frac{\frac{2}{(3a)^{2/3}}-\frac{3}{(3a)^{2/3}}}{\left(\frac{1}{4a}\right)^{2/3}-4\left(\frac{1}{4a}\right)^{2/3}}=\frac{\frac{1}{(3a)^{2/3}}}{\frac{1}{(4a)^{2/3}}}\frac{(-1)}{(-3)}$$

$$\Rightarrow \left(\frac{4}{3}\right)^{2/3}\frac{1}{3}\Rightarrow\frac{2}{3}\left(\frac{2}{9}\right)^{1/3}$$

21. [8] $\lim\limits_{x\to0}\frac{1}{x^8}\left\{1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}.\cos\frac{x^2}{4}\right\}$

$$=\lim_{x\to0}\frac{\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)}{x^4}\frac{}{x^4}$$

$$=\lim_{x\to0}\frac{2\sin^2\frac{x^2}{4}}{\frac{x^4}{16}\times16}\times\frac{2\sin^2\frac{x^2}{8}}{\frac{x^4}{64}\times64}$$

$$=\frac{4}{16\times64}=2^{-8}$$

$\therefore\ k=8$

22. [1] Here $\lim\limits_{x\to0}\left|\frac{1-x+|x|}{\lambda-x+[x]}\right|=L$

$$\text{Here L.H.L.}=\lim_{h\to0}\left|\frac{1+h+h}{\lambda+h-1}\right|=\left|\frac{1}{\lambda-1}\right|$$

$$\text{R.H.L.}=\lim_{h\to0}\left|\frac{1-h+h}{\lambda+h-0}\right|=\left|\frac{1}{\lambda}\right|$$

$\because$ Limit exists. Hence L.H.L. = R.H.L.

$\Rightarrow |\lambda-1|=|\lambda|$

$\Rightarrow \lambda=\frac{1}{2}$ and $L=2$

23. (*c*) $\lim\limits_{x\to0}\frac{x\left(e^{\frac{\sqrt{1+x^2+x^4}-1}{x}}-1\right)}{\sqrt{1+x^2+x^4}-1}=\lim\limits_{x\to0}\frac{e^{\frac{\sqrt{1+x^2+x^4}-1}{x}}-1}{\frac{\sqrt{1+x^2+x^4}-1}{x}}$

$$=\lim_{x\to0}\frac{e^{\frac{x+x^3}{1+\sqrt{1+x^2+x^4}}}-1}{\frac{x+x^3}{1+\sqrt{1+x^2+x^4}}}$$

$$=\lim_{x\to0}\frac{1+\frac{x+x^3}{1+\sqrt{1+x^2+x^4}}+\frac{1}{2!}\left(\frac{x+x^3}{1+\sqrt{1+x^2+x^4}}\right)^2....\infty-1}{\frac{x+x^3}{1+\sqrt{1+x^2+x^4}}}=1$$

24. (*c*) Eq.P $(x)=x^2-x-2=0$ $\Rightarrow x=2,-1$

$\Rightarrow \alpha=2$

$$\text{Now }\lim_{x\to2^+}\frac{\sqrt{1-\cos(x^2-x-2)}}{x-2}$$

$$=\lim_{x\to2^+}\frac{\sqrt{2}\left|\sin\left(\frac{x^2-x-2}{2}\right)\right|}{x-2}$$

$$=\lim_{x\to2^+}\frac{\sqrt{2}\frac{\sin(x^2-x-2)}{2}}{\left(\frac{x^2-x-2}{2}\right)}\times\frac{(x^2-x-2)}{2(x-2)}$$

$$=\frac{1}{\sqrt{2}}\lim_{x\to2^+}\left(\frac{\sin\left(\frac{(x^2-x-2)}{2}\right)}{\frac{x^2-x-2}{2}}\right)\times\lim_{x\to2^+}\frac{(x-2)(x+1)}{(x-2)}$$

$$=\frac{1}{\sqrt{2}}\times1\times3=\frac{3}{\sqrt{2}}$$

25. (Bonus)

$$\lim_{x\to1}\left(\frac{\int_0^{(x-1)^2}t\cos(t^2)dt}{(x-1)\sin(x-1)}\right)$$

$$=\lim_{x\to1}\frac{\frac{1}{2}\sin(x-1)^4}{(x-1)\sin(x-1)}$$

$$\therefore\ \text{L.H.L.}=\lim_{h\to0}\frac{\frac{1}{2}\sin(h^4)}{h\cdot\sin h}=0$$

$$\text{R.H.L.}=\lim_{h\to0}\frac{\frac{1}{2}\sin(h^4)}{h\sin h}=0$$

$\therefore$ L.H.L. = R.H.L. = 0

$$\therefore\ \lim_{x\to1}\frac{\int_0^{(x-1)^2}.t\cos(t^2)\,dt}{(x-1)\sin(x-1)}=0$$

(No option is correct)

26. [7]

$$f(x)=x^6+2x^4+x^3+2x+3$$

$$\lim_{x\to1}\frac{x^n f(1)-f(x)}{x-1}=44$$

$$\lim_{x\to1}\frac{9x^n-\left(x^6+2x^*+x^3+2x+3\right)}{x-1}=44$$

$$\lim_{x\to1}\frac{9nx^{n-1}-\left(-\left(xx^5+2x^3+3x^2+2\right)\right)}{1}=44$$

$\Rightarrow 9n-(19)=44$

$\Rightarrow 9n=63$

$\Rightarrow n=7$

27. (c) $\lim\limits_{h\to0}2\times2\left\{\frac{\sin\left(\frac{\pi}{6}+h-\frac{\pi}{6}\right)}{2\sqrt{3}h(\cos\left(h+\frac{\pi}{6}\right)}\right\}$

$$=\frac{2}{\sqrt{3}}\times\frac{2}{\sqrt{3}}=\frac{4}{3}$$

28. [5] $\lim\limits_{x\to 0}\dfrac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}$

Applying L' Hospital Rule

$$\lim_{x\to 0}\frac{a-4e^{4x}}{a\left(e^{4x}-1\right)+ax(4e^{4x})}$$ So a = 4

Applying L' Hospital Rule lim

$$\lim_{x\to 0}\frac{-16e^{4x}}{a\left(4e^{4x}\right)+a(4e^{4x})+ax\left(16e^{4x}\right)}$$

$$\frac{-16}{4a+4a}=\frac{-16}{32}=-\frac{1}{2}=b$$

$$a-2b=4-2\left(\frac{-1}{2}\right)=4+1=5$$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*b*) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right)=4$

$$\lim_{x\to\infty}\left(\frac{x^2(1-a)+x(1-a-b)+(1-b)}{x+1}\right)=4$$

Limit is finite

It exists when $1-a=0\Rightarrow a=1$

then $\lim\limits_{x\to\infty}\left(\dfrac{1-a-b+\frac{1-b}{x}}{1+\frac{1}{x}}\right)=4 \quad \therefore 1-a-b=4$

$\Rightarrow b=-4$

2. (*b*) $((1+a)^{1/3}-1)x^2+((a+1)^{1/2}-1)x+((a+1)^{1/6}-1)=0$

let $a+1=t^6$

$\therefore\ (t^2-1)x^2+(t^3-1)x+(t-1)=0$

$(t+1)x^2+(t^2+t+1)x+1=0$

As $a\to 0$, $t\to 1$

$2x^2+3x+1=0\Rightarrow x=-1$ and $x=-\dfrac{1}{2}$

3. (0) $\lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\dfrac{1}{4}$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{1+\sqrt{x}}=\frac{1}{4}$$

Hence $\lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{(x-1)+\sin(x-1)}\right\}^{1+\sqrt{x}}=\dfrac{1}{4}$

put $x=1+h$,

$$\lim_{h\to 0}\left\{\frac{-ah+\sinh}{h+\sinh}\right\}^{1+\sqrt{1+h}}=\frac{1}{4}$$

or $\dfrac{-a+1}{2}=\dfrac{1}{2}$ or $-\dfrac{1}{2}\Rightarrow a=0$ or 2

But at a = 2, $\dfrac{-ah+\sinh}{h+\sinh}$ tends to negative value

So correct Answer is a = 0

However a = 2 may be accepted if this is not considered

4. (*a,b,c*) $f(x)=\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$

Let $\dfrac{\pi}{2}\sin x=\theta\because \theta\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

$\therefore\ f(x)=\sin\left(\dfrac{\pi}{6}\sin\theta\right)$

Let $\dfrac{\pi}{6}\sin\theta=\phi.\therefore\ \phi\in\left[-\dfrac{\pi}{6},\dfrac{\pi}{6}\right]$

$\therefore\ f(x)=\sin\phi\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

$\therefore$ (a)Now $fog(x)=\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin\dfrac{\pi}{2}\sin x\right)\right)$

Clearly, range of fog is also $\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

$\therefore$ (b)

Now, $\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$

$$=\lim_{x\to 0}\frac{2}{\pi}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\times\frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\sin x}{x}\times x}$$

$$=\lim_{x\to 0}\frac{2}{\pi}\times\frac{\pi}{6}\times\frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}\times\frac{\frac{\pi}{2}\sin x}{x}$$

$$\frac{1}{3}\times\frac{\pi}{2}=\frac{\pi}{6}$$

$\therefore$ (c)

Now, gof(x) = 1

$$\Rightarrow \frac{\pi}{2}\sin\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)=1$$

$$\Rightarrow \sin\left(\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)=\frac{2}{\Pi}\cong\frac{2}{3.14}>\frac{1}{2}$$

5. (*b*) $\because\ m \geq 2$ and $n \geq 2$

$$= \lim_{\alpha\to 0} \frac{e(e^{\cos(\alpha^n)-1}-1)}{(\cos(\alpha^n)-1)} \times \left(\frac{\cos(\alpha^n)-1}{(\alpha^n)^2}\right)\frac{\alpha^{2n}}{\alpha^m}$$

$$= e \times \lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)-1}-1)}{\cos(\alpha^n)-1)}\right) \times \lim_{\alpha\to 0}\left(\frac{\cos(\alpha^n)-1}{\alpha^{2n}}\right) \times \lim_{\alpha\to 0}\alpha^{2n-m}$$

$$= e \times 1 \times -\frac{1}{2} \times \lim_{\alpha\to 0}\alpha^{2n-m}$$

Now $\lim_{\alpha\to 0}\alpha^{2n-m}$ must be equal to 1.

i.e. $2n - m = 0$

$$\frac{m}{n} = 2$$

6. [7] $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$

$$\lim_{x\to 0}\frac{x^3\beta}{\alpha x - \sin x} = 1$$

$$\lim_{x\to 0}\frac{3x^2\beta}{\alpha - \cos x} = 1$$

$\therefore \alpha - 1 = 0 \qquad \Rightarrow \alpha = 1$

$$\lim_{x\to 0}\frac{6x\beta}{\sin x} = 1 \ \Rightarrow 6\beta = 1$$

$\therefore 6(\alpha+\beta) = 7$

7. (*c*,*d*)

$$f(1^+) = \lim_{h\to 0}\frac{1-(1+h)(1+h)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{1-(1+h)^2}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{-h^2-2h}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}(-h-2)\cos\frac{1}{h}$$

$\Rightarrow \lim_{h\to 0} f(1^+)$ does not exist

$$f(1^-) = \lim_{h\to 0}\frac{1-(1-h)(1+h)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{1-(1-h^2)}{h}\cos\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{h^2}{h}\cos\frac{1}{h} \qquad = \lim_{h\to 0} h\cos\frac{1}{h} = 0$$

8. [*b*,*d*]

P-1 :

$\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ = exist and finite

(*b*) $f(x) = x^{2/3}$, $\lim_{h\to 0}\frac{h^{2/3}-(0)}{\sqrt{|h|}} = \lim_{h\to 0}\frac{|h|^{2/3}}{\sqrt{|h|}} = 0$

(*d*) $f(x) = |x|$, $\lim_{h\to 0}\frac{|h|-0}{\sqrt{|h|}} \Rightarrow \lim_{h\to 0}\sqrt{|h|} = 0$

P-2 :

$\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h^2|}}$ = exist and finite

(*a*) $f(x) = x|x|$, $\lim_{h\to 0}\frac{h(h)-0}{h^2} = \begin{cases} \text{RHL} = \lim_{x\to\infty}\frac{h^2}{h^2} = 1 \\ \text{LHL} = \lim_{x\to\infty}\frac{-h^2}{h^2} = -1 \end{cases}$

(*c*) $f(x) = \sin x$ $\lim_{x\to\infty}\frac{\sinh - 0}{h^2}$ = DNE

9. [1.00]

$$\lim_{x\to 0^+}\frac{e^{\left(\frac{\ell n(1-x)}{x}\right)} - \frac{1}{e}}{x^a}$$

$$= \lim_{x\to 0^+}\frac{1}{e}\frac{e^{\left(1+\frac{\ell n(1-x)}{x}\right)}-1}{x^a}$$

$$= \frac{1}{e}\lim_{x-0^+}\frac{1+\frac{\ell n(1-x)}{x}}{x^a}$$

$$= \frac{1}{e}\lim_{x-0^+}\frac{\ell n(1-x)+x}{x^{(a+1)}}$$

$$= \frac{1}{e}\lim_{x-0^+}\frac{\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-....\right)+x}{x^{a+1}}$$

Thus, a=1

10. [8]

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}\cdot 2\sin 2x\cos x}{2\sin 2x\sin\frac{3x}{2}+\left(\cos\frac{5x}{2}-\cos\frac{3x}{2}\right)-\sqrt{2}(1+\cos 2x)}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2 x}{2\sin 2x\left(\sin\frac{3x}{2}-\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2 x}{4\sin x\cos x\left(2\cos x\cdot\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x}{8\sin x\cdot\sin\frac{x}{2}-2\sqrt{2}} = 8$$

4 Continuity and Derivability Methods of Differentiation

Topicwise Questions

1. (*a*) For continuous $\lim_{x\to 2} f(x) = f(2) = k$

$\Rightarrow \quad k = \lim_{x\to 2} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}$

$= \lim_{x\to 2} \dfrac{(x^2 - 4x + 4)(x+5)}{(x-2)^2} = 7$.

2. (*b*) $f(x) = \dfrac{x+1}{(x-3)(x+4)}$.

Hence the points are
$3, -4$.

3. (*b*) $\lim_{x\to 1} f(x) = \lim_{x\to 1} x + 1 = 2 = k.$

4. (*b*) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{2\sin 2x}{2x.5} = \dfrac{2}{5} = k.$

5. (*a*) $\lim_{x\to 2-} f(x) = 3, \lim_{x\to 2+} f(x) = 3$ and $f(2) = 3$

6. (*c*) Here $f\left(\dfrac{3\pi}{4}\right) = 1$ and $\lim_{x\to 3\pi/4-} f(x) = 1$

$\lim_{x\to 3\pi/4+} f(x) = \lim_{h\to 0} 2\sin\dfrac{2}{9}\left(\dfrac{3\pi}{4} + h\right) = 2\sin\dfrac{\pi}{6} = 1$.

Hence $f(x)$ is continuous at $x = \dfrac{3\pi}{4}$.

7. (*b*) $\lim_{x\to 1-} f(x) = 1, \lim_{x\to 1+} f(x) = 6.$

8. (*d*) By definition of continuity, we know that

$\lim_{x\to 3+} f(x) = f(3) = \lim_{x\to 3-} f(x)$

$\Rightarrow \lim_{x\to 3-} f(x) = 4$ or $\lim_{h\to 0} 3 - h + \lambda = 4$

$\Rightarrow 3 + \lambda = 4 \Rightarrow \lambda = 1$

9. (*c*) f(x) is continuous at $x = \dfrac{\pi}{2}$, then

$\lim_{x\to \pi/2} f(x) = f(0)$

or $\lambda = \lim_{x\to \pi/2} \dfrac{1 - \sin x}{\pi - 2x}$, $\left(\dfrac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule,

$\lambda = \lim_{x\to \pi/2} \dfrac{-\cos x}{-2}$ Þ $\lambda = \lim_{x\to \pi/2} \dfrac{\cos x}{2} = 0.$

10. (*d*) If f(x) is continuous at x= 0, then

$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{2 - \sqrt{x+4}}{\sin 2x}$, $\left(\dfrac{0}{0}\text{ form}\right)$

Using L–Hospital's rule,

$f(0) = \lim_{x\to 0} \dfrac{\left(-\dfrac{1}{2\sqrt{x+4}}\right)}{2\cos 2x} = -\dfrac{1}{8}.$

11. (*c*) L.H.L. $= \lim_{x\to 0^-} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x} = k$

R.H.L. $= \lim_{x\to 0^+} (2x^2 + 3x - 2) = -2$

Since it is continuous, L.H.L = R.H.L $\Rightarrow k = -2$.

12. (*c*) $|x|$ is continuous at $x = 0$ and $\dfrac{|x|}{x}$ is also discontinuous at

$\therefore$ is discontinuous at x=0 .

$f(x) = |x| + \dfrac{|x|}{x}$ is discontinuous at x = 0.

13. (*a*) $f(5) = \lim_{x\to 5} f(x) = \lim_{x\to 5} \dfrac{x^2 - 10x + 25}{x^2 - 7x + 10}$

$= \lim_{x\to 5} \dfrac{(x-5)^2}{(x-2)(x-5)} = \dfrac{5-5}{5-2} = 0$.

14. (*d*) As $Lf'(2) \neq Rf'(2)$.

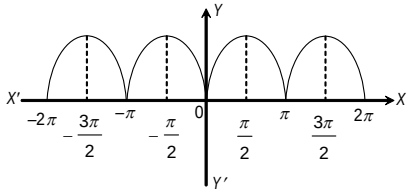
15. (*d*) Here, when $-1 \le x \le 1$, $0 \le x \sin \pi x < 1$

$\Rightarrow f(x) = [x \sin \pi x] = 0$ for $-1 \le x \le 1$,

i.e., $f(x)$ is constant function (equal to zero) in $[-1, 1]$.

$\Rightarrow f(x)$ is differentiable in $(-1, 1)$.

16. (*d*) It can be easily seen from the graph of $f(x) = |\sin x|$ that it is every where continuous but not differentiable at integer multiples of π and at $x = 0$.



17. (*d*) $\lim_{x\to 0} f(x) = x^2 \sin\left(\frac{1}{x}\right)$,

but $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and $x \to 0$

$\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$

Therefore f(x) is continuous at x = 0. Also, the function

$f(x) = x^2 \sin\frac{1}{x}$ is differentiable because

$$Rf'(x) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0,$$

$$Lf'(x) = \lim_{h\to 0} \frac{h^2 \sin(1/-h)}{-h} = 0.$$

18. (*b*) By definition, $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{2(1+h)-5} - \left(\frac{-1}{3}\right)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{2h-3} + \frac{1}{3}\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{3+2h-3}{3h(2h-3)}\right) = \lim_{h\to 0}\left(\frac{2h}{3h(2h-3)}\right)$$

$$= \lim_{h\to 0} \frac{2}{3(2h-3)} = \frac{2}{3(-3)} = \frac{-2}{9}.$$

19. (*c*) It is fundamental concept.

20. (*c*) $\frac{|f(x) - f(y)|}{x - y} \le (x - y)$

$$\lim_{x\to y} \frac{|f(x) - f(y)|}{x - y} \le \lim_{x\to y} (x - y)$$

$f'(x) = 0$

$\Rightarrow f(x) = C$

$\Rightarrow f(0) = 0$

$\Rightarrow C = 0$ $\qquad f(1) = 0$

21. (*c*) $y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

Therefore,

$$x^2 . \frac{dy}{dx} - xy + 2 = x^2\left(1 - \frac{1}{x^2}\right) - x\left(1 + \frac{1}{x}\right) + 2 = 0$$

22. (*a*) $\log|x| = \log x$, if $x > 0 = \log(-x)$, if $x < 0$

Hence $\frac{d}{dx}\{\log|x|\} = \frac{1}{x}$, if $x > 0$

$= \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$, if $x < 0$

Thus $\frac{d}{dx}\{\log|x|\} = \frac{1}{x}$, if $x \neq 0$.

23. (*b*) $f(x) = \sqrt{1 + \cos^2(x^2)}$

$$f'(x) = \frac{1}{2\sqrt{1+\cos^2(x)^2}} . (2\cos x^2).(-\sin x^2).(2x)$$

$$f'(x) = \frac{-x \sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$$

At $x = \frac{\sqrt{\pi}}{2}, f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{-\frac{\sqrt{\pi}}{2}.\sin\frac{2\pi}{4}}{\sqrt{1+\cos^2\frac{\pi}{4}}} = \frac{-\frac{\sqrt{\pi}}{2}.1}{\sqrt{\frac{3}{2}}}$

$\therefore \quad f'\left(\frac{\sqrt{\pi}}{2}\right) = -\sqrt{\frac{\pi}{6}}$.

24. (*a*) $\frac{d}{dx}(e^x \log\sin 2x) = e^x \log\sin 2x + 2e^x \frac{1}{\sin 2x}\cos 2x$

$= e^x \log\sin 2x + e^x 2\cot 2x = e^x(\log\sin 2x + 2\cot 2x)$.

25. (*b*) $\frac{d}{dx}[\sqrt{\sec^2 x + \operatorname{cosec}^2 x}] = \frac{d}{dx}\left[\sqrt{\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)}\right]$

$= \frac{d}{dx}[2\operatorname{cosec} 2x] = -4\operatorname{cosec} 2x \cot 2x$.

26. (*c*) $y = \log x . e^{(\tan x + x^2)}$

$$\therefore \frac{dy}{dx} = e^{(\tan x + x^2)} . \frac{1}{x} + \log x . e^{(\tan x + x^2)}(\sec^2 x + 2x)$$

$$= e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$$

27. (*a*) $y = a\sin^4\theta \Rightarrow \frac{dy}{d\theta} = 4a\sin^3\theta\cos\theta$

and $x = a\cos^4\theta \Rightarrow \frac{dx}{d\theta} = -4a\cos^3\theta\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin^2\theta}{\cos^2\theta} = -\tan^2\theta$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{3\pi}{4}} = -\tan^2\left(\frac{3\pi}{4}\right) = -1$$

28. (*a*) $y = (\sin x)^{(\sin x)^{(\sin x).....\infty}}$

$\Rightarrow \quad y = (\sin x)^y \Rightarrow \log_e y = y \log \sin x$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}[\log \sin x + y \cot x]$

$\therefore \frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \sin x}$

29. (*c*) Let $y = \cos^{-1}\sqrt{x}$ and $z = \sqrt{1-x}$

$$\therefore \frac{dy}{dz} = \frac{\frac{-1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}}{-\frac{1}{2\sqrt{1-x}}} = \frac{1}{\sqrt{x}}.$$

30. (*c*) The differential coefficient of $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ with respect to $\tan^{-1}x = \frac{\frac{d}{dx}\left(\frac{\tan^{-1}x}{1+\tan^{-1}x}\right)}{\frac{d}{dx}(\tan^{-1}x)}$

$$= \frac{(1+\tan^{-1}x)\frac{d}{dx}\tan^{-1}x - \tan^{-1}x\frac{d}{dx}(1+\tan^{-1}x)}{(1+\tan^{-1}x)^2.\frac{d}{dx}\tan^{-1}x}$$

$$= \frac{1}{(1+\tan^{-1}x)^2}$$

31. (*c*) $y = e^{x^3}$, $z = \log x \Rightarrow \frac{dy}{dx} = e^{x^3}.(3x^2) = 3x^2e^{x^3}$ .....(i)

and $\frac{dz}{dx} = \frac{1}{x}$ ....(ii)

$\Rightarrow \quad \frac{dy}{dz} = \frac{3x^2e^{x^3}}{(1/x)} = 3x^3e^{x^3}$

32. (*c*) Putting $x = \sin A$ and $\sqrt{x} = \sin B$

$y = \sin^{-1}(\sin A\sqrt{1-\sin^2 B} + \sin B\sqrt{1-\sin^2 A})$

$= \sin^{-1}[\sin(A+B)] = A + B = \sin^{-1}x + \sin^{-1}\sqrt{x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$

33. (*c*) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$

$$= \tan^{-1}\frac{5x-x}{1+5x.x} + \tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}.x}$$

$$= \tan^{-1}5x - \tan^{-1}x + \tan^{-1}\frac{2}{3} - \tan^{-1}x$$

$\therefore \quad \frac{dy}{dx} = \frac{5}{1+25x^2}$.

34. (*a*) $f(x) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - \sqrt{\frac{1+x}{2}}\right)\right] + x^x$

$$f(x) = \frac{\pi}{2} - \sqrt{\frac{1+x}{2}} + x^x$$

$$\therefore f'(x) = -\frac{1}{\sqrt{2}}.\frac{1}{2\sqrt{1+x}} + x^x(1+\log x)$$

$$f'(1) = -\frac{1}{4} + 1 = \frac{3}{4}$$

35. (*b*) $y = \cos^{-1}\cos(x-1), \quad x > 0$

$\Rightarrow y = x-1, \quad x > 0$ and $0 < x-1 < p$

we have, $1 < \frac{5\pi}{4} < p+1$

$\therefore \quad y = x-1, \ 1 \le x \le p+1$ and $\frac{5\pi}{4} \in [1, p+1]$

$$\left.\frac{dy}{dx}\right|_{x=\frac{5\pi}{4}} = 1\Big|_{x=\frac{5\pi}{4}} = 1$$

36. (*d*) $y = \cos^{-1}\sqrt{1-t^2} = \sin^{-1}t$

and $x = \sin^{-1}(3t - 4t^3) = 3\sin^{-1}t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{1}{\sqrt{1-t^2}}\right)}{3\left(\frac{1}{\sqrt{1-t^2}}\right)} \Rightarrow \frac{dy}{dx} = \frac{1}{3}.$$

37. (*a*) $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$; Put $x^x = \tan\theta$

$\therefore \quad y = f(x) = \cot^{-1}\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right)$

$= \cot^{-1}(-\cot 2\theta) = \pi - \cot^{-1}(\cot 2\theta)$

$\Rightarrow \quad y = \pi - 2\theta = \pi - 2\tan^{-1}(x^x)$

$\frac{dy}{dx} = \frac{-2}{1+x^{2x}}.x^x(1+\log x) \Rightarrow f'(1) = -1$

38. (*c*) Since $g(x)$ is the inverse of $f(x)$, therefore

$f(x) = y \Leftrightarrow g(y) = x$

Now, $g'(f(x)) = \frac{1}{f'(x)}, \forall x \Rightarrow g'(f(x)) = 1 + x^3, \ \forall x$

$\Rightarrow \ g'(y) = 1 + (g(y))^3$

[using $f(x) = y \Leftrightarrow x = g(y)$]

$\Rightarrow \ g'(x) = 1 + (g(x))^3$ (replacing *y* by *x*).

39. (*c*) $\frac{dx}{d\theta} = a\cos\theta$ and $\frac{dy}{d\theta} = -b\sin\theta$

$\Rightarrow \ \frac{dy}{dx} = \frac{-b}{a}\tan\theta$ and $\frac{d^2y}{dx^2} = \frac{-b}{a}\sec^2\theta\frac{d\theta}{dx}$

$\Rightarrow \ \frac{d^2y}{dx^2} = \frac{-b}{a}\sec^2\theta\frac{1}{a\cos\theta} = \frac{-b}{a^2}\sec^3\theta$.

40. (*a*) $y = e^{\tan^{-1}x} \Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1+x^2).\frac{e^{\tan^{-1}x}}{(1+x^2)} - e^{\tan^{-1}x}(2x)}{(1+x^2)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1-2x)e^{\tan^{-1}x}}{(1+x^2)^2}$

$\Rightarrow \frac{d^2y}{dx^2}(1+x^2) = (1-2x)\frac{dy}{dx}$.

Learning Plus

1. (*a*) $\because$ $g(x)$ is discontinuous at $x = \pm 1$

and $fog(x) = \frac{1}{2-x^2}$

hence fog discontinuous at $x = \pm 1, \pm\sqrt{2}$

2. (*b*) $f(x) = \begin{cases} \frac{\sqrt{1+px} - \sqrt{1-px}}{x}, & -1 \le x \le 0 \\ \frac{2x+1}{x+2}, & 0 \le x \le 1 \end{cases}$

since it is cont, so,

$\lim\limits_{x\to 0^-} f(x) = f(0)$

$\lim\limits_{h\to 0} \frac{\sqrt{1+p(-h)} - \sqrt{1-p(-h)}}{-h} = -\frac{1}{2}$

$\lim\limits_{h\to 0} \frac{(1-ph)-(1+ph)}{-h\left\{\sqrt{1-ph}+\sqrt{1+ph}\right\}} = -\frac{1}{2}$

$\frac{+2p}{2} = -\frac{1}{2}$

$p = -1/2$

3. (*c*) Given $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$\Rightarrow \lim\limits_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{e^{\infty}-1}{e^{\infty}+1} = -1$

$\Rightarrow \lim\limits_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{1-e^{-\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \frac{1-e^{-\infty}}{1+e^{\infty}} = 1$

So, $\lim\limits_{x\to 0} f(x)$ exists at $x = 0$, but at it is not continuous.

4. (*d*) $f(x) = |x-1| + |x-2| + \cos x$

All three functions are cont. in [0, 4] so sum of all these functions is also a cont. funs.

5. (*d*) $f(x) = \frac{|x-3|}{|x-2|} + \frac{1}{1+[x]}$

$x \ne 2$ $\quad 1 + [x] = 0$

$[x] \ne -1$, $\quad x \in [1, 0)$

And [x] will be disoint. at every integer

So $x \in R - \{(-1, 0) \cup n, n \in I\}$

6. (*b*) $f(x)$ should be a constant function.

7. (*c*) $f\left(\frac{\pi^+}{4}\right) = \pi\left(\frac{\pi^+}{4}\right) + 1\pi\times 0 + 1 = 1$

$f\left(\frac{\pi}{4^-}\right) = f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

So jump $= 1 - \frac{\pi}{4}$

8. (*d*) RHL $= \lim\limits_{h\to 0} \sin[\ell nh] = [-1, 1]$

LHL $= \lim\limits_{h\to 0} \sin[\ell n\, h] = [-1, 1]$

So DNE

9. (*b*) $\lim\limits_{x\to\infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$

if $0 < x < 1$

then $x^n \to 0$ when $n \to \infty$

$f(x) = -1$

if $x > 1$

$$\lim_{x\to\infty} \frac{x^n - \sin x^n}{x^n + \sin x^n} = \frac{x^n + \sin x^n - 2\sin x^n}{x^n + \sin x^n}$$

$$= 1 - \frac{2\sin x^n}{x^n + \sin x^n}$$

Limit if $x > 1$

$$\lim_{n\to\infty} = 1$$

so has a finite discontinuity at $x = 1$

10. (*c*) $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases}$; $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h) = 0$

and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} (0+h)^2 = 0$

$\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

Hence f(x) is continuous function at $x = 0$.

$$Lf'(x) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-0}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$Rf'(x) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 - 0}{h} = 0$$

$\Rightarrow Lf'(x) = Rf'(x)$

Hence f(x) is differentiable at $x = 0$

Now $f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \ge 0 \end{cases}$;

$\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{h\to 0} f'(0-h) = 0$

and $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{h\to 0} f'(0+h) = \lim\limits_{h\to 0} 2(0+h) = 0$

$\Rightarrow \lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^+} f'(x) = 0$

Hence f(x) is continuous function at $x = 0$.

Now

$$Lf''(x) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$Rf''(x) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{2(0+h)-0}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

$\Rightarrow Lf''(x) \ne Rf''(x)$

Hence $f'(x)$ is not differentiable at $x = 0$.

11. (*b*) $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$

$$f(0^+) = \lim_{h\to 0} \frac{h\left(\sqrt{h} - \sqrt{h+1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h-h-1}{\sqrt{h}+\sqrt{h+1}} = -1$$

12. (*b*) $f(x) = \begin{cases} x\dfrac{(3e^{1/x}+4)}{2-e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$f'(0^+) = \lim_{h\to 0} \frac{h\dfrac{(3e^{1/h}+4)}{2-e^{1/h}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{3\left(1+\dfrac{1}{h}\right)+4}{2-1-\dfrac{1}{h}}$$

$$= \lim_{h\to 0} \frac{7h+3}{h-1} = 3$$

$$f'(0^-) = \lim_{h\to 0} \frac{-h\dfrac{(3e^{-1/h}+4)}{2-e^{-1/h}} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{3\left(1-\dfrac{1}{h}\right)+4}{2-\left(1-\dfrac{1}{h}\right)}$$

$$= \frac{7h-3}{h+1} = -3$$

so, not diff. at $x = 0$

13. (*b*) $f(x) = \begin{cases} \sqrt{x}\left(1+\sin\dfrac{1}{x}\right), & x > 0 \\ -\sqrt{x}\left(1+\sin\dfrac{1}{x}\right), & x < 0 \\ 0, & x = 0 \end{cases}$

$f'(0^+) = \dfrac{f(0+h)-f(0)}{h} \Rightarrow \lim\limits_{h\to 0}$ d

$= \lim_{h\to 0} \dfrac{\left(1+\sin\frac{1}{h}\right)}{\sqrt{h}}$ = not defined

$\Rightarrow$ not differentiable

$f(0^+) = \lim_{h\to 0} \sqrt{h}\left(1+\sin\frac{1}{h}\right)$

$= \lim_{h\to 0} \frac{h}{\sqrt{h}}\left(1+\sin\frac{1}{h}\right)$

$= \lim_{h\to 0} \left(\frac{1}{\sqrt{h}}\right)\frac{\left(1+\sin\frac{1}{h}\right)}{(1/h)}$

$= \lim_{h\to 0} \frac{1}{\sqrt{h}} \left\{h + h\sin\frac{1}{h}\right\}$

$= 0$

$f(0^-) = \lim_{h\to 0} -\sqrt{h}\left(1-\sin\frac{1}{h}\right)$

$= 0$

14. (*b*) Function is not differentiable at 2 points between

$x = -\frac{\pi}{2}$ & $x = \frac{\pi}{2}$ not continuity at

$x = -\frac{\pi}{2}, \frac{\pi}{2}$ also $\therefore$ 4 points

15. (*b*) $f'(2^-) = f'(2^+) = 2$

& $f'(3^+) = f'(3^-) = \frac{21}{4}$

16. (*d*) $f'(O^+) = p + q$ ....(1)
$f'(O^-) = -p + q$ ....(2)
$f'(O^+) = f'(O^-) \Rightarrow p + q = 0, r \in R$

17. (*b*) If f is differentiable everywhere.
then |f| will also be diff. everywhere.
and if two fns. are diff. then sum of then will also be diff. everywhere

18. (*b*) $f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = \ldots\ldots = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0$

Since there are infinitely many points in $x \in (0,1)$

where $f(x) = 0$ and $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \Rightarrow f(0) = 0$

And since there are infinitely many points in the neighbourhood of x = 0 such that
$\Rightarrow$ f(x) remains constant in the neighbourhood of x = 0
$\Rightarrow f'(0) = 0$

19. (*c*) Put $y = 0 \Rightarrow f\left(\frac{x}{3}\right) = \frac{f(x)}{3} \Rightarrow f(3x) = 3f(x)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{f(3x) + f(3h)}{3} - f(x)}{h} = \lim_{h\to 0}$

$\frac{\frac{8f(x) + 3f(x)}{3} - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0)$

$f'(x) = 3 \Rightarrow f(x) = 3x + c \Rightarrow f(x) = 3x$

20. (*d*) $f(x + y) = f(x).f(y)$
differentiate w.r.t. x
$f'(x + y) = f'(x) \cdot f(y)$
put x = 0, y = 5
$f'(5) = f'(0) \cdot f(5)$
$= 3.2$
$\therefore f'(5) = 6$

21. (*a*) $f'(x) = \sqrt{2x^2 - 1}$, $y = f(x^2)$

$f'(x^2) = \sqrt{2x^4 - 1}$, $\frac{dy}{dx} = 2x.f'(x^2)$

$\frac{dy}{dx} = 2x.\sqrt{2x^4 - 1}$

$\left(\frac{dy}{dx}\right)_{x=1} = 2$

22. (*c*) $y = x^3 - 8x + 7$ and $x = f(t)$

$\frac{dy}{dt} = 2$ & $x = 3$ at $t = 0$

$\because \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \frac{dx}{dt} = \frac{dy/dt}{dy/dx}$

$\Rightarrow \frac{dx}{dt} = \frac{2}{3x^2 - 8}$

$\because$ at t = 0, x = 3

$\therefore \frac{dx}{dt}$ (at t = 0) $= \frac{2}{19}$

23. (*c*) $y = f(x)$
$f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x)$
$f'(3) = f'(-3) = -2$

24. (*c*) $\frac{d}{dx}\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right) = ax + b$

$\Rightarrow \frac{d}{dx}\left(\frac{(x^2 + x + 1)\cdot(x^2 - x + 1)}{(x^2 + x + 1)}\right) = ax + b$

$\frac{d}{dx}(x^2 - x + 1) = ax + b$

$\Rightarrow 2x - 1 = ax + b$

25. (*c*) $\tan y = \frac{2t}{1-t^2}$ .....(i)

and $\sin x = \frac{2t}{1+t^2}$ .....(ii)

From (i), differentiating *w.r.t. t* of *y*, we get,

$$\sec^2 y \frac{dy}{dt} = \frac{2\left(1+t^2\right)}{\left(1-t^2\right)^2}$$

and $\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{(1+\tan^2 y)}$

or $\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{\left[1+\left(\frac{2t}{1-t^2}\right)^2\right]} = \frac{2}{1+t^2}$ .....(iii)

and from (ii), differentiating *w.r.t. t* of *x*, we get

$$\cos x \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

or $\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \frac{1}{\sqrt{1-\frac{(2t)^2}{(1+t^2)^2}}} = \frac{2}{1+t^2}$ .....(iv)

Hence $\frac{dy}{dx} = 1$.

Differentiating with respect to *t*, we get

$$\frac{dx}{dt} = \frac{(1+t^2)(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

and $\frac{dy}{dt} = \frac{(1+t^2)2a-2at(2t)}{(1+t^2)^2} = \frac{2a(1-t^2)}{(1+t^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(1-t^2)}{-2t}$; $\therefore \frac{dy}{dx} = \frac{a(t^2-1)}{2t}$.

26. (*a*) Given $y = (\sin x)^{\tan x}$; $\log y = \tan x . \log \sin x$

Differentiate with respect to *x*,

$$\frac{1}{y}.\frac{dy}{dx} = \tan x.\cot x + \log \sin x.\sec^2 x$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \log \sin x.\sec^2 x].$$

27. (*d*) $f(x) = |x|^{|\sin x|}$

at $x = \pi/4$, $|x| = x$ and $|\sin x| = \sin x$

$\therefore$ $f(x) = x^{\sin x}$

$\Rightarrow$ $\ell n(f(x)) = \sin x \, . \, \ell nx$

$\Rightarrow \frac{1}{f(x)} f'(x) = \cos x \, \ell nx + \frac{\sin x}{x}$ $\Rightarrow f'(\pi/4) = \left(\frac{\pi}{4}\right)^{1/\sqrt{2}} \left(\frac{\sqrt{2}}{2}\ell n\frac{\pi}{4} + \frac{2\sqrt{2}}{\pi}\right)$

28. (*a*) $u = \sec^{-1} \frac{1}{(2x^2-1)}$; $v = \sqrt{1-x^2}$

$u = \cos^{-1}(2x^2 - 1)$;

differentiating w.r.t. to x

$$\frac{du}{dx} = \frac{-1\times(4x)}{\sqrt{1-(2x^2-)^2}} \quad \& \quad \frac{du}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{-4x}{\sqrt{-4x^4+4x^2}} \times \frac{\sqrt{1-x^2}}{-x} = \frac{4}{2x}$$

$$\left|\frac{du}{dv}\right|_{x=1/2} = \frac{4}{2(1/2)} = 4$$

29. (*d*) $y = \sqrt{\sin x + y}$

squaring both side

$y^2 = \sin x + y$

$2yy' = \cos x + y'$

differentiating w.r.t. to x

$$y' = \frac{\cos x}{2y-1}$$

30. (*c*) Put $x = \cos\theta$

$$\therefore y = \tan^{-1}\frac{\cos\theta}{1+\sin\theta} + \sin\left[2\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right]$$

$$= \tan^{-1}\frac{\sin\phi}{1+\cos\phi} + \sin\left[2\tan^{-1}\tan\left(\frac{\theta}{2}\right)\right],$$

{where $\phi = 90° - \theta$ }

$$= \tan^{-1}\tan\left(\frac{\phi}{2}\right) + \sin\left(2.\frac{\theta}{2}\right) = \left(\frac{\phi}{2}\right) + \sin\theta$$

$$= \frac{\pi}{4} - \frac{\theta}{2} + \sqrt{1-\cos^2\theta} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x + \sqrt{1-x^2}$$

31. (*c*) Putting $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= 2\theta + 2\theta = 4\tan^{-1}x$$

$$\Rightarrow y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(4\tan^{-1}x\right) = \frac{4}{1+x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1-2x}{2\sqrt{1-x^2}}$$

32. (*d*) $y = \sin^{-1}(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2})$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x(1-x)}} + p$$

$$y = \sin^{-1}(x) + \sin^{-1}(\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

33. (*b*) $g(x) = f^{-1}(x)$ $\qquad g'(f(x)) = \frac{1}{f'(x)}$

$$g'(f(x)) \cdot f'(x) = 1 = \frac{1+x^4}{x^5}$$

$$g'(f(g(2))) = \frac{1+a^4}{a^5}$$

34. (*b*)

$$F'(x) = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ g' & g' & h' \\ f''' & g''' & h''' \end{vmatrix} =$$

0

35. (*c*) $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 3x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

differentiating w.r.t. to x

$$f¢(x) = \begin{vmatrix} \sin x & \cos x & -\sin x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix} +$$

$$\begin{vmatrix} \cos x & \sin x & \cos x \\ -2\sin 2x & 2\cos 2x & -4\sin 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix} +$$

$$\begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & 3\cos 3x & -9\sin 3x \end{vmatrix}$$

$$f'\left(\frac{\pi}{2}\right) = \begin{vmatrix} -1 & 0 & -1 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{vmatrix} +$$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 3 & 0 & 9 \end{vmatrix}$$

$= -1(-2) + 0 - 1(1) + 0 - 1(-3) + 0$
$= 2 - 1 + 3 = 4$

36. (*b*) From the given relation $\frac{y}{x} = \log x - \log(a+bx)$

Differentiating we get

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{x} - \frac{1}{a+bx}b = \frac{a}{x(a+bx)}$$

$$\therefore x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad \text{.....(i)}$$

Differentiating again *w.r.t. x*, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax.b}{(a+bx)^2}$$

$$\Rightarrow x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(x\frac{dy}{dx} - y\right)^2, \text{ [by (i)].}$$

37. (*d*) $y = f(e^x)$

$y' = f'(e^x) \cdot e^x \Rightarrow y'' = f''(e^x)\, e^{2x} + e^x f'(e^x)$

38. (*a*) Given that $y^2 = ax^2 + bx + c$

$$2y.\frac{dy}{dx} = 2ax + b \qquad \text{...(i)}$$

$$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 2a$$

$$\Rightarrow y.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = a \qquad \text{...(ii)}$$

(i) and (ii) $\frac{yd^2y}{dx^2} = a - \left(\frac{2ax+b}{2y}\right)^2$

put $y^2 = ax + bx + c$

$$\Rightarrow y^3\frac{d^2y}{dx^2} = \frac{1}{4}\left(4a\left(ax^2+bx+c\right) - \left(4a^2x^2 + b^2 + 4abx\right)\right)$$

$$= \frac{1}{4}\left(4a^2x^2 + 4abx + 4ac - 4a^2x^2 - b^2 - 4abx\right)$$

$$= \frac{1}{4}\left(4ac - b^2\right) = \text{constant.}$$

39. (*d*) $y = e^{2x}$

$$\frac{dy}{dx} = 2e^{2x} \quad \left|\ \frac{dx}{dy} = \frac{1}{2}e^{-2x},\right.$$

$$\frac{d^2y}{dx^2} = 4e^{2x} \quad \left|\ \frac{d}{dy}\left(\frac{dx}{dy}\right) = -e^{-2x} \times \frac{1}{2e^{2x}}\right.$$

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = -\frac{4e^{2x} \times e^{-4x}}{2} = -2e^{-2x}$$

40. (*d*)

Advanced Level Multiconcept Questions

1. (*a*,*b*,*c*)

$f(x) = [x]$ and $g(x) = \begin{cases} 0 & , \quad x \in I \\ x^2 & , \quad x \in R - I \end{cases}$

$\lim_{x\to 1} g(x) = \lim_{x\to 1} x^2 = 1$, but $g(1) = 0$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} [x]$

does not exist since LHL = 0 and RHL = 1

$gof(x) = g([x]) = 0$

$\Rightarrow$ gof(x) is continous for all values of x

$fog = \begin{cases} 0 & , \quad x \in I \\ [x^2] & , \quad x \in R - I \end{cases}$

$fog(1) = 0$, $\lim_{x\to 1^-} fog(x) = 0$,

$\lim_{x\to 1^+} fog(x) = 1$

fog is not continous at x = 1

2. (*b*,*c*,*d*)

(*a*) f(x) is continuous no where

(*b*) g(x) is continuous at x = 1/2

(*c*) h(x) is continuous at x = 0

(*d*) k(x) is continuous at x = 0

3. (*b*,*d*)

$y = f(x) = \begin{cases} (\sin^{-1} x)^2 \cos 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

f(x) can be discontinuous only at x = 0 in [–1,1]

So we check only at x = 0

LHD (x = 0) =

$$\lim_{h\to 0} \frac{(\sin^{-1}-h)^2 \cos\left(-\frac{1}{n}\right) - 0}{-h}$$

$$\lim_{h\to 0} -\left(\frac{\sin^{-1}-h}{n}\right)^2 . h\cos\left(\frac{1}{n}\right) = -1.\ 0.$$

[finite quantity between [–1,1]] = 0

RHD (x = 0) is $\lim_{h\to 0} h\frac{\left(\sin^{-1}-h\right)^2}{n^2}.\cos\left(\frac{1}{n}\right) = 0$

Hence f(x) is differentiable as well as continuous in [–1,1]

4. (*b*,*d*)

$f(x) = (\sin^{-1} x)^2 . \cos\left(\frac{1}{x}\right)$ if $x \neq 0$

$= 0$ if $x = 0$

LHL = RHL = $\lim_{x\to 0} (\sin^{-1} x)^2 \cos\left(\frac{1}{x}\right)$

= 0 × [a finite quantity b/w (–1, 1)]

= 0

$f'(0^+) = \lim_{h\to 0} \frac{(\sin^{-1}h)^2 \cos\left(\frac{1}{h}\right)}{h}$

$= \lim_{h\to 0} \left(\frac{\sin^{-1}h}{h}\right) (\sin^{-1} h) \cos(1/h)$

= 1 × (0) × (a finite quantity)

= 0

$f'(0^-) = 0$

5. (*a*,*b*,*d*)

$f(x) = x^2 \sin\left(\frac{1}{x}\right)$, $x \neq 0$

$= 0$, $x = 0$

cont. at x = 0

$f(0^+) = f(0^-) = f(0) = 0$

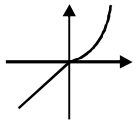
$f'(0^+) = \lim_{h\to 0} \frac{h^2 \sin 1/h}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0$

$f'(0^-) = 0$

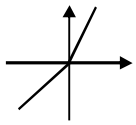
Diff. at x = 0

6. (*a*,*b*,*d*)

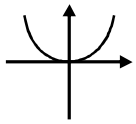
(*a*)



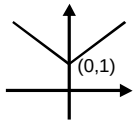
(*b*)



(*c*) $h(x) = x^2 \quad x \geq 0 = -x^2 \quad x < 0$



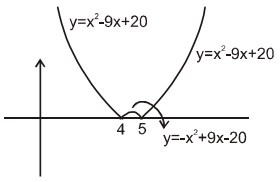
(*d*)



7. (*b*,*c*,*d*)

$f(x) = |(x-4)(x-5)|$

$f'(x) = 2x - 9$, $x > 5$



$f'(x) = -2x + 9$, $4 < x < 5$

Not defined at x = 4, 5 by graph

8. (*a,d*)

$x = \cos t,\ y = \ell n\ t$

$$\frac{dy}{dx} = \frac{1}{t}.\frac{1}{-\sin t}$$

at $t = \frac{\pi}{2}$ | at $t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{-2}{\pi} \qquad \frac{dy}{dx} = -\frac{12}{\pi}$$

9. (*a,b,c,d*)

$u = e^x \sin x\ ,\ v = e^x \cos x$

$$\frac{du}{dx} = e^x \cos x + e^x \sin x$$

$$\frac{dv}{dx} = e^x \cos x - e^x \sin x$$

Adding : $\frac{du}{dx} + \frac{dv}{dx} = 2V = [D]$

$$\frac{d^2u}{dx^2} = e^x \cos x - e^x \sin x + e^x \cos x + e^x \sin x$$

$= 2V \Rightarrow [B]$

$$v\,.\,\frac{du}{dx} - u\,.\,\frac{du}{dx} = e^{2x}\cos^2 x + e^{2x}\sin x \cos x$$

$- e^{2x}\sin x \cos x + e^{2x}\sin^2 x$

$= u^2 + v^2 \rightarrow [A]$

10. (*a,b,d*)

$x^p y^q = (x + y)^{p+q}$

taking log both side

$P\ell nx + q\ell ny = (p + q)\ \ell n\ (x + y)$

$$\frac{P}{x} + \frac{q}{y}\ .\ y' = \frac{p+q}{x+y}\ (1 + y')$$

$$\frac{P}{x} + \left(\frac{q}{y} - \frac{p+q}{x+y}\right) y' = \frac{p+q}{x+y}$$

$$\frac{P}{x} + \left\{\frac{qx + qy - py - qy}{(x+y)y}\right\} y' = \frac{p+q}{x+y}$$

$$\left\{\frac{(qx - py)}{(x+y)y}\right\} y' = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\left\{\frac{qx - py}{(x+y)y}\right\}\ y' = \frac{px + qx - px - py}{x(x+y)}$$

$$y' = \frac{(qx - py)}{(qx - py)} \times \frac{(x+y)y}{(x+y)x}$$

$$y' = \frac{y}{x}$$

11. (*a,b*)

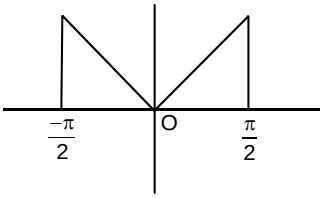
$\sin^{-1}\frac{t}{\sqrt{1-t^2}}$ w.r.t. $\cos^{-1}\frac{t}{\sqrt{1+t^2}}$

Put $t = \tan q - \frac{\pi}{2} < q < \frac{\pi}{2}$

$$u = \sin^{-1}\left(\frac{\tan\theta}{|\sec\theta|}\right)$$

$\Rightarrow u = \sin^{-1}(\sin q)$

$u = q \Rightarrow \tan^{-1} t$



$$\frac{du}{dt} = \frac{1}{(1+t^2)}$$

$$V = \cos^{-1}\frac{1}{|\sec\theta|}$$

$\Rightarrow \cos^{-1}(\cos\theta)$

$\theta \quad \theta > 0$ $\qquad -\theta,\ \theta < 0$

$\tan^{-1} > 0$ $\qquad t < 0$

$t > 0$

$V = \tan^{-1} t$

$$\left.\begin{aligned}\frac{dv}{dt} &= \frac{1}{1+t^2}\ ,\ t > 0\\ \frac{dv}{dt} &= \frac{1}{1+t^2}\ ,\ t < 0\end{aligned}\right\} \begin{matrix}\text{for } t > 0,\\ 1\\ \text{\& for } t < 0\\ -1\end{matrix}$$

12. (*a,b,c,d*)

$2^x + 2^y = 2^{x+y}$ ... (i)

diff. both sides w.r.t.x

$$2^x \cdot \ell n2 + 2^y\ .\ \ell n2\ \frac{dy}{dx} = 2^{x+y}\ .\ \ell n2\left(1 + \frac{dy}{dx}\right)$$

$$2^x - 2^{x+y} = (2^{x+y} - 2^y)\ \frac{dy}{dx} \quad \text{... (ii)}$$

$$\frac{2^x(1-2^y)}{2^y(2^x - 1)} = \frac{dy}{dx}$$

from (i) & (ii)

$$2^x - 2^x - 2^y = (2^x + 2^y - 2^y)\ \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2^y}{2^x}$$

$$\frac{dy}{dx} = \frac{2^x - 2^{x+y}}{2^{x+y} - 2^y} = \frac{-2^y}{2^y(2^x - 1)} = \frac{1}{1-2^x} = 1 - 2^y$$

13. (*b,c*)

$\because\ t = \frac{1}{2}\ \ell n(x^2 + y^2)$

$$\Rightarrow \frac{1}{2}\ell n(x^2+y^2) = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

Case-I : When $x \geq 0$

$$\Rightarrow \ell n(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{x^2+y^2}(2x+2yy') = \frac{2}{1+\frac{y^2}{x^2}}\left(\frac{xy'-y}{x^2}\right)$$

$$\Rightarrow xy' - yy' = x + y$$

$$\Rightarrow y' = \frac{x+y}{x-y}$$

Let $y = x\tan\theta\ ; \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \sin^{-1}\left(\frac{x\tan\theta}{|x\sec\theta|}\right)$$

$$= \sin^{-1}\left(\frac{x}{|x|}\sin\theta\right) = \begin{cases} \theta = \tan^{-1}\left(\frac{y}{x}\right), & x \geq 0 \\ -\theta = -\tan^{-1}\left(\frac{y}{x}\right), & x < 0 \end{cases}$$

Case-II : When $x < 0$

$$\ell n(x^2+y^2) = -2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{(2x+2yy')}{x^2+y^2} = \frac{-2}{1+\frac{y^2}{x^2}}\left(\frac{xy'-y}{x^2}\right)$$

$$y'(x+y) = y - x$$

$$y' = \frac{y-x}{y+x}$$

14. (*b,c,d*)

$f(x) = \sin 2x\{\sin(x+x^2).\sin(x-x^2) + \cos(x+x^2)\cos(x-x^2)\}$
$+ \sin 2x^2\{\cos(x+x^2)\cos(x-x^2)$
$-\sin(x-x^2)\sin(x+x^2)\}$
$\Rightarrow f(x) = \sin 2x.\cos 2x^2 + \cos 2x.\sin 2x^2$
$\Rightarrow f(x) = \sin(2x+2x^2)$
$\Rightarrow f'(x) = (2+4x)\cos(2x+2x^2)$
Now

$$f'\left(-\frac{1}{2}\right) = (2-2)\cos\left(-1+\frac{1}{2}\right) = 0$$

$f'(-1) = -2\cos 0 = -2$
$f''(x) = 4\cos(2x+2x^2) - (2+4x)^2\sin(2x+2x^2)$
$f''(0) = 4 - 0 = 4$

15. (*c,d*)

$f''(x) = -f(x)$ ... (i)
$f'(x) = g(x)$ ... (ii)
$h'(x) = (f(x))^2 + (g(x))^2$ ... (iii)
$h(0) = 2,\ h(1) = 4$
Differentiating equation (ii) w.r.t. x
$f''(x) = g'(x) = -f(x)$
Differentiating equation (iii) w.r.t. x
$h''(x) = 2f(x).f'(x) + 2g(x).g'(x)$
$= 2f(x).f'(x) - 2f'(x).f(x) = 0$ {Q $g'(x) = -f(x)$}
$\Rightarrow h'(x)$ is constant
$\Rightarrow h(x)$ is linear function
$\therefore\ h(0) = 2 \Rightarrow h(x)$ not passing through (0, 0)
Let $y = h(x) = ax + b$
at $x = 0$
$y = 2 = b \Rightarrow y = ax + 2$
at $x = 1$
$a + 2 = 4$
$a = 2$
$\Rightarrow$ curve is $y = 2x + 2$

16. (*c*)

17. (*c*)

18. (*c*)

(16,17,18)

$$g(t) = \lim_{x\to 0}(1 + a\tan x)^{t/x}$$

$$g(t) = e^{\lim_{x\to 0}\frac{t}{x}a\tan x} = e^{\lim_{x\to 0}ta\frac{\tan x}{x}}$$

$g(t) = e^{ta} = e^{ta}$
$g(x) = e^{ax}$
$\therefore\ a = 2,\ g(x) = e^{2x}$
$g(2) = e^4$

$$f(x) = \begin{cases} x\,e^{ax} & , \ x \leq 0 \\ x + ax^2 - x^3 & , \ x > 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + ax^2 - x^3 = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\,e^{ax} = 0$$

$f(0) = 0$
$a \hat{I} (0, \infty)$

$$f'(0^+) = \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+}\frac{h + ah^2 - h^3}{h} = 1$$

$$f'(0^-) = \lim_{h\to 0^-}\frac{f(o-h)-f(o)}{-h} = \lim_{h\to 0^-}\frac{he^{-ah}}{h} = 1$$

19. (*b*)
20. (*a*)
21. (*c*)
(19 to 21)

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2x+5y-2}{5x+2y+1} = \frac{-5}{8} \text{ at } (1, 1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{(5x+2y+1)(2+5\frac{dy}{dx}) - (2x+5y-2)(5+2\frac{dy}{dx})}{(5x+2y+1)^2}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(1,1)} = \frac{111}{256}$$

For question 20

Slope of normal at $(1, 1) = -\frac{dx}{dy} = \frac{8}{5}$

Equation of normal

$$y - 1 = \frac{8}{5}(x-1) \Rightarrow 5y - 5 = 8x - 8$$

$$\Rightarrow 8x - 5y - 3 = 0$$

22. (*a*) → (p,q,r), (*b*) → (p,r,s), (*c*) → (p,r,s), (*d*) → (p,r,s)
(*a*) $f(x) = |x^3|$ is continuous and differentiable

(*b*) $f(x) = \sqrt{|x|}$ is continuous

$$f'(x) = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|} \{\text{does not exist at } x = 0\}$$

(*c*) $f(x) = |\sin^{-1} x|$ is continuous

$$f'(x) = \frac{\sin^{-1} x}{|\sin^{-1} x|} \cdot \frac{1}{\sqrt{1-x^2}} \quad \{\text{does not exist at } x = 0\}$$

(*d*) $f(x) = \cos^{-1}|x|$ is continuous

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x}{|x|} \{\text{does not exist at } x = 0\}$$

23. (*a*) → (s); (*b*) → (r); (*c*) → (s); (*d*) → (p)
(*a*) $y = f(x^3)$

$$\therefore \frac{dy}{dx} = f'(x^3) \cdot 3x^2$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=1} = f'(1) \cdot 3 = 9$$

(*b*) $f(xy) = f(x) + f(y)$
$f(1) = f(1) + f(1)$
$\therefore f(1) = 0$

$$Q\ f(1) = f(e) + f\left(\frac{1}{e}\right)$$

$$\therefore f(e) + f\left(\frac{1}{e}\right) = 0$$

(*c*) $f^2(x) = -f(x), f'(x) = g(x)$
$\therefore g'(x) = f''(x) = -f(x)$
$h(x) = (f(x))^2 + (g(x))^2$
$\therefore h'(x) = 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)$
$= 2f(x) \cdot g(x) + 2g(x)(-f(x)) = 0$
$\therefore h(x) = c, x \in R$
$\therefore h(10) = h(5) = 9.$

(*d*) $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x), \frac{\pi}{2} < x < p$

$$\frac{dy}{dx} = \frac{-\text{cosec}^2 x}{1+\cot^2 x} + \frac{-1}{1+\tan^2 x} \cdot \sec^2 x$$
$$= -1 - 1 = -2$$

24. (0001)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x)+f(h)-2xh-1\}-f(x)}{h}$$

(Using the given relation)

$$= \lim_{h\to 0} -2x + \lim_{h\to 0}\frac{f(h)-1}{h} = \lim_{h\to 0} -2x + \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

[Putting $x = 0 = y$ in the given relation we find $f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$]

$\therefore f'(x) = -2x + f'(0) \Rightarrow f(x) = -x^2 - \sin a.x + C$
$f(0) = -0 - 0 + C \Rightarrow C = 1$
$\therefore f(x) = -x^2 - \sin a .x + 1$

So, $f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1$

$\therefore f\{f'(0)\} = 1$

25. (0000)
Given that f(x) is a function satisfying
$f(-x) = f(x), \forall x \in R$ … (1)
Also $f'(0)$ exists
$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$
Now, $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = (0)$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h} = f'(0) \quad \ldots (2)$$

Again $Lf'(0) = f'(0)$

$$\Rightarrow \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = f'(0)$$

$\Rightarrow \lim_{h\to0}\frac{f(-h)-f(0)}{-h}=f'(0)$

$\Rightarrow \lim_{h\to0}\frac{f(h)-f(0)}{h}=-f'(0)$ … (3)

[Using eq. (1)]

From equations (2) and (3) we get, Þ f′(0) = 0

26. [4] Differentiability at x = 1

f′(1⁻)

$=\lim_{h\to0}\frac{\frac{\sin[(1-h)^2]\,\pi}{(1-h)^2-3(1-h)+8}+a(1-h)^3+b-(a+b)}{-h}$

$=\lim_{h\to0}\frac{a(1-h)^3-a}{-h}\left(\frac{0}{0}\text{ form}\right)=\lim_{h\to0}\frac{3a(1-h)^2}{1}$

f′(1⁻) = 3a

$f'(1^+)=\lim_{h\to0}\frac{2\cos(1+h)\,\pi+\tan^{-1}(1+h)-a-b}{h}$

$=\lim_{h\to0}\frac{(-2\cos\pi h+\tan^{-1}(1+h)-a-b)}{h}$

Function is differentiable

$\therefore\ -2+\frac{\pi}{4}=a+b$ ....(1)

$=\lim_{h\to0}\frac{-2\cos\pi h+\tan^{-1}(1+h)-2+\pi/2}{h}$

$=\lim_{h\to0}2p\sin ph+\frac{1}{1+(1+h)^2}=\frac{1}{2}$

Now f′(1⁻) = f′(1⁺)

$3a=\frac{1}{2}$

$a=\frac{1}{6}$ ....(2)

by (1) and (2) $b=\frac{\pi}{4}-\frac{13}{6}$

27. [7] $\lim_{x\to1^-}f(x)=\lim_{x\to1}x^2e^{2(x-1)}=1$

f(1) = 1

$\lim_{x\to1^+}f(x)=\lim_{x\to1}a\,\text{sgn}(x+1)\cos2(x-1)+bx^2$

$= a.1.1+b$

for continuity a + b = 1

LHD (x = 1) is $\lim_{h\to0}\frac{(1-h)^2e^{-2h}-1}{h}$

$=\lim_{h\to0}2e^{-2h}+he^{-2h}+\left(\frac{e^{-2h}-1}{h}\right)$

= 2 + 0 + 2 = 4

RHD (x = 1) is $\lim_{h\to0}\frac{a\,\text{sgn}(2+h)\cos2h+b(1+h)^2-1}{h}=$

$\lim_{h\to0}\frac{a\cos2h+b+bh^2+2bh-(a+b)}{h}$

$=\lim_{h\to0}a\left(\frac{\cos2h-1}{h}\right)+bh+2b=2b$

f(x) is differentiable at x = 1 if 2b = 4

$\Rightarrow$ b = 2 a = –1

28. [0] As $0<\{e^x\}<1$

$\therefore\ \lim_{n\to\infty}\frac{\{e^x\}^n-1}{\{e^x\}^n+1}=-1\Rightarrow f(x)=-1\ \forall\ x\in R$

29. (*a*) $f(x)=x+\cfrac{1}{2x+\cfrac{1}{2x+\frac{1}{2x+....}}}$

$x+f(x)=2x+\cfrac{1}{2x+\cfrac{1}{2x+.....}}$

$x+f(x)=2x+\cfrac{1}{2x+f(x)}$

$(x+f(x))(2x+f(x))=2x[2x+f(x)]+1$

on differentiation and puting x = 50

$\frac{f(50)f'(50)}{50}=1$

30. [1] $(x^2+y^2)^2=\left(t-\frac{1}{t}\right)^2=t^2+\frac{1}{t^2}-2$

$\Rightarrow x^4+y^4+2x^2y^2=x^2y^4-2\Rightarrow x^2y^2=-1$

differenciating with respect to x

$\Rightarrow x^2.2y\frac{dy}{dx}+y^2.2x=0\Rightarrow x^3y\frac{dy}{dx}=1$

31. [1] $\log(x+y)=2xy;\ \text{When}\ x=0;y=1$

$\frac{1}{x+y}\left[1+\frac{dy}{dx}\right]=2\left[x\frac{dy}{dx}+y\right]$

at (0,1)

$\Rightarrow\frac{dy}{dx}=1$

32. [2] Taking log on both sides

$n\log(x+y)=\log x+\log y$

differenciating wrt x $\frac{n}{x+y}\left[1+\frac{dy}{dx}\right]=\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{y}{x}\left[\frac{x+y-nx}{ny-x-y}\right]\Rightarrow n=2$ satisfies

33. [9] $y=\frac{2x^2}{(x-2)(x-3)(x-4)}+\frac{3x+x^2-3x}{(x-3)(x-4)}=\frac{x^3}{(x-2)(x-3)(x-4)}$

$\log y = 3\log x - \log(x-2) - \log(x-3) - \log(x-4)$

$\frac{dy}{\frac{dx}{y}}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x}-\frac{1}{x-3}+\frac{1}{x}-\frac{1}{x-4}$

$=\frac{-2}{x(x-2)}-\frac{3}{x(x-3)}-\frac{4}{x(x-4)}$

$\frac{x}{y}\frac{dy}{dx}=\frac{2}{2-x}+\frac{3}{3-x}+\frac{4}{4-x}\Rightarrow a+b+c=9$

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEAR'S

1. (*d*) $f(x)=|x-p|\ (e^{|x|}-1)\sin|x|$

we check differentiability at x = p & x = 0 at x = p

R.H.D.= $\lim_{h\to0^+}\frac{|\pi+h-\pi|\left(e^{|\pi+h|}-1\right)\sin|\pi+h|-0}{h}=0$

L.H.D. $\lim_{h\to0^+}\frac{|\pi-h-\pi|\left(e^{|\pi+h|}-1\right)\sin|\pi-h|-0}{-h}=0$

$\because$ RHD = LHD so function is differentiable at x = p

at x = 0

R.H.D. = $\lim_{h\to0^+}\frac{|h-\pi|\left(e^{|h|}-1\right)\sin|h|-0}{h}=0$

L.H.D. $\lim_{h\to0^+}\frac{|-h-\pi|\left(e^{|-h|}-1\right)\sin|-h|-0}{-h}=0$

$\because$ RHD = LHD so function is differentiable at x = 0

set S is empty set, f

2. (*d*) For x = 1

R.H.L = a + b

L.H.L = 5

So to be continuous at x = 1

a + b = 5 ...(i)

for x = 3

R.H.L = b + 15

L.H.L = a + 3b

b + 15 = a + 3b

a + 2b = 15 ...(ii)

for x = 5

RHL = 30

LHL = b + 25

b + 25 = 30

b = 5

From equation (ii)

a = 10

but a = 10 and b = 5 does not satisfied equation (i)

So f(x) is discontinuous for $a\in R$ and $b\in R$

3. (*b*) From the graph we can easily conclude that f (x) is non – derivable at x = – 2, –1,0,1,2

4. (*a*) At x = p + or x = p –

f(x) = sin x – x + 2(x – p) cos x

f(x) is differentiable

For x = 0 +

f(x) = sin x – x + 2 (x – p) cos x

f '(x) = cos x – 1 + 2 cos x – 2(x – p) sin x

f '(0 +) = 2

For x = 0 –

f(x) = – sin x + x + 2(x – p) cos x

f '(x) = – cos x + 1 + 2 cos x – 2(x – p) sin x

f '(0 –) = 2

LHD = RHD

$\therefore$ f is differentiable at x = 0.

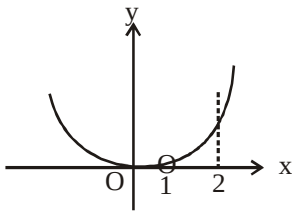
5. (*d*) $|f(x)|=\begin{cases}1 & -2\le x<0\\ |x^2-1| & 0\le x\le 2\end{cases}$

$f(|x|)=\begin{cases}-1 & ;\ -2\le|x|<0\\ |x^2-1| & ;\ 0\le x\le 2\end{cases}$

$\Rightarrow f(|x|)=\{x^2-1\,;-2\le x\le 2\}$

$g(x)=|f(x)|+f(|x|):\begin{cases}1+x^2-1 & ;\ -2\le x<0\\ |x^2-1|+x^2-1; & 0\le x\le 2\end{cases}$

$g(x)=\begin{cases}x^2 & ;\ -2<x<0\\ 0 & ;\ 0\le x<1\\ 2(x^2-1) & ;\ 1\le x<2\end{cases}$



$\therefore$ g(x) is not differentiable at x = 1

6. (*d*) $f(x)=\begin{cases}-(x+1) & , \quad -1\le x<0\\ x & , \quad 0\le x<1\\ 2x & , \quad 1\le x<2\\ x+2 & , \quad 2\le x<3\\ x+3 & , \quad x=3\end{cases}$

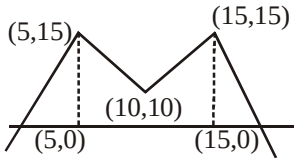
function discontinuous at x = 0,1,3

7. (*c*) $f(x)=15-|x-10|,\ x\in R$

$f(f(x))=15-|f(x)-10|$

$=15-|15-|x-10|-10|$

$=15-|5-|x-10||$



x = 5, 10, 15 are points of non differentiability

Aliter :

At x = 10 *f*(x) is non differentiable

also, when $15-|x-10|=10$

$\Rightarrow x=5,15$

$\therefore$ non differentiability points are {5, 10, 15}

8. (*a*) $\therefore$ function should be continuous at $x=\frac{\pi}{4}$

$$\therefore \lim_{x\to\frac{\pi}{4}} f(x)=f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x-1}{\cot x-1}=k$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{-\sqrt{2}\sin x}{-\text{cosec}^2 x}=k \text{ (Using L' Hospital)}$$

$$\lim_{x\to\frac{\pi}{4}}\sqrt{2}\sin^3 x=k \Rightarrow k=\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^3=\frac{1}{2}$$

9. (*a*) f(x) → $a|\pi-x|+1;\ x\ge 5$; → $b|\pi-x|+3;\ x>5$

$a|\pi-5|+1=b|5-\pi|+3$

$a(5-\pi)+1=b(5-\pi)+3$

$(a-b)(5-\pi)=2$

$$a-b=\frac{2}{5-\pi}$$

10. (*d*) $f(x)=[x]-\left[\frac{x}{4}\right]$

$$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\left(\left([x]-\left[\frac{x}{4}\right]\right)\right)=4-1=3$$

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}\left([x]-\frac{x}{4}\right)=3-0=3$$

$f(x)=3$

$\therefore$ continuous at x = 4

11. (*d*) $\text{RHL}=\lim_{x\to 0^+}\frac{\sqrt{x+x^2}-\sqrt{x}}{x^{\frac{3}{2}}}=\lim_{x\to 0^+}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2}$

$$\text{LHL}=\lim_{x\to 0}\frac{\sin(p+1)x+\sin x}{x}=(p+1)+1=p+2$$

for continuity LHL = RHL = f(0)

$\Rightarrow (p,q)=\left(\frac{-3}{2},\frac{1}{2}\right)$

12. (*a*) $g'(c)=\lim_{h\to 0}\frac{|f(c+h)|-|f(c)|}{h}$

$$\lim_{h\to 0}\frac{|f(c+h)|}{h}=\lim_{h\to 0}\frac{|f(c+h)-f(c)|}{h}$$

$$=\lim_{h\to 0}\left|\frac{f(c+h)-f(c)}{h}\right|\frac{|h|}{h}$$

$$=\lim_{h\to 0}|f'(c)|\frac{|h|}{h}=0, \text{ if } f'(c)=0$$

i.e. g(x) is differentiable at x = c, if f′(c) = 0

13. [3] $\because$ f(x) is not differentiable at x = 1, 3, 5

$$\sum f(f(x))=f(f(1))+f(f(3))+f(f(5))$$

$$=1+1+1=3.$$

14. [5] $\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(\frac{1}{x}\ell n\left(\frac{1+3x}{1-2x}\right)\right)=$

$$\lim_{x\to 0}\left(\frac{\ell n(1+3x)}{x}-\frac{\ell n(1-2x)}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{3\ell n(1+3x)}{3x}-\frac{2\ell n(1-2x)}{-2x}\right)=3+2=5$$

$\therefore$ f(x) will be continuous if $f(0)=\lim_{x\to 0} f(x)$

15. (BONUS)

If we consider f(x) = constant function, the option (2) becomes 0 < 0 which is false so answer (2) cannot be answer.

16. (*b*) LHL = a + 3
f(0) = b

$$\text{RHL} = \lim_{h\to 0}\left(\frac{(1+3h)^{\frac{1}{3}}-1}{h}\right)=1$$

$\therefore$ a = – 2
b = 1
$\therefore$ a + 2b = 0

17. (*b*) $\lim_{x\to 0} x\left(\frac{4}{x}-\left\{\frac{4}{x}\right\}\right) = A \Rightarrow \lim_{x\to 0} 4 - x\left\{\frac{4}{x}\right\} = A \Rightarrow 4 - 0$
=A
check when

(*a*) $x = \sqrt{A+1} \Rightarrow x = \sqrt{5} \Rightarrow$ discontinuous

(*b*) $x = \sqrt{A+21} \Rightarrow x = 5 \Rightarrow$ continuous

(*c*) $x = \sqrt{A} \Rightarrow x = 2 \Rightarrow$ continuous

(*d*) $x = \sqrt{A+5} \Rightarrow x = 3 \Rightarrow$ continuous

18. (*b*) $\because$ f(x) is continuous function
$\therefore$ $f(1^-) = f(1^+)$

$$\Rightarrow ae + \frac{b}{e} = c \ldots.(i)$$

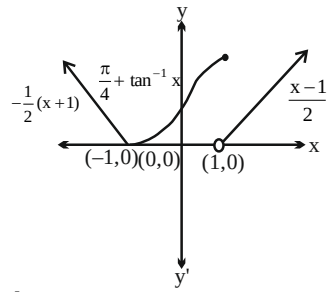
and $f(3^-) = f(3^+) \Rightarrow 9c = 9a + 6c$
$\Rightarrow$ c = 3a .....(ii)
For f'(0) + f'(2) =e
a – b + 4c = e
$a - 3ae + ae^2 + 12a = e$
$13a - 3ae + ae^2 = e$

$$\therefore a = \frac{e}{e^2 - 3e + 13}$$

19. (*b*) Contrapositive statement will be
"If a function is not continuous at 'a', then it is not differentiable at 'a'.

20. (*d*)
$$f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x, & |x| \le 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$$

$\therefore$
$$f(x) = \begin{cases} \frac{-x-1}{2}, & x < -1 \\ \frac{\pi}{4} + \tan^{-1} x, & -1 \le x \le 1 \\ \frac{1}{2}(x-1), & x > 1 \end{cases}$$



f(x) is discontinuous at x = 1.
f(x) is non differentiable at x = –1, 1.

21. (*d*) $\lim_{t\to x} \frac{t^2 f^2(x) - x^2 f^2(t)}{t - x} = 0$

$$\Rightarrow \lim_{t\to x} \frac{2tf^2(x) - 2x^2 f(t).f'(t)}{1} = 0$$

(Using L'Hospital's Rule)

$$\Rightarrow f(x) = xf'(x) = \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating w.r.t x, we get
$\Rightarrow$ In f(x) = In x + In C
$\Rightarrow$ f(x) = Cx
$\because$ f(1) = e
$\Rightarrow$ C = e; So f(x) = ex
When f(x) = 1 = ex

$$\Rightarrow x = \frac{1}{e}$$

22. (*b*)
$$f(x) = \begin{cases} k_1(x-\pi)^2 - 1 & ; x \le \pi \\ k_2 \cos x, & ; x > \pi \end{cases}$$

$$f'(x) = \begin{cases} 2k_1(x-\pi) & ; x \le \pi \\ -k_2 \sin x, & ; x > \pi \end{cases}$$

$$f''(x) = \begin{cases} 2k_1 & ; x \le \pi \\ -k_2 \cos x, & ; x > \pi \end{cases}$$

$\because$ f (x) is twice differentiable at x = π, then

(i) $\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^-} f(x) \Rightarrow -1 = -k_2 \Rightarrow k_2 = 1$

(ii) $\lim_{x\to\pi^+} f''(x) = \lim_{x\to\pi^-} f''(x) \Rightarrow k_2 = 2k_1 \Rightarrow k_1 = \frac{1}{2}$

23. [8] $f(x) = x\left[\frac{x}{2}\right]$ may be discontinuous where $\frac{x}{2}$ an integer.
So possible points of discontinuity are;
x = ± 2, ± 4, ± 6, ± 8 and 0
but at x = 0

$$\lim_{x\to 0^+} f(x) = 0 = f(0) = \lim_{x\to 0^-} f(x)$$

So f(x) will be discontinuous at x = ± 2, ± 4, ± 6 and ± 8

24. [5] If $g(x) = x^5 \sin\left(\frac{1}{x}\right)$ and $h(x) = x^5 \cos\left(\frac{1}{x}\right)$
Then g'' (0) = 0 and h''(0) = 0
So f'' (0^+) = g'' (0^+) + 10 = 10
and f'' (0^-) = h''(0^-) + 2λ = f'' (0^+)
$\Rightarrow$ 2λ = 10
λ = 5

25. (*b*)

$$g'(x) = \frac{|\log 2 - \sin|\log 2 - \sin x||}{(\log 2 - \sin|\log 2 - \sin x|)} \times [-\cos|\log 2 - \sin x|] \times \frac{|\log 2 - \sin x|}{(\log 2 - \sin x)} \times [-\cos x]$$

$$\Rightarrow g'(0) = \cos(\log 2)$$

26. (*b*) Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ where $x \in \left(0, \frac{1}{4}\right)$,

$$= \tan^{-1}\left(\frac{2.(3x^{3/2})}{1-(3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$$

As $3x^{3/2} \in \left(0, \frac{3}{8}\right)$

$\therefore \quad \frac{dy}{dx} = 2 \times \frac{1}{1+9x^3} \times 3 \times \frac{3}{2} \times x^{1/2}$

$= \frac{9}{1+9x^3}\sqrt{x}$

$\therefore \quad g(x) = \frac{9}{1+9x^3}$

27. (*d*) $\frac{dx}{dt} = 3\sec^2 t$

$\frac{dy}{dt} = 3\sec t \tan t$

$\frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t$

$\frac{d^2y}{dx^2} = \cos t \frac{dt}{dx}$

$= \frac{\cos t}{3\sec^2 t} = \frac{\cos^2 t}{3} = \frac{1}{3.2\sqrt{2}} = \frac{1}{6\sqrt{2}}$

28. (*c*) $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$f''(x) = 6x + 2a$

$f'''(x) = 6a = f'(1) = 3 + 2a + b \Rightarrow a + b = -3$

$b = f''(x) = 12x + 2a \Rightarrow 2a - b = -12$

$c = f'''(3)$

$\Rightarrow$ c = 6 and a = – 5, b = 2

$\Rightarrow f(x) = x^2 - 5x^2 + 2x + 6$

$\Rightarrow f(2) = 8 - 20 + 4 + 6 = -2$

29. (*c*) $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})} = \frac{1}{\left(x^m + \frac{1}{x^m}\right)\left(y^n + \frac{1}{y^n}\right)}$

Put $x^m + \frac{1}{x^m} \geq 2$

$\Rightarrow \quad \frac{1}{\left(x^m + \frac{1}{x^m}\right)} \leq \frac{1}{2}$

$\Rightarrow$ Maximum Value $= \frac{1}{4}$

30. (*a*) $2y\ln 2x - \ln 4 + 2x - 2y$

$2y(1 + \ln 2x) = \ln 4 + 2x$

$y = \frac{\ln 2x - \frac{\ln 2}{x}}{(1+\ln 2x)^2}(1 + \ln 2x)^2$

$= \ln 2x - \frac{\ln 2}{x}$

$= \frac{x\ln(2x) - \ln 2}{x}$

31. (*b*) $x.\frac{1}{\ell nx} \cdot \frac{1}{x} + \log_e(\log_e x) - 2x + 2y \cdot \frac{dy}{dx} = 0$

Put x = e

$1 - 2e + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2e-1}{2y} \quad (1)$

Put x = e in original equation

$O - e^2 + y^2 = 4$

$y^2 = 4 + e^2$

Now put in (i)

$\frac{dy}{dx} = \frac{2e-1}{2\sqrt{4+e^2}}$

32. (*d*) Consider $\cot^{-1}\left(\frac{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x}{\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\sin x}\right)$

$= \cot^{-1}\left(\frac{\sin\left(x + \frac{\pi}{3}\right)}{\cos\left(x + \frac{\pi}{3}\right)}\right)$

$= \cot^{-1}\left(\tan\left(x + \frac{\pi}{3}\right)\right) = \frac{\pi}{2} - \tan^{-1}\left(\tan\left(x + \frac{\pi}{3}\right)\right)$

$\begin{cases} \frac{\pi}{2} - \left(x + \frac{\pi}{3}\right) = \left(\frac{\pi}{6} - x\right), 0 < x < \frac{\pi}{6} \\ \frac{\pi}{2} - \left(\left(x - \frac{\pi}{3}\right) - \pi\right) = \left(\frac{7\pi}{6} - x\right); \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$

$\therefore \quad 2y = \begin{cases} \left(\frac{\pi}{6} - x\right)^2; 0 < x < \frac{\pi}{6} \\ \left(\frac{7\pi}{6} - x\right)^2; \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$

$\therefore \; 2\frac{dy}{dx} = \begin{cases} 2\left(\frac{\pi}{6} - x\right).(-1); & 0 < x < \frac{\pi}{6} \\ 2\left(\frac{7\pi}{6} - x\right).(-1); & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$

33. (*b*) $y = f(f(f(x))) + (f(x))^2$

$\frac{dy}{dx} = f'(f(f(x)))\, f'(f(x)) f'(x) + 2f(x) f'(x)$

$= f'(1) f'(1) f'(1) + 2f(1) f'(1)$

$= 3 \times 5 \times 3 + 2 \times 1 \times 3$

$= 27 + 6$

$= 33$

34. (*a*) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n$

Diff. w.r.t. x

$\Rightarrow n(1 + x)^{n-1} = {}^nC_1 + {}^nC_2 (2x) + \ldots\ldots + {}^nC_n n(x)^{n-1}$

multiply by x both side

$\Rightarrow nx(1 + x)^{n-1} = {}^nC_1 x + {}^nC_2 (2x^2) + \ldots\ldots + {}^nC_n (nx^n)$

Diff w.r.t x

$\Rightarrow$ $n[(1+x)^{n-1}+(n-1)x(1+x)^{n-2}]$
$= {}^nC_1 + {}^nC_2 2^2 x + \ldots\ldots {}^nC_n (n^2) x^{n-1}$
Put $x = 1$ and $n = 20$
$\Rightarrow$ ${}^{20}C_1 + 2^2\,{}^{20}C_2 + 3^2\,{}^{20}C_3 + \ldots + (20)^2\,{}^{20}C_{20}$
$= 20 \times 2^{18}[2+19] = 420\,(2^{18}) = A(2^{\beta})$

35. (*d*) $f(x) = \tan^{-1}\left(\dfrac{\sin x - \cos x}{\sin x + \cos x}\right)$

$= \tan^{-1}\left(\dfrac{\tan x - 1}{\tan x + 1}\right) = \tan^{-1}\left(x - \dfrac{\pi}{4}\right)$

$\because x - \dfrac{\pi}{4} \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

$\therefore f(x) = x - \dfrac{\pi}{4}$

$\Rightarrow$ its derivative w.r.t. $\dfrac{x}{2}$ is $\dfrac{1}{1/2} = 2$

36. (*a*) $y = \sqrt{\dfrac{2\cos^2\alpha}{\sin\alpha\cos\alpha} + \dfrac{1}{\sin^2\alpha}} = \sqrt{2\cot\alpha + \operatorname{cosec}^2\alpha}$
$= |1 + \cot\alpha| = -1 - \cot\alpha$

$\dfrac{dy}{d\alpha} = \operatorname{cosec}^2\alpha \Rightarrow \left(\dfrac{dy}{d\alpha}\right)$ at $\alpha = \dfrac{5\pi}{6}$ will be $= 4$.

37. (*b*) $kx^{k-1} + ky^{k-1}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{k-1} \Rightarrow \dfrac{dy}{dx} + \left(\dfrac{x}{y}\right)^{k-1} = 0$

$k - 1 = \dfrac{-1}{3}$

$k = 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

38. (*b*) $x = \dfrac{1}{2}$, $y = \dfrac{-1}{4} \Rightarrow xy = \dfrac{-1}{8}$

$y.\dfrac{1.(2x)}{2\sqrt{-x^2}} + y'.\sqrt{1-x^2} = -\left\{1.\sqrt{1-y^2} + \dfrac{x.(-2y)}{2\sqrt{1-y^2}}y'\right\}$

$-\dfrac{xy}{\sqrt{1-x^2}} + y'\sqrt{1-x^2} = -\sqrt{1-y^2} + \dfrac{xy.y'}{\sqrt{1-y^2}}$

$y'\left(\sqrt{1-x^2} - \dfrac{xy}{\sqrt{1-y^2}}\right) = \dfrac{xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}$

$y'\left(\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{1}}{8.\dfrac{\sqrt{15}}{3}}\right) = \dfrac{-1}{8.\sqrt{\dfrac{3}{2}}} - \dfrac{\sqrt{15}}{4}$

$y'\left(\dfrac{\sqrt{45}+1}{2\sqrt{15}}\right) = -\dfrac{\left(1+\sqrt{45}\right)}{4\sqrt{3}}$

$y' = -\dfrac{\sqrt{5}}{2}$

39. (*d*) $f'(x) = \tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right)$

$= \tan^{-1}\left(\dfrac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right) = \tan^{-1}\left(\dfrac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right) = \dfrac{\pi}{4}+\dfrac{x}{2}$

$(f'(x)dx) = \dfrac{\pi}{4} + \dfrac{x}{2}dx$

$f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4} + c$

$f(0) = c = 0 \Rightarrow f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4}$

So $f(1) = \dfrac{\pi+1}{4}$

40. (*b*) $\dfrac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta$

$\dfrac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\sin 2\theta - \sin\theta}{\cos\theta - \cos 2\theta}$

$= \dfrac{2\sin\frac{\theta}{2}.\cos\frac{3\theta}{2}}{2\sin\frac{\theta}{2}\sin\frac{3\theta}{2}} = \cos t\,\dfrac{3\theta}{2}$

$\dfrac{d^2y}{dx^2} = \dfrac{-3}{2}\operatorname{cosec}^2\dfrac{3\theta}{2}.\dfrac{d\theta}{dx}$

$\dfrac{d^2y}{dx^2} = \dfrac{-\frac{3}{2}\operatorname{cosec}^2\frac{3\theta}{2}}{2(\cos\theta - \cos 2\theta)}$

$\left.\dfrac{d^2y}{dx^2}\right|_{\theta=\pi} = -\dfrac{3}{4(-1-1)} = \dfrac{3}{8}$

41. (*a*) $f(g(x)) = x$
$\Rightarrow$ $f'(g(x)).g'(x) = 1$
Put $x = a$
$\Rightarrow$ $f'(g(a))\,g'(a) = 1$
$\Rightarrow$ $f'(b) \times 5 = 1$

$\Rightarrow f'(b) = \dfrac{1}{5}$

42. [91]

$$y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$$

Let $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

$$\therefore\ y = \sum_{k=1}^{6} k\cos^{-1}\{\cos\alpha\cos kx - \sin\alpha\sin kx\}$$

$$= \sum_{k=1}^{6} k\cos^{-1}(\cos(kx+\alpha))$$

$$= \sum_{k=1}^{6} k(kx+\alpha) = \sum_{k=1}^{6}(k^2x + \alpha k)$$

$$\frac{dy}{dx} = \sum_{k=1}^{6} k^2 = \frac{6(7)(13)}{6} = 91$$

43. (*d*) $y^2 + 2\ln(\cos x) = y$ ....(i)

$\Rightarrow 2yy' - 2\tan x = y'$ ....(ii)

From (1) $y(0) = 0$ or 1

$\therefore\ y'(0) = 0$

Again differentiating (2) we get

$2(y')^2 + 2yy'' - 2\sec^2 x = y''$

gives $|y''(0)| = 2$

44. (10)

$f(x + y) = f(x) + f(y) + xy^2 + x^2y$

Differentiate w.r.t. x ;

$f'(x + y) = f'(x) + y^2 + 2xy$

Put $y = -x$

$f'(0) = f'(x) + x^2 - 2x^2$ .....(i)

As $\lim_{x\to 0}\frac{f(x)}{x} = 1 \Rightarrow f(0) = 0$

$\therefore\ f'(0) = 1$ ....(ii)

From (i), (ii);

$f'(x) = (x^2 + 1)$

$f'(3) = 10$

45. (*c*) $(a + \sqrt{2}b\cos x)(a - \sqrt{2}b\cos y) = a^2 - b^2$

Differentiating both sides

$$(-\sqrt{2}b\sin x)(a - \sqrt{2}\cos y) + (a + \sqrt{2}b\cos x)$$

$$(\sqrt{2}b\sin y)y' = 0$$

at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$-b(a - b) + (a + b)by' = 0$

$$\frac{dy}{dx} = \frac{a-b}{a+b}$$

$$\Rightarrow\ \frac{dx}{dy} = \frac{a+b}{a-b}$$

46. (*c*)

$$\frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{d\left(\tan^{-1}\left(\frac{2x\sqrt{1+x^2}}{1-2x^2}\right)\right)} = \frac{\frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{dx}}{\frac{d\left(\tan^{-1}\left(\frac{2x\sqrt{1+x^2}}{1-2x^2}\right)\right)}{dx}}$$

Simplifying $\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)$ Put $x = \tan\theta$

$$\Rightarrow\ \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-(1-2\sin^2\theta/2)}{2\sin\theta/2\cos\theta/2}\right) = \frac{\theta}{2}$$

$$\therefore\ \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\tan^{-1}x}{2}$$

& similarly $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ Put $x = \sin\theta$

$$\Rightarrow\ \tan^{-1}\left(\frac{\sin 2\theta}{\cos 2\theta}\right) = 2\theta = 2\sin^{-1}x$$

Hence required derivative

$$\left.\frac{\frac{1}{\frac{2(1+x^2)}{2}}}{\frac{\sqrt{1-x^2}}{4(1+x^2)}}\right|_{x=\frac{1}{2}} = \frac{\sqrt{3}}{10}$$

47. [14]

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x-1), & \text{for } x \le 0 \\ \frac{\tan 2x - \sin 2x}{bx^3}, & \text{for } x > 0 \end{cases}$$

For continuity at '0'

$$\lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+}\frac{\tan 2x - \sin 2x}{b\,x^3} = -a$$

$$\Rightarrow \lim_{x\to 0^+}\frac{\frac{8x^3}{3} + \frac{8x^3}{3!}}{bx^3} = -a$$

$$\Rightarrow 8\left(\frac{1}{3} + \frac{1}{3!}\right) = -ab \Rightarrow 4 = -ab$$

$$\Rightarrow 10 - ab = 14$$

48. [0]

We have, $f(x+y)=f(x)\cdot f(y)$ for all x, y

$\therefore \quad f(0)=f(0)\cdot f(0)\Rightarrow f(0)\{f(0)-1\}=0$

$\Rightarrow \quad f(0)=1 \quad [\because f(0)\neq 0$

$f'(0)=2\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$\Rightarrow \lim_{h\to 0}\frac{f(h)-1}{h}=2 \quad [\because f(0)=1]\ldots$ (i) Also,

$'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$

[using, $f(x+y)=f(x)\cdot f(y)$]

$=f(x)\left\{\lim_{h\to 0}\frac{f(h)-1}{h}\right\}$

$\therefore \quad f'(x)=2f(x)$ [from Eq. (i)] $\Rightarrow \quad \frac{f'(x)}{f(x)}=2$

On integrating both sides between 0 to x, we get

$\int_0^x \frac{f'(x)}{f(x)}dx=2x$

$\Rightarrow \log_e|f(x)|-\log_e|f(0)|=2x$

$\Rightarrow \log_e|f(x)|=2x[\because f(0)=1$

$\Rightarrow f(x)=e^{2x}\Rightarrow \lim_{h\to 0}\frac{(e^{2h}-1)}{2}=0$

49. [4]

$$\lim_{x\to 0}\frac{\left\{a\left(1+x+\frac{x^2}{2!}+\ldots.\right)-b\left(1-\frac{x^2}{2!}+\ldots.\right)+c\left(1-x+\frac{x^2}{2!}\ldots..\right)\right\}}{\left(\frac{x\sin x}{x}\right)x}=2$$

For this limit to exist

$a-b+c=0,\ a-c=0$

$\frac{a}{2}+\frac{b}{2}+\frac{c}{2}=2$

$\Rightarrow a+b+c=4$

JEE-ADVANCED PREVIOUS YEAR'S

1. (*b*) (i) for derivability at x = 0

$$\text{L.H.D.}=f'(0^-)=\lim_{h\to 0^+}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0^+}\frac{h^2.\left|\cos\left(-\frac{\pi}{h}\right)\right|-0}{-h}=\lim_{h\to 0^+}-h.\left|\cos\frac{\pi}{h}\right|=0$$

$$\text{RHD } f'(0^+)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h^2.\left|\cos\left(\frac{\pi}{h}\right)\right|-0}{h}=0$$

So f(x) is derivable at x = 0

(ii) check for derivability at x = 2

$$\text{RHD}=f'(2^+)=\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0^+}\frac{(2+h)^2.\left|\cos\left(\frac{\pi}{2+h}\right)\right|-0}{h}$$

$$=\lim_{h\to 0^+}\frac{(2+h)^2.\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$=\lim_{h\to 0^+}\frac{(2+h)^2.\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)}{h}$$

$$=\lim_{h\to 0^+}\frac{(2+h)^2.\sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi}{2(2+h)}\right)h}.\frac{\pi}{2(2+h)}$$

$=(2)^2.\frac{\pi}{2(2)}=\pi$

$$\text{LHD}=\lim_{h\to 0^+}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0^+}\frac{(2-h)^2.\left|\cos\left(\frac{\pi}{2-h}\right)\right|-0}{-h}$$

$$=\lim_{h\to 0^+}\frac{(2-h)^2.\left(-\cos\left(\frac{\pi}{2-h}\right)\right)-0}{-h}=\lim_{h\to 0^+}\frac{(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{h}$$

$$=\lim_{h\to 0^+}\frac{(2-h)^2.\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}=$$

$$\lim_{h\to 0^+}\frac{(2-h)^2.\sin\left(-\frac{\pi h}{2(2-h)}\right)}{\left(-\frac{\pi h}{2(2-h)}\right)}.\frac{-\pi}{2(2-h)}=-\pi$$

So f(x) is not derivable at x = 2

2. (*b*,*d*)

$$\left.\begin{array}{l} f(2n) = a_n \\ f(2n^+) = a_n \\ f(2n^-) = b_n + 1 \end{array}\right\} \quad \begin{array}{l} a_n = b_n + 1 \\ a_n - b_n = 1 \end{array}$$

So B is correct

$$\left.\begin{array}{l} f(2n+1) = a_n \\ f((2n+1)^-) = a_n \\ f((2n+1)^+) = b_{n+1} - 1 \end{array}\right\} \quad \begin{array}{l} a_n = b_{n+1} - 1 \\ a_n - b_{n+1} = -1 \\ a_{n-1} - b_n = -1 \end{array}$$

So D is correct

3. (*a*,*d*)

Consider

$h(x) = f(x) - g(x)$ Assume $a < b$

$h(a) = \lambda - g(a) > 0$

$h(b) = f(b) - \lambda < 0$

else if $a > b$ $h(a) < 0$ and $h(b) > 0$.

By intermediate value theorem $\Rightarrow h(c) = 0$.....(1)

(*a*) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$

$(f(c) - g(c))\,(f(c) + g(c) + 3) = 0$

So there exist a 'c' : $f(c) - g(c)$

from (1).

Hence *a* is correct.

(*d*) Similarly $(f(c))^2 = (g(c))^2$

$(f(c) - g(c))\,(f(c) + g(c)) = 0$

$\Rightarrow$ (*d*) is correct.

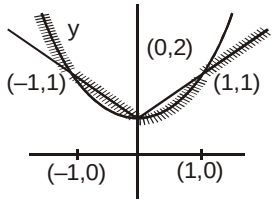
B & C are wrong as by counter eg

If $f(x) = g(x) = \lambda \neq 0$, then

$B \to \lambda^2 + \lambda = \lambda^2 + 3\lambda$ is not possible.

$C \to \lambda^2 + 3\lambda = \lambda^2 + \lambda$ is not possible.

4. (*c*) $f(x) = |x| + 1 = \begin{cases} x + 1 & x \geq 0 \\ -x + 1 & x < 0 \end{cases}$



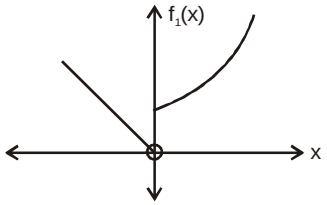
$g(x) = x^2 = 1$

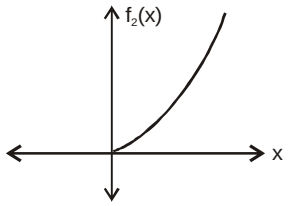
Number of Non-differential points 3.

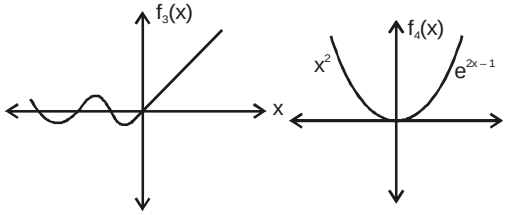
5. (*d*)

$$f_2(f_1(x)) = (f_1(x))^2 - \begin{cases} x^2 & x < 0 \\ e^{2x} & x \geq 0 \end{cases}$$

$$f_4(x) \begin{cases} x^2 & x < 0 \\ e^{2x} - 1 & x \geq 0 \end{cases}$$







$f_4(x)$ is many-one onto, continuous and non-derivable

$f_3(x)$ is many-one, into, continuous and derivable

$f_2(x)$ is one-one, into, differentiable

Hence $R \to 2$

so (*d*)

$p \to 1,\ q \to 3,\ R \to 2,\ S \to 4$

6. (*a*,*d*)

$g(0) = 0,\ g'(0) = 0\ \ g'(1) \neq 0$

$$f(x) \begin{cases} g(x) & ; \ x > 0 \\ -g(x) & ' \ x < 0 \\ 0 & ; \ 0 \end{cases} \quad h(x) = e^{|x|}$$

$$f(h(x)) = g(e^{|x|}),\ \ h(f(x)) = e^{|g(x)|}$$

$$R(f'(0)) = \lim_{x \to 10} \frac{g(x) - g(0)}{x - 0} = g'(0) = 0$$

$$L(f'(0)) = \lim_{x \to 10} \frac{-g(x) - g(0)}{x - 0} = g'(0) = 0$$

$$R(h'(0)) = 1 \ \& \ L(h'(0)) = -1$$

So h(x) is non derivable. at x = 0

Now $\lim_{x \to 0} \frac{f(h(x)) - f(h(0))}{x} = \lim_{x \to 0} \frac{g(e^{|x|}) - g(1)}{x}$

$$R(f'(h(x))) = \lim_{x \to 0^+} \frac{g(e^x) - g(1)}{x}$$

$$= \lim_{x \to 0^+} \frac{g(e^x) - g(1)}{e^x - 1} \cdot \frac{e^x - 1}{x} = g'(1)$$

$L\left(f'(h(x))\right) = -g'(1)$ Hence $f(h(x))$ is non derivable at $x = 0$

Since x = 0 is repeated root of g(x)

So $e^{|g|x||}$ is differetiable at x = 0

hence (*a*), (*d*)

7. (*a,b,d*)

$f(x) = x\cos(\pi(x + [x])$

Check continuity at x = n

$f(n) = n\cos 2n\pi = n$

$f(n^+) = n\cos 2n\pi = n$

$f(n^-) = n\cos(2n - 1)\pi = -n$

It is discontinous at all integer points except 0

8. (*b*) $P(x, y) : f(x + y) = f(x)f'(y) + f'(x)f'(y)\ \forall x, y \in R$

$P(0, 0) : f(0) = f(0)\, f'(0) + f'(0)\, f(0)$

$\Rightarrow 1 = 2f'(0)$

$\Rightarrow f'(0) = \frac{1}{2}$

$P(x, 0) : f(x) = f(x) \cdot f'(0) + f'(x).f(0)$

$\Rightarrow f(x) = \frac{1}{2}f(x) + f'(x)$

$\Rightarrow f'(x) = \frac{1}{2}f(x)$

$\Rightarrow f(x) = e^{\frac{1}{2}x} \quad \Rightarrow \ln(f(4)) = 2$

9. (*d*) (i) $f(x) = \sin\sqrt{1 - e^{-x^2}}$

$$f_1'(x) = \cos\sqrt{1 - e^{-x^2}} \cdot \frac{1}{2\sqrt{1 - e^{-x^2}}}\left(0 - e^{-x^2} \cdot (-2x)\right)$$

at x = 0 $f_1'(x)$ does not exist

So. $P \to 2$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x \to 0^+} \frac{\sin x}{x}\frac{x}{\tan^{-1} x} = 1$$

$\Rightarrow f_2(x)$ does not continuous at x = 0

So $Q \to 1$

(iii) $f_3(x) = \left[\sin \ell n(x + 2)\right] = 0$

$1 < x + 2 < e^{\pi/2}$

$\Rightarrow 0 < \ell n(x + 2) < \frac{\pi}{2}$

$\Rightarrow 0 < \sin(\ell n(x + 2) < 1$

$\Rightarrow f_3(x) = 0$

So $R \to 4$

(iv) $f_4(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

So $S \to 3$

10. (*c*) f(x) is a non-periodic, continuous and odd function

$$f(x) = \begin{cases} -x^2 + x\sin x, & x < 0 \\ x^2 - x\sin x, & x \geq 0 \end{cases}$$

$$f(-\infty) = \lim_{x \to -\infty}\left(-x^2\right)\left(1 - \frac{\sin x}{x}\right) = -\infty$$

$$f(-\infty) = \lim_{x \to \infty} x^2\left(1 - \frac{\sin x}{x}\right) = \infty$$

$\Rightarrow$ Range of f(x) =R

$\Rightarrow$ f(x) is an onto function .....(1)

$$f'(x) = \begin{cases} -2x + \sin x + x\cos x, & x < 0 \\ 2x - \sin x - x\cos x, & x \geq 0 \end{cases}$$

For $(0, \infty)$

$f'(x) = (x - \sin x) + x(1 - \cos x)$

always +ve always +ve

or 0 or 0

$\Rightarrow f'(x) > 0$

$\Rightarrow f'(x) \geq 0,\ \forall x \in (-\infty, \infty)$

equality at x = 0

$\Rightarrow$ f(x) is one – one function ....(2)

From (1) & (2), f(x) is both one-one & onto.

11. (*a,c*)

$f : R \to R\ \ f(x) = (x^2 + \sin x)(x - 1)\ \ f(1^+) = f(1^-) = f(1) = 0$

$fg(x) : f(x).g(x) \qquad fg : R \to R$

let $fg(x) = h(x) = f(x).g(x) h : R \to R$

option (c) $h'(x) = f'(x)g(x) + f(x)\, g'(x)$

$h'(1) = f'(1)\, g(1) + 0,$

(as f(1) = 0, g'(x) exists}

$\Rightarrow$ if g(x) is differentiable then h(x) is also differentiable

(true) option (A) If g(x) is continuous at

x = 1 then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h \to 0^+} \frac{h(1 + h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h \to 0^+} \frac{f(1 + h)g(1 + h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h \to 0^+} \frac{f(1 - h)g(1 - h) - 0}{-h} = f'(1)g(1)$$

Soh(x)=f(x).g(x) is differentiable

at x=1 (True)

Option (*b*) (*d*) $h'(1^+) = \lim_{h\to 0^+} \frac{h(1+h)-h(1)}{-h}$

$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$

$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$

$\Rightarrow g(1^+) = g(1^-)$

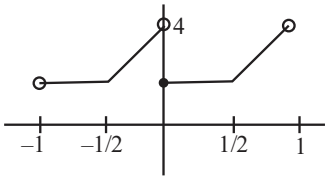
So we cannot comment on the continuity and differentiability of the function.

12. (*d*) f(x)=|2x–1|+|2x+1|

g(x)={x}

f(g(x))=|2{x}–1|+|2{x}+1|

$= \begin{cases} 2 & \{x\} \le 1/2 \\ 4\{x\} & \{x\} > 1/2 \end{cases}$



discontinuous at $x = 0 \Rightarrow c = 1$

Non differential at $x = -\frac{1}{2}, 0, \frac{1}{2}$ $\Rightarrow d = 3$

$\therefore\ c + d = 4$

13. (*a,b,d*)

Since $f(x)=xg(x)$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} xg(x)$

$\lim_{x\to 0} f(x) = \left(\lim_{x\to 0} x\right)\cdot\left(\lim_{x\to 0} g(x)\right)$

$\lim_{x\to 0} f(x) = 0\times 1 = 0$ ....(1)

$f(x+y)=f(x)+f(x)+f(x)f(y)$

Now we check continuity of $f(x)$

at x=a

$\lim_{h\to 0} f(a+h) = f(a)+f(b)+f(a)+f(h)$

$\lim_{x\to 0} (f(a)+f(h)(1+f(a)))$

$\lim_{h\to 0} f(a+h)=f(a)$

$\therefore f(x)$ is continuous $\forall x \in R$

$\lim_{x\to 0} f(x)=f(0)=0 \left(\lim_{x\to 0} f(x) = 0\right)$

$\therefore f(0)=0$

and $\lim_{x\to 0} \frac{f'(x)}{1} = 1$

$\therefore f'(0) = 1$

Now

$f(x+y)=f(x)+f(y)+f(x)f(y)$

using partical derivative (w.r.t. y)

$f'(x+y)+f'(y)+f(x)+f'(y)$

put y=0

$f'(x)=f'(0)+f(x)f'(0)$

$f'(x)=1+f(x)$

$\int \frac{f'(x)}{1+f(x)}dx = \int 1dx$

$\ell n\,|(1+f(x))| = x + C$

$f(0)=0;\ c=0 \quad \therefore |1+f(x)|=e^x$

$1+f(x)=\pm e^x$ or $f(x) = \pm e^x - 1$

Now $f(0)=0 \quad \therefore f(x)=e^x-1$

$\therefore f(x) = x^e - 1$

option (A) is correct

and $f'(x)=e^x$

$f'(0) = 1$ option (D) is correct

$g(x) = \frac{f(x)}{x} = \begin{cases} \frac{e^x-1}{x}; & x \ne 0 \\ 1; & x = 0 \end{cases}$

$g'(0+h) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$

$= \lim_{h\to 0} \frac{\frac{e^h-1}{h}-1}{h} = \frac{1}{2}$

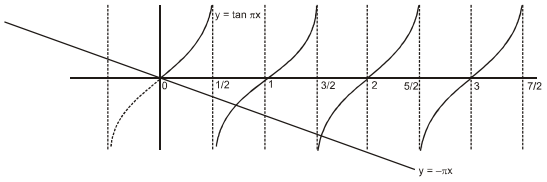
option (*b*) is correct

14. (*b,c*)

$f(x) = x \sin \pi x,\ x > 0$

$f'(x) = \sin \pi x + \pi x \cos \pi x = 0$

$\tan \pi x = -\pi x$



15. (*b,c*)

$f: R \to R$

$f(x) = x^3 + 3x + 2$

f is invertible.

Since $g(f(x)) = x$

$\Rightarrow f(x) = g^{-1}(x)$ or $f^{-1}(x) = g(x)$

(*a*) Since $g(f(x)) = x$

$\Rightarrow g'(f(x)).f'(x) = 1$

For $f(x) = 2, x = 0$ so, $g'(2) = \frac{1}{f'(0)} \Rightarrow g'(2) = \frac{1}{3}$

(*b*,*d*) Now, $h\left(g\left(g\left(x\right)\right)\right) = x$

$\Rightarrow h\left(g\left(g\left(f\left(x\right)\right)\right)\right) = f\left(x\right)$

$\Rightarrow h\left(g\left(x\right)\right) = f\left(x\right)$

$\Rightarrow h\left(g\left(3\right)\right) = f\left(3\right) = 38$

Again, $h\left(g\left(x\right)\right) = f\left(x\right)$

$\Rightarrow h\left(g\left(f\left(x\right)\right)\right) = f\left(f\left(x\right)\right)$

$\Rightarrow h\left(x\right) = f\left(f\left(x\right)\right)$

$\Rightarrow h'\left(x\right) = f'\left(f\left(x\right)\right).f'\left(x\right)$

$\Rightarrow h'\left(1\right) = f'\left(f\left(1\right)\right).f'\left(1\right) = f'\left(6\right).f'\left(1\right)$

$= 111 \times 6 = 666$

(*c*) $g\left(g\left(x\right)\right) = 0 \quad \therefore g\left(x\right) = g^{-1}\left(0\right)$

$\Rightarrow g\left(x\right) = f\left(0\right) = 2$

$\Rightarrow x = g^{-1}\left(2\right) = f\left(2\right) = 16$

So, $h\left(g\left(g\left(x\right)\right)\right) = x \qquad \Rightarrow h\left(0\right) = 16$

Hence, (b, c)

16. (*b*,*c*,*d*)

$\lim\limits_{t\to x} \frac{f\left(x\right)\sin t - f\left(t\right)\sin x}{t - x} = \sin^2 x$

$\lim\limits_{t\to x} \frac{f\left(x\right)\cos t - f'\left(t\right)\sin x}{1} = \sin^2 x$

$\Rightarrow f\left(x\right)\cos x - f'\left(x\right)\sin x = \sin^2 x$

$\Rightarrow -\left(\frac{f'\left(x\right)\sin x - f\left(x\right)\cos x}{\sin^2 x}\right) = 1$

$\Rightarrow -d\left(\frac{f\left(x\right)}{\sin x}\right) = 1$

$\Rightarrow \frac{f\left(x\right)}{\sin x} = x + c$

Put $x = \frac{\pi}{6}$ & $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$\therefore c = 0 \Rightarrow f(x) = -x\sin x$

(*a*) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4}\frac{1}{\sqrt{2}}$

(*b*) $f(x) = -x\sin x$

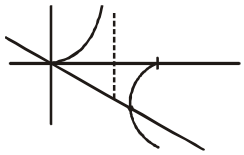
as $\sin x > x - \frac{x^3}{6}, -x\sin < -x^2 + \frac{x^4}{6}$

$\therefore f\left(x\right) < -x^2 + \frac{x^4}{6} \forall x \in \left(0, \pi\right)$

(*c*) $f'(x) = -\sin x - x\cos x$

$f'\left(x\right) = 0 \Rightarrow \tan - x \Rightarrow$ there exist $\alpha \in \left(0, \pi\right)$ for which

$f'\left(\alpha\right) = 0$



(*d*) $f''\left(x\right) = -2\cos x + x\sin x$

$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

PRAYAS1.0: t.me/realprayasians
PRAYAS2.0: t.me/realprayasians2

12 INVERSE TRIGONOMETRIC FUNCTIONS

INTRODUCTION

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. represents angles or real numbers whose values of sine, cosine and tangent is 'x', provided that the value in numerical form is smallest. These can be written as arc sin x, arc cos x etc. If two angles whose modulus is equal, in which one is positive and other is negative then we take positive sign.

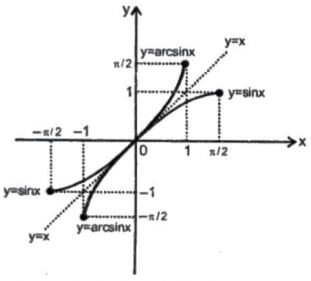
It is worthwhile noting that the functions sinx, cosx etc. are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-odomain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

DOMAIN & PRINCIPLE VALUE RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

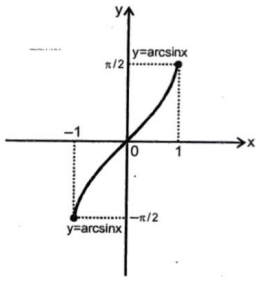
S.No.	Function	Domain	Principle value range (PVR)
1.	$y = \sin^{-1}x$	$x \in [-1, 1]$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1}x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1}x$	$x \in R$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1}x$	$x \in R$	$y \in (0, \pi)$
5.	$y = \sec^{-1}x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \text{cosec}^{-1}x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS

(a) $f: [-\pi/2, \pi/2] \to [-1, 1]$
$f(x) = \sin x$



$f^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$
$f^{-1}(x) = \sin^{-1}x$

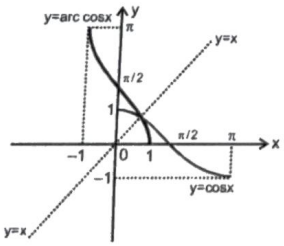


Note : $\sin^{-1}x$ is θ, then θ is the numerically smallest angle, either negative or positive whose sine is equal to x.

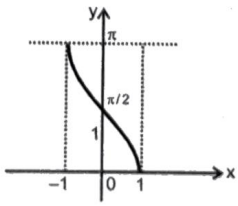
Remarks :

(i) $\sin^{-1}x$ is monotonic increasing in its domain
(ii) It is a bounded function
(iii) It is an odd function
(iv) It is aperiodic
(v) It is continuous

(b) $f: [0, \pi] \to [-1, 1]$
$f(x) = \cos x$



$f^{-1}: [-1, 1] \to [0, \pi]$
$f^{-1}(x) = \cos^{-1}x$



Note : $\cos^{-1}x$ is θ, then θ is the angle $\in [0, \pi]$, whose cosine is equal to x.

Remarks :

(i) $\cos^{-1} x$ is monotonic decreasing in its domain

(ii) It is a bounded function

(iii) It is aperiodic

(iv) It is neither even nor odd

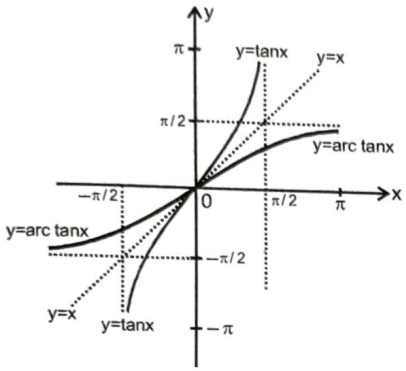
(v) It is continuous

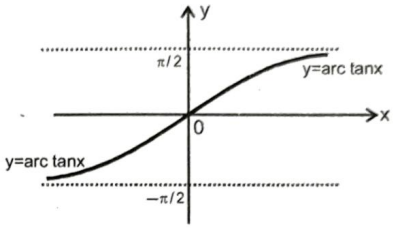
(c) $f: (-\pi/2, \pi/2) \to R$

$f(x) = \tan x$

$f^{-1} : R \to (-\pi/2, \pi/2)$

$f^{-1}(x) = \tan^{-1} x$





Note : $\tan^{-1}x$ is θ, then θ is the angle $\in (-\pi/2, \pi/2)$, whose tangent is equal to x.

Remarks :

(i) $\tan^{-1} x$ is monotonic increasing in its domain

(ii) It is an odd function

(iii) It is a continuous function

(iv) It is aperiodic

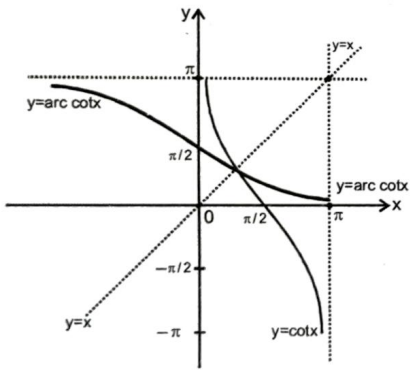
(v) It is a bounded function

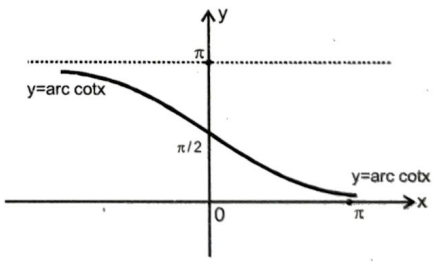
(d) $f: (0, \pi) \to R$

$f(x) = \cot x$

$f^{-1} : R \to (0, \pi)$

$f^{-1}(x) = \cot^{-1} x$





Note : $\cot^{-1}x$ is θ, then θ is the angle $\in (0, \pi)$, whose cotangent is equal to x.

Remarks :

(i) $\cot^{-1} x$ is monotonic decreasing in its domain

(ii) It is neither even nor odd function

(iii) It is a continuous function

(iv) It is aperiodic

(v) It is a bounded function

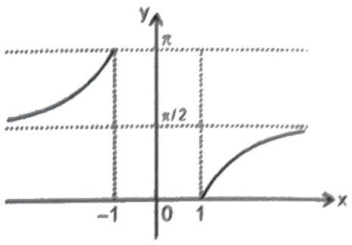
(e) $f:[0,\pi/2)\cup(\pi/2,\pi]\to(-\infty,-1]\cup[1,\infty)$

$f(x)=\sec x$

$f^{-1}:(-\infty,-1]\cup[1,\infty)\to[0,\pi/2)\cup(\pi/2,\pi]$

$f^{-1}(x)=\sec^{-1}x$

Note : $\sec^{-1}x$ is θ, then θ is the angle $\in[0,\pi/2)\cup(\pi/2,\pi]$, whose secant is equal to x.



Remarks :

(i) $\sec^{-1}x$ is increasing whenever it is defined

(ii) It is neither even nor odd function

(iii) It is aperiodic

(iv) It is a bounded function

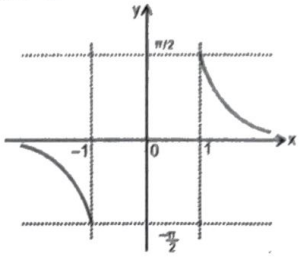
(f) $f:[-\pi/2,0)\cup(0,\pi/2]\to(-\infty,-1]\cup[1,\infty)$

$f(x)=\operatorname{cosec}x$

$f^{-1}:(-\infty,-1]\cup[1,\infty)\to[-\pi/2,0)\cup(0,\pi/2]$

$f^{-1}(x)=\operatorname{cosec}^{-1}x$

Note : $\operatorname{cosec}^{-1}x$ is θ, then θ is the angle $\in[-\pi/2,0)\cup(0,\pi/2]$, whose cosecant is equal to x.



Remarks :

(i) $\operatorname{cosec}^{-1}x$ is monotonic decreasing

(ii) It is an odd function

(iii) It is aperiodic

(iv) It is a bounded function

Note : Basis on the above discussion we get following results :

(i) All inverse trigonometric functions shows angle.

(ii) If $x\geq 0$ then all six trigonometric functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\cot^{-1}x$ shows acute angle.

(iii) If $x<0$ then $\sin^{-1}x$, $\tan^{-1}x$ and $\operatorname{cosec}^{-1}x$, shows angle between $-\pi/2$ to 0 (IV quadrant)

(iv) If $x<0$ then $\cos^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}x$ shows obtuse angle (II quadrant)

(v) III quadrant never used in inverse trigonometric functions.

TRAIN YOUR BRAIN

Example 1: (a) $\tan^{-1}(-1)$, (b) $\cot^{-1}(-1)$, (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Sol. (a) $\tan\left(-\frac{\pi}{4}\right)=-1$,

$\therefore\ \tan^{-1}(-1)=-\frac{\pi}{4}\left\{\because -\frac{\pi}{4}\in\text{range of }\tan^{-1}x\right\}$

(b) $\cot\left(\frac{3\pi}{4}\right)=-1$

$\therefore\ \cot^{-1}(-1)=\frac{3\pi}{4}\left\{\because \frac{3\pi}{4}\in\text{range of }\cot^{-1}x\right\}$

(c) $\sin\left(\frac{-\pi}{3}\right)=\frac{-\sqrt{3}}{2}$,

$\therefore\ \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\frac{-\pi}{3}\left\{\because \frac{-\pi}{3}\in\text{range of }\sin^{-1}x\right\}$

Example 2: Simplify

$$\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{-1}{2}\right)-\tan^{-1}(-\sqrt{3})+\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

Sol. The value $=\frac{-\pi}{4}+\frac{2\pi}{3}-\left(\frac{-\pi}{3}\right)+\left(\frac{2\pi}{3}\right)$

$$=-\frac{\pi}{4}+\frac{2\pi}{3}+\frac{\pi}{3}+\frac{2\pi}{3}=\frac{17\pi}{12}$$

Example 3: Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Sol. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

$=\tan\left[\frac{\pi}{3}+\left(-\frac{\pi}{6}\right)\right]=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$

Self Practice Problems

(1) Find the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

(2) Find the value of $\operatorname{cosec}[\sec^{-1}(-\sqrt{2}) + \cot^{-1}(-1)]$

(3) Find the domain of $y = \sec^{-1}(x^2 + 3x + 1)$

(4) Find the domain of $y = \cos^{-1}\left(\frac{x^2}{1+x^2}\right)$

(5) Find the domain of $y = \tan^{-1}(\sqrt{x^2 - 1})$

Answers :

(1) 1

(2) -1

(3) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$

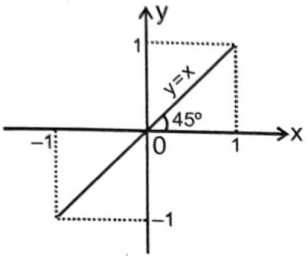
(4) R

(5) $(-\infty, -1] \cup [1, \infty)$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

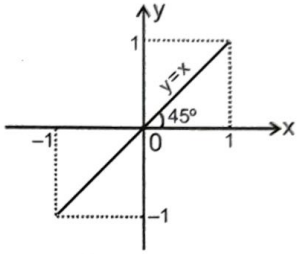
P-1 **(i)** $y = \sin(\sin^{-1} x) = x$

$x \in [-1, 1], y \in [-1, 1]$



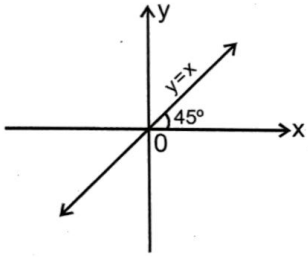
(ii) $y = \cos(\cos^{-1} x) = x$

$x \in [-1, 1], y \in [-1, 1]$



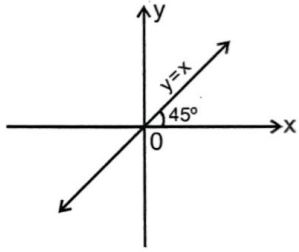
(iii) $y = \tan(\tan^{-1} x) = x$

$x \in R, y \in R$



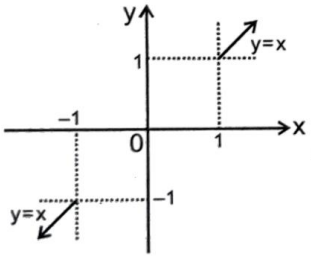
(iv) $y = \cot(\cot^{-1} x) = x$

$x \in R, y \in R$



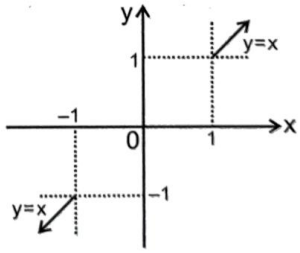
(v) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$

$|x| \geq 1, |y| \geq 1$



(vi) $y = \sec(\sec^{-1} x) = x$

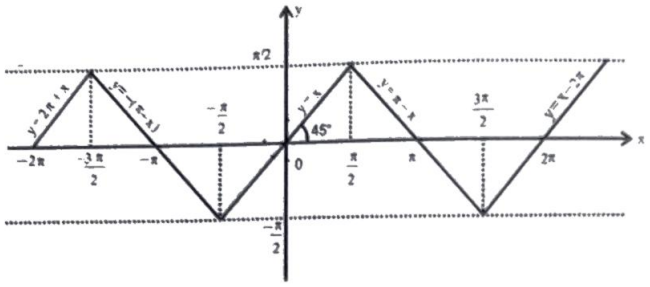
$|x| \geq 1 ; |y| \geq 1$



P-2

(i) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is periodic function with period 2π and it is an odd function

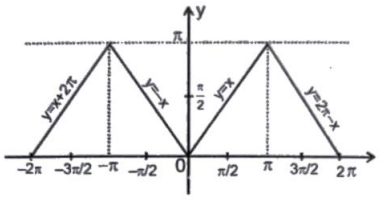
$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\pi \leq x \leq -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$



e.g. $\sin^{-1}(\sin 1) = 1; \sin^{-1}(\sin 2) = \pi - 2; \sin^{-1}(\sin 3) = \pi - 3; \sin^{-1}(\sin 4) = \pi - 4; \sin^{-1}(\sin 5) = 5 - 2\pi$

(ii) $y = \cos^{-1}(\cos x), x \in R, y \in [0, \pi]$, is periodic function with period 2π and it is an even function

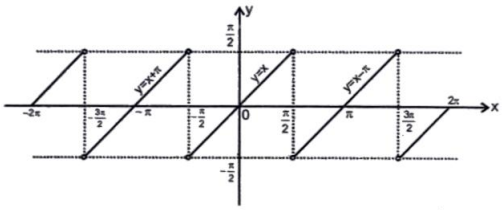
$$\cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \\ x - 2\pi, & 2\pi < x \le 3\pi \end{cases}$$



e.g. $\cos^{-1}(\cos 1) = 1; \cos^{-1}(\cos 2) = 2; \cos^{-1}(\cos 3) = 3; \cos^{-1}(\cos 4) = 2\pi - 4; \cos^{-1}(\cos 5) = 2\pi - 5$

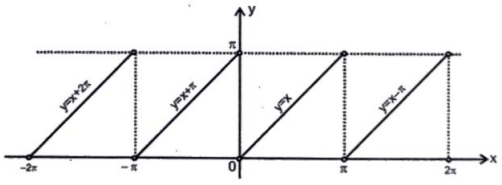
(iii) $y = \tan^{-1}(\tan x), x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}; y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is periodic function with period π and it is an odd function

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi & ; -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x & ; -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & ; \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

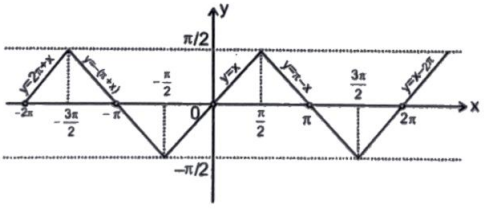


(iv) $y = \cot^{-1}(\cot x), x \in R - \{n\pi, n \in I\}, y \in [0, \pi]$, is periodic function with period π and it is neither an even nor odd function

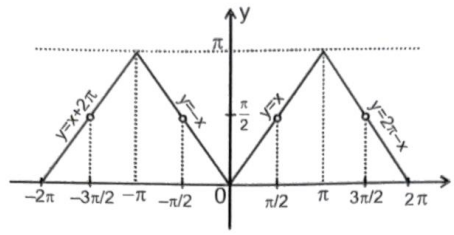
$$\cot^{-1}(\cot x) = \begin{cases} x + \pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \\ x - \pi & ; \pi < x < 2\pi \end{cases}$$



(v) $y = \text{cosec}^{-1}(\text{cosec } x), x \in R - \{n\pi, n \in I\}, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is periodic function with period 2π and it is an odd function



(vi) $y = \sec^{-1}(\sec x)$, $x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is periodic function with period 2π and it is an even function



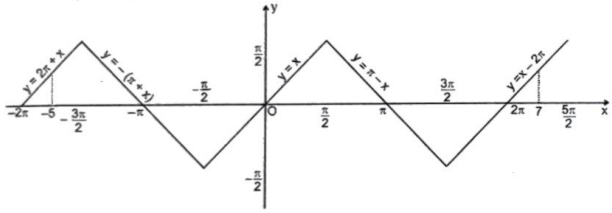
TRAIN YOUR BRAIN

Example 4 Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Sol. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \because 2\pi < 7 < \frac{5\pi}{2}$

aph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore$ $\sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because$ $-2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that $\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example 5: Evaluate following

(i) $\sin(\cos^{-1} 3/5)$ **(ii)** $\cos(\tan^{-1} 3/4)$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Sol. **(i)** Let $\cos^{-1} 3/5 = \theta$ then $\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore$ $\sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$ then $\tan\theta = 3/4 \Rightarrow \cos\theta = 4/5$

$\therefore$ $\cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right)$

$= \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

TRAIN YOUR BRAIN

Example 6: Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol. Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ .....(i)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

(i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because$ $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ $\quad$ if $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \Rightarrow y = -\frac{3}{2}$

Example 7: Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Sol. $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Example 8: Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$.

If y simplifies to $a\pi + b$ then find $(a - b)$.

Sol. $\sin^{-1}(\sin 8) = \sin^{-1}(\sin(3\pi - 8)) = 3\pi - 8$

$\tan^{-1}(\tan 10) = \tan^{-1}(\tan(10 - 3\pi)) = 10 - 3\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

$\sec^{-1}(\sec 9) = \sec^{-1}(\sec(9 - 2\pi)) = 9 - 2\pi$

$\cot^{-1}(\cot 6) = \cot^{-1}(\cot(6 - \pi)) = 6 - \pi$

$\text{cosec}^{-1}(\text{cosec } 7) = \text{cosec}^{-1}(\text{cosec}(7 - 2\pi)) = 7 - 2\pi$

$y = (3\pi - 8) + (3\pi - 10) + (4\pi - 12) + (2\pi - 9) + (-\pi + 6) + (2\pi - 7) = 13\pi - 40$

$\Rightarrow$ $a = 13$ and $b = -40 \Rightarrow a - b = 13 - (-40) = 53$

P-3 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $|x| \geq 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $|x| \geq 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x}, & x > 0 \\ \pi + \tan^{-1}\frac{1}{x}, & x < 0 \end{cases}$

P-4 (i) $\sin^{-1}(-x) = -\sin^{-1}x$; $-1 \leq x \leq 1$

(ii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$; $|x| \geq 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$; $x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$; $x \in R$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1}x$; $-1 \leq x \leq 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x$; $|x| \geq 1$

P-5 (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$; $|x| \leq 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$; $x \in R$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$; $|x| \geq 1$

Note : $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$

Example 9: Find the value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$.

Sol. $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$ and $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$

$= -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$

P-6 (i) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), x > 0, y > 0 \text{ and } xy < 1 \text{ (acute angle)} \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), x > 0, y > 0 \text{ and } xy > 1 \text{ (obtuse angle)} \end{cases}$$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$ $x > 0, y > 0$

Note : (i) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ (remember)

(ii) $\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 = \frac{\pi}{2}$ (remember)

(iii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z$

$$= \begin{cases} \tan^{-1}\left[\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right], \text{where } x > 0, y > 0, z > 0 \text{ and } xy+yz+zx < 1 \\ \pi + \tan^{-1}\left[\frac{x+y+z+xyz}{1-(xy+yz+zx)}\right], \text{where } x > 0, y > 0, z > 0 \text{ and } xy+yz+zx > 1 \end{cases}$$

(I) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(II) Parallelly we have $\sin^{-1}x = \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), x > 0; y > 0$

and $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right), x > 0, y > 0, x < y$

Example 10: Prove that,

$$\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3\sqrt{11}} + \sin^{-1}\frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

Sol. $\tan^{-1}\frac{1}{2\sqrt{2}} + \tan^{-1}\frac{1}{7\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}}$

$= \tan^{-1}\frac{\frac{1}{2\sqrt{2}} + \frac{1}{7\sqrt{2}}}{1 - \frac{1}{28}} + \tan^{-1}\frac{3}{\sqrt{2}}$

$= \tan^{-1}\frac{9\sqrt{2}}{27} + \tan^{-1}\frac{3}{\sqrt{2}}$

$= \tan^{-1}\frac{\sqrt{2}}{3} + \tan^{-1}\frac{3}{\sqrt{2}}$

$= \cot^{-1}\frac{3}{\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}} = \frac{\pi}{2}$

Example 11: Find the value of $\sin^{-1}\frac{3}{\sqrt{73}} + \cos^{-1}\frac{11}{\sqrt{146}} + \cot^{-1}\sqrt{3}$.

Sol. $\tan^{-1}\frac{3}{8} + \tan^{-1}\frac{5}{11} + \cot^{-1}\sqrt{3} = \tan^{-1}\left[\frac{\frac{3}{8} + \frac{5}{11}}{1 - \frac{3}{8}\cdot\frac{5}{11}}\right]$

$+ \cot^{-1}\sqrt{3} = \tan^{-1}(1) + \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

Example 12: If $\cos^{-1}x/2 + \cos^{-1}y/3 = \theta$, prove that $9x^2 + 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Sol. Let $\cos^{-1}x/2 = \alpha$, and $\cos^{-1}y/3 = \beta$

$\therefore$ $\cos\alpha = x/2$ and $\cos\beta = y/3$.

Given, $\alpha + \beta = \theta$ $\therefore$ $\cos(\alpha + \beta) = \cos\theta$

or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \theta$

or $\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}} = \cos\theta$

or $\frac{xy}{6} - \frac{\sqrt{4-x^2}.\sqrt{9-y^2}}{6} = \cos\theta$

or $(xy - 6\cos\theta)^2 = (4 - x^2)(9 - y^2)$

or $x^2y^2 + 36\cos^2\theta - 12xy\cos\theta = 36 - 9x^2 - 4y^2 + x^2y^2$

or $9x^2 - 12y\cos\theta + 4y^2 = 36(1 - \cos^2\theta)$

or $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$.

Example 13: Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$.

Sol. Let $\cos^{-1}\frac{a}{b} = \theta \therefore \cos\theta = \frac{a}{b}$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} + \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$$

$$= \frac{\left(1 + \tan\frac{\theta}{2}\right)^2 + \left(1 - \tan\frac{\theta}{2}\right)^2}{1 - \tan^2\frac{\theta}{2}}$$

$$= \frac{2\left(1 + \tan^2\frac{\theta}{2}\right)}{1 - \tan^2\frac{\theta}{2}} = \frac{2}{\cos\theta} = \frac{2b}{a} = \text{R.H.S.}$$

Example 14: Evaluate $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

Sol. Let $z = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

$\because \sin^{-1}\frac{4}{5} = \frac{\pi}{2} - \cos^{-1}\frac{4}{5}$

$\therefore z = \cos^{-1}\frac{12}{13} + \left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right) - \tan^{-1}\frac{63}{16}$.

$$z = \frac{\pi}{2} - \left(\cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}\right) - \tan^{-1}\frac{63}{16} \quad \text{.....(i)}$$

$\because \frac{4}{5} > 0, \frac{12}{13} > 0$ and $\frac{4}{5} < \frac{12}{13}$

$$\therefore \cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}$$

$$= \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \sqrt{1 - \frac{16}{25}}\sqrt{1 - \frac{144}{169}}\right]$$

$$= \cos^{-1}\left(\frac{63}{65}\right)$$

$\therefore$ equation (i) can be written as

$$z = \frac{\pi}{2} - \cos^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$$

$$z = \sin^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right) \quad \text{.....(ii)}$$

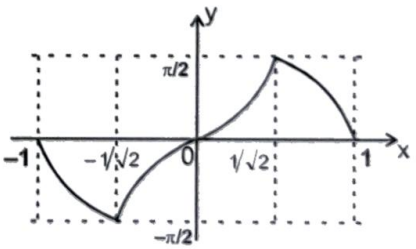
$\because \sin^{-1}\left(\frac{63}{65}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

$\therefore$ from equation (ii), we get

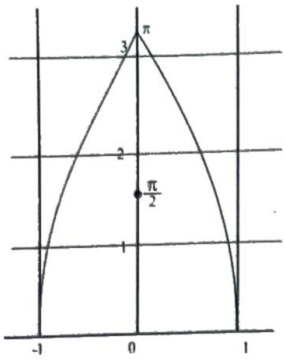
$\therefore z = \tan^{-1}\left(\frac{63}{16}\right) - \tan^{-1}\left(\frac{63}{16}\right) \Rightarrow z = 0$

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

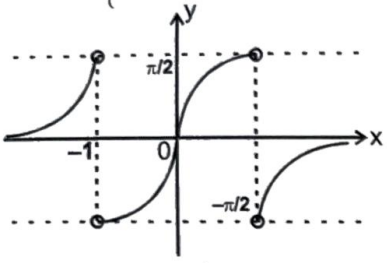
(i) $\sin^{-1}2x\sqrt{1-x^2} = \begin{cases} -\pi - 2\sin^{-1}x & ; -1 \le x \le \frac{-1}{\sqrt{2}} \\ 2\sin^{-1}x & ; \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & ; \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$



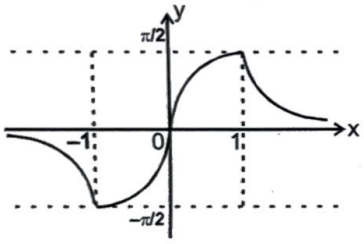
(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & ; \quad 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & ; -1 \le x \le 0 \end{cases}$



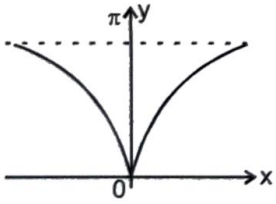
(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & ; \ x < -1 \\ 2\tan^{-1}x & ; \ -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; \ x > 1 \end{cases}$



(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1}x & ; \ x < -1 \\ 2\tan^{-1}x & ; \ -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & ; \ x > 1 \end{cases}$



(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & ; \ x \ge 0 \\ -2\tan^{-1}x & ; \ x \le 0 \end{cases}$



Example 15: Show that $\cos^{-1}\dfrac{1}{7} + 2\cot^{-1}\dfrac{1}{3} = \dfrac{5\pi}{4}$

Sol. $\cot^{-1}\dfrac{1}{7} + 2\cot^{-1}\dfrac{1}{3}$

$= \dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{7} + 2\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{3}\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + 2\tan^{-1}\dfrac{1}{3}\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + 2\tan^{-1}\dfrac{2.1/3}{1-(1/3)^2}\right)\left(\because \dfrac{1}{3} < 1\right)$

$= \dfrac{3\pi}{2} - \left(\tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{3}{4}\right)$

$= \dfrac{3\pi}{2} - \tan^{-1}\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}}\left(\because \dfrac{1}{7}\cdot\dfrac{3}{4} < 1\right)$

$= \dfrac{3\pi}{2} - \tan^{-1}1 = \dfrac{3\pi}{2} - \dfrac{\pi}{4} = \dfrac{5\pi}{4}$

Example 16: Evaluate **(i)** $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{2}{3}\right)$ **(ii)** $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right)$

Sol. **(i)** Let $\cos^{-1}\dfrac{2}{3} = \theta$. Then $\cos\theta = \dfrac{2}{3}$ and $0 < \theta < \dfrac{\pi}{2}$

Now $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{2}{3}\right) = \tan\left(\dfrac{1}{2}\theta\right)$

$= \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} = \dfrac{1-2/3}{\sqrt{1-(2/3)^2}} = \dfrac{1/3}{\frac{\sqrt{9-4}}{3}} = \dfrac{1}{\sqrt{5}}$

Alter : $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-2/3}{1+2/3} = \dfrac{1}{5}$

$\therefore \ \tan\dfrac{\theta}{2} = \dfrac{1}{\sqrt{5}}. \left[\text{As} \ \ 0 < \dfrac{\theta}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}\right]$

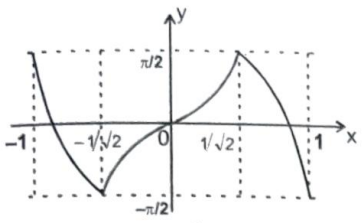
(ii) Let $\tan^{-1}\dfrac{1}{5} = \theta$. Then $\tan\theta = \dfrac{1}{5}$ and $0 < \theta < \dfrac{\pi}{2}$

Now $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right) = \tan\left(2\theta - \dfrac{\pi}{4}\right)$

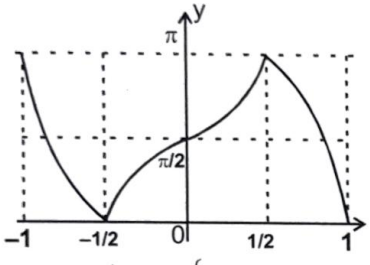
$= \dfrac{\tan 2\theta - 1}{1+\tan 2\theta} = \dfrac{\frac{2\tan\theta}{1-\tan^2\theta} - 1}{1+\frac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{2\tan\theta - 1 + \tan^2\theta}{1 - \tan 2\theta + 2\tan\theta} = \dfrac{\frac{2}{5} - 1 + \frac{1}{25}}{1 - \frac{1}{25} + \frac{2}{5}} = \dfrac{10 - 25 + 1}{25 - 1 + 10} = \dfrac{-14}{34} = \dfrac{-7}{17}$.

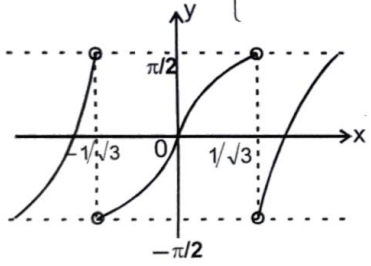
(vi) $\sin^{-1}(3x - 4x^3) = \begin{cases} -\pi - 3\sin^{-1}x & ; \ -1 \le x \le \dfrac{-1}{2} \\ 3\sin^{-1}x & ; \ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & ; \ \dfrac{1}{2} \le x \le 1 \end{cases}$



(vii) $\cos^{-1}(4x^3-3x) = \begin{cases} -2\pi + 3\cos^{-1}x & ; \ -1 \le x \le \dfrac{-1}{2} \\ 2\pi - 3\cos^{-1}x & ; \ -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x & ; \ \dfrac{1}{2} \le x \le 1 \end{cases}$



(viii) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \begin{cases} \pi + 3\tan^{-1}x & ; \ x < \dfrac{-1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \ \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \ x > \dfrac{1}{\sqrt{3}} \end{cases}$



EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

TRAIN YOUR BRAIN

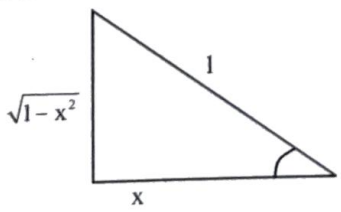
Example 17: Solve $\cos^{-1}x\sqrt{3} + \cos^{-1}x = \pi/2$.

Sol. Given, $\cos^{-1}x\sqrt{3} + \cos^{-1}x = \pi/2$ ..(1)

or $\cos^{-1}x\sqrt{3} = \pi/2 - \cos^{-1}x$

or $\cos\cos^{-1}x\sqrt{3} = \cos(\pi/2 - \cos^{-1}x)$

or $x\sqrt{3} = \sin\cos^{-1}x$ or $x\sqrt{3} = \sin\sin^{-1}\sqrt{1-x^2}$



or $x\sqrt{3} = \sqrt{1-x^2}$ Squaring we get $3x^2 = 1 - x^2$

or $4x^2 = 1 \Rightarrow x = \pm 1/2$

Verification : When $x = 1/2$

L.H.S. of equation $= \cos^{-1}(\sqrt{3}/2) + \cos^{-1}(1/2)$

$= \pi/6 + \pi/3 + \pi/2 =$ R.H.S. of equation

When $x = -1/2$.

L.H.S. of equation $= \cos^{-1}(-\sqrt{3}/2) + \cos^{-1}(-1/2)$

$= \pi - \cos^{-1}(\sqrt{3}/2) + \pi - \cos^{-1}(1/2)$

$= \pi - \pi/6 + \pi - \pi/3 = 3\pi/2 \neq$ R.H.S. of equation

$\therefore$ $x = 1/2$ is the only solution

Example 18: Solve for x : $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$.

Sol. We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x.\cot^{-1}x = \dfrac{5\pi^2}{8}$

$\Rightarrow \left(\dfrac{\pi}{2}\right)^2 - 2\tan^{-1}x.(\pi/2 - \tan^{-1}x) = \dfrac{5\pi^2}{8}$

$\Rightarrow \dfrac{\pi^2}{4} - 2.\dfrac{\pi}{2}.\tan^{-1}x + 2(\tan^{-1}x)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \dfrac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1}x = -\pi/4, 3\pi/4 \Rightarrow \tan^{-1}x = -\pi/4$; $x = -1$ {neglecting $\tan^{-1}x = 3\pi/4$ as $\tan^{-1}x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ }

INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

TRAIN YOUR BRAIN

Example 19: Find the interval in which $\cos^{-1}x > \sin^{-1}x$.

Sol. We have, $\cos^{-1}x > \sin^{-1}x$ {for $\cos^{-1}x$ to be real; $x \in [-1, 1]$}

$\Rightarrow 2\cos^{-1}x > \pi/2$

$\Rightarrow \cos^{-1}x > \pi/4$ or $\cos(\cos^{-1}x) < \cos\pi/4$

$\Rightarrow x < \dfrac{1}{\sqrt{2}}$ $\therefore x \in \left(-1, \dfrac{1}{\sqrt{2}}\right)$

Example 20: Find the solution set of the inequation $\sin^{-1}(\sin 5) > x^2 - 4x$

Sol. $\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

$\Rightarrow x \in \left(2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi}\right)$

TRAIN YOUR BRAIN

Example 21: Sum the series, $\tan^{-1}\frac{4}{1+3\cdot4}+\tan^{-1}\frac{6}{1+8\cdot9}+\tan^{-1}\frac{8}{1+15\cdot16}+\ldots\ldots$ to 'n' terms.

Sol. $T_n=\tan^{-1}\frac{2(n+1)}{1+\{(n+1)^2-1\}\{(n+1)^2\}}$

$=\tan^{-1}\frac{2n+2}{1+(n^2+2n)(n+1)^2}$

$=\tan^{-1}\left[\frac{2n+2}{1+n(n+2)(n+1)(n+1)}\right]$

$=\tan^{-1}\left[\frac{(n+1)(n+2)-n(n+1)}{1+\{n(n+1)\}\{(n+1)(n+2)\}}\right]$

$=\tan^{-1}(n+1)(n+2)-\tan^{-1}n(n+1)$

Put $n=1, 2, 3, \ldots\ldots, n$ and add, we get $S_n=\tan^{-1}(n+1)(n+2)-\tan^{-1}2$

Example 22: Sum the series upto 'n' terms,

$\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{2}{16}+\tan^{-1}\frac{2}{25}+\ldots..$ to n terms.

Also show that, $S_\infty=\tan^{-1}3$.

Sol. $T_n=\tan^{-1}\frac{2}{n^2+2n+1}$

$=\tan^{-1}\frac{(n+2)-n}{1+n(n+2)}$

$=\tan^{-1}(n+2)-\tan^{-1}(n)$

Hence,

$S_n=\tan^{-1}(n+2)+\tan^{-1}(n+1)-(\tan^{-1}1+\tan^{-1}2)$

$S_\infty=\lim_{n\to\infty}S_n=\frac{\pi}{2}+\frac{\pi}{2}-\left(\pi+\tan^{-1}\left(\frac{1+2}{1-2}\right)\right)=\tan^{-1}3$

Topicwise Questions

GRAPHS, DOMAIN & PRINCIPLE VALUE RANGE OF ITF

1. The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

(a) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (b) $\frac{-\pi}{3}$

(c) $\frac{-2\pi}{3}$ (d) $\frac{5\pi}{3}$

PROPERTIES OF ITF (I - VI)

2. The value of $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$ is

(a) -2 (b) $8\pi - 26$

(c) $4\pi + 2$ (4) 2

3. If $\tan^{-1} x - \tan^{-1} y = \tan^{-1} A$, then $A =$

(a) $x - y$ (b) $x + y$

(c) $\frac{x-y}{1+xy}$ (d) $\frac{x+y}{1-xy}$

4. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) =$

(a) $\frac{7\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{2}$

5. If $\sin^{-1} x = \theta + \beta$ and $\sin^{-1} y = \theta - \beta$, then $1 + xy =$

(a) $\sin^2\theta + \sin^2\beta$ (b) $\sin^2\theta + \cos^2\beta$

(c) $\cos^2\theta + \cos^2\beta$ (d) $\cos^2\theta + \sin^2\beta$

6. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right] =$

(a) $\frac{\pi}{4} - \frac{x}{2}$ (b) $\frac{\pi}{4} + \frac{x}{2}$

(c) $\frac{x}{2}$ (d) $\frac{\pi}{4} - x$

7. $\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3) =$

(a) 5 (b) 13

(c) 15 (d) 6

8. The value of x which satisfies the equation $\tan^{-1} x = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(a) 3 (b) –3

(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

9. $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] =$

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

10. $\cot^{-1} 3 + \text{cosec}^{-1}\sqrt{5} =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

11. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\cos\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$

(c) $\frac{10\pi}{3}$ (d) 0

12. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{12}$ (d) $\frac{\pi}{6}$

13. The value of $\tan\left(\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}\right)$ is

(a) 5/6 (b) 7/6

(c) 1/6 (d) 1/7

14. $\cos\left[2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right] =$

(a) $\frac{2\sqrt{6}}{5}$ (b) $-\frac{2\sqrt{6}}{5}$

(c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

15. $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}=$ π

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) 2π

SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

16. $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) π

17. $\tan(\cos^{-1}x)$ is equal to

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{1+x^2}$

(c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\sqrt{1-x^2}$

18. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=$

(a) $\frac{17}{7}$ (b) $-\frac{17}{7}$

(c) $\frac{7}{17}$ (d) $-\frac{7}{17}$

19. $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=$

(a) $\cot^{-1}\sqrt{x}$ (b) $\tan^{-1}\sqrt{x}$

(c) $\tan^{-1}x$ (d) $\cot^{-1}x$

20. $\sin\left(4\tan^{-1}\frac{1}{3}\right)=$

(a) $\frac{12}{25}$ (b) $\frac{24}{25}$

(c) $\frac{1}{5}$ (d) $\frac{25}{24}$

21. The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)+\cos(\tan^{-1}2\sqrt{2})=\frac{16}{15}$

(a) $\frac{16}{15}$ (b) $\frac{14}{15}$

(c) $\frac{12}{15}$ (d) $\frac{11}{15}$

22. The solution of $\sin^{-1}x-\sin^{-1}2x=\pm\frac{\pi}{3}$ is

(a) $\pm\frac{1}{3}$ (b) $\pm\frac{1}{4}$

(c) $\pm\frac{\sqrt{3}}{2}$ (d) $\pm\frac{1}{2}$

23. $\sin\left[3\sin^{-1}\left(\frac{1}{5}\right)\right]=$

(a) 71/125 (b) 74/125

(c) 3/5 (d) 1/2

EQUATIONS, INEQUATIONS INVOLVING ITF, SUMMATION OF SERIES

24. A solution of the equation $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$ is

(a) $x=1$ (b) $x=-1$

(c) $x=0$ (d) $x=\pi$

25. If $\tan(x+y)=33$ and $x=\tan^{-1}3$, then y will be

(a) 0.3 (b) $\tan^{-1}(1.3)$

(c) $\tan^{-1}(0.3)$ (d) $\tan^{-1}\left(\frac{1}{18}\right)$

26. If $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$, then $x=$

(a) $\sqrt{2}$ (b) 3

(c) $\sqrt{3}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

27. $\sin^{-1}x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$,

(a) 0 (b) $\frac{1}{\sqrt{5}}$

(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

28. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=$

(a) 0 (b) 1

(c) $\frac{1}{xyz}$ (d) xyz

29. For the equation $\cos^{-1}x+\cos^{-1}2x+\pi=0$, the number of real solution is

(a) 1 (b) 2

(c) 0 (d) ∞

30. The equation $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$ has

(a) No solution (b) Only one solution

(c) Two solutions (d) Three solutions

Learning Plus

1. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is equal to

(*a*) $\frac{5\pi}{6}$ (*b*) $\frac{7\pi}{12}$

(*c*) $\frac{5\pi}{12}$ (*d*) $\frac{3\pi}{5}$

2. Domain of $f(x)=\cos^{-1}x+\cot^{-1}x+\operatorname{cosec}^{-1}x$ is

(*a*) $[-1,1]$ (*b*) R

(*c*) $(-\infty,-1]\cup[1,\infty)$ (*d*) $\{-1,1\}$

3. Range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$ is

(*a*) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ (*b*) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

(*c*) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (*d*) none of these

4. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}]=y$, then

(*a*) $y=\frac{4}{5}$ (*b*) $y=\frac{2}{\sqrt{5}}$

(*c*) $y=-\frac{2}{\sqrt{5}}$ (*d*) $y^2=\frac{10}{11}$

5. If $\sum_{i=1}^{n}\cos^{-1}\alpha_i=0$, then $\sum_{i=1}^{n}\alpha_i$

(*a*) n (*b*) –n

(*c*) 0 (*d*) None of these

6. If $\cot^{-1}\frac{n}{\pi}>\frac{\pi}{6}$, $n\in N$, then the maximum value of 'n' is

(*a*) 1 (*b*) 5

(*c*) 9 (*d*) None of these

7. $\operatorname{cosec}^{-1}(\cos x)$ is real if

(*a*) $x\in[-1,1]$

(*b*) $x\in R$

(*c*) x is an odd multiple of $\frac{\pi}{2}$

(*d*) x is an integral multiple of π

8. If $\pi\le x\le 2\pi$, then $\cos^{-1}(\cos x)$ is equal to

(*a*) x (*b*) $\pi-x$

(*c*) $2\pi+x$ (*d*) $2\pi-x$

9 The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to

(*a*) 3/4 (*b*) –3/4

(*c*) 1/16 (*d*) 1/4

10. The value of $\sin^{-1}[\cos\{\cos^{-1}(\cos x)+\sin^{-1}(\sin x)\}]$, where $x\in\left(\frac{\pi}{2},\pi\right)$ is

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{\pi}{4}$

(*c*) $-\frac{\pi}{4}$ (*d*) $-\frac{\pi}{2}$

11. If $x\ge 0$ and $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x$, then

(*a*) $\frac{\pi}{2}\le\theta\le\frac{3\pi}{4}$ (*b*) $0<\theta\le\frac{\pi}{4}$

(*c*) $0<\theta\le\frac{\pi}{2}$ (*d*) $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

12. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then $\cos^{-1}x+\cos^{-1}y$ is equal to

(*a*) $\frac{2\pi}{3}$ (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{6}$ (*d*) π

13. The equation $\sin^{-1}x-\cos^{-1}x=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has :

(*a*) no solution
(*b*) unique solution
(*c*) infinite number of solutions
(*d*) none of these

14. The solution of the equation $\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)-\frac{\pi}{6}=0$ is

(1) x=2 (*b*) x=–4

(*c*) x=4 (*d*) none of these

15. If $x<0$ then value of $\tan^{-1}(x)+\tan^{-1}\left(\frac{1}{x}\right)$ is equal to

(*a*) $\frac{\pi}{2}$ (*b*) $-\frac{\pi}{2}$

(*c*) 0 (*d*) none of these

16. The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(*a*) $\frac{6}{17}$ (*b*) $\frac{7}{16}$

(*c*) $\frac{5}{7}$ (*d*) $\frac{17}{6}$

17. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right), x\neq 0$ is equal to

(a) x (b) 2x

(c) $\frac{2}{x}$ (d) $\frac{x}{2}$

18. $\tan^{-1}a+\tan^{-1}b$, where $a>0, b>0, ab>1$, is equal to

(a) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)-\pi$

(c) $\pi+\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (d) $\pi-\tan^{-1}\left(\frac{a+b}{1-ab}\right)$

19. $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) none of these

20. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}, \frac{\pi}{2}<x<\pi$, is

(a) $\pi-\frac{x}{2}$ (b) $\frac{\pi}{2}+\frac{x}{2}$

(c) $\frac{x}{2}$ (d) $2\pi-\frac{x}{2}$

21. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $0\le x\le 1$ are

(a) $0,\pi$ (b) $0,\frac{\pi}{4}$

(c) $-\frac{\pi}{4},\frac{\pi}{4}$ (d) $\frac{\pi}{4},\frac{\pi}{2}$

22. Which one of the following correct ?

(a) $\tan 1>\tan^{-1}1$ (b) $\tan 1<\tan^{-1}1$

(c) $\tan 1=\tan^{-1}1$ (d) None of these

23. The number of solutions of the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is

(a) 0 (b) 1

(c) 2 (d) 3

24. $\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{5}{13}\right)$ is equal to

(a) $\cos^{-1}\left(\frac{33}{65}\right)$ (b) $\cos^{-1}\left(-\frac{33}{65}\right)$

(c) $\cos^{-1}\left(\frac{64}{65}\right)$ (d) none of these

25. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then $\tan\theta$ is equal to

(a) 1/3 (b) 3

(c) 1 (d) -1

26. The set of values of 'x' for which the formula $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$ is true, is

(a) $(-1,0)$ (b) $[0,1]\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$

(c) $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ (d) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

27. $\cos^{-1}\left\{\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right\}=\cos^{-1}\frac{x}{2}-\cos^{-1}x$ holds for

(a) $|x|\le 1$ (b) $x\in R$

(c) $0\le x\le 1$ (d) $-1\le x\le 0$

28 All possible values of p and q for which $\cos^{-1}\sqrt{p}+\cos^{-1}(\sqrt{1-p})+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}$ holds, is

(a) $p=1, q=\frac{1}{2}$ (b) $q>1, p=\frac{1}{2}$

(c) $0\le p\le 1, q=\frac{1}{2}$ (d) None of these

29. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then a value of x is

(a) 1 (b) 3

(c) 4 (d) 5

30. The value of x satisfying $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$ are

(a) $0,\frac{1}{2}$ (b) 0

(c) $1,-1$ (d) None of these

31. The number of solution(s)of the equation $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, is / are

(a) 0 (b) 1

(c) 2 (d) more than two

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for

(a) $\sin^{-1}\alpha$ (b) $\cos^{-1}\alpha$

(c) $\sec^{-1}\alpha$ (d) $\operatorname{cosec}^{-1}\alpha$

2 α, β and γ are three angles given by $\alpha = 2\tan^{-1}(\sqrt{2}-1)$, $\beta = 3\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right)$ and $\gamma = \cos^{-1}\frac{1}{3}$. Then

(a) $\alpha > \beta$ (b) $\beta > \gamma$

(c) $\alpha < \gamma$ (d) $\alpha > \gamma$

3. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then

(a) $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = 0$

(b) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(c) $x^{50} + y^{25} + z^5 = 0$

(d) $\frac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

4. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(-\frac{14\pi}{5}\right)\right\}\right]$ is:

(a) $\cos\left(-\frac{7\pi}{5}\right)$ (b) $\sin\left(\frac{\pi}{10}\right)$

(c) $\cos\left(\frac{2\pi}{5}\right)$ (d) $-\cos\left(\frac{3\pi}{5}\right)$

5. If $6\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \pi$, then

(a) $x=1$ (b) $x=2$

(c) $x=3$ (d) $x=4$

6. $\sin^{-1}x > \cos^{-1}x$ holds for

(a) all values of x (b) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

(c) $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ (d) $x = 0.75$

7. Let $f(x) = \sin^{-1}x + \cos^{-1}x$. Then $\frac{\pi}{2}$ is equal to

(a) $f\left(\frac{1}{2}\right)$ (b) $f(k^2 - 2k + 3), k \in R$

(c) $f\left(\frac{1}{1+k^2}\right), k \in R$ (d) $f(-2)$

8. If $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, then x may be

(a) 1 (b) $-\frac{1}{2}$

(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

9. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to:

(a) $\frac{1}{2}\cos^{-1}x$ (b) $\cos^{-1}\sqrt{\frac{1+x}{2}}$

(c) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (d) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

10. If the numerical value of $\tan(\cos^{-1}(4/5) + \tan^{-1}(2/3))$ is a/b in the simplest form then

(a) $a+b=23$ (b) $a-b=11$

(c) $3b=a+1$ (d) $2a=3b$

11. $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4 - 2n^2 + 2}$ is equal to:

(a) $\tan^{-1}2 + \tan^{-1}3$ (b) $4\tan^{-1}1$

(c) $\pi/2$ (d) $\sec^{-1}\left(-\sqrt{2}\right)$

12. For the equation $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$, which of the following is invalid ?

(a) $a^2x + 2a = x$ (b) $a^2 + 2ax + 1 = 0$

(c) $a \neq 0$ (d) $a \neq -1, 1$

13 If $a = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ and $b = \tan^{-1}(-\sqrt{3}) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$, then

(a) $a-b = 17\pi/12$ (b) $a+b = 17\pi/12$

(c) $a+b = -7\pi/12$ (d) $a-b = \pi/12$

14. If $\cos^{-1}x = \tan^{-1}x$, then

(a) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$

(b) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(c) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

(d) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

Comprehension – 1 (No. 15 to 17)

$$\tan^{-1}(\tan\theta)=\begin{cases}\pi+\theta & , -\frac{3\pi}{2}<\theta<-\frac{\pi}{2}\\ \theta & , -\frac{\pi}{2}<\theta<\frac{\pi}{2}\\ -\pi+\theta & , \frac{\pi}{2}<\theta<\frac{3\pi}{2}\end{cases},$$

$$\sin^{-1}(\sin\theta)=\begin{cases}-\pi-\theta & , -\frac{3\pi}{2}\le\theta<-\frac{\pi}{2}\\ \theta & , -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ \pi-\theta & , \frac{\pi}{2}<\theta\le\frac{3\pi}{2}\end{cases},$$

$$\cos^{-1}(\cos\theta)=\begin{cases}-\theta & , -\pi\le\theta<0\\ \theta & , 0\le\theta\le\pi\\ 2\pi-\theta & , \pi<\theta\le 2\pi\end{cases}$$

Based on the above results, answer each of the following :

15. $\cos^{-1}x$ is equal to

(*a*) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<1$

(*b*) $-\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*c*) $\sin^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*d*) $\sin^{-1}\sqrt{1-x^2}$ if $0<x<1$

16. $\sin^{-1}x$ is equal to

(*a*) $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<0$

(*b*) $\cos^{-1}\sqrt{1-x^2}$ if $-1<x<1$

(*c*) $\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$

(*d*) $-\cos^{-1}\sqrt{1-x^2}$ if $0<x<1$

17. $\cos^{-1}x$ is equal to

(*a*) $-\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

(*b*) $\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

(*c*) $-\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $0<x<1$

(*d*) $\pi+\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1<x<0$

Comprehension – 2 (No. 18 to 20)

The domain and range of inverse circular function are defined as follows

	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\operatorname{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-(\pi)$
$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$

18. $\sin^{-1}x<\frac{3\pi}{4}$ then solution set of x is

(*a*) $\left(\frac{1}{\sqrt{2}},1\right]$ (*b*) $\left(-\frac{1}{\sqrt{2}},-1\right]$

(*c*) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ (*d*) None of these

19. $\sin^{-1}x+\operatorname{cosec}^{-1}x$ at $x=-1$ is

(*a*) π (*b*) 2π

(*c*) 3π (*d*) $-\pi$

20. If $x\in[-1, 1]$, then range of $\tan^{-1}(-x)$ is

(*a*) $\left[\frac{3\pi}{4},\frac{7\pi}{4}\right]$ (*b*) $\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$

(*c*) $[-\pi, 0]$ (*d*) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

21. Match the column

Column - I	Column - II
(*a*) $\sin^{-1}x-\cos^{-1}x$ is maximum at	(p) $x=-1$
(*b*) $(\sin^{-1}x)^2+(\cos^{-1}x)^2$ is minimum at	(q) $x=-\frac{1}{\sqrt{2}}$
(*c*) $(\tan^{-1}x)^2+(\cot^{-1}x)^2$ is minimum at	(r) $x=0$
(*d*) $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ is maximum at	(s) $x=\frac{1}{\sqrt{2}}$
	(t) $x=1$

22. Match the Column

Column - I	Column - II
(a) The value of $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$ is	(p) $\frac{17}{6}$
(b) The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ is	(q) $\frac{-7}{17}$
(c) The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is	(r) $\frac{3}{4}$
(d) The value of $\left(\tan^{-1}\left(\sin\left(\cot^{-1}\left(\frac{1}{2}\right)\right)\right)\right)$ is	(s) $\frac{\sqrt{5}}{3}$

23. Match the column

Column – I	Column – II
(a) $f(x)=\sin^{-1}\left(\frac{2}{\lvert\sin x-1\rvert+\lvert\sin x+1\rvert}\right)$	(P) f(x) is many one
(b) $f(x)=\cos^{-1}(\lvert x-1\rvert-\lvert x-2\rvert)$	(Q) Domain of f(x) is R
(c) $f(x)=\sin^{-1}\left(\frac{\pi}{\lvert\sin^{-1}x-(\pi/2)\rvert+\lvert\sin^{-1}x+(\pi/2)\rvert}\right)$	(R) Range contain only irrational number
(d) $f(x)=\cos(\cos^{-1}\lvert x\rvert)$ $+\sin^{-1}(\sin x)-\cos ec^{-1}(\cos ecx)+\cos ec^{-1}\lvert x\rvert$	(S) f(x) is even.

NUMERICAL BASED QUESTIONS

24. If $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$, then x is equal to_____.

25. The sum of roots of the function

$$\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{1+4x}=\tan^{-1}\frac{2}{x^2}$$

26. Solution of equation $\cot^{-1}x+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$ is

27. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals

28. The value of $\left\{\tan\left(\cos^{-1}\left(-\frac{2}{7}\right)-\pi/2\right)\right\}$ is $\frac{a}{b\sqrt{c}}$ then find value of a + b + c

29. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. If $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}, x>\frac{3}{4}$ then x is equal to :

[JEE Main - 2019 (January)]

(a) $\frac{\sqrt{145}}{12}$ (b) $\frac{\sqrt{145}}{10}$

(c) $\frac{\sqrt{146}}{12}$ (d) $\frac{\sqrt{145}}{11}$

2. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to :

[JEE Main - 2019 (January)]

(a) π (b) 7π

(c) 0 (d) 10

3. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is :

[JEE Main - 2019 (January)]

(a) $\frac{21}{19}$ (b) $\frac{19}{21}$

(c) $\frac{22}{23}$ (d) $\frac{23}{22}$

4. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval:

[JEE Main - 2019 (January)]

(a) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

(b) $(\cot 2, \infty)$

(c) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

(d) $(\cot 5, \cot 4)$

5. Considering only the principal values of inverse functions, the set

$$A = \left\{x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}\right\}$$

[JEE Main - 2019 (January)]

(a) contains two elements

(b) contains more than two elements

(c) is a singleton

(d) is an empty set

6. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$,

[JEE Main - 2019(April)]

where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to :

(a) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\tan^{-1}\left(\frac{9}{14}\right)$

(c) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (d) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

7. The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to :

[JEE Main - 2019(April)]

(a) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$ (b) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

(c) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$ (d) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$

8. If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, $x \leq \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to

[JEE Main - 2019(April)]

(a) $4\sin^2\alpha - 2x^2y2$ (b) $4\cos^2\alpha + 2x^2y2$

(c) $4\sin^2\alpha$ (d) $2\sin^2\alpha$

9. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ is equal to :

[JEE Main-2020 (September)]

(a) $\frac{\pi}{2}$ (b) $\frac{7\pi}{4}$

(c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{4}$

10. If S is the sum of the first 10 terms of the series

[JEE Main-2020 (September)]

$$S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \ldots,$$

then tan (S) is equal to :

(a) $-\frac{6}{5}$ (b) $\frac{5}{11}$

(c) $\frac{10}{11}$ (d) $\frac{5}{6}$

11. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy [JEE Main-2021 (March)]

$$\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$$ is equal to :

(a) 2 (b) 0

(c) 3 (d) 1

12. The sum of possible values of x for

$$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$$ is :

[JEE Main-2021 (March)]

(a) $-\frac{32}{4}$ (b) $-\frac{33}{4}$

(c) $-\frac{31}{4}$ (d) $-\frac{30}{4}$

13. If $\cot^{-1}(\alpha) = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots\ldots$ upto 100 terms, then α is : [*JEE Main-2021 (March)*]

(*a*) 1.02 (*b*) 1.03

(*c*) 1.01 (*d*) 1.00

14. The number of solutions of the equation

$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1,1]$, and [x] denotes the greatest integer less than or equal to x, is : [*JEE Main-2021 (March)*]

(*a*) 0 (*b*) Infinite

(*c*) 2 (*d*) 4

JEE-ADVANCED PREVIOUS YEARS

1. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is [*JEE Advanced-2013*]

(*a*) $\frac{23}{25}$

(*b*) $\frac{25}{23}$

(*c*) $\frac{23}{24}$

(*d*) $\frac{24}{23}$

2. Match List I with List II and select the correct answer using the code given below the lists : [*JEE Advanced- 2013*]

List - I	List - II
P. $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$ takes value	1. $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\frac{x-y}{2}$ is	2. $\sqrt{2}$
R. If $\cos\left(\frac{\pi}{4}-x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4}+x\right)\cos 2x$ then possible value of sec x is	3. $\frac{1}{2}$
S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then possible value of x is	4. 1

Codes :

	P	Q	R	S
(*a*)	4	3	1	2
(*b*)	4	3	2	1
(*c*)	3	4	2	1
(*d*)	3	4	1	2

3. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is [*JEE Advanced-2014*]

4. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) [*JEE Advanced -2015*]

(*a*) $\cos\beta > 0$ (*b*) $\sin\beta < 0$

(*c*) $\cos(\alpha+\beta) > 0$ (*d*) $\cos\alpha < 0$

5. The number of real solutions of the equation

$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is _____ (Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

[*JEE Advanced - 2018*]

6. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \Sigma_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0,\infty)$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TURE?

[*JEE Advanced - 2018*]

(*a*) $\Sigma_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$)

(*b*) $\Sigma_{j=1}^{10}\left(1+f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(*c*) For any fixed positive integer n, $\lim_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(*d*) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

7. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes value $\sin\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$ and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

[*JEE Advanced - 2018*]

List-I

(P) The range of f is

(Q) The range of g contains

(R) The domain of f contains

(S) The domain of g is

List-II

(1) $\left(-\infty, \frac{1}{1-e}\right] \cup \leq \left[\frac{e}{e-1}, \infty\right)$

(2) $(0, 1)$

(3) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(4) $(-\infty, 0) \cup (0, \infty)$

(5) $\left(-\infty, \frac{e}{e-1}\right]$

(6) $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

(*a*) (P) $\to$ 4 ; (Q) $\to$ 2 ; (R) $\to$ 1 ; (S) $\to$ 1

(*b*) (P) $\to$ 3 ; (Q) $\to$ 3 ; (R) $\to$ 6 ; (S) $\to$ 5

(*c*) (P) $\to$ 4 ; (Q) $\to$ 2 ; (R) $\to$ 1 ; (S) $\to$ 6

(*d*) (P) $\to$ 4 ; (Q) $\to$ 3 ; (R) $\to$ 6 ; (S) $\to$ 5

8. The value of

$$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right) \text{ in the interval } \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right] \text{ equals}$$

[*JEE Advanced-2019*]

ANSWER KEY

Topicwise Questions

1. (*b*)	**2.** (*b*)	**3.** (*c*)	**4.** (*b*)	**5.** (*b*)	**6.** (*a*)	**7.** (*c*)	**8.** (*a*)	**9.** (*c*)	**10.** (*b*)
11. (*a*)	**12.** (*d*)	**13.** (*d*)	**14.** (*b*)	**15.** (*a*)	**16.** (*a*)	**17.** (*a*)	**18.** (*d*)	**19.** (*b*)	**20.** (*b*)
21. (*b*)	**22.** (*d*)	**23.** (*a*)	**24.** (*c*)	**25.** (*c*)	**26.** (*c*)	**27.** (*b*)	**28.** (*b*)	**29.** (*c*)	**30.** (*a*)

Learning Plus

1. (*b*)	**2.** (*d*)	**3.** (*c*)	**4.** (*b*)	**5.** (*a*)	**6.** (*b*)	**7.** (*d*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*d*)	**12.** (*b*)	**13.** (*b*)	**14.** (*c*)	**15.** (*b*)	**16.** (*d*)	**17.** (*c*)	**18.** (*c*)	**19.** (*a*)	**20.** (*b*)
21. (*b*)	**22.** (*a*)	**23.** (*c*)	**24.** (*b*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*c*)	**29.** (*b*)	**30.** (*a*)
31. (*b*)									

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*c,d*)	**2.** (*b,c*)	**3.** (*a,b*)	**4.** (*b, c, d*)	**5.** (*b, d*)	**6.** (*c, d*)	**7.** (*a,c*)	**8.** (*a,c,d*)	**9.** (*a, b, c*)	**10.** (*a,b,c*)
11. (*a, d*)	**12.** (*b,c*)	**13.** (*a,c*)	**14.** (*a, c*)	**15.** (*d*)	**16.** (*c*)	**17.** (*d*)	**18.** (*d*)	**19.** (*a*)	**20.** (*b*)

21. (*a*) → (t), (*b*) → (s), (*c*) → (t), (*d*) → (p) **22.** (*a*) → (p), (*b*) → (q), (*c*) → (r) (*d*) → (s)

23. (*a*) → (p, q, r, s); (*b*) → (p); (*c*) → (P,R,S); (*d*) → (p) **24.** (13) **25.** (3) **26.** (3) **27.** (1) **28.** (10)

29. (2)

JEE Mains & Advanced Past Years Questions

JEE-MAINS PREVIOUS YEARS

1. (*a*)	**2.** (*a*)	**3.** (*a*)	**4.** (*b*)	**5.** (*c*)	**6.** (*a*)	**7.** (*c*)	**8.** (*c*)	**9.** (*c*)	**10.** (*d*)
11. (*c*)	**12.** (*a*)	**13.** (*c*)	**14.** (*a*)						

PRAYAS 1.0 t.me/realprayasians.

JEE-ADVANCED PREVIOUS YEARS

1. (*b*)	**2.** (*b*)	**3.** [3]	**4.** (*bcd*)	**5.** (2)	**6.** (*d*)	**7.** (*a*)	**8.** (0.00)

PRAYAS 2.0 t.me/realprayasians2

23 APPLICATION OF DERIVATIVES

DERIVATIVE AS RATE OF CHANGE

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of change of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition: The average rate of change of a function f(x) with respect to x over an interval [a, a + h] is defined as

$$\frac{f(a+h)-f(a)}{h}.$$

The instantaneous rate of change of f(x) with respect to x is defined as $f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists.

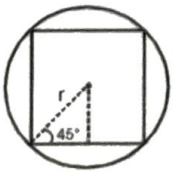
Note: To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

TRAIN YOUR BRAIN

Example 1: If area of circle increases at a rate of $2cm^2$/sec, then find the rate at which area of the inscribed square increases.

Sol. Area of circle, $A_1=\pi r^2$. Area of square, $A_2=2r^2$

(see figure)



$$\frac{dA_1}{dt}=2\pi r\frac{dr}{dt},\ \frac{dA_2}{dt}=4r.\frac{dr}{dt}$$

$$\because\quad 2=2\pi r.\frac{dr}{dt}\Rightarrow r\frac{dr}{dt}=\frac{1}{\pi}$$

$$\therefore\ \frac{dA_2}{dt}=4.\frac{1}{\pi}=\frac{4}{\pi}\ cm^2/sec$$

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm^2/sec.

Example 2: The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Sol. Let at some time t, the length of edge is x cm.

$$v=x^3\Rightarrow\frac{dv}{dt}=3x^2\frac{dx}{dt}\qquad\left(\text{but }\frac{dv}{dt}=7\right)$$

$$\Rightarrow\frac{dx}{dt}=\frac{7}{3x^2}\ cm/sec.$$

Now $S=6x^2$

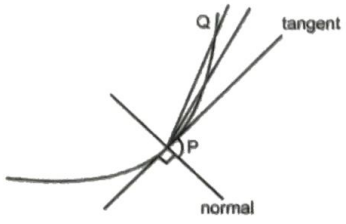
$$\frac{dS}{dt}=12x\frac{dx}{dt}\Rightarrow\frac{dS}{dt}=12x.\frac{7}{3x^2}=\frac{28}{x}$$

when x = 4 cm, $\frac{dS}{dt}=7\ cm^2/sec.$

TANGENT AND NORMAL

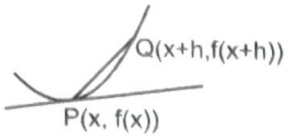
Tangent : The tangent to the curve at P is the line through P whose slope is limit of the slop of the sequence as $Q\to P$ from either sides.

Normal : Line which is perpendicular to tangent at the point of contact is called normal to the curve at that point



Geometrical Meaning of $\frac{dy}{dx}$

As $Q \to P$, $h \to 0$ and slope of chord PQ tends to slope of tangent at P (see figure).



$$\text{Slope of chord PQ} = \frac{f(x+h)-f(x)}{h}$$

$$\lim_{Q \to P} \text{slope of chord PQ} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \text{slope of tangent at P} = f'(x) = \frac{dy}{dx}$$

EQUATION OF TANGENT AND NORMAL

Tangent at a point is a line which touches the curve at that point and normal at a point is a line which is perpendicular to the tangent at that point.

Equation of tangent at $A(x_1, y_1)$:

$$y - y_1 = \left(\frac{dy}{dx}\right)_{\text{at A}} (x - x_1)$$

Equation of the normal at $A(x_1, y_1)$:

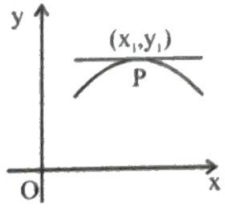
$$y - y_1 = \frac{1}{\left(\frac{dy}{dx}\right)_{\text{at A}}}(x - x_1)$$

Remark :

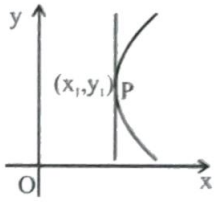
(i) If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

(ii) If $\lim_{h \to 0} \frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

(iii) (a) If $\left.\frac{dy}{dx}\right|_{P(x_1,y_1)} = 0 \Rightarrow$ tangent at P is parallel to x-axis and converse.



(b) $\left.\frac{dy}{dx}\right|_{P(x_1,y_1)} \to \infty \Rightarrow$ tangent at P is parallel to y-axis.



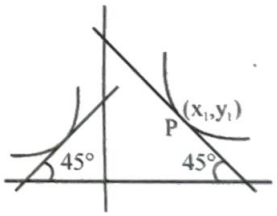
(iv) (a) If tangent is parallel is parallel to $ax + by + c = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{a}{b}$$

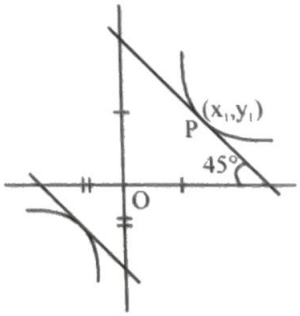
(b) If tangent is perpendicular to $ax + by + c = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} \cdot \left(-\frac{a}{b}\right) = -1$$

(v) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes $\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \pm 1$



(vi) If the tangent at $P(x_1, y_1)$ makes equal non zero intercepts on the coordinate axes $\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -1$



(vii) If the tangent at $P(x_1, y_1)$ to the curve cuts equal distances on the coordiantes axis from the origin

$$\Rightarrow \frac{dy}{dx} = \pm 1$$

Parametric Form : If the equation of the curve is respresented parametrically i.e. $x = f(t)$ and $y = g(t)$,

where $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$, $\{f'(t) \neq 0\}$

then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$

and the equation of the tangent is

$$y - g(t) = \frac{dy/dt}{dx/dt}\{(x - f(t))\}$$

Special Cases:

(i) If the tangent at P meeting the curve $y = f(x)$ again at Q

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1} \quad .....(i)$$

& $y_1 = f(x)$ .....(ii)

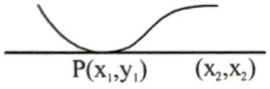
from (i) & (ii) we get equation of tangent.



(ii) If the form external point tangent is drawn to the curve

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1} \quad .....(i)$$

& $y_1 = f(x)$ .....(ii)



Some Common Parametric Coordinates on a Curve:

(i) For $x^2 + y^2 = r^2$, take parametric coordiante $x = r\cos\theta$ & $y = r\sin\theta$.

(ii) For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, take parametric coordiante $x = a\cos\theta$ & $y = b\sin\theta$.

(iii) For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, take parametric coordiante $x = a\sec\theta$ & $y = b\tan\theta$.

(iv) For $y^2 = 4ax$, take parametric coordiante $x = at^2$ & $y = 2at$.

(v) For $x^{2/3} + y^{2/3} = a^{2/3}$, take parametric coordiante $x = a\cos^3\theta$ & $y = a\sin^3\theta$.

(vi) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$, take $x = a\cos^4\theta$ & $y = a\sin^4\theta$.

(vii) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$, take $x = a(\sin\theta)^{2/n}$ & $y = b(\sin\theta)^{2/n}$.

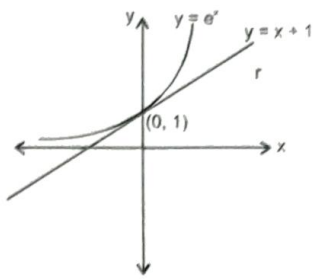
(viii) For $c^2(x^2 + y^2) = x^2y^2$, take $x = c\sec\theta$ and $y = \text{cosec}\theta$.

TRAIN YOUR BRAIN

Example 3: Find equation of tangent to $y = e^x$ at $x = 0$. Hence draw graph

Sol. At $x = 0 \Rightarrow y = e^0 = 1$

Hence point of tangent is (0, 1)



$$\frac{dy}{dx} = e^x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$$

Hence equation of tangent is

$1(x - 0) = (y - 1) \quad \Rightarrow \quad y = x + 1$

Example 4: Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Sol. In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is $(-2, 2)$

$$\frac{dy}{dx} = 2x + 1 \Rightarrow \left.\frac{dy}{dx}\right|_{x=-2} = -3.$$

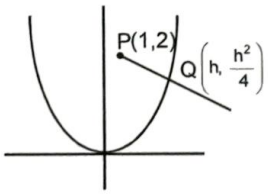
So the slope of normal at $(-2, 2)$ is $\frac{1}{3}$.

Hence equation of normal is $\frac{1}{3}(x + 2) = y - 2$.

$\Rightarrow 3y = x + 8$.

Example 5: Find the equation of all possible normal/s to the parabola $x^2 = 4y$ drawn from point (1, 2).

Sol.



Let point Q be $\left(h, \frac{h^2}{4}\right)$ on parabola $x^2 = 4y$ as shown in figure

Now, m_{PQ} = slope of normal at Q.

Slope of normal $= -\left.\frac{dx}{dy}\right|_{x=h} = -\frac{2}{h}$

$$\Rightarrow \frac{\frac{h^2}{4}-2}{h-1} = -\frac{2}{h}$$

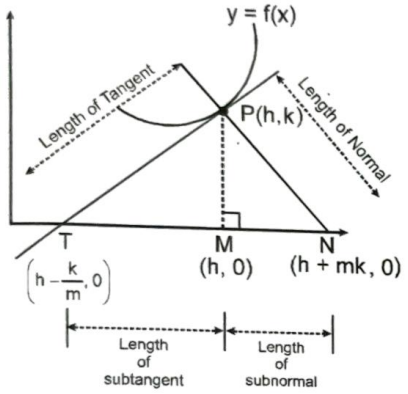
$$\Rightarrow \frac{h^3}{4} - 2h = -2h + 2$$

$$\Rightarrow h^3 = 8 \Rightarrow h = 2$$

Hence coordinates of point Q is (2, 1) and so equation of required normal becomes $x + y = 3$.

LENGTHS OF TANGENT, NORMAL SUBTANGENT AND SUBNORMAL

Let P (h, k) be any point on curve $y = f(x)$. Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N. Then the length PT is called the length of tangent and PN is called length of normal. (as shown in figure)



Projection of segment PT on x-axis, TM, is called the subtangent and similarly projection of line segment PN on x axis, MN is called subnormal.

Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is $m(x-h) = (y-k)$.

Putting $y = 0$, we get x - intercept of tangent is $x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is $x = h + km$

Now, length PT, PN, TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

TRAIN YOUR BRAIN

Example 6: Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.

Sol. Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$$\frac{dy}{dx} = 3x^2 + 6x + 4 \Rightarrow m = 4$$

and, $k = y(0) \Rightarrow k = -1$

$$\ell = |k|\sqrt{1 + \frac{1}{m^2}} \Rightarrow \ell = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$$

Example 7: Find the length of tangent, normal, subtangent and subnormal to the curve $y = \frac{x}{1-x^2}$ at the point having abscissa $\sqrt{2}$.

Sol. At $x = \sqrt{2}$, $y = -\sqrt{2}$

$\Rightarrow$ Point is $P(\sqrt{2}, -\sqrt{2})$

Now, $\frac{dy}{dx} = \frac{(1-x^2)-x(-2x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$

$\therefore \left.\frac{dy}{dx}\right|_{at\,p} = \frac{1+2}{(1-2)^2} = 3 = m(\text{let})$

Equation of tangent : $y + \sqrt{2} = 3(x - \sqrt{2})$

It intersects x-axis at $T\left(\frac{4\sqrt{2}}{3}, 0\right)$

Length of tangent

$$= PT = \sqrt{\left(\frac{4\sqrt{2}}{3} - \sqrt{2}\right)^2 + (0+\sqrt{2})^2} = \frac{2\sqrt{5}}{3}$$

Perpendicular dropped from P on x-axis is $M(\sqrt{2}, 0)$

$\therefore$ Subtangent $= MT = \frac{4\sqrt{2}}{3} - \sqrt{2} = \frac{\sqrt{2}}{3}$

Equation of normal : $y + \sqrt{2} = -\frac{1}{3}(x - \sqrt{2})$

It intersects x-axis at $N(-2\sqrt{2}, 0)$

Length of normal $= PN = \sqrt{20}$ and subnormal $= MN = 3\sqrt{2}$.

2nd Method :

Length of tangent

$$= \left|\frac{y\sqrt{1+m^2}}{m}\right| = \left|\frac{(-\sqrt{2})\sqrt{1+9}}{3}\right| = \frac{2\sqrt{5}}{3}$$

Length of normal

$$= \left|y\sqrt{1+m^2}\right| = \left|-\left(\sqrt{2}\right)\times\sqrt{10}\right| = 2\sqrt{5}$$

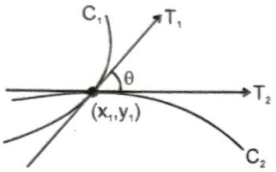
$$\text{Length of subtangent} = \left|\frac{y}{m}\right| = \left|\frac{-\sqrt{2}}{3}\right| = \frac{\sqrt{2}}{3}$$

$$\text{Length of subnormal} = |y\times m| = \left|-\sqrt{2}\times 3\right| = 3\sqrt{2}$$

ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$



where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).

Notes:

(*i*) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or graphically.

(*ii*) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(*iii*) Two curves are said to be orthogonal if angle between them at **each** point of intersection is right angle. i.e. $m_1 m_2 = -1$.

TRAIN YOUR BRAIN

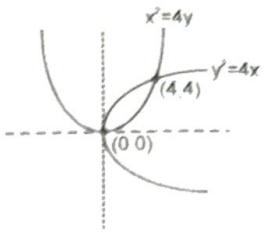
Example 8: Find angle between $y^2 = 4x$ and $x^2 = 4y$. Are these two curves orthogonal?

Sol. $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4) (see figure).

$C_1 : y^2 = 4x$ $\qquad$ $C_2 : x^2 = 4y$

$$\frac{dy}{dx} = \frac{2}{y} \qquad \frac{dy}{dx} = \frac{x}{2}$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} \to \infty \qquad \left.\frac{dy}{dx}\right|_{(0,0)} = 0$$



Hence $\tan\theta = 90°$ at point (0, 0)

$$\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(4,4)} = 2$$

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}\right| = \frac{3}{4}$$

These two curves are not orthogonal because angle between them at (4, 4) is not 90°.

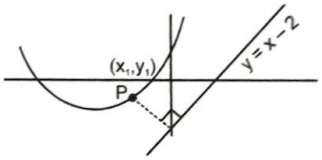
SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

TRAIN YOUR BRAIN

Example 9: Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol.



Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$

then $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of line

$\Rightarrow 2x_1 + 3 = 1 \Rightarrow x_1 = -1 \Rightarrow y_1 = 0$

Hence point $(-1, 0)$ is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest

distance $\Rightarrow p = \frac{3}{\sqrt{2}}$.

ERROR AND APPROXIMATION

Let $y = f(x)$. Then, we know that

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$$

$$\therefore \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)+\varepsilon$$

where $\varepsilon \to 0$, when $\delta x \to 0$

So, $f(x+\delta x)-f(x) = f'(x).\delta x + \varepsilon.\delta x$

or $f(x+\delta x)-f(x) = f'(x).\delta x$ (approximately)

or $\delta y = f'(x).\delta x$ $\quad [\because f(x+\delta x)-f(x) = \delta y]$

Thus, if δx is an error in x, then the corresponding error δy in y can be calculated.

- δx and δy are known as differentials.
- If $y = f(x)$ and δy is an increment in y, corresponding to an increment δx in x, then we have $\delta y = \frac{dy}{dx}.\delta x$.
- The error δx in x is called an absolute error.
- $\frac{\delta x}{x}$ is called the relative error and $\left(\frac{\delta x}{x}\times 100\right)$ is called the percentage error.

TRAIN YOUR BRAIN

Example 10: Find the approximate value of $\sqrt{0.037}$.

Sol. Let $f(x) = \sqrt{x}$

Now, $f(x+\delta x)-f(x) = f'(x).\delta x$

or $f(x+\delta x)-f(x) = \frac{1}{2\sqrt{x}}.\delta x$

We may write, $0.037 = (0.04 - 0.003)$

Taking $x = 0.04$ and $\delta x = -0.003$, we have

$$f(0.037)-f(0.04) = \frac{1}{2\sqrt{0.04}}.(-0.003)$$

$$\text{or } f(0.037) = f(0.04) - \frac{0.003}{2\times 0.2}$$

$$= \left(\sqrt{0.04} - \frac{3}{400}\right) = \left(0.2 - \frac{3}{400}\right) = \frac{77}{400}$$

$$\therefore \sqrt{0.037} = \frac{77}{400} = 0.1925$$

MONOTONICITY

Let f be a real valued function having domain D($D \subset R$) and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes:

(*i*) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non decreasing). But converse need not be true.

(*ii*) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non increasing). Again, converse need not be true

(*iii*) If $f(x) =$ constant in S, then f is increasing as well as decreasing in S

(*iv*) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(*v*) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

(*vi*) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

APPLICATION OF DIFFERENTIATION FOR DETECTING MONOTONICITY

Let I be an interval (open or closed or semi open and semi closed)

(*i*) If $f'(x) > 0\ \forall\ x \in I$, then f is strictly increasing in I

(*ii*) If $f'(x) < 0\ \forall\ x \in I$, then f is strictly decreasing in I

Basic Definition Test

The function f(x) is said to be strictly increasing at a point x_0 if for a sufficiently small $h > 0$ the condition (Fig. 1) $f(x_0 - h) < f(x_0) < f(x_0 + h)$ is fulfilled.

The function f(x) is said to be strictly decreasing at a point x_0 if for a sufficiently small $h > 0$ the condition (Fig. 2) $f(x_0 - h) > f(x_0) > f(x_0 + h)$ is fulfilled.

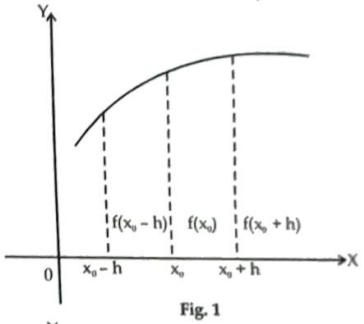


Fig. 1

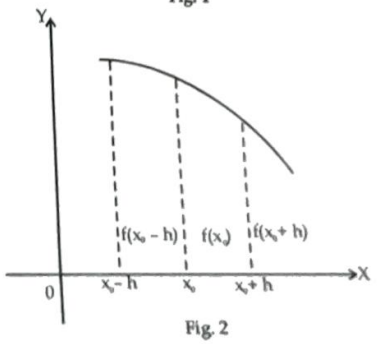


Fig. 2

A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

Sufficiency Test

If the derivative function $f'(x)$ in an interval (a, b) is every where positive, then the function $f(x)$ in this interval is Increasing. If $f'(x)$ is every where negative, then $f(x)$ is Decreasing.

Note: The test (criterion) also holds true when the derivative takes on zero values in the interval (a, b) so long as $f(x)$ does not identically become zero throughout the interval (a, b) or in some interval (a', b') comprising a part of (a, b). The function $f(x)$ would be a constant on such an interval.
If $f'(a) = 0$ then examine the sign of $f'(a^-)$ and $f'(a^+)$
(a) If $f'(a^-) > 0$ and $f'(a^+) > 0$ then strictly increasing
(b) If $f'(a^-) < 0$ and $f'(a^+) < 0$ then strictly decreasing

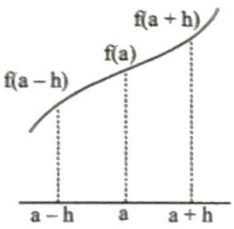
Note:

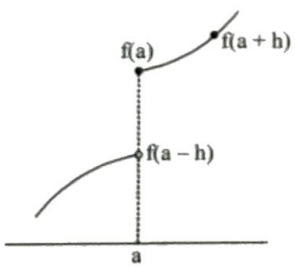
(*i*) If a function is invertible it has to be either increasing or decreasing.
If a function is continuous in the intervals in which it rises and falls may be separated by points at which its derivative is zero or it fails to exist.

(*ii*) Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0$, $\forall\ x \in I$, except for countably many points where $f'(x) = 0$, then $f(x)$ is strictly increasing in I.
$\{f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I $\}$

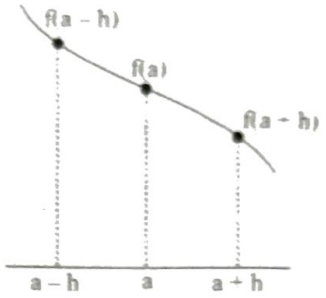
MONOTONICITY OF FUNCTION ABOUT A POINT

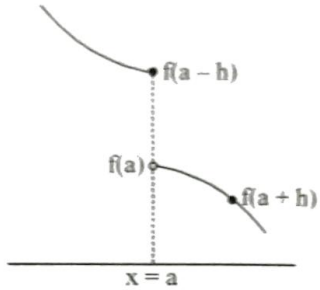
1. A function $f(x)$ is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



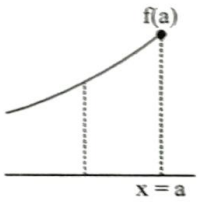


2. A function $f(x)$ is called a strictly decreasing function about a point $x = a$, if it is strictly decreasing in an open interval containing a (as shown in figure)

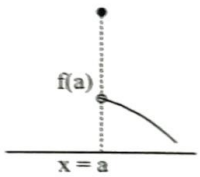




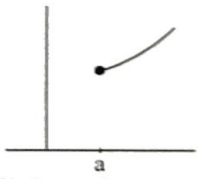
Note: If $x = a$ is a boundary point then use the appropriate one sided inequality to test monotonicity of $f(x)$.



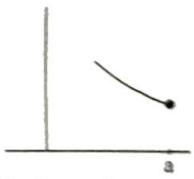
f is increasing at x = a



f is decreasing at x = a



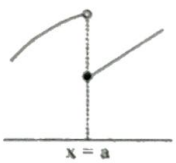
f is increasing at x = a



f is decreasing at x = a

e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at $x = a$.

(i)

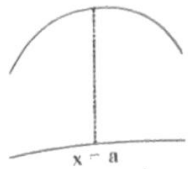


neither increasing nor decreasing at x = a

(ii)



Increasing at x = a



neither increasing nor decreasing at x = a



decreasing at x = a

TEST FOR INCREASING AND DECREASING FUNCTIONS ABOUT A POINT

Let f(x) be differentiable.

(1) If $f'(a) > 0$ then f(x) is increasing at $x = a$.

(2) If $f'(a) < 0$ then f(x) is decreasing at $x = a$.

(3) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

(i) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at $x = a$.

(ii) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at $x = a$.

(iii) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at $x = a$.

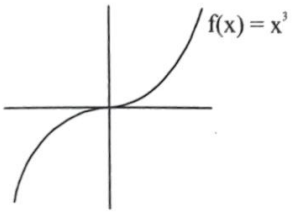
Note: If a function f(x) is increasing in (a, b) and f(x) is continuous in [a, b], then f(x) is increasing on [a, b]

TRAIN YOUR BRAIN

Example 11: Let $f(x) = x^3$. Find the intervals of monotonicity.

Sol. $f'(x) = 3x^2$

$f'(x) > 0$ everywhere except at $x = 0$. Hence f(x) will be strictly increasing function for $x \in R$ {see figure}



Example 12: Find the intervals of monotonocity of the following functions:

(a) $f(x) = \dfrac{|x-1|}{x^2}$

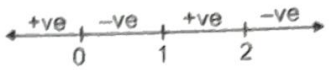
(b) $f(x) = 2x^2 - \ell n\,|x|$

(c) $f(x) = \dfrac{x^3}{x^4+27}$

Sol. **(a)** We have $f(x) = \dfrac{1-x}{x^2}, x < 1$; $f(x) = \dfrac{x-1}{x^2}, x \geq 1$

and $f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3}, x < 1$;

$f'(x) = \dfrac{2-x}{x^3}, x > 1$



Now, from the sign scheme for $f'(x)$, we have

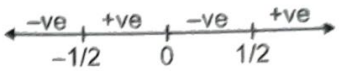
$\Rightarrow$ f(x) strictly increases in $(-\infty, 0)$

strictly decreases in (0, 1) ; strictly increases in (1, 2); strictly decreases in $(2, \infty)$

Ans. : Increases in $(-\infty, 0)$, (1, 2); Decreases in (0, 1), $(2, \infty)$

(b) We have $f(x) = 2x^2 - \ell n\,|x|$ and

$$f'(x) = 4x - \frac{1}{x} = \frac{4x^2-1}{x}$$



Now, from the sign scheme for $f'(x)$, we have

$\Rightarrow$ f(x) strictly decreases in $(-\infty, -1/2)$

strictly increases in $(-1/2, 0)$; strictly decreases in $(0, 1/2)$; strictly increases in $(-1/2, \infty)$

Ans : Increases in $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \infty\right)$;

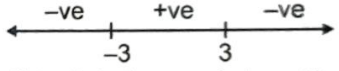
Decreases in $\left(-\infty, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)$

(c) We have $f(x) = \dfrac{x^3}{x^4+27}$ and

$$f'(x) = \frac{(x^4+27)(3x^2) - x^3(4x^3)}{(x^4+27)^2} = \frac{-x^2(x^4-81)}{(x^4+27)^2}$$

$$= \frac{x^2(x^2+9)(x+3)(x-3)}{(x^4+27)^2}$$

Now, from the sign scheme for $f'(x)$, we have

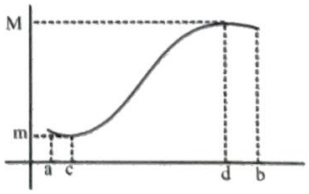


$\Rightarrow$ f(x) strictly decreases in $(-\infty, -3)$; strictly increases in $(-3, 3)$; strictly decreases in $(3, \infty)$.

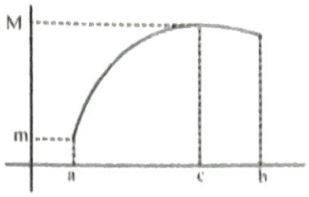
Ans : Increases in $(-3, 3)$; Decreases in $(-\infty, -3)$, $(3, \infty)$

GREATEST AND LEAST VALUE OF A FUNCTION

(i) Extreme value Theorem : If f is continuous on [a, b], then f takes on, a least value m and a greatest value M on this interval.

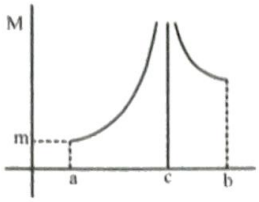


Minimum value 'm' occurs at $x = c$ and maximum value M occurs at $x = d, c, d \in (a, b)$

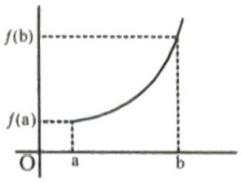


Minimum value 'm' occurs at the end point x = c and maximum valueM occurs at x = c, c ∈ (a, b)

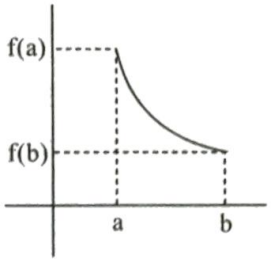
Note: Continuity through the interval [a, b] is essential for the validity of this theorem. There is a discontinuity at x = c, c ∈ [a, b]. The function has a minimum value at the x = a and has no maximum value.



(ii) If a continuous fuction y = $f(x)$ is increasing in the closed interval [a, b] then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a,b].



(iii) If a continuous function y = $f(x)$ is decreasing in [a, b] then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a,b].



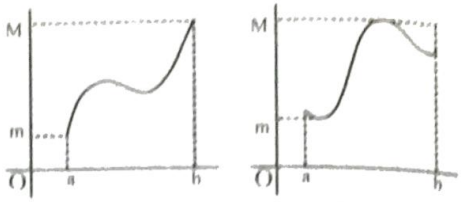
(iv) If function is continuous & non-monotonic in [a,b], then greatest & least value of function exist where

(i) $f'(x)$ is zero

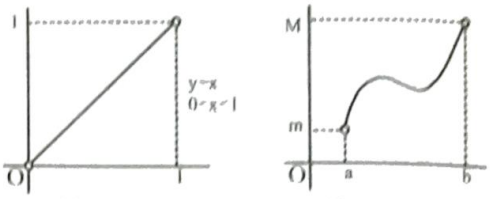
(ii) $f'(x)$ does not exist

(iii) At the end points of the intervel

(v) If a continuous function y = $f(x)$ is non monotonic in [a, b], then range is [m, M].



(vi) In an open interval, a continuous function need not have either a maximum or a minimum value. The function f(x) = x has neither a largest nor a smallest value in (0, 1).



Monotonic Non-monotonic

USE OF MONOTONICITY FOR PROVING INEQUALITIES

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of h(x) = f(x) − g(x)

Note: In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

TRAIN YOUR BRAIN

Example 13: For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Sol. Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function

$g(x) = x - \tan x \quad \Rightarrow g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right) \quad \Rightarrow g(x)$ is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \quad \Rightarrow x < \tan x$

$\sin x < x < \tan x$ Hence proved

Example 14: Prove that, $2x \sec x + x > 3 \tan x$ for $0 < x < \pi/2$.

Sol. $f(x) = 2x \sec x + x - 3 \tan x$

$f'(x) = 2 \sec x + 2x \sec x \tan x + 1 - 3 \sec^2 x = \sec^2 x$

$[2 \cos x + 2x \sin x + \cos^2 x - 3]$

Consider $g(x) = 2 \cos x + 2x \sin x + \cos^2 x - 3$

$g'(x) = -2 \sin x + 2x \cos x + 2 \sin x - 2 \sin x \cos x = 2 \cos x (x - \sin x) > 0$ for $x \in (0, \pi/2)$

ROLLE'S THEOREM

If a function f defined on [a, b] is

(*i*) Continuous on [a, b]

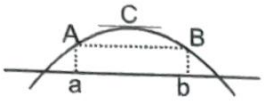
(*ii*) Derivable on (a, b) and

(*iii*) f(a) = f(b),

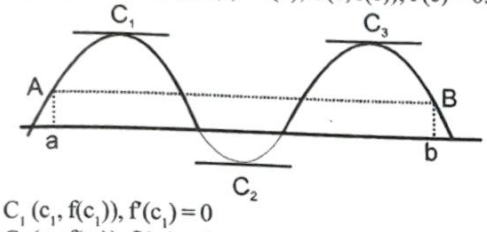
then there exists at least one real number c between a and b (a < c < b) such that f'(c) = 0

Geometrical Explanation of Rolle's Theorem

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), f'(c) = 0.



$C_1(c_1, f(c_1))$, $f'(c_1) = 0$

$C_2(c_2, f(c_2))$, $f'(c_2) = 0$

$C_3(c_3, f(c_3))$, $f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

Algebraic Interpretation of Rolle's Theorem:

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of f'(x)

TRAIN YOUR BRAIN

Example 15: Verify Rolle's theorem for $f(x) = (x-a)^n (x-b)^m$, where m, n are positive real numbers, for $x \in [a, b]$.

Sol. Being a polynomial function f(x) is continuous as well as differentiable. Also f(a) = f(b)

$\Rightarrow f'(x) = 0$ for some $x \in (a, b)$

$n(x-a)^{n-1}(x-b)^m + m(x-a)^n (x-b)^{m-1} = 0$

$\Rightarrow (x-a)^{n-1}(x-b)^{m-1}\,[(m+n)x - (nb+ma)] = 0$

$\Rightarrow x = \dfrac{nb+ma}{m+n}$, which lies in the interval (a, b), as $m, n \in R^+$.

LAGRANGE'S MEAN VALUE THEOREM (LMVT)

If a function f defined on [a, b] is

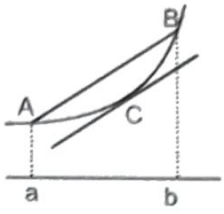
(*i*) continuous on [a, b] and

(*ii*) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

Geometrical Interpretation of LMVT

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), f'(c) = slope of AB.

TRAIN YOUR BRAIN

Example 16: Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Sol. f(1) = –2 ; f(–1) = –10

$$\Rightarrow f'(c) = \frac{f(1)-f(-1)}{1-(-1)} \Rightarrow -2c + 4 = 4 \Rightarrow c = 0$$

Example 17: Using Lagrange's mean value theorem, prove that

if $b > a > 0$, then $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$

Sol. Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \text{ for } a < c < b \text{ and}$$

$$f'(x) = \frac{1}{1+x^2},$$

Now f'(x) is a monotonically decreasing function

Hence if $a < c < b \Rightarrow f'(b) < f'(c) < f'(a)$

$$\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

Hence proved

CURVE SKETCHING

The following checklist is intended as a guide to sketching a curve y = f(x). Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

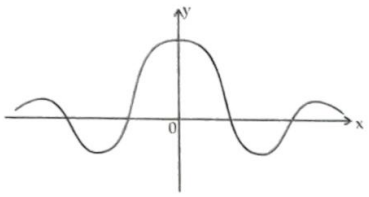
I. **Domain:** It's often useful to start by determining the domain D of f, that is, the set of values of x for which f(x) is defined.

II. **Intercepts:** The y-intercept is f(0) and this tells us where the curve intersect the y-axis. To find the x-intercepts, we set y = 0 and solve for x. (You can omit this step if the equation is difficult to solve.)

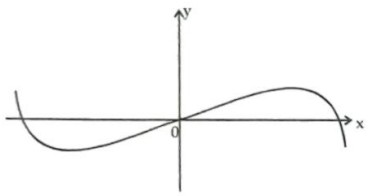
III. Symmetry:

(a) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an even function and the curve is symmetric about the y-axis. This means that out work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples : $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(b) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$. [Rotate 180° about the origin; see Figure (b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.



(a) Even function : reflectional symmetry



(b) Odd function : rotational symmetry

(c) If $f(x + p) = f(x)$ for all x in D, where p is positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph lookslike in an interval of length p, then we can use translation to sketch the entire graph (see Figure).

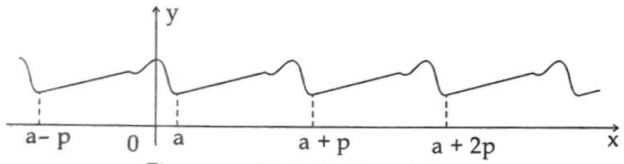


Figure Periodic Function : translational symmetry

IV. Asypmtotes:

(a) Horizontal Asymptotes. If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(b) Vertical Asymptotes. The line $x = a$ is a vertical asymptote if at least one of the following statements is true :

$\lim_{x \to a^+} f(x) = \infty$ $\quad$ $\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$ $\quad$ $\lim_{x \to a^-} f(x) = -\infty$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (ii) is true. If f(a) is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a} f(x)$ or $\lim_{x \to a^-} f(x)$, whether or not this limit is infinite,

V. Interval of Increase / Decrease: Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

VI. Local Maximum and Minimum Value: Find the critical numbers of f [the number c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then f(c) is a local maximum. If f' changes from negative to positive at c, then f(c) is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if c is a critical number such that $f''(c) \neq 0$. Then $f''(c) > 0$ implies that f(c) is a local minimum, whereas $f''(c) < 0$ implies that f(c) is a local maximum.

VII. Concavity and Points of Inflection: Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

VIII. Sketch the Curve: Using the information in items A – G, draw the graph. Sketch the asymptotes as dashed lines, Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

TRAIN YOUR BRAIN

Example 18: Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Sol. I. Domain $= \{x | x + 1 > 0\} = \{x | x > -1\} = (-1, \infty)$

II. The x- and y-intercepts are both 0.

III. Symmetry : None

IV. Since $\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$ there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to -1^+$ and f(x) is always positive, we have $\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$ and so the line $x = -1$ is a vertical asymptote.

V. $$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot 1/(2\sqrt{x+1})}{x+1} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

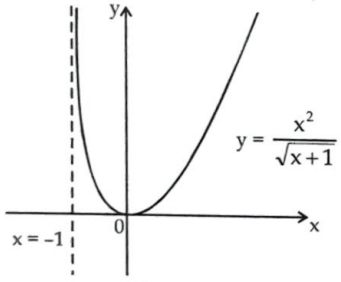
VI. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

VII. $$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3}$$

$$= \frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

Note: that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus, $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

VIII. The curve is sketched in Figure.



Figure

TRAIN YOUR BRAIN

Example 19: Sketch the graph of $f(x) = xe^x$

Sol. I. The domain is R.

II. The x- and y-intercepts are both 0.

III. Symmetry : None

IV. Because both x and e^x become large as $x \to \infty$, we have $\lim_{x \to \infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of L'Hospital's Rule :

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} (-e^x) = 0$$

Thus, the x-axis is horizontal asymptote.

V. $f'(x) = xe^x + e^x = (x+1)e^x$. Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(X) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

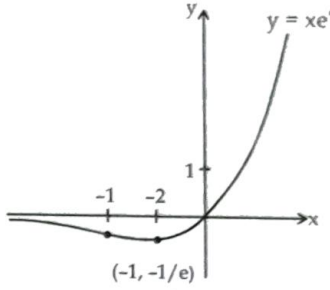
VI. Because $f'(-1) = 0$ and f changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum.

VII. $f''(x) = (x+1)e^x + e^x = (x+2)e^x$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$.

f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection points in $(-2, -2e^{-2})$.

VIII. We use this information to sketch the curve in Figure.



Figure

MAXIMA AND MINIMA

A function f(x) is said to have a local maximum at $x = a$ if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of $x = a$. Symbolically

$$\left.\begin{aligned} f(a) &> f(a+h) \\ f(a) &> f(a-h) \end{aligned}\right] \Rightarrow x = a \text{ gives local maxima for a}$$

sufficiently small positive h.

Similarly, a function f(x) is said to have a local minimum value at $x = b$ if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at $x = b$. Symbolically if

$$\left.\begin{aligned} f(b) &< f(b+h) \\ f(b) &< f(b-h) \end{aligned}\right] \Rightarrow x = b \text{ gives local minima for a}$$

sufficiently small positive h.

Note:

(*i*) The local maximum & minimum values of a function are also known as relative maxima or relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(*ii*) The term 'extremum' or 'turning point' is used both for local maximum or minimum values.

(*iii*) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(*iv*) A function can have several local maximum & minimum values & a local minimum value may even be greater than a local maximum value.

Global Maximum: A function f(x) is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \le f(c)$ for all $x \in E$.

We say global maximum occurs at $x = c$ and global maximum (or global maximum value) is f(c).

Local Maxima: A function f(x) is said to have a local maximum at $x = c$ if f(c) is the greatest value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \le f(c)$.

i.e. $f(c - \delta) \le f(c) \ge f(c + \delta)$, $0 < \delta < h$

Note: If $x = c$ is a boundary point then consider $(c - h, c)$ or $(c, c + h)$ $(h > 0)$ appropriately.

Global Minimum: A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \ge f(c)$ for all $x \in E$.

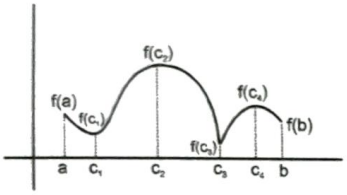
Local Minima: A function f(x) is said to have a local minimum at $x = c$ if f(c) is the least value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \ge f(c)$.

i.e. $f(c - \delta) \ge f(c) \le f(c + \delta)$, $0 < \delta < h$

Extrema: A maxima or a minima is called an extrema.

Explanation : Consider graph of $y = f(x)$, $x \in [a, b]$



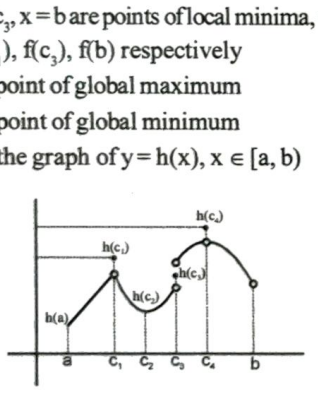
$x = a, x = c_2, x = c_4$ are points of local maxima, with maximum values $f(a), f(c_2), f(c_4)$ respectively.

$x = c_1, x = c_3, x = b$ are points of local minima, with minimum values $f(c_1), f(c_3), f(b)$ respectively

$x = c_2$ is a point of global maximum

$x = c_3$ is a point of global minimum

Consider the graph of $y = h(x)$, $x \in [a, b]$



$x = c_1, x = c_4$ are points of local maxima, with maximum values $h(c_1), h(c_4)$ respectively.

$x = a, x = c_2$ are points of local minima, with minimum values $h(a), h(c_2)$ respectively.

$x = c_3$ is neither a point of maxima nor a minima.

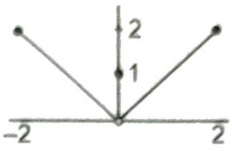
Global maximum is $h(c_4)$

Global minimum is h(a)

TRAIN YOUR BRAIN

Example 20: Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at $x = 0$.

Sol. f(x) has local maxima at $x = 0$ (see figure).



MAXIMA, MINIMA FOR DIFFERENTIABLE FUNCTIONS

A necessary condition for an extrema: Let f(x) be differentiable at $x = c$.

Theorem: A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow f'(c) = 0$

Note:

$f'(c) = 0$ is only a necessary condition but not sufficient

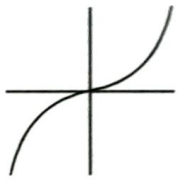
i.e. $f'(c) = 0$

$\Rightarrow$ f(c) is extremum.

Consider $f(x) = x^3$

$f'(0) = 0$

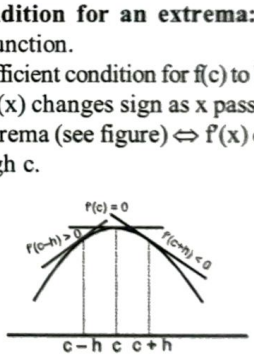
but f(0) is not an extremum (see figure).



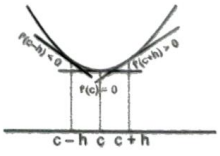
Sufficient condition for an extrema: Let f(x) be a differentiable function.

Theorem: A sufficient condition for f(c) to be an extremum of f(x) is that f'(x) changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ f'(x) changes sign as x passes through c.



$x = c$ is a point of maxima. f'(x) changes sign from positive to negative.



$x = c$ is a point of local minima (see figure), $f'(x)$ changes sign from negative to positive.

Stationary points: The points on graph of function $f(x)$ where $f'(x) = 0$ are called stationary points.

Rate of change of $f(x)$ is zero at a stationary point.

TRAIN YOUR BRAIN

Example 21: Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Sol. $f'(x) = 12x^2 - 12x - 24$

$f'(x) = 0 \Rightarrow x = -1, 2$

$f(-1) = 23, f(2) = -31$

$(-1, 23), (2, -31)$ are stationary points

Example 22: If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at $x = -1$ and $x = 3$. Find a, b, c.

Sol. Extreme values basically mean maximum or minimum values, since $f(x)$ is differentiable function so $f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow \quad a = -3, b = -9, c \in R$

FIRST DERIVATIVE TEST

Let $f(x)$ be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

Step - III. Observe change of sign

(*i*) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(*ii*) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(*iii*) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

TRAIN YOUR BRAIN

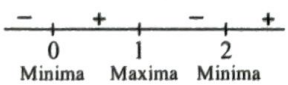
Example 23: Find the points of maxima or minima of $f(x) = x^2 (x-2)^2$.

Sol. $f(x) = x^2 (x-2)^2$

$f'(x) = 4x (x-1) (x-2)$

$f'(x) = 0 \Rightarrow x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence $x = 1$ is point of maxima, $x = 0, 2$ are points of minima.

MAXIMA, MINIMA FOR CONTINUOUS FUNCTIONS

Let $f(x)$ be a continuous function.

Critical points: The points where $f'(x) = 0$ or $f(x)$ is not differentiable are called critical points.

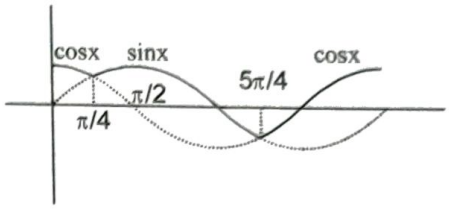
Stationary points $\subseteq$ Critical points.

Important Note: For $f(x)$ defined on a subset of R, points of extrema (if exists) occur at critical points

TRAIN YOUR BRAIN

Example 24: Find critical points of $f(x) = \max(\sin x, \cos x)$ $\forall\, x \in (0, 2\pi)$.

Sol.



From the figure it is clear that $f(x)$ has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

Example 25: Find the possible points of Maxima/Minima for $f(x) = |x^2 - 2x|$ $(x \in R)$

Sol. $f(x) = \begin{cases} x^2 - 2x & x \geq 2 \\ 2x - x^2 & 0 < x < 2 \\ x^2 - 2x & x \leq 0 \end{cases}$

$f'(x) = \begin{cases} 2(x-1) & x > 2 \\ 2(1-x) & 0 < x < 2 \\ 2(x-1) & x < 0 \end{cases}$

$f'(x) = 0$ at $x = 1$ and $f'(x)$ does not exist at $x = 0, 2$. Thus these are critical points.

Example 26: Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of $f(x)$ at $x = 0$.

Sol. $f(x)$ is continuous at $x = 0$.

$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$

$f'(0^+) = 3$ and $f'(0^-) = -10$ thus $f(x)$ is non-differentiable at $x = 0 \Rightarrow x = 0$ is a critical point.

Also derivative changes sign from negative to positive, so $x = 0$ is a point of local minima.

Global Extrema for Continuous Functions

(*i*) Function defined on closed interval

Let $f(x)$, $x \in [a, b]$ be a continuous function

Step - I : Find critical points. Let it be $c_1, c_2, \ldots\ldots, c_n$

Step - II : Find $f(a), f(c_1)\ldots\ldots\ldots, f(c_n), f(b)$

Let $M = \max \cdot \{f(a), f(c_1), \ldots\ldots\ldots, f(c_n), f(b)\}$

$m = \min \cdot \{f(a), f(c_1), \ldots\ldots f(c_n), f(b)\}$

Step-III : M is global maximum. m is global minimum.

(*ii*) Function defined on open interval.

Let $f(x)$, $x \in (a, b)$ be continuous function.

Step - I Find critical points . Let it be $c_1, c_2, \ldots\ldots c_n$

Step - II Find $f(c_1), f(c_2), \ldots\ldots, f(c_n)$

Let $M = \max \cdot \{f(c_1), \ldots\ldots f(c_n)\}$

$m = \min \cdot \{f(c_1), \ldots\ldots\ldots, f(c_n)\}$

Step-III $\lim\limits_{x \to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x \to b^-} f(x) = \ell_2$ (say).

Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$

Step - IV

(*i*) If $m \le \ell$ then m is global minimum

(*ii*) If $m > \ell$ then $f(x)$ has no global minimum

(*iii*) If $M \ge L$ then M is global maximum

(*iv*) If $M < L$, then $f(x)$ has no global maximum

Example 27: Find the greatest and least values of $f(x) = x^3 - 12x$, $x \in [-1, 3]$

Sol. The possible points of maxima/minima are critical points and the boundary points.

for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$

$x = 2$ is the only critical point.

Examining the value of $f(x)$ at points $x = -1, 2, 3$. We can find greatest and least values.

x	f(x)
−1	11
2	−16
3	−9

$\therefore$ Minimum $f(x) = -16$ & Maximum $f(x) = 11$.

MAXIMA, MINIMA BY HIGHER ORDER DERIVATIVES

Second derivative test : Let $f(x)$ have derivatives up to second order

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$. Let $x = c$ be a solution

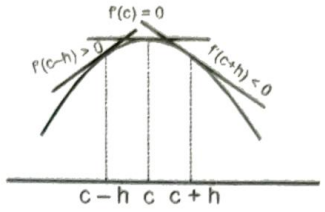
Step - III. Find $f''(c)$

Step - IV.

(*i*) If $f''(c) = 0$ then further investigation is required

(*ii*) If $f''(c) > 0$ then $x = c$ is a point of minima.

(*iii*) If $f''(c) < 0$ then $x = c$ is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).

$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

TRAIN YOUR BRAIN

Example 28: Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

Sol. Let $y = \left(\frac{1}{x}\right)^x$

$$\log y = x \log_e \left(\frac{1}{x}\right) = -x \log_e x \qquad \ldots(i)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = -\left[x.\frac{1}{x} + \log_e x\right] = -(1 + \log_e x) \qquad \ldots(ii)$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{1}{x}\right)^x (1 + \log_e x)$$

Differentiating again

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{1}{x} \qquad \ldots(iii)$$

From (ii) $\frac{dy}{dx} = -y(1 + \log_e x) = -\left(\frac{1}{x}\right)^x (1 + \log_e x)$

For maximum or minimum values of y, $\frac{dy}{dx} = 0$

$\therefore \left(\frac{1}{x}\right)^x (1 + \log_e x) = 0$; but $\left(\frac{1}{x}\right)^x \neq 0$ for any x

$\therefore 1 + \log_e x = 0$; $\log_e x = -1$; $x = e^{-1} = \frac{1}{e}$ $\qquad \ldots(iv)$

when $x = \frac{1}{e}$ equation (iii) gives

$\frac{1}{y}\frac{d^2y}{dx^2} - 0 = -e$ $\quad \therefore$ $\frac{d^2y}{dx^2} = -e(e)^{1/e} < 0$

Hence y is maximum when $x = \frac{1}{e}$ and the maximum value of $y = (e)^{1/e}$

Example 29: Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Sol. $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0 \Rightarrow$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0 \Rightarrow$ Minima at $x = \frac{5\pi}{6}$

n^{th} Derivative test :

Let $f(x)$ have derivatives up to nth order

If $f'(c) = f''(c) = \ldots\ldots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$ then we have following possibilities

(*i*) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima

(*ii*) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.

(*iii*) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$

(*iv*) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

TRAIN YOUR BRAIN

Example 30: Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Sol. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

Now, $f''(1) < 0 \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \Rightarrow$ Minima at $x = 3$

and, $f''(0) = 0 \Rightarrow$ IInd derivative test fails

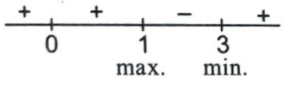
so, $f'''(x) = 30(2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Note: It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at $x = 0$

$f'(x) = 5x^2 (x-1)(x-3)$



APPLICATION OF MAXIMA, MINIMA

For a given problem, an objective function can be constructed in terms of one parameter and then extremum value can be evaluated by equating the differential to zero. As discussed in n^{th} derivative test maxima/minima can be identified.

Useful Formulae of Mensuration to Remember

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r \ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
    (Note that lateral surfaces of a prism are all rectangle).
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
    (Note that slant surfaces of a pyramid are triangles).

TRAIN YOUR BRAIN

Example 31: Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Sol. $x + y = 60$

$\Rightarrow x = 60 - y \Rightarrow xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)y^3$; $y \in (0, 60)$

for maximizing $f(y)$ let us find critical points

$f'(y) = 3y^2 (60 - y) - y^3 = 0$

$f'(y) = y^2 (180 - 4y) = 0$

$\Rightarrow y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at $y = 45$.

So $x = 15$ and $y = 45$.

Example 32: Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

Sol. Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow A = xy$.

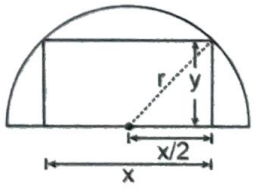
Here x and y are not independent variables and are related by Pythogorus theorem with r.

$\frac{x^2}{4} + y^2 = r^2$

$\Rightarrow y = \sqrt{r^2 - \frac{x^2}{4}}$

$\Rightarrow A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$

$\Rightarrow A(x) = \sqrt{x^2 r^2 - \frac{x^4}{4}}$



Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $x \in (0, r)$

A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$

$\Rightarrow x = r\sqrt{2}$

also $f'(r\sqrt{2}^+) < 0$ and $f'(r\sqrt{2}^-) > 0$

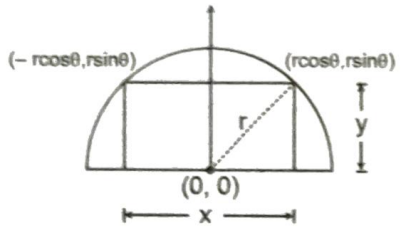
confirming at f(x) is maximum when $x = r\sqrt{2}$ &

$y = \frac{r}{\sqrt{2}}$.

2nd Method

Let us choose coordinate system with origin as centre of circle (as shown in figure).

$A = xy$



$\Rightarrow A = 2(r\cos\theta)(r\sin\theta)$

$\Rightarrow A = r^2 \sin 2\theta, \theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4}$

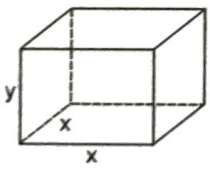
$\Rightarrow x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example 33: A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Sol. Let length of base be x meter and height be y meter (as shown in figure).

$V = x^2y$

again x and y are related to surface area of this tank which is equal to 40 m^2.



$\Rightarrow x^2 + 4xy = 40$

$y = \frac{40 - x^2}{4x}$ $x \in (0, \sqrt{40})$

$\Rightarrow V(x) = x^2\left(\frac{40 - x^2}{4x}\right)$

$V(x) = \frac{(40x - x^3)}{4}$

maximizing volume, $V'(x) = \frac{(40 - 3x^2)}{4} = 0$

$\Rightarrow x = \sqrt{\frac{40}{3}}$ m

and $V''(x) = -\frac{3x}{2} \Rightarrow V''\left(\sqrt{\frac{40}{3}}\right) < 0$.

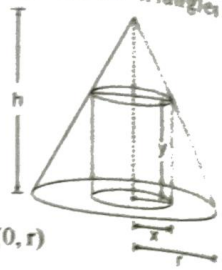
Confirming that volume is maximum at $x = \sqrt{\frac{40}{3}}$ m

Example 34: If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. Let x be the radius of cylinder and y be its height

$v = \pi x^2 y$

x, y can be related by using similar triangles (as shown in figure).



$\frac{y}{r - x} = \frac{h}{r}$

$\Rightarrow y = \frac{h}{r}(r - x)$

$\Rightarrow v(x) = \pi x^2 \frac{h}{r}(r - x)\, x \in (0, r)$

$\Rightarrow v(x) = \frac{\pi h}{r}(rx^2 - x^3)$

$v'(x) = \frac{\pi h}{r} x\,(2r - 3x)$

$v'\left(\frac{2r}{3}\right) = 0$ and $v''\left(\frac{2r}{3}\right) < 0$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

MAXIMUM AND MINIMUM FOR DISCRETE VALUED FUNCTIONS

TRAIN YOUR BRAIN

Example 35: Find the largest term in the sequence

$a_n = \frac{n}{n^2 + 10}$ $(n \in N)$.

Sol. Consider the function $f(x) = \frac{x}{x^2 + 10}$, $x > 0$.

Then $f'(x) = \frac{(x^2 + 10) - 2x^2}{(x^2 + 10)^2}$

$= \frac{-(x + \sqrt{10})(x - \sqrt{10})}{(x^2 + 10)^2} > 0 \; \forall \; 0 < x < \sqrt{10}$

$< 0 \; \forall \; x > \sqrt{10}$

$\Rightarrow$ f(x) strictly increases in $(0, \sqrt{10})$ strictly decreases in $(\sqrt{10}, \pi)$

$\Rightarrow$ f(x) has greatest value at $x = \sqrt{10}$

Hence, the given sequence has greatest value at $n = 3$ or $n = 4$.

Now, we have $a_3 = \frac{3}{19}$ and $a_4 = \frac{4}{26}$. Hence, $a_3 = \frac{3}{19}$ is the largest term of the given sequence

CONCAVITY, CONVEXITY, POINT OF INFLECTION

A function f(x) is concave in (a, b) if tangent drawn at every point $(x_0, (f(x_0))$, for $x_0 \in (a, b)$ lie below the curve. f(x) is convex in (a,b) if tangent drawn at each point $(x_0, f(x_0))$, $x_0 \in$ (a, b) lie above the curve.

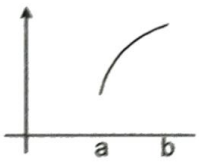
A point (c, f(c)) of the graph y = f(x) is said to be a point of inflection of the graph, if f(x) is concave in $(c - \delta, c)$ and convex in $(c, c + \delta)$ (or vice verse), for some $\delta \in R^+$.

Results :

1. If $f''(x) > 0 \ \forall \ x \in (a, b)$, then the curve y = f(x) is concave in (a, b)



2. If $f''(x) < 0 \ \forall \ x \in (a, b)$ then the curve y = f(x) is convex in (a, b)



3. If f is continuous at x = c and f''(x) has opposite signs on either sides of c, then the point (c, f(c)) is a point of inflection of the curve
4. If f'(c) = 0 and f''(c) $\neq$ 0, then the point (c, f(c)) is a point of inflection

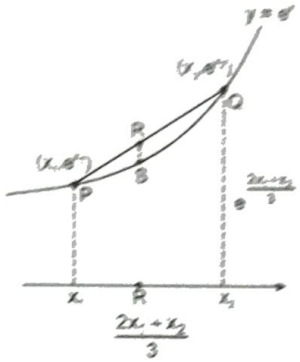
Proving Inequalities using curvature: Generally these inequalities involve comparison between values of two functions at some particular points.

TRAIN YOUR BRAIN

Example 36: Prove that for any two numbers x_1 & x_2, $\frac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}$

Sol. Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve $y = e^x$

Let R be another point which divides $\overline{PQ}$ in ratio 1 : 2.



y coordinate of point R is $\frac{2e^{x_1} + e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1 + x_2}{3}}$. Since $f(x) = e^x$ is concave up, the point R will always be above the point S.

$\Rightarrow \quad \frac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}$

Alternate : Above inequality could also be easily proved using AM and GM.

Example 37: Find the points of inflection of the function $f(x) = \sin^2 x, x \in [0, 2\pi]$

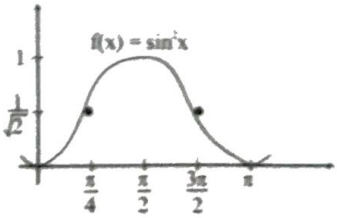
Sol. $f(x) = \sin^2 x$

$f'(x) = \sin 2x$

$f''(x) = 2\cos 2x$

$f''(0) = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

both these points are inflection points as sign of f''(x) change on either sides of these points.



Example 38: Find the intervals of concavity of the graph of the function $y = x^5 + 5x - 6$.

Sol. We have $y' = 5x^4 + 5$, $y'' = 20x^3$. If $x < 0$, then $y'' < 0$ and the curve is concave down ; now if $x > 0$, then $y'' > 0$ and the curve is concave up. Thus we see that the curve is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$.

Example 39: Find the inflection points of the curve

$$y = (x - 5)^{5/3} + 2.$$

Sol. We find $y' = \frac{5}{3}(x - 5)^{2/3}$, $y'' = \frac{10}{9\sqrt[3]{(x-5)}}$.

The second derivative does not vanish for any value of x and does not exist at $x = 5$. The value $x = 5$ is the abscissa of the inflection point since $y''(5 - h) < 0$, $y''(5 + h) > 0$. Thus, (5, 2) is the inflection point. Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist.

Topicwise Questions

Derivative as Rate of Change

1. A particle moves in a straight line in such a way that its velocity at any point is given by $v^2 = 2 - 3x$, where x is measured from a fixed point. The acceleration is
   (a) Uniform (b) Zero
   (c) Non-uniform (d) Indeterminate
2. A particle is moving in a straight line. Its displacement at time t is given by $s = -4t^2 + 2t$, then its velocity and acceleration at time $t = \frac{1}{2}$ second are
   (a) $-2, -8$ (b) $2, 6$
   (c) $-2, 8$ (d) $2, 8$

Equation of Tangent & Normal

3. The point of the curve $y^2 = 2(x-3)$ at which the normal is parallel to the line $y - 2x + 1 = 0$ is
   (a) (5,2) (b) $\left(-\frac{1}{2}, -2\right)$
   (c) (5,–2) (d) $\left(\frac{3}{2}, 2\right)$
4. The line $x + y = 2$ is tangent to the curve $x^2 = 3 - 2y$ at its point
   (a) (1, 1) (b) (–1, 1)
   (c) $(\sqrt{3}, 0)$ (d) (3,–3)
5. The angle between curves $y^2 = 4x$ and $x^2 + y^2 = 5$ at (1, 2) is
   (a) $\tan^{-1}(3)$ (b) $\tan^{-1}(2)$
   (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$
6. For the curve $by^2 = (x + a)^3$ the square of subtangent is proportional to
   (a) $(\text{Subnormal})^{1/2}$ (b) Subnormal
   (c) $(\text{Subnormal})^{3/2}$ (d) $(\text{Subnormal})^2$
7. The equation of tangent at (–4, – 4) on the curve $x^2 = -4y$ is
   (a) $2x + y + 4 = 0$ (b) $2x - y - 12 = 0$
   (c) $2x + y - 4 = 0$ (d) $2x - y + 4 = 0$
8. The normal to the curve $x = a(1 + \cos\theta), y = a\sin\theta$ at 'θ' always passes through the fixed point
   (a) (a, a) (b) (0, a)
   (c) (0, 0) (d) (a, 0)

Length of Tangent, Normal

9. The length of the normal at point 't' of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is
   (a) $a\sin t$ (b) $2a\sin^3(t/2)\sec(t/2)$
   (c) $2a\sin(t/2)\tan(t/2)$ (d) $2a\sin(t/2)$

Angle between Curves

10. The angle of intersection of the curves $y = x^2$ and $x = y^2$ at (1, 1) is
    (a) $\tan^{-1}\left(\frac{4}{3}\right)$ (b) $\tan^{-1}(1)$
    (c) 90° (d) $\tan^{-1}\left(\frac{3}{4}\right)$

Monotonocity

11. The interval in which the function x^2e^{-x} is non decreasing, is
    (a) $(-\infty, 2]$ (b) $[0, 2]$
    (c) $[2, \infty)$ (d) $(-\infty, 2)$
12. For the every value of x the function $f(x) = \frac{1}{5^x}$ is
    (a) Decreasing
    (b) Increasing
    (c) Neither increasing nor decreasing
    (d) Increasing for $x > 0$ and decreasing for $x < 0$
13. If $f(x) = \sin x - \frac{x}{2}$ is increasing function, then
    (a) $0 < x < \frac{\pi}{3}$ (b) $-\frac{\pi}{3} < x < 0$
    (c) $-\frac{\pi}{3} < x < \frac{\pi}{3}$ (d) $x = \frac{\pi}{2}$
14. If $f(x) = x^3 - 10x^2 + 200x - 10$, then
    (a) f(x) is decreasing in $]-\infty, 10]$ and increasing in $[10, \infty[$
    (b) f(x) is increasing in and decreasing in $[10, \infty[$
    (c) f(x) is increasing throughout real line
    (d) f(x) is decreasing throughout real line

15. If $f(x)=\sin x-\cos x$, the function decreasing in $0\le x\le 2\pi$ is

(a) $[5\pi/6, 3\pi/4]$ (b) $[\pi/4, \pi/2]$

(c) $[3\pi/2, 5\pi/2]$ (d) $[3\pi/4, 7\pi/4]$

Greatest & Least Value of a Function

16. The maximum value of $2x^3-24x+107$ in the interval $[-3, 3]$ is

(a) 75 (b) 89

(c) 125 (d) 139

17. The minimum value of $2x^2+x-1$ is

(a) $-\frac{1}{4}$ (b) $\frac{3}{2}$

(c) $\frac{-9}{8}$ (d) $\frac{9}{4}$

Inequalities

18. The function x^x is increasing, when

(a) $x>\frac{1}{e}$ (b) $x<\frac{1}{e}$

(c) $x<0$ (d) For all real x

Rolle's Thearom

19. For which interval, the function $\frac{x^2-3x}{x-1}$ satisfies all the conditions of Rolle's theorem

(a) [0, 3] (b) [−3, 0]

(c) [1.5, 3] (d) For no interval

20. The function $f(x)=x^3-6x^2+ax+b$ satisfy the conditions of Rolle's theorem in [1, 3]. The values of a and b are

(a) 11, −6 (b) −6, 11

(c) −11, 6 (d) 6, −11

L.M.V.T

21. If $f(x)=\cos x, 0\le x\le\frac{\pi}{2}$, then the real number '$c$' of the mean value theorem is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\sin^{-1}\left(\frac{2}{\pi}\right)$ (d) $\cos^{-1}\left(\frac{2}{\pi}\right)$

Maxima/minima for differentiable function first and second derivative test

22. The function $x^5-5x^4+5x^3-1$ is

(a) Maximum at $x=3$ and minimum at $x=1$

(b) Minimum at $x=0$

(c) Neither maximum nor minimum at $x=0$

(d) Maximum at $x=0$

23. The value of a so that the sum of the squares of the roots of the equation $x^2-(a-2)x-a+1=0$ assume the least value, is

(a) 2 (b) 1

(c) 3 (d) 0

24. The real number x when added to its inverse gives the minimum value of the sum at x equal to

(a) −2 (b) 2

(c) 1 (d) −1

25. The function $f(x)=2x^3-15x^2+36x+4$ is maximum at

(a) x=2 (b) x=4

(c) x=0 (d) x=3

26. The function $f(x)=2x^3-3x^2-12x+4$ has

(a) No maxima and minima

(b) One maximum and one minimum

(c) Two maxima

(d) Two minima

27. The function $f(x)=x^{-x}, (x\in R)$ attains a maximum value at $x=$

(a) 2 (b) 3

(c) $1/e$ (d) 1

28. The minimum value of $2x+3y$, when $xy=6$, is

(a) 12 (b) 9

(c) 8 (d) 6

2nd Direction and Point of Intersection

29. Divide 20 into two parts such that the product of one part and the cube of the other is maximum. The two parts are

(a) (10, 10) (b) (5, 15)

(c) (13, 7) (d) (10, 15)

30. The ratio of height of cone of maximum volume inscribed in a sphere to its radius is

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$

(c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Learning Plus

1. Gas is being pumped into a spherical balloon at the rate of 30 ft^3/min. Then the rate at which the radius increases when it reaches the value 15 ft is

   (a) $\frac{1}{30\pi}$ ft/ min. (b) $\frac{1}{15\pi}$ ft/ min.

   (c) $\frac{1}{20}$ ft/ min. (d) $\frac{1}{25}$ ft/ min.

2. The equation of tangent to the curve $y = 2\cos x$ at $x = \frac{\pi}{4}$ is

   (a) $y - \sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right)$

   (b) $y + \sqrt{2} = \sqrt{2}\left(x + \frac{\pi}{4}\right)$

   (c) $y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$

   (d) $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$

3. Co-ordinates of a point on the curve $y = x\log x$ at which the normal is parallel to the line $2x - 2y = 3$ are

   (a) (0,0) (b) (e, e)

   (c) $(e^2, 2e^2)$ (d) $(e^{-2}, -2e^{-2})$

4. If normal to the curve $y = f(x)$ is parallel to x-axis, then correct statement is

   (a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} = 1$

   (c) $\frac{dx}{dy} = 0$ (d) $\frac{dy}{dx} = 2$

5. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \pi/2$ is

   (a) 2a (b) a / 2

   (c) $\sqrt{2}$ a (d) a / $\sqrt{2}$

6. An equation of the tangent to the curve $y = x^4$ from the point (2, 0) not on the curve is

   (a) $y = 0$ (b) $x = 0$

   (c) $x + y = 0$ (d) $x - 1 = 0$

7. At what point on the curve $x^3 - 8a^2y = 0$, the slope of the normal is $\frac{-2}{3}$

   (a) (a, a) (b) (2a, –a)

   (c) (2a, a) (d) (–a, a)

8. The angle of intersection of curves $y = x^2$, $6y = 7 - x^3$ at (1, 1) is

   (a) $\pi/4$ (b) $\pi/3$

   (c) $\pi/2$ (d) π

9. For which interval the given function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is decreasing

   (a) $(-2, \infty)$ (b) $(-2, -1)$

   (c) $(-\infty, -1)$ (d) $(-\infty, -2)$ & $(-1, \infty)$

10. Function $f(x) = x^4 - \frac{x^3}{3}$ is

    (a) Increasing for $x > \frac{1}{4}$ and decreasing for $x < \frac{1}{4}$

    (b) Increasing for every value of x

    (c) Decreasing for every value of x

    (d) Increasing for $x < \frac{1}{4}$ and decreasing for $x > \frac{1}{4}$

11. The interval of the decreasing function $f(x) = x^3 - x^2 - x - 4$ is

    (a) $\left(\frac{1}{3}, 1\right)$ (b) $\left(-\frac{1}{3}, 1\right)$

    (c) $\left(-\frac{1}{3}, \frac{1}{3}\right)$ (d) $\left(-1, -\frac{1}{3}\right)$

12. Function $f(x) = \frac{\lambda\sin x + 6\cos x}{2\sin x + 3\cos x}$ is monotonic increasing, if

    (a) $\lambda > 1$ (b) $\lambda < 1$

    (c) $\lambda < 4$ (d) $\lambda > 4$

13. The function $f(x) = \frac{\log x}{x}$ is increasing in the interval

    (a) (1, 2e) (b) (0, e)

    (c) (2, 2e) (d) (1/e, 2e)

14. If $f(x) = \frac{1}{x+1} - \log(1+x), x > 0$, then f is

    (a) An increasing function

    (b) A decreasing function

    (c) Both increasing and decreasing function

    (d) Neigther increasing nor decreasing function

15. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is

    (a) 0 (b) 1

    (c) 2 (d) Infinite

16. Consider the function $f(x) = e^{-2x}\sin 2x$ over the interval $\left(0, \frac{\pi}{2}\right)$. A real number $c \in \left(0, \frac{\pi}{2}\right)$, as guaranteed by Rolle's theorem, such that $f'(c) = 0$ is

    (a) $\pi/8$ (b) $\pi/6$

    (c) $\pi/4$ (d) $\pi/3$

17. In the Mean-Value theorem $\frac{f(b)-f(a)}{b-a}=f'(c)$, if $a=0, b=\frac{1}{2}$ and $f(x)=x(x-1)(x-2)$, the value of c is

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$

(c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

18. If the function $f(x)=x^3-6ax^2+5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve $y=f(x)$ at $x=\frac{7}{4}$ is parallel to the chord that joins the points of intersection of the curve with the ordinates $x=1$ and $x=2$. Then the value of a is

(a) $\frac{35}{16}$ (b) $\frac{35}{48}$

(c) $\frac{7}{16}$ (d) $\frac{5}{16}$

19. The function $x^2\log x$ in the interval $(1, e)$ has

(a) A point of maximum (b) A point of minimum

(c) Points of maximum as well as of minimum

(d) Neither a point of maximum nor minimum

20. If $f(x)=2x^3-3x^2-12x+5$ and $x\in[-2,4]$, then the maximum value of function is at the following value of x

(a) 2 (b) –1

(c) –2 (d) 4

21. What are the minimum and maximum values of the function $x^5-5x^4+5x^3-10$

(a) $-37,-9$

(b) $10,0$

(c) It has 2 min. and 1 max. values

(d) It has 2 max. and 1 min. values

22. If x is real, then greatest and least values of $\frac{x^2-x+1}{x^2+x+1}$ are

(a) $3,-\frac{1}{2}$ (b) $3,\frac{1}{3}$

(c) $-3,-\frac{1}{3}$ (d) $-3,-\frac{1}{2}$

23. The minimum value of $\frac{\log x}{x}$ in the interval $[2,\infty)$ is

(a) $\frac{\log 2}{2}$ (b) Zero

(c) $\frac{1}{e}$ (d) Does not exist

24. The function $y=a(1-\cos x)$ is maximum when $x=$

(a) π (b) $\pi/2$

(c) $-\pi/2$ (d) $-\pi/6$

25. The minimum value of $4e^{2x}+9e^{-2x}$ is

(a) 11 (b) 12

(c) 10 (d) 14

26. The maximum value of xy when $x+2y=8$ is

(a) 20 (b) 16

(c) 24 (d) 8

27. The area of a rectangle will be maximum for the given perimeter, when rectangle is a

(a) Parallelogram (b) Trapezium

(c) Square (d) Rhombus

28. The maximum value of function $x^3-12x^2+36x+17$ in the interval [1, 10] is

(a) 17 (b) 177

(c) 77 (d) 117

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. For the curve $x=t^2+3t-8$, $y=2t^2-2t-5$, at point $(2,-1)$

(a) length of subtangent is 7/6.

(b) slope of tangent = 6/7

(c) length of tangent = $\sqrt{(85)}/6$

(d) None of these

2. If tangent at point (1, 2) on the curve $y=ax^2+bx+\frac{7}{2}$ be parallel to normal at (–2, 2) on the curve $y=x^2+6x+10$, then

(a) $a=1$

(b) $a=-1$

(c) $b=-5/2$

(d) $b=5/2$

3. The co-ordinates of the point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign is
(a) (2, 8/3) (b) (3, 7/2)
(c) (1, 5/6) (d) None of these

4. The co-ordinates of a point on the parabola $2y = x^2$ which is nearest to the point (0, 3) is
(a) (2, 2) (b) $(-\sqrt{2}, 1)$
(c) $(\sqrt{2}, 1)$ (d) (–2, 2)

5. If the curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ & $y^3 = 16x$ intersect at right angles, then values of a is/are
(a) $\frac{2}{\sqrt{3}}$ (b) 2
(c) $-\frac{2}{\sqrt{3}}$ (d) not possible

6. Which of the following pair(s) of curves is/are orthogonal
(a) $y^2 = 4ax$; $y = e^{-x/2a}$ (b) $y^2 = 4ax$; $x^2 = 4ay$
(c) $xy = a^2$; $x^2 - y^2 = b^2$ (d) $y = ax$; $x^2 + y^2 = c^2$

7. The function $f(x) = 2\ln(x-2) - x^2 + 4x + 1$ increases in the intervals
(a) (1, 2) (b) (2, 3)
(c) $\left[\frac{5}{2}, 3\right]$ (d) (2, 4)

8. Let $g(x) = 2f(x/2) + f(1-x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g(x)
(a) decreases in $\left[0, \frac{2}{3}\right]$
(b) decreases $\left[\frac{2}{3}, 1\right]$
(c) increases in $\left[0, \frac{2}{3}\right]$
(d) increases in $\left[\frac{2}{3}, 1\right]$

9. The function $f(x) = 3x^4 + 4x^3 - 12x^2 - 7$ is
(a) $\uparrow$ in $(-2, 0)$ & $(1, \infty)$
(b) $\downarrow$ in $(-\infty, -2)$ & $(0, 1)$
(c) $\downarrow$ in $(-2, 0)$ & $(1, \infty)$
(d) $\uparrow$ in $(-\infty, -2)$ & $(0, 1)$

10. The curve $y = \frac{x+1}{x^2+1}$ has
(a) x = 1, as point of inflection
(b) $x = -2 + \sqrt{3}$, as point of inflection
(c) x = – 1, as point of minimum
(d) $x = -2 - \sqrt{3}$, as point of inflection

11. Let $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$, consider the following statement about f(x).
(a) f(x) has local minima at x = 0
(b) f(x) has local maxima at x = 0
(c) absolute maximum value of f(x) is not defined
(d) f(x) is local maxima at x = –3, x = 1

12. The sum of the legs of a triangle is 9 cm. When the triangle rotates about one of the legs, a cone results which has the maximum volume. Then
(a) slant height of such a cone is $3\sqrt{5}$
(b) maximum volume of the cone is 32π
(c) curved surface of the cone is $18\sqrt{5}\pi$
(d) semi vertical angle of cone is $\tan^{-1}\sqrt{2}$

Comprehension–1 (No. 13 to 15)

Let a(t) be a function of t such that $\frac{da}{dt} = 2$ for all values of t and a = 0 when t = 0. Further $y = m(t)x + c(t)$ is tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then

13. If the rate of change of distance of vertex of $y = x^2 - 2ax + a^2 + a$ from the origin with respect to t is k, then k =
(a) 2 (b) $2\sqrt{2}$
(c) $\sqrt{2}$ (d) $4\sqrt{2}$

14. If the rate of change of c(t) with respect to t, when t = k, is ℓ, then ℓ
(a) $16\sqrt{2} - 2$ (b) $8\sqrt{2} + 2$
(c) $10\sqrt{2} + 2$ (d) $16\sqrt{2} + 2$

15. The rate of change of m(t), with respect to t, at $t = \ell$ is
(a) –2 (b) 2
(c) – 4 (d) 4

Comprehension–2 (No. 16 & 17)

Consider a function f defined by $f(x) = \sin^{-1}\sin\left(\frac{x + \sin x}{2}\right)$, $\forall x \in [0, \pi]$, which satisfies
$f(x) + f(2\pi - x) = \pi$, $\forall x \in [\pi, 2\pi]$ and $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$, then

16. If α is the length of the largest interval on which f(x) is increasing, then α =
(a) $\frac{\pi}{2}$ (b) π
(c) 2π (d) 4π

17. If f(x) is symmetric about $x = \beta$, then $\beta =$

(a) $\frac{\alpha}{2}$ (b) α

(c) $\frac{\alpha}{4}$ (d) 2α

18. **Match the column:**

Column-I

(a) If θ is angle between the curves $y = [|\sin x| + |\cos x|]$, ($[\cdot]$ denote GIF) and $x^2 + y^2 = 5$ then $\text{cosec}^2\theta$ is

(b) Length of subnormal to $x = \sqrt{2}\cos t, y = -3\sin t$ at $t = \frac{-\pi}{4}$ is

(c) If $[a, b]$, $(b < 1)$ is largest interval in which $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 19$ is strictly increasing then $\frac{a}{b}$ is

(d) If $a + b = 8$, $a, b > 0$ then minimum value of $\frac{a^3 + b^3}{48}$ is

Column-II

(p) $\frac{5}{4}$ (q) 2

(r) $\frac{8}{3}$ (s) $\frac{9}{2}$

19. **Match the column:**

Column – I

(a) A rectangle is inscribed in an equilateral triangle of side 4cm. Square of maximum area of such a rect angle is

(b) The volume of a rectangular closed box is 72 and the base sides are in the ratio 1 : 2. The least total surface area is

(c) Maximum value of $\left(\sqrt{-3 + 4x - x^2} + 4\right)^2 + (x - 5)^2$ (where $1 \le x \le 3$) is

(d) The sides of a rectangle of greatest perimeter which is inscribed in a semicircle of radius $\sqrt{5}$ are a and b. Then $a^3 + b^3 =$

Column – II

(p) 65 (q) 36

(r) 12 (s) 108

NUMERICAL BASED QUESTIONS

20. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then absolute value of b is

21. If the rate of change in the circumference of a circle is 0.3 cm/sec, then the rate of change in the area of the circle when the radius is 5 cm is

22. Roll's theorem holds for the function $x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point 4/3, the values of b + c are

23. Abscissa of a point on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes is

24. If the tangent at (a,b) to the curve $x^3 + y^3 = c^3$ meet the curve again at (a_1, b_1) and $\frac{a_1}{a} + \frac{b_1}{b} = -k$ then k is

25. If a particle moves in a straight line according to the law $S = \frac{t^3}{3} - \frac{t^2}{2} - 6t + 5$, then its acceleration where velocity is zero is

26. The tangent at any point on the curve $x^4 + y^4 = a^4$ cuts intercepts p and q on the axes then $p^{-4/3} + q^{-4/3} = \frac{1}{\left(a^k\right)^{1/3}}$ then k is .....

27. If a variable tangent to the curve $x^2y = c^3$ makes intercepts of length a & b on x & y axis respectively then $\frac{a^2b}{c^3} = \mu$. then $[\mu]$ is (where $[.]$ is greatest integer function.)

28. Height of a cylinder of maximum volume that can be inscribed is a sphere of radius 12cm is $8\sqrt{K}$ then K is .....

29. The number of real roots of the equation $x^3 - 3x + 1 = 0$ is

JEE-MAIN PREVIOUS YEARS

1. Consider $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right)$. A normal to $y=f(x)$ at $x=\frac{\pi}{6}$ also passes through the point:

[JEE Main-2016]

(a) $(0,0)$ (b) $\left(0,\frac{2\pi}{3}\right)$

(c) $\left(\frac{\pi}{6},0\right)$ (d) $\left(\frac{\pi}{4},0\right)$

2. The normal to the curve $y(x-2)(x-3)=x+6$ at the point where the curve intersects the y-axis passes through the point:

[JEE Main- 2017]

(a) $\left(\frac{1}{2},\frac{1}{3}\right)$ (b) $\left(-\frac{1}{2},-\frac{1}{2}\right)$

(c) $\left(\frac{1}{2},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},-\frac{1}{3}\right)$

3. Twenty meters of wire is available for fencing off a flower bed in the form of a circular sector. Then the maximum are (in sq. m) of the flower bed, is:

[JEE Main- 2017]

(a) 30 (b) 12.5
(c) 10 (d) 25

4. The radius of a circle, having minimum area, which touches the cure $y=4-x^2$ and the lines, $y=|x|$ is :

(a) $4\left(\sqrt{2}+1\right)$ (b) $2\left(\sqrt{2}+1\right)$

(c) $2\left(\sqrt{2}-1\right)$ (d) $4\left(\sqrt{2}-1\right)$

5. Tangents are drawn to the hyperbola $4x^2-y^2=36$ at the point P and If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is -

[JEE Main - 2018]

(a) $54\sqrt{3}$ (b) $60\sqrt{3}$
(c) $36\sqrt{5}$ (d) $45\sqrt{5}$

6. Let $f(x)=x^2+\frac{1}{x^2}$ and $g(x)=x-\frac{1}{x}, x\in R-\{-1,0,1\}$. If $h(x)=\frac{f(x)}{g(x)}$ then the local minimum value of $h(x)$ is :

[JEE Main - 2018]

(a) -3 (b) $-2\sqrt{2}$
(c) $2\sqrt{2}$ (d) 3

7. The maximum volume (in cu.m) of the right circular cone having slant height 3 m is

[JEE Main - 2019 (January)]

(a) 6π (b) $3\sqrt{3}\pi$

(c) $\frac{4}{3}\pi$ (d) $2\sqrt{3}\pi$

8. If θ denotes the acute angle between the curves, $y=10-x^2$ and $y=2+x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to

[JEE Main - 2019 (January)]

(a) $\frac{4}{9}$ (b) $\frac{8}{15}$

(c) $\frac{7}{17}$ (d) $\frac{8}{17}$

9. Let $f:(-1,1)\to R$ be a function defined by $f(x)=\max\left\{-|x|,-\sqrt{1-x^2}\right\}$. if K be the set of all points at which f is not differentiable, then K has exactly:

[JEE Main - 2019 (January)]

(a) five elements (b) one element
(c) three elements (d) two elements

10. The tangent to the curve, $y=xe^{x^2}$ passing, through the point (1, e) also passes through the point :

[JEE Main - 2019 (January)]

(a) $(2,3e)$ (b) $\left(\frac{4}{3},2e\right)$

(c) $\left(\frac{5}{3},2e\right)$ (d) $(3,6e)$

11. A helicopter is flying along the curve given by $y=x^{3/2}-2, (x\geq 0)$. A solider positioned at the point $\left(\frac{1}{2},7\right)$ wants to shoot down the helicopter when it is nearest to him. then this nearest distance is;

[JEE Main - 2019 (January)]

(a) $\frac{\sqrt{5}}{6}$ (b) $\frac{1}{3}\sqrt{\frac{7}{3}}$

(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

12. Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$, $x \in R$, where a, b and d are non - zero real constant. Then,

[JEE Main - 2019 (January)]

(a) f is an increasing function of x
(b) f is a decreasing function of x
(c) f is not a continuous function of x
(d) f is neither increasing nor decreasing function of x

13. The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$ on the set $S = \{x \in R : x^2 + 30 \le 11x\}$ is

[JEE Main - 2019 (January)]

(a) -122 (b) -222
(c) 122 (d) 222

14. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is:

[JEE Main - 2019 (January)]

(a) 36 (b) $20\sqrt{2}$
(c) 32 (d) $18\sqrt{3}$

15. Let P(4, –4) and Q(9, 6) be two points on the parabola, $y^2 = 4x$ and let X be any point on the arc POQ of this parabola, where O is the vertex of this parabola, such that the area of ΔPXQ is maximum. Then this maximum Area (in sq. units) is:

[JEE Main - 2019 (January)]

(a) $\frac{75}{2}$ (b) $\frac{125}{4}$
(c) $\frac{625}{4}$ (d) $\frac{125}{2}$

16. The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point :

[JEE Main - 2019 (January)]

(a) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{8}, -7\right)$
(c) $\left(-\frac{1}{8}, 7\right)$ (d) $\left(\frac{1}{4}, \frac{7}{2}\right)$

17. If the function f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x) - 14}{(x-1)^2} = 0 (x \ne 1)$ is:

[JEE Main - 2019 (January)]

(a) –7 (b) 5
(c) 7 (d) 6

18. Let $f : [0, 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then ϕ is

[JEE Main - 2019 (April)]

(a) decreasing on (0, 2)
(b) decreasing on (0, 1) and increasing on (1, 2)
(c) increasing on (0, 2)
(d) increasing on (0, 1) and decreasing on (1, 2)

19. The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is

[JEE Main - 2019 (April)]

(a) $2\sqrt{3}$ (b) $\sqrt{3}$
(c) $\sqrt{6}$ (d) $\frac{2}{3}\sqrt{3}$

20. If f(x) is a non-zero polynomial of degree four, having local extreme points at x = –1, 0, 1 ; then the set $S = \{x \in R : f(x) = f(0)\}$ Contains exactly :

[JEE Main - 2019 (April)]

(a) four irrational numbers
(b) two irrational and one rational number.
(c) four rational numbers
(d) two irrational and two rational number.

21. Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$, then S is equal to :

[JEE Main - 2019 (April)]

(a) $\left\{-\frac{1}{3}, -1\right\}$ (b) $\left\{\frac{1}{3}, -1\right\}$
(c) $\left\{-\frac{1}{3}, 1\right\}$ (d) $\left\{\frac{1}{3}, 1\right\}$

22. A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 cubic meter per minute. The the rate (in m/min.), at which the level of water is rising at the instant when the depth of water in the tank is 10 m; is :-

[JEE Main - 2019 (April)]

(a) $2/\pi$ (b) $1/5\pi$
(c) $1/10\pi$ (d) $1/15\pi$

23. A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/ min. When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is :

[JEE Main - 2019 (April)]

(a) $\frac{1}{9\pi}$ (b) $\frac{5}{6\pi}$
(c) $\frac{1}{18\pi}$ (d) $\frac{1}{36\pi}$

24. Let $a_1, a_2, a_3, \ldots$ be an A.P. with $a_6 = 2$. Then the common difference of this A.P. which maximise the produce $a_1a_4a_5$, is : [JEE Main - 2019(April)]

(a) $\frac{6}{5}$ (b) $\frac{8}{5}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

25. If the tangent to the curve $y = \frac{x}{x^2-3}$ $x \in R$, $\left(x \neq \pm\sqrt{3}\right)$, at a point $(\alpha,\beta) \neq (0,0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then : [JEE Main - 2019(April)]

(a) $|6\alpha + 2\beta| = 19$ (b) $|2\alpha + 6\beta| = 11$.

(c) $|6\alpha + 2\beta| = 9$ (d) $|2\alpha + 6\beta| = 19$

26. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$. Then the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is : [JEE Main - 2019(April)]

(a) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (b) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

(c) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$ (d) $[0, \infty)$

27. If the tangent to the curve, $y = x^3 + ax - b$ at the point $(1, -5)$ is perpendicular to the line, $-x+y+4 = 0$, then which one of the following points lies on the curve ? [JEE Main - 2019(April)]

(a) $(-2, 2)$ (b) $(2, -2)$

(c) $(2, -1)$ (d) $(-2, 1)$

28. Let the function $f : [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \leq 2$ for all $x \in (-7, 0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval [JEE Main-2020 (January)]

(a) $[-6, 20]$ (b) $(-\infty, 20]$

(c) $(-\infty, 11]$ (d) $[-3, 11]$

29. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is: [JEE Main-2020 (January)]

(a) $\frac{4-\sqrt{5}}{3}$ (b) $\frac{\sqrt{7}-2}{3}$

(c) $\frac{2}{3}$ (d) $\frac{4-\sqrt{7}}{3}$

30. Let $f(x)$ be a polynomial of degree 5 such that $x = \pm 1$ are its critical points. If $\lim\limits_{x\to 0}\left(2 + \frac{f(x)}{x^3}\right) = 4$, then which one of the following is not true? [JEE Main-2020 (January)]

(a) $x = 1$ is a point of minima and $x = -1$ is a point of maxima of f.

(b) $f(1) - 4f(-1) = 4$.

(c) $x = 1$ is a point of maxima and $x = -1$ is a point of minimum of f.

(d) f is an odd function.

31. Let $f'(x) = x\cos^{-1}(-\sin|x|)$, $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true? [JEE Main-2020 (January)]

(a) $f'(0) = \frac{-\pi}{2}$

(b) f' is decreasing in $\left(\frac{-\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

(c) f is not differentiable at $x = 0$

(d) f' is increasing in $\left(\frac{-\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

32. If c is a point at which Rolle's Theorem holds for the function, $f(x) = \log_e\left(\frac{x^2+\alpha}{7x}\right)$ in the interval $[3, 4]$, where $\alpha \in R$, then $f''(c)$ is equal to [JEE Main-2020 (January)]

(a) $\frac{\sqrt{3}}{7}$ (b) $\frac{-1}{24}$

(c) $\frac{-1}{12}$ (d) $\frac{1}{12}$

33. Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to [JEE Main-2020 (January)]

34. Let $f : (1, 3) \to R$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is : [JEE Main-2020 (January)]

(a) $\left(\frac{3}{4}, \frac{4}{5}\right)$ (b) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

(c) $\left(\frac{2}{5}, \frac{3}{5}\right] \cup \left(\frac{3}{4}, \frac{4}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{4}{5}\right]$

35. The length of the perpendicular from the origin on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at the point (2, 2) is

[JEE Main-2020 (January)]

(a) $4\sqrt{2}$ (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) 2

36. Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at $x =$

[JEE Main-2020 (January)]

37. Let f be any function continuous on [a, b] and twice differentiable on (a, b). If for all $x \in (a, b)$, $f'(x) > 0$ and $f''(x) < 0$, then for any $c \in (a, b)$, $\frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than

[JEE Main-2020 (January)]

(a) 1 (b) $\frac{b-c}{c-a}$

(c) $\frac{c-a}{b-c}$ (d) $\frac{b+a}{b-a}$

38. Let a function $f: [0, 5] \to R$ be continuous, $f(1) = 3$ and F be defined as: $F(x) = \int_1^x t^2 g(t)\,dt$, where $g(t) = \int_1^t f(u)du$. Then for the function F, the point $x = 1$ is :

[JEE Main-2020 (January)]

(a) a point of local minima
(b) a point of focal maxima
(c) not a critical point
(d) a point of inflection

39. Let $f : (-1, \infty) \to R$ be defined by $f(0) = 1$ and $f(x) = \frac{1}{x}\log_e(1+x), x \neq 0$. Then the function f:

[JEE Main-2020 (September)]

(a) increases in $(-1, \infty)$
(b) increases in $(-1, 0)$ and decreases in $(0, \infty)$
(c) decreases in $(-1, 0)$ and increases in $(0, \infty)$
(d) decreases in $(-1, \infty)$

40. The equation of the normal to the curve

[JEE Main-2020 (September)]

$y = (1 + x)^{2y} + \cos^2(\sin^{-1}x)$ at $x = 0$ is :

(a) $y = 4x + 2$ (b) $y + 4x = 2$

(c) $x + 4y = 8$ (d) $2y + x = 4$

41. If the tangent to the curve $y = x + \sin y$ at a point (a, b) is parallel to the line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$, then :

[JEE Main-2020 (September)]

(a) $b = a$ (b) $b = \frac{\pi}{2} + a$

(c) $|a + b| = 1$ (d) $|b - a| = 1$

42. Let P(h, k) be a point on the curve $y = x^2 + 7x + 2$, nearest to the line, $y = 3x - 3$. Then the equation of the normal to the curve at P is :

[JEE Main-2020 (September)]

(a) $x - 3y - 11 = 0$ (b) $x - 3y + 22 = 0$

(c) $x + 3y - 62 = 0$ (d) $x + 3y + 26 = 0$

43. If p(x) be a polynomial of degree three that has a local maximum value 8 at $x = 1$ and a local minimum value 4 at $x = 2$; then $p(0)$ is equal to :

[JEE Main-2020 (September)]

(a) 12 (b) 6

(c) –24 (d) –12

44. If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape; then the rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10 cm, is:

[JEE Main-2020 (September)]

(a) 9 (b) 20

(c) 10 (d) 18

45. Suppose $f(x)$ is a polynomial of degree four, having critical points at –1, 0, 1. If $T = \{x \in R \mid f(x) = f(0)\}$, then the sum of squares of all the elements of T is :

[JEE Main-2020 (September)]

(a) 4 (b) 2

(c) 6 (d) 8

46. The function, $f(x) = (3x - 7)x^{2/3}$, $x \in R$, is increasing for all x lying in:

[JEE Main-2020 (September)]

(a) $(-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$ (b) $(-\infty, 0) \cup \left(\frac{3}{7}, \infty\right)$

(c) $\left(-\infty, -\frac{14}{15}\right) \cup (0, \infty)$ (d) $\left(-\infty, \frac{14}{15}\right)$

47. The area (in sq. units) of the largest rectangle ABCD whose vertices A and B lie on the x-axis and vertices C and D lie on the parabola, $y = x^2 - 1$ below the x-axis is

[JEE Main-2020 (September)]

(a) $\frac{4}{3\sqrt{3}}$ (b) $\frac{1}{3\sqrt{3}}$

(c) $\frac{2}{3\sqrt{3}}$ (d) $\frac{4}{3}$

48. Let f be a twice differentiable function on (1, 6). If $f(2) = 8$, $f'(2) = 5$, $f'(x) \geq 1$ and $f''(x) \geq 4$, for all $x \in (1, 6)$, then

[JEE Main-2020 (September)]

(a) $f(5) + f'(5) \geq 28$ (b) $f'(5) + f''(5) \leq 20$

(c) $f(5) \leq 10$ (d) $f(5) + f''(5) \leq 26$

49. If the lines $x + y = a$ and $x - y = b$ touch the curve $y = x^2 - 3x + 2$ at the points where the curve intersects the x-axis, then $\frac{a}{b}$ is equal to _____.

[JEE Main-2020 (September)]

50. If $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$, then :

[JEE Main-2020 (September)]

(a) $x = 1$ is a local maxima and $x = -\frac{2}{3}$ is a local minima of f.

(b) $x = 1$ and $x = -\frac{2}{3}$ are local maxima of f.

(c) $x = 1$ and $x = -\frac{2}{3}$ are local minima of f.

(d) $x = 1$ is a local minima and $x = -\frac{2}{3}$ is a local maxima of f.

51. Which of the following points lies on the tangent to the curve $x^4e^y + 2\sqrt{y+1} = 3$ at the point $(1, 0)$?

[JEE Main-2020 (September)]

(a) $(2, 2)$ (b) $(-2, 4)$

(c) $(2, 6)$ (d) $(-2, 6)$

52. Awire of length 20m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then the length of the side (in meters) of the hexagon, so that the combined area of the square and the hexagon is minimum, is:

[JEE Main-2021 (August)]

(a) $\frac{5}{2+\sqrt{3}}$ (b) $\frac{10}{2+3\sqrt{3}}$

(c) $\frac{5}{3+\sqrt{3}}$ (d) $\frac{10}{3+2\sqrt{3}}$

53. Let 'a' be a real number such that the function $f(x) = ax^2 + 6x - 15, x \in R$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then the function $g(x) = ax^2 - 6x + 15, x \in R$ has a:

[JEE Main-2021]

(a) local maximum at $x = -\frac{3}{4}$

(b) local minimum at $x = -\frac{3}{4}$

(c) local maximum at $x = \frac{3}{4}$

(d) local minimum at $x = \frac{3}{4}$

54. Let slope of the tangent line to a curve at any point $P(x, y)$ be given by $\frac{xy^2 + y}{x}$. If the curve intersects the line $x + 2y = 4$ at $x = -2$, then the value of y, for which the point $(3, y)$ lies on the cune, is:

[JEE Main-2021 (February)]

(a) $-\frac{18}{19}$ (b) $-\frac{4}{3}$

(c) $\frac{18}{35}$ (d) $-\frac{18}{11}$

JEE-ADVANCED PREVIOUS YEARS

1. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contacts of the tangents on the hyperbola are

[IIT JEE - 2012]

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (d) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

2. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ $p(3) = 2$, then $p'(0)$ is

[IIT JEE-2012]

3. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is

[IIT JEE-2012]

4. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is [JEE Advanced - 2013]

(a) 6 (b) 4

(c) 2 (d) 0

5. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are

[JEE Advanced - 2013]

(a) 24 (b) 32

(c) 45 (d) 60

6. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

[JEE Advanced - 2013]

7. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x =

[JEE Advanced - 2013]

(a) -2 (b) $\frac{-2}{3}$

(c) 2 (d) $\frac{2}{3}$

Comprehension – (No. 8 & 9)

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$. [JEE Advanced-2013]

8. Which of the following is true for $0 < x < 1$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

9. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true ?

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$

(d) $f'(x) < f(x), \frac{3}{4} < x < 1$

10. A line L : $y = mx + 3$ meets y - axis at E(0, 3) and the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum Match List I with List II and select the correct answer using the code given below the lists :

[JEE Advanced-2013]

List - I	List - II
P. m=	1. $\frac{1}{2}$
Q. Maximum area of ΔEFG is	2. 4
R. y_0 =	3. 2
S. y_1 =	4. 1

Codes :	P	Q	R	S
(a)	4	1	2	3
(b)	3	4	1	2
(c)	1	3	2	4
(d)	1	3	4	2

Comprehension (No. 11 & 12)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$. [IIT JEE-2013]

11. Length of chord PQ is

(a) 7a (b) 5a

(c) 2a (d) 3a

12. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

(a) $\frac{2}{3}\sqrt{7}$ (b) $\frac{-2}{3}\sqrt{7}$

(c) $\frac{2}{3}\sqrt{5}$ (d) $\frac{-2}{3}\sqrt{5}$

13. Let $a \in R$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then

[JEE Advanced - 2014]

(a) f(x) has three real roots if $a > 4$

(b) f(x) has only one real root if $a > 4$

(c) f(x) has three real roots if $a < -4$

(d) f(x) has three real roots if $-4 < a < 4$

14. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is

[JEE Advanced - 2014]

15. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral PQRS is

[JEE Advanced -2014]

(a) 3 (b) 6

(c) 9 (d) 15

Comprehension – (No. 16 & 17)

Let a, r, s, t be nonzero real numbers. Let $P(at^2, 2at)$, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0)

[JEE Advanced-2014]

16. The value of r is:

(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$

(c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

17. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$

(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

18. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

[JEE Advanced -2015]

19. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is

[JEE Advanced-2015]

20. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P? [JEE Advanced - 2015]

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$

(C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$

21. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is.

[JEE Advanced - 2015]

22. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

[JEE Advanced -2015]

(a) $e_1^2 + e_2^2 = \frac{43}{40}$ (b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(c) $|e_1^2 - e_2^2| = \frac{5}{8}$ (d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

23. Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

[JEE Advanced-2015]

(a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(b) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

24. A cylindrical container is to be made from certain solid material with the following constraints: It has fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

[JEE Advanced-2015]

25. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

[JEE Advanced - 2015]

26. Let $f, g: [-1, 2] \to R$ be continuous function which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is (are)

[JEE Advanced-2015]

(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

27. Let $f: R \to (0, 1)$ be a continuous function. Then which of the following function(s) has (have) the value zero at some point in the interval $(0, 1)$?

[JEE Advanced -2017]

(a) $e^x - \int_0^x f(t)\sin t \, dt$ (b) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t \, dt$

(c) $x - \int_0^{\frac{\pi}{2}} f(t)\cos t \, dt$ (d) $x^9 - f(x)$

28, 29 and **30** by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x\log_e x,\ x \in (0, \infty)$

Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.

Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.

Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	**(i)** $\lim_{x\to\infty} f(x) = 0$	**(P)** f is increasing in $(0, 1)$
(II) $f'(x) = 0$ for some $x \in (1, e)$	**(ii)** $\lim_{x\to\infty} f(x) = -\infty$	**(Q)** f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	**(iii)** $\lim_{x\to\infty} f'(x) = -\infty$	**(R)** f' is increasing in $(0, 1)$
(IV) $f''(x) = 0$ for some $x \in (1, e)$	**(iv)** $\lim_{x\to\infty} f''(x) = 0$	**(S)** f' is decreasing in (e, e^2)

28. Which of the following options is the INCORRECT combination?

[*JEE Advanced -2017*]

(*a*) (I) (iii) (P) (*b*) (II) (iv) (Q)

(*c*) (II) (iii) (P) (*d*) (III) (i) (R)

29. Which of the folllowing options is the only CORRECT combination ?

[*JEE Advanced -2017*]

(*a*) (I) (iii) (R) (*b*) (III) (iv) (P)

(*c*) (II) (iii) (S) (*d*) (IV) (i) (S)

30. Which of the folllowing options is the only CORRECT combination ?

[*JEE Advanced -2017*]

(*a*) (III) (iii) (R) (*b*) (IV) (iv) (S)

(*c*) (II) (ii) (Q) (*d*) (I) (i) (P)

31. If $f : R \to R$ is a twice differentiable function such that $f'(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1$, then

[*JEE Advanced -2017*]

(*a*) $f'(1) \le 0$ (*b*) $f'(1) > 1$

(*c*) $0 < f'(1) \le \frac{1}{2}$ (*d*) $\frac{1}{2} < f'(1) \le 1$

32. If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$, then

[*JEE Advanced -2017*]

(*a*) $f(x) > e^{2x}$ in $(0, \infty)$

(*b*) $f(x) < e^{2x}$ in $(0, \infty)$

(*c*) $f(x)$ is increasing in $(0, \infty)$

(*d*) $f(x)$ is decreasing in $(0, \infty)$

33. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

[*JEE Advanced -2017*]

(*a*) $f(x)$ attains its minimum at $= 0$

(*b*) $f(x)$ attains its maximum at $x = 0$

(*c*) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

(*d*) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

34. For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? [*JEE Advanced - 2018*]

(*a*) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)

(*b*) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(*c*) $\lim_{x\to\infty} f(x) = 1$

(*d*) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(a) \ne 0$

35. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ? [*JEE Advanced - 2018*]

(*a*) $f(2) < 1 - \log_e 2$

(*b*) $f(2) > 1 - \log_e 2$

(*c*) $g(1) > 1 - \log_e 2$

(*d*) $g(1) < 1 - \log_e 2$

36. Let $f : R \to R$ be given by

$$\begin{cases} x^5 + 5x^4 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

Then which of the following options is/are correct ?

[*JEE Advanced -2019*]

(*a*) f has a local maximum at $x = 1$

(*b*) f is onto

(*c*) f is increasing on $(-\infty, 0)$

(*d*) f' is NOT differntiable at $x = 1$

37. Let $f : R \to R$ be given by $f(x) = (x-1)(x-2)(x-5)$. Define $F(x) = \int_0^x f(t)dt$, $x > 0$. Then which of the following options is/are correct? [*JEE Advanced -2019*]

(*a*) F has a local minimum at $x = 1$

(*b*) F has a local maximum at $x = 2$

(*c*) $F(x) \ne 0$ for all $x \in (0,5)$

(*d*) F has two local maxima and one local minimum in $(0, \infty)$

38. Let $f(x)=\frac{\sin \pi x}{x^2}$, $x>0$ **[JEE Advanced -2019]**

Let $x_1<x_2<x_3<.....<x_n<....$ be all the points of local maximum off and $y_1<y_2<y_3<.....<y_n<..$ be all the points of local minimum off.

Then which of the following options is/are correct ?

(*a*) $|x_n-y_n|>1$ for every n

(*b*) $x_1<y_1$

(*c*) $x_n \in \left(2n, 2n+\frac{1}{2}\right)$ for every n

(*d*) $x_{n+1}-x_n>2$ for every n

39. Consider all rectangles lying in the region

$$\left\{(x,y)\in \mathbb{R}\times\mathbb{R}: 0\le x\le \frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

[JEE(Advanced) - 2020]

(*a*) $\frac{3\pi}{2}$ (*b*) π

(*c*) $\frac{\pi}{2\sqrt{3}}$ (*d*) $\frac{\pi\sqrt{3}}{2}$

40. For a polynomial g(x) with real coefficient, let m_g denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficient defined by

$S=\{(x^2+1^2)(a_0+a_1x+a_2x^2+a_3x^3): a_0, a_1, a_2, a_3\in\mathbb{R}\}$.

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'}+m_{f''})$, where $f\in S$, is ....

[JEE(Advanced) - 2020]

41. Let b be a nonzero real number. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f(0)=1$. If the derivative f' of f satisfies the equation $f'(x)=\frac{f(x)}{b^2+x^2}$

for all $x\in\mathbb{R}$, then which of the following statements is/are TRUE? **[JEE(Advanced) - 2020]**

(*a*) If $b>0$, then f is an increasing function

(*b*) If $b<0$, then f is a decreasing function

(*c*) $f(x)f(-x)=1$ for all $x\in\mathbb{R}$

(*d*) $f(x)-f(-x)=0$ for all $x\in\mathbb{R}$

42. Let the function $f:(0,\pi)\to\mathbb{R}$ be defined by $f(\theta)=(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^4$

Suppose the function f has a local minimum at θ precisely when $\theta\in\{\lambda_1\pi,....\lambda_r\pi\}$, where $0<\lambda_1<.....<\lambda_r<1$. Then the value of $\lambda_1+....+\lambda_r$ is .....

[JEE(Advanced) - 2020]

ANSWER KEY

Topicwise Questions

1. (*a*)	**2.** (*a*)	**3.** (*c*)	**4.** (*a*)	**5.** (*a*)	**6.** (*b*)	**7.** (*d*)	**8.** (*d*)	**9.** (*c*)	**10.** (*d*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*d*)	**16.** (*d*)	**17.** (*c*)	**18.** (*a*)	**19.** (*d*)	**20.** (*a*)
21. (*c*)	**22.** (*c*)	**23.** (*b*)	**24.** (*c*)	**25.** (*a*)	**26.** (*b*)	**27.** (*c*)	**28.** (*a*)	**29.** (*b*)	**30.** (*b*)

Learning Plus

1. (*a*)	**2.** (*c*)	**3.** (*d*)	**4.** (*c*)	**5.** (*c*)	**6.** (*a*)	**7.** (*c*)	**8.** (*c*)	**9.** (*d*)	**10.** (*a*)
11. (*b*)	**12.** (*d*)	**13.** (*b*)	**14.** (*b*)	**15.** (*b*)	**16.** (*a*)	**17.** (*c*)	**18.** (*b*)	**19.** (*d*)	**20.** (*d*)
21. (*a*)	**22.** (*b*)	**23.** (*d*)	**24.** (*a*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*b*)		

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING\NUMERICAL

1. (*a,b,c*)	**2.** (*a,c*)	**3.** (*a,b*)	**4.** (*a,d*)	**5.** (*a,c*)	**6.** (*a,c,d*)	**7.** (*b,c*)	**8.** (*b,c*)	**9.** (*a,b*)	**10.** (*a,b,d*)
11. (*a,c,d*)	**12.** (*a,c*)	**13.** (*b*)	**14.** (*d*)	**15.** (*c*)	**16.** (*c*)	**17.** (*b*)	**18.** (*a*)-(p), (*b*)-(s), (*c*)-(q), (*d*)-(r)		
19. (*a*)-(r), (*b*)-(s), (*c*)-(q), (*d*)-(p)			**20.** (3)	**21.** (1.5)	**22.** (3)	**23.** (4)	**24.** (1)	**25.** (5)	**26.** (4)
27. [6]	**28.** (3)	**29.** (3)							

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*b*)	**2.** (*c*)	**3.** (*d*)	**4.** (Bonus/4)	**5.** (*d*)	**6.** (*c*)	**7.** (*d*)	**8.** (*b*)	**9.** (*c*)	**10.** (*b*)
11. (*c*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*b*)	**16.** (*b*)	**17.** (*c*)	**18.** (*b*)	**19.** (*a*)	**20.** (*b*)
21. (*c*)	**22.** (*b*)	**23.** (*c*)	**24.** (*b*)	**25.** (*a*)	**26.** (*b*)	**27.** (*b*)	**28.** (*b*)	**29.** (*d*)	**30.** (*c*)
31. (*b*)	**32.** (*d*)	**33.** (4)	**34.** (*b*)	**35.** (*c*)	**36.** (3)	**37.** (*c*)	**38.** (*a*)	**39.** (*d*)	**40.** (*c*)
41. (*d*)	**42.** (*d*)	**43.** (*d*)	**44.** (*a*)	**45.** (*a*)	**46.** (*a*)	**47.** (*a*)	**48.** (*a*)	**49.** (0.50)	**50.** (*d*)
51. (*d*)	**52.** (*d*)	**53.** (*a*)	**54.** (*a*)						

JEE-ADVANCED PREVIOUS YEARS

1. (*a,b*)	**2.** (9)	**3.** (5)	**4.** (*c*)	**5.** (*a,c*)	**6.** (9)	**7.** (*a,b*)	**8.** (*d*)	**9.** (*c*)	**10.** (*a*)
11. (*b*)	**12.** (*d*)	**13.** (*b,d*)	**14.** (8)	**15.** (*d*)	**16.** (*d*)	**17.** (*b*)	**18.** (4)	**19.** (2)	**20.** (*a,d*)
21. (4)	**22.** (*a,b*)	**23.** (*a,b,d*)	**24.** (4)	**25.** (3)	**26.** (*b,c*)	**27.** (*c,d*)	**28.** (*d*)	**29.** (*c*)	**30.** (*c*)
31. (*b*)	**32.** (*a,c*)	**33.** (*b,c*)	**34.** (*a,b,d*)	**35.** (*b,c*)	**36.** (*a,b,d*)	**37.** (*a,b,c*)	**38.** (*a,c,d*)	**39.** (*c*)	**40.** (5.00)
41. (*a,c*)	**42.** (0.50)								

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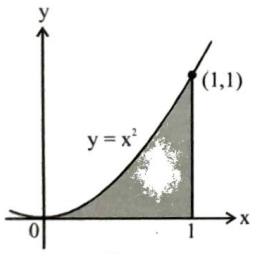
25 DEFINITE INTEGRATION

DEFINITION

A definite integral is denoted by $\int_a^b f(x)\,dx$ which represents the algebraic area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

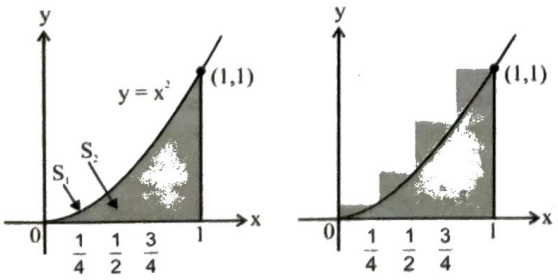
THE AREA PROBLEM

Use rectangles to estimate the area under the parabola $y = x^2$ from, 0 to 1 We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$ and $x = \frac{3}{4}$ as in Figure (a).



Figure

We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure (b)]. In other words, the heights of these rectangle are the values of the function $f(x) = x^2$ at the right end points of the subintervals $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.



(a) (b)

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}.1^2 = \frac{15}{32}$$

$$= 0.46875$$

From the Figure (b) we see that the area A of S is less than R_4, so $A < 0.46875$

Instead of using the rectangles in Figure (b) we could use the smaller rectangles in Figure (c) whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its heights is 0). The sum of the areas of these approximating rectangles is

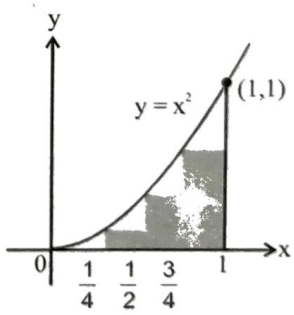


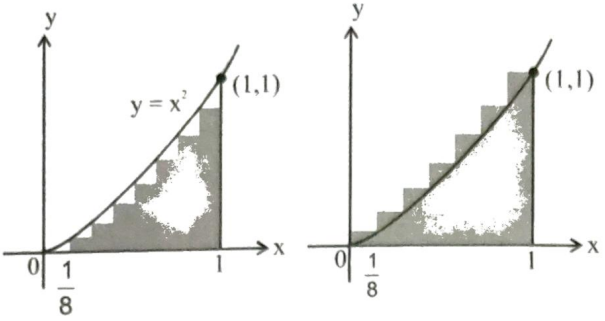
Figure (c)

$$L_4 = \frac{1}{4}.0^2 + \frac{1}{4}.\left(\frac{1}{4}\right)^2 + \frac{1}{4}.\left(\frac{1}{2}\right)^2 \frac{1}{4} + \frac{1}{4}.\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for

$A\,0.21875 < A < 0.46875$

We can repeat this procedure with a larger number of strips. Figure (d), (e) shows what happens when we divide the region S into eight strips of equal width.



(d) Using left endpoint (e) Using right endpoint

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A : $0.2734375 < A < 0.3984375$

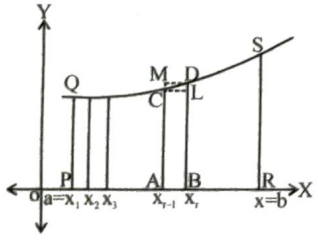
So one possible answer the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375. We coluld obtain better estimates by increasing the number of strips.

DEFINITE INTEGRAL AS THE LIMIT OF SUM

If $f(x) > 0$ for all $x \in [a, b]$ then the definite integral $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the ordinate $x = a$, $x = b$ and x-axis.

To evaluate area of the region PRSQP between the curve x-axis and the ordinates $x = a$ and $s = b$. divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1], [x_1, x_2], \ldots [x_{r-1}, x_r] \ldots [x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $\ldots x_r = a + rh$ and $x_n = b = a + nh$ or $h = \dfrac{b-a}{n}$.

We note that as $n \to \infty$, $h \to 0$.



The region PRSQP under consideration is the sum of n subregions where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots n$.

From figure, we have arc of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ...(1)

Evidently as $x_r - x_{r-1} \to$ i.e. $h \to 0$ all the three area shown in (1) become nearly equal to each other. Now we from the following sums.

$$s_n = h|f(x_0) + \ldots + f(x_{n-1})| = h\sum_{r=0}^{n-1} f(x_1)$$

and $$S_n = h|f(x_1) + \ldots + f(x_2) + \ldots + f(x_n)| = h\sum_{r=1}^{n} f(x_r)$$

Hence s_n and S_n denote the sum of areas of all lower rectangles and upper rectangle raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$ respectively.

In view of the inequalituy (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, wer have

$x_n <$ area of the region PRSQP $< S_n$

As $n \to$ strip become narrower and narrower. It is assumed that the limiting value of (2) and (3) are same in both cases and the common limiting vlaues is the required area under the curve. Symbolically, we have

$$\lim_{x\to\infty} S_n = \lim_{x\to\infty} s_n$$

$$= \text{area of the region PRSQP} = \int_a^b f(x)\,dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangle below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)\,dx = \lim_{h\to 0} [hf(a) + f(a+h) + \ldots + f[(a) + (n-1)h]]$$

$$\int_a^b f(x)\,dx = (b-a)\lim_{h\to 0}\frac{1}{n}[f(a) + f(a+h) + \ldots\ldots\ldots\ldots\ldots\ldots +$$

$$f[(a) + (n-1)h]] \qquad \ldots (6)$$

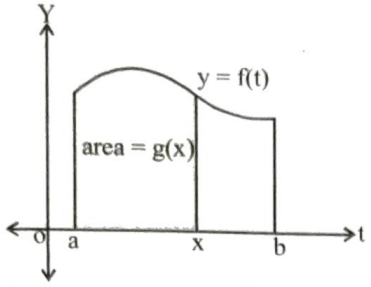
where $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

The above expression (6) is known as the definition of definite integral as the limit of sum.

Evaluating a definite integral by evaluating the limit of a sum is called evaluating definite integral by first principle or by ab-inito method.

Remark : The value of definite integral of a function over any particular interval dependeds on the function and the inerval but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a dummy variable.

THE FUNDAMENTAL THEOREM OF CALCULUS



The fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose fromt he tangent problem, where integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Issac Barrow (1630 – 1677), discovlered that thse two problem are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundament Theorem of Calculus given the precise inverse relationship between the derivative and the integral. It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.

The Fundamental Theorem of Calculus

Part-1 : If f is continuous on [a, b], then the function g defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$ is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

Part-2 : If f is continuous on [a, b], then $\int_b^a f(x)\cdot dx = F(b) - F(a)$ where F is any antiderivative of f, that is a function such that F' = I.

PROPERTIES OF DEFINITE INTEGRAL

P-1 : CHANGE OF VARIABLE : The definite integral $\int_a^b f(x)\,dx$ is a number, it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral ;

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr .$$

P-2 : CHANGE OF LIMIT: When we defined the definite integral $\int_a^b f(x)\,dx$, we implicity assumed that a < b. But the definition as a limit of sum makes sense even if a > b. Notice that if we reverse a and b, then Δx changes from (b – a)/n to (a – b)/n.

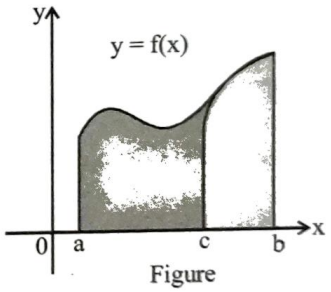
Therefore $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P-3 : ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION :

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This can be easily interpreted from the geometric interpretation in Figure : The area under y = f(x) from a to c plus the area from c to b is equal to the total area from a to b.

This property is useful when f(x) is not continuous in [a, b] because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the subinterval.



Figure

TRAIN YOUR BRAIN

Example 1: If it is known that

$\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Sol. We have $\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$, So

$$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$

Example 2: Find $\int_1^3 (3 - 2x + x^2)\,dx$.

Sol. The function f defined by $f(x) = 3 - 2x + x^2$ is continuous and has antiderivative g defined by

$g(x) = 3x - x^2 + \frac{1}{3}x^3$. Therefore, by the fundamental theorem of the calculus,

$$\int_1^3 (3 - 2x + x^2)\,dx = g(3) - g(1) = (9 - 9 + 9) - (3 - 1 + \frac{1}{3}) = \frac{20}{3}.$$

Example 3: If $f(x) = \begin{cases} x+3 & : \ x < 3 \\ 3x^2 + 1 & : \ x \geq 3 \end{cases}$, then find $\int_2^5 f(x)\,dx$.

Sol.
$$\int_2^5 f(x)dx = \int_2^3 f(x)dx + \int_3^5 f(x)dx$$

$$= \int_2^3 (x+3)dx\ dx + \int_3^5 (3x^2+1)\,dx = \left[\frac{x^2}{2} + 3x\right]_2^3 + \left[x^3 + x\right]_3^5$$

$$= \frac{9-4}{2} + 3(3-2) + 5^3 - 3^3 + 5 - 3 = \frac{211}{2}$$

Example 4: Show that $\int_0^2 (2x+1)\,dx = \int_0^5 (2x+1)\,dx + \int_5^2 (2x+1)\,dx$

Sol. L.H.S. $= [x^2 + x]_0^2 = 4 + 2 = 6$

R.H.S. $= 25 + 5 - 0 + (4 + 2) - (25 + 5) = 6$

$\therefore$ L.H.S. = R.H.S

P-4 :

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$$

Proof :

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^a f(x)\,dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\,du$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

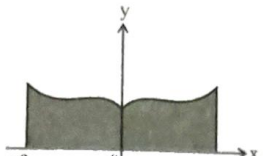
(a) If f is even, then

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

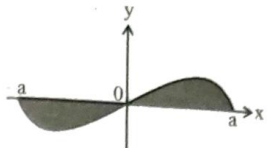
(b) If f is odd, then

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$$

Theorem is illustrated by Figure(a, b) For the case where f is positive and even, part(a) says that the area under y= f(x) from –a to a is twice the area from 0 to a because of symmetry. Thus, part (b) says the integral is 0 because the areas cancel.



f is Even



f is odd

e.g., Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_0^2 (x^6 + 1)\,dx$$

$$= \left[\frac{1}{7}x^7 + x\right]_0^2 = 2\left(\frac{128}{7} + 2\right) = \frac{284}{7}$$

e.g., Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so $\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$

TRAIN YOUR BRAIN

Example 5: Evaluate $\int_{-1}^{1} \frac{e^x + e^{-x}}{1 + e^x}\,dx$

Sol.
$$\int_{-1}^{1} \frac{e^x + e^{-x}}{1 + e^x}\,dx = \int_0^1 \left(\frac{e^x + e^{-x}}{1 + e^x} + \frac{e^{-x} + e^x}{1 + e^{-x}}\right)dx$$

$$= \int_0^1 \left(\frac{e^x + e^{-x}}{1 + e^x} + \frac{e^x(e^{-x} + e^x)}{e^x + 1}\right)dx$$

$$= \int_0^1 (e^x + e^{-x})\,dx = e - 1 + \frac{(e^{-1} - 1)}{-1} = \frac{e^2 - 1}{e}$$

Example 6: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$.

Sol. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\,dx = 2$ ($\because \cos x$ is even function)

Example 7: Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right)dx$.

Sol. Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right)$

$$\Rightarrow f(-x) = \log_e\left(\frac{2+x}{2-x}\right)$$

$$= -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$$

i.e. f(x) is odd function

$$\therefore \int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$$

P-5 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

OR

Further $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Example 8: Prove that

$$\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\,dx = \frac{\pi}{4}.$$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx$ ... (i)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\,dx \quad \text{.... (ii)}$$

on adding eq. (i) & (ii), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)} + \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \Rightarrow I = \frac{\pi}{4}$$

P-6 :

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$$

[if $f(2a-x) = f(x)$]

$= 0$ [if $f(2a-x) = -f(x)$]

TRAIN YOUR BRAIN

Example 9: Evaluate $\int_0^{\pi} \sin^3 x \cos^3 x\,dx$.

Sol. Let $f(x) = \sin^3 x \cos^3 x \Rightarrow f(\pi - x) = -f(x)$

$$\therefore \int_0^{\pi} \sin^3 x \cos^3 x\,dx = 0$$

Example 10: Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$.

Sol. Let $f(x) = \frac{1}{1+2\sin^2 x}$

$\Rightarrow f(\pi - x) = f(x)$

$$\Rightarrow \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin^2 x}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{1+\tan^2 x+2\tan^2 x}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{1+3\tan^2 x} = \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi/2}$$

$\because \tan\frac{\pi}{2}$ is undefined, we take limit

$$= \frac{2}{\sqrt{3}}\left[\lim_{x\to\frac{\pi}{2}^-} \tan^{-1}(\sqrt{3}\tan x) - \tan^{-1}(\sqrt{3}\tan 0)\right]$$

$$= \frac{2}{\sqrt{3}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

2nd Method: $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x+2}\,dx$

$$= \int_0^{\pi} \frac{\operatorname{cosec}^2 x\,dx}{\cot^2 x+3}$$

Observe that we are not converting in terms of tan x as it is not continuous in $(0, \pi)$

$$= -\frac{1}{3}\left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\sqrt{3}}\left[\lim_{x\to\pi^-} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) - \lim_{x\to 0^+} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]$$

$$= -\frac{1}{\sqrt{3}}\left[-\frac{\pi}{2}-\frac{\pi}{2}\right] = \frac{\pi}{\sqrt{3}}$$

P-7 : $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$ **; where 'a' is the period of the function i.e. f(a + x) = f(x)**

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx, n \in z$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx, n \in z, a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx, m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)\,dx, n \in z, a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx, n \in z, a, b \in R$

Remark : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

TRAIN YOUR BRAIN

Example 11: Evaluate $\int_{-1}^{2} e^{\{x\}}\,dx$.

Sol. $\int_{-1}^{2} e^{\{x\}}\,dx = \int_{-1}^{-1+3} e^{\{x\}}\,dx = 3\int_0^1 e^{\{x\}}\,dx = 3\int_0^1 e^{\{x\}}\,dx = 3(e-1)$

Example 12: Evaluate $\int_0^{n\pi+v} |\cos x|\,dx$, $\frac{\pi}{2} < v < \pi$ and $n \in z$.

Sol. $\int_0^{n\pi+v} |\cos x|\,dx = \int_0^{v} |\cos x|\,dx + \int_v^{n\pi+v} |\cos x|\,dx$

$= \int_0^{\frac{\pi}{2}} \cos x - \int_{\pi/2}^{v} \cos x\,dx + n\int_0^{\pi} |\cos x|\,dx$

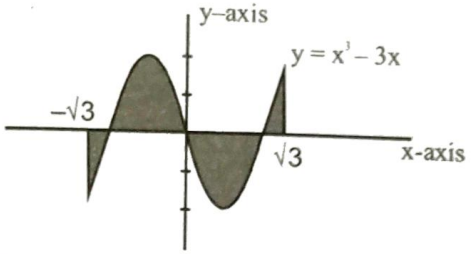
$= (1-0) - (\sin v - 1) + 2n\int_0^{\frac{\pi}{2}} \cos x\,dx = 2 - \sin v + 2n(1-0)$

$= 2n + 2 - \sin v$

Example 13: Evaluate $\int_{-2}^{2} (x^3 - 3x)\,dx$.

Sol. The integrand, $f(x) = x^3 - 3x$, is an an odd function; i.e., the equation f(–x) is satisifed for every x. Its graph, drwan in Figure, is therefore symmetric under reflection first about the x-axis and then about the y-axis. If follows that the region above the x-axis has the same areas as the region below it. We conclude that

$\int_{-2}^{2} (x^3 - 3x)\,dx = 0.$



P-8 : If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

P-9: If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and if f(x) is monotonically increasing in (a, b), then $f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

P-10 : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

P-11 : If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x)\,dx \ge 0$

TRAIN YOUR BRAIN

Example 14: For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{\sqrt{4-x^2}}$, $f_2(x) = \frac{1}{\sqrt{4-2x^2}}$ and $f_3(x) = \frac{1}{\sqrt{4-x^2-x^3}}$ in ascending order and hence prove that

$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$.

Sol. $\because 0 < x^3 < x^2 \Rightarrow x^2 < x^2 + x^3 < 2x^2$

$\Rightarrow -2x^2 < -x^2 - x^3 < -x^2$

$\Rightarrow 4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

$\Rightarrow \sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2}$

$\Rightarrow f_1(x) < f_3(x) < f_2(x)$ for $x \in (0, 1)$

$$\Rightarrow \int_0^1 f_1(x)\,dx < \int_0^1 f_3(x)\,dx < \int_0^1 f_2(x)\,dx$$

$$\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$$

Example 15: Estimate the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$.

Sol. Let $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$$

$\Rightarrow$ f(x) is monotonically decreasing function.

f(0) is not defined, so we evaluate

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1. \text{ Take } f(0) = \lim_{x\to 0^+} f(x) = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{2}{\pi}\cdot\left(\frac{\pi}{2} - 0\right) < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < 1\cdot\left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow 1 < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$$

DERIVATIVE OF ANTIDERIVATIVE (NEWTON LEIBNITZ RULE)

If h(x) & g(x) are differentiable function of x and f(x) is continuous then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)]\cdot h'(x) - f[g(x)]\cdot g'(x)$$

TRAIN YOUR BRAIN

Example 16: Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Sol. Since $f(t) = \sqrt{1+t^2}$ is continuous, therefore

$g'(x) = \sqrt{1+x^2}$

Example 17: If $F(t) = \int_0^t \frac{1}{x^2+1}\,dx$, find F'(1), F'(2), and F'(x).

Sol. The integrand in this example is the continuous function

f defined by $f(x) = \frac{1}{x^2+1}$.

$F'(t) = f(t) = \frac{1}{t^2+1}$.

In particular, $F'(1) = \frac{1}{1^2+1} = \frac{1}{2}$, $F'(2) = \frac{1}{2^2+1} = \frac{1}{5}$,

Example 18: Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$.

Sol. Let $u = x^4$.

Then $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$

$= \frac{d}{dx}\int_1^u \sec t\,dt$

$= \frac{d}{du}\left(\int_1^u \sec t\,dt\right)\frac{du}{dx} = \sec u\,\frac{du}{dx} = \sec(x^4)\cdot 4x^3.$

Example 19: FInd the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\,dt$

Sol. $F'(x) = \frac{dF}{du}\frac{du}{dx} = \frac{d}{du}\left[\int_{\pi/2}^u \cos t\,dt\right]\frac{du}{dx}$

$= (\cos u)(3x^2) = (\cos x^3)(3x^2)$

$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2}$

$= (\sin x^3) - 1$ $F'(x) = (\cos x^3)(3x^2)$.

Example 20: Evaluate $\int_{-\infty}^0 xe^x\,dx$.

Sol. $\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$;

$$\int_t^0 xe^x\,dx = xe^x\Big|_t^0 - \int_t^0 e^x\,dx = -te^t - 1 + e^t$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hopital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}(-e^t) = 0$$

Therefore $\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}(-te^t - 1 + e^t)$

$= -0 - 1 + 0 = -1$

Example 21: If $f(x)=\int_{\ell nx}^{x}\frac{dt}{x+t}$, then find $f'(x)$.

Sol. $f'(x)=\int_{\ell n\,x}^{x}\frac{-1}{(x+t)^2}dt+1\cdot\frac{1}{2x}-\frac{1}{x}\frac{1}{(x+\ell n\,x)}$

$$=\frac{1}{(x+t)}\Big]_{\ell n\,x}^{x}+\frac{1}{2x}-\frac{1}{x(x+\ell n\,x)}$$

$$=\frac{1}{2x}-\frac{1}{x+\ell n\,x}+\frac{1}{2x}-\frac{1}{x(x+\ell n\,x)}$$

$$=\frac{1}{x}-\frac{x+1}{x(x+\ell n\,x)}=\frac{\ell n\,x-1}{x(x+\ell n\,x)}$$

2nd Method : $f(x)=\int_{\ell nx}^{x}\frac{dt}{x+t}=\left[\ell n(x+t\right]_{\ell nx}^{x}$

(treating 't' as constant)

$f(x)=\ell n\,2x-\ell n\,(x+\ell n\,x)$

$$f'(x)=\overline{n-1}-\frac{1}{(x+\ell n\,x)}\left(1+\frac{1}{x}\right)=\frac{\ell n\,x-1}{x(x+\ell n\,x)}$$

Example 22: Evaluate $\int_0^1\frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}dx$, 'a' being parameter.

Sol. Let $I(a)=\int_0^1\frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}dx$

$$\frac{dI(a)}{da}=\int_0^1\frac{x}{(1+a^2x^2)}\frac{1}{x\sqrt{1-x^2}}$$

$$dx=\int_0^1\frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$$

Put $x=\sin t\Rightarrow dx=\cos t\,dt$

L.L. : $x=0\Rightarrow t=0$

U.L. : $x=1\Rightarrow t=\frac{\pi}{2}$

$$\frac{dI(a)}{da}=\int_0^{\frac{\pi}{2}}\frac{1}{1+a^2\sin^2 t}\frac{1}{\cos t}\cos t\,dt=\int_0^{\frac{\pi}{2}}\frac{dt}{1+a^2\sin^2 t}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2 t\,dt}{1+(1+a^2)\tan^2 t}=\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\sqrt{1+a^2}\tan t\right)\Big]_0^{\frac{\pi}{2}}$$

$$I'(a)=\frac{1}{\sqrt{1+a^2}}\cdot\frac{\pi}{2}$$

$$\Rightarrow I(a)=\frac{\pi}{2}\ell n\left(a+\sqrt{1+a^2}\right)+c$$

But $I(0)=0\Rightarrow c=0$

$$\Rightarrow I(a)=\frac{\pi}{2}\ell n\left(a+\sqrt{1+a^2}\right)$$

DEFINITE INTEGRAL AS LIMIT OF A SUM

$$\int_a^b f(x)\,dx=\lim_{x\to\infty}h[f(a)+f(a+h)+f(a+2h)+.....$$

$$+f(a+(n-1)h)]$$

$$=\lim_{x\to 0}h\sum_{r=0}^{n-1}f(a+rh)\quad\text{where } b-a=nh$$

If $a=0$ & $b=1$ then, $\lim_{n\to\infty}h\sum_{r=0}^{n-1}f(rh)=\int_0^1 f(x)\,dx$; where $nh=1$

or $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=0}^{n-1}f\left(\frac{r}{n}\right)=\int_0^1 f(x)\,dx$

Remark : The symbol $\int$ was introduced by Leibnitz and is called integral sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called integration.

Steps to express the limit of sum as definte integral

Step 1. Express the given series in the form $\sum\frac{1}{n}f\left(\frac{r}{n}\right)$

Step 2. Then the limit is its sum when $n\to\infty$, i.e. $\lim_{n\to\infty}\frac{1}{n}f\left(\frac{r}{n}\right)$

Step 3. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{Lt}\ \Sigma$ by $\int$

Step 4. Evaluate $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\lim_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right)=\int_0^p f(x)\,dx$ $\left(\because \lim_{n\to\infty}\left(\frac{r}{n}\right)\Big|_{r=1}\right.$

$=0,\ \lim_{n\to\infty}\left(\frac{r}{n}\right)\Big|_{r=np}=p\Big)$

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Example 23: Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.....+\frac{1}{2n}\right]$$

Sol. $$\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.....+\frac{1}{2n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r+n}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\left(\frac{r}{n}\right)+1}$$

$$=\int_0^1\frac{dx}{x+1}=[\ell n(x+1)]_0^1=\ell n\,2$$

Example 24: Evaluate

$$\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+.....+\frac{3}{5n}\right].$$

Sol. $$\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n+r}{n^2+r^2}=\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\,\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$\because \lim_{n\to\infty}\left(\frac{r}{n}\right)=0$, when $r=1$, lower limit $=0$

and $\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\left(\frac{2n}{n}\right)=2$, when $r=2n$, upper limit $=2$

$$\int_0^2\frac{1+x}{1+x^2}\,dx=\int_0^2\frac{1}{1+x^2}\,dx+\frac{1}{2}\int_0^2\frac{2x}{1+x^2}\,dx$$

$$=[\tan^{-1}x]_0^2+\left[\frac{1}{2}\log_e(1+x^2)\right]_0^2$$

$$=\tan^{-1}2+\frac{1}{2}\ell n\,5$$

Example 25: Evaluate $\int_{-1}^{2}7x-5\,dx$ as a limit of sums.

Sol. Here $a=-1$, $b=2$, and $h=\frac{2+1}{n}$, i.e. $nh=3$ and $f(x)=7x-5$

Now, we have

$$\int_{-1}^{2}(7x-5)dx=\lim_{h\to0}[f(-1)+f(-1+h)+f(-1+2h)+.....+f(-1+(n-1)h)]$$

Note that

$f(-1)=-7-5=-12$

$f(-1+h)=-7+7h-5=12+7h$

$f(-1+(n-1)h)=7(n-1)h-12$

Therefore,

$$\int_{-1}^{2}(7x-5)dx=\lim_{h\to0}h[(-12)+(7h-12)+(14h-12)+....+(7(n-1)h-12)].$$

$$=\lim_{h\to0}h[7h[1+2+....+(n-1)]-12n]$$

$$=\lim_{h\to0}\left[7h\frac{(n+1)n}{2}-12n\right]$$

$$=\lim_{h\to0}\left[\frac{7}{2}(nh)(nh-h)-12nh\right]$$

$$=\frac{7}{3}(3)(3-0)-12\times3$$

$$=\frac{7\times9}{2}-36=-\frac{9}{2}$$

WALLI'S FORMULA & REDUCTION FORMULA

$$\int_0^{\pi/2}\sin^n x.\cos^m x\,dx=\frac{[(n-1)(n-3)(n-5)....1\text{ or }3][(m-1)(m-3)....1\text{ or }2]}{(m+n)(m+n-2)(m+n-4)...1\text{ or }2}K$$

Where $K=\frac{\pi}{2}$ if both m and n are even $(m, n\in N)$; $=1$ otherwise

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Example 26: Prove that $\int_0^{\pi/4}(\cos2\theta)^{3/2}\cos\theta\,d\theta=\frac{3\pi}{16\sqrt{2}}$

Sol. L.H.S. $=\int_0^{\pi/4}(\cos2\theta)^{3/2}\cos\theta\,d\theta$

$=\int_0^{\pi/4}(1-2\sin^2\theta)^{3/2}\cos\theta\,d\theta$

(Put $\sqrt{2}\sin\theta=\sin t\Rightarrow\cos\theta\,d\theta=\frac{\cos t}{\sqrt{2}}\,dt$)

when $\theta \to 0$ then $t \to 0$; $\theta \to \pi/4$ then $t \to \pi/2$

$\therefore$ L.H.S. $= \int_0^{\pi/2} \frac{\cos^4 t}{\sqrt{2}}\, dt = \frac{1}{\sqrt{2}} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

$= \frac{3\pi}{16\sqrt{2}}$ = R.H.S. (From Walli's formula)

Example 27: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x(\sin x + \cos x)\, dx$.

Sol. Given integral

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cos^2 x\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx$

$= 0 + 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx$ ($\because \sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$= 2 . \frac{1.2}{5.3.1} = \frac{4}{15}$

Example 28: Evaluate $\int_0^{\pi} x \sin^5 x \cos^6 x\, dx$.

Sol. Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x\, dx$

$I = \int_0^{\pi} (\pi - x) \sin^5(\pi - x) \cos^6(\pi - x)\, dx$

$= \pi \int_0^{\pi} \sin^5 . \cos^6 x\, dx - \int_0^{\pi} x \sin^5 x . \cos^6 x\, dx$

$\Rightarrow 2I = \pi . 2 \int_0^{\frac{\pi}{2}} \sin^5 x . \cos^6 x\, dx$

$I = \pi \frac{4.2.5.3.1}{11.9.7.5.3.1}$

$I = \frac{8\pi}{693}$

Topicwise Questions

FUNDAMENTAL DEFINITE INTEGRATION, DEFINITE INTEGRATION BY SUBSTITUTION

1. $\int_0^1 \tan^{-1} x\,dx =$

(a) $\frac{\pi}{4}-\frac{1}{2}\log 2$ (b) $\pi-\frac{1}{2}\log 2$

(c) $\frac{\pi}{4}-\log 2$ (d) $\pi-\log 2$

2. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}dx =$

(a) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (b) $\frac{\pi}{4}-\frac{1}{2}\log 2$

(c) $\frac{\pi}{2}+\log 2$ (d) $\frac{\pi}{2}-\log 2$

3. $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx =$

(a) $\frac{1}{20}\log 3$ (b) $\log 3$

(c) $\frac{1}{20}\log 5$ (d) $\frac{1}{10}\log 5$

4. $\int_0^{\pi/6} \frac{\sin x}{\cos^3 x}dx =$

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$

(c) 2 (d) $\frac{1}{3}$

5. $\int_{-\pi/4}^{\pi/2} e^{-x}\sin x\,dx =$

(a) $-\frac{1}{2}e^{-\pi/2}$ (b) $-\frac{\sqrt{2}}{2}e^{-\pi/4}$

(c) $-\sqrt{2}(e^{-\pi/4}+e^{-\pi/4})$ (d) 0

6. $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$ equals

(a) $\left(\frac{\pi}{2}-1\right)$ (b) $\left(\frac{\pi}{2}+1\right)$

(c) $\frac{\pi}{2}$ (d) $(\pi+1)$

7. $\int_1^x \frac{\log x^2}{x}dx =$

(a) $(\log x)^2$ (b) $\frac{1}{2}(\log x)^2$

(c) $\frac{\log x^2}{2}$ (d) $\frac{1}{3}\log x^2$

8. $\int_3^8 \frac{2-3x}{x\sqrt{(1+x)}}dx$ is equal to

(a) $2\log\left(3/2e^3\right)$ (b) $\log(3/e^3)$

(c) $4\log(3/e^3)$ (d) $\log(2/3\,e^3)$

Properties of definite integration

9. $\int_0^{\pi} x f(\sin x)\,dx =$

(a) $\pi\int_0^{\pi} f(\sin x)\,dx$ (b) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

(c) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)\,dx$ (d) $\frac{\pi}{2}\int_0^{2\pi} f(\sin x)\,dx$

10. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx =$

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

11. $\int_0^{\pi} x\log\sin x\,dx =$

(a) $\frac{\pi}{2}\log\frac{1}{2}$ (b) $\frac{\pi^2}{2}\log\frac{1}{2}$

(c) $\pi\log\frac{1}{2}$ (d) $\pi^2\log\frac{1}{2}$

12. $\int_0^{\pi/2} \log\tan x\,dx =$

(a) $\frac{\pi}{2}\log_e 2$ (b) $-\frac{\pi}{2}\log_e 2$

(c) $\pi\log_e 2$ (d) 0

13. $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)d\theta =$

(a) 0 (b) 1

(c) 2 (d) 1/2

14. $\int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta =$

(a) 1 (b) 2

(c) $\frac{\pi}{4}$ (d) 0

15. $\int_{-1/2}^{1/2} (\cos x)\left[\log\left(\frac{1-x}{1+x}\right)\right] dx =$

(a) 0 (b) 1

(c) $e^{1/2}$ (d) $2e^{1/2}$

16. For any integer n, the integral $\int_0^{\pi} e^{\sin^2 x} \cos^3(2n+1)x\, dx =$

(a) -1 (b) 0

(c) 1 (d) π

17. $\int_{-3}^{3} \frac{x^2 \sin x}{1+x^6} dx =$

(a) 4 (b) 2

(c) 0 (d) 3

18. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is

(a) 0 (b) 1

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

19. If $I = \int_0^{100\pi} \sqrt{(1-\cos 2x)}\, dx$, then the value of I is

(a) $100\sqrt{2}$ (b) $200\sqrt{2}$

(c) $50\sqrt{2}$ (d) $25\sqrt{2}$

20. The value of $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(a) 2 (b) $\frac{3}{4}$

(c) 0 (d) 1/4

21. If $f(x)$ is an even function of x, then $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos x)\, dx$ is equal to

(a) 0 (b) $\int_0^{\frac{\pi}{2}} f(\cos x)\, dx$

(c) $2\int_0^{\frac{\pi}{2}} f(\sin x)\, dx$ (d) $\int_0^{\pi} f(\cos x)\, dx$

22. $\int_{-1}^{1} \log(x + \sqrt{x^2+1})\, dx =$

(a) 0 (b) $\log 2$

(c) $\log\frac{1}{2}$ (d) $\log\frac{1}{3}$

23. Find the value of $\int_0^9 [\sqrt{x} + 2] dx$, where [.] is the greatest integer function

(a) 31 (b) 22

(c) 23 (d) 11

24. $\int_0^{\pi/2} \sin 2x \log\tan x\, dx$ is equal to

(a) π (b) $\pi/2$

(c) 0 (d) 2π

25. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx =$

(a) $\frac{\pi}{2}\log 2$ (b) $\pi \log 2$

(c) $-\frac{\pi}{2}\log 2$ (d) $-\pi \log 2$

26. $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} =$

(a) $\pi/12$ (b) $\pi/2$

(c) $\pi/6$ (d) $\pi/4$

Summation of series by definite Integration, Gamma function, Leibnitz's rule

27. $\lim_{n\to\infty} \frac{1^{99} + 2^{99} + 3^{99} + \ldots n^{99}}{n^{100}} =$

(a) $\frac{9}{100}$ (b) $\frac{1}{100}$

(c) $\frac{1}{99}$ (d) $\frac{1}{101}$

28. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals

(a) $1+\sqrt{5}$ (b) $-1+\sqrt{5}$

(c) $-1+\sqrt{2}$ (d) $1+\sqrt{2}$

29. $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots \frac{1}{2n}\right] =$

(a) 0 (b) $\log_e 4$

(c) $\log_e 3$ (d) $\log_e 2$

1. If $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$, then x can be equal to:

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) 2 (d) $\frac{\sqrt{3}}{2}$

2. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, where $0 < \alpha < \frac{\pi}{2}$, is equal to:

(a) $\sin\alpha$ (b) $\alpha\sin\alpha$

(c) $\frac{\alpha}{2\sin\alpha}$ (d) $\frac{\alpha}{2}\sin\alpha$

3. If $f(x) = \begin{cases} x & x<1 \\ x-1 & x\geq 1 \end{cases}$, then $\int_0^2 x^2 f(x)\,dx$ is equal to:

(a) 1 (b) $\frac{4}{3}$

(c) $\frac{5}{3}$ (d) $\frac{5}{2}$

4. Suppose for every integer n, $\int_n^{n+1} f(x)\,dx = n^2$. The value of $\int_{-2}^{4} f(x)\,dx$ is

(a) 16 (b) 14

(c) 19 (d) 21

5. $\int_0^{\pi} |1+2\cos x|\,dx$ equals to:

(a) $\frac{2\pi}{3}$ (b) π

(c) 2 (d) $\frac{\pi}{3}+2\sqrt{3}$

6. The value of $\int_{-1}^{3} (|x-2|+[x])\,dx$ is equal to (where [*] denotes greatest integer function)

(a) 7 (b) 5

(c) 4 (d) 3

7. If f(0) = 1, f(2) = 3, f'(2) = 5 and f'(0) is finite, then $\int_0^1 x\cdot f''(2x)\,dx$ is equal to

(a) zero (b) 1

(c) 2 (d) 3

8. If $\int_{-1}^{3/2} |x\sin\pi x|\,dx = \frac{k}{\pi^2}$, then the value of k is

(a) $3\pi+1$ (b) $2\pi+1$

(c) 1 (d) 4

9. $\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx$ equals to:

(a) $\frac{\pi}{4}+\frac{1}{2}$ (b) $\frac{\pi}{4}-\frac{1}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}+\frac{1}{2}$

10. $\int_{\log\pi-\log 2}^{\log\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}\,dx$ is equal to

(a) $\sqrt{3}$ (b) $-\sqrt{3}$

(c) $\frac{1}{\sqrt{3}}$ (d) $-\frac{1}{\sqrt{3}}$

11. If $I_1 = \int_e^{e^2} \frac{dx}{\ell n\,x}$ and $I_2 = \int_1^2 \frac{e^x}{x}\,dx$, then

(a) $I_1 = I_2$ (b) $2I_1 = I_2$

(c) $I_1 = 2I_2$ (d) $I_1 = 3I_2$

12. The value of $\int_0^{\pi/2} \ell n|\tan x+\cot x|\,dx$ is equal to:

(a) $\pi\,\ell n\,2$ (b) $-\pi\,\ell n\,2$

(c) $\frac{\pi}{2}\,\ell n\,2$ (d) $-\frac{\pi}{2}\,\ell n\,2$

13. $\int_0^{(\pi/2)^{1/3}} x^5.\sin x^3\,dx$ equals to

(a) 1 (b) 1/2

(c) 2 (d) 1/3

14. $\int_0^{\infty} [2e^{-x}]\,dx$ is equal to

(a) 0 (b) $\ell n\ 2$

(c) e^2 (d) $2e^{-1}$

15. If $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, then $\int_0^{\infty} e^{-ax^2}\,dx$ where $a > 0$ is

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\sqrt{\pi}}{2a}$

(c) $2\frac{\sqrt{\pi}}{a}$ (d) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

16. Let $A = \int_0^1 \frac{e^t\,dt}{1+t}\,dt$ then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\,dt$ has the value

(a) Ae^{-a} (b) $-Ae^{-a}$

(c) $-ae^{-a}$ (d) Ae^{a}

17. The value of $\int_1^2 ([x^2]-[x]^2)\,dx$ is equal to (where [*] denotes the greatest integer function)

(a) $4+\sqrt{2}-\sqrt{3}$ (b) $4-\sqrt{2}+\sqrt{3}$

(c) $4-\sqrt{3}-\sqrt{2}$ (d) $4+\sqrt{3}+\sqrt{2}$

18. The solution for x of the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$ is

(a) $-\sqrt{2}$ (b) π

(c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{2}$

19. The value of $\int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}\,dx$ is

(a) 0 (b) 1

(c) 2 (d) 3

20. If $\frac{d}{dx} f(x) = g(x)$ for $a \le x \le b$, then $\int_a^b f(x)g(x)dx$ equals to:

(a) f(b) – f(a) (b) g(b) – g(a)

(c) $\frac{[f(b)]^2-[f(a)]^2}{2}$ (d) $\frac{[g(b)]^2-[g(a)]^2}{2}$

21. If [x] denotes the greatest integer less than or equal to x, then the value of $\int_1^5 [|x-3|]\,dx$ is

(a) 1 (b) 2

(c) 4 (d) 8

22. $\int_0^2 (x-\log_2 a)\,dx = 2\log_2\left(\frac{2}{a}\right)$, if

(a) $a > 0$ (b) $a > 2$

(c) $a = 4$ (d) $a = 8$

23. $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx =$

(a) $\frac{1}{c}\int_a^b f(x)\,dx$ (b) $\int_a^b f(x)\,dx$

(c) $c\int_a^b f(x)\,dx$ (d) $\int_{ac^2}^{bc^2} f(x)\,dx$

24. If $\int_{\ell n\,2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, then x =

(a) 4 (b) $\ell n\ 8$

(c) $\ell n\ 4$ (d) $\ell n\ 2$

25. $\int_{5/2}^{5} \frac{\sqrt{(25-x^2)^3}}{x^4}\,dx$ equals to

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

26. $\int_{2-\log 3}^{3+\log 3} \frac{\log(4+x)}{\log(4+x)+\log(9-x)}\,dx$

(a) cannot be evaluated

(b) is equal to $\frac{5}{2}$

(c) is equal to $1+2\log 3$

(d) is equal to $\frac{1}{2}+\log 3$

27. If $\int_0^{11} \frac{11^x}{11^{[x]}}\,dx = \frac{k}{\log 11}$ then value of k is (where [*] denotes greatest integer function)

(a) 11 (b) 101

(c) 110 (d) 111

28. If f(x) is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx$ equals

(a) $\sin\theta + \operatorname{cosec}\theta$ (b) $\sin^2\theta$

(c) $\operatorname{cosec}^2\theta$ (d) 0

29. If f(x) and g(x) are continuous functions satisfying f(x) = f(a − x) and g(x) + g(a − x) = 2, then $\int_0^a f(x)\,g(x)\,dx$ is equal to

(a) $\int_0^a g(x)\,dx$ (b) $\int_0^a f(x)\,dx$

(c) 0 (d) $\int_0^{2a} f(x)\,dx$

30. $\int_{-\pi/2}^{\pi/2} \frac{|x|\,dx}{8\cos^2 2x + 1}$ has the value

(a) $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{12}$

(c) $\frac{\pi^2}{24}$ (d) $\frac{\pi^2}{18}$

31. If f(a + b − x) = f(x), then $\int_a^b x\,f(x)\,dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ (b) $\frac{a+b}{2}\int_a^b f(x)\,dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\,dx$ (d) $\frac{a+b}{2}\int_a^b f(a+b+x)\,dx$

32. $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is equal to

(a) 0 (b) $\pi/2$

(c) $\pi/3$ (d) $\pi/4$

33. $\int_{-1}^{1} \frac{x^4}{1+e^{x^7}}\,dx$ is

(a) $\frac{1}{2}$ (b) 0

(c) $\frac{1}{5}$ (d) 1

34. $\lim_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{n}{r^2 - n^2}$ is equal to

(a) $\log\sqrt{\frac{2}{3}}$

(b) $\log\sqrt{\frac{3}{2}}$

(c) $\log\frac{2}{3}$

(d) $\log\frac{3}{2}$

35. The value of

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$ is

(a) $\frac{e^{\pi/2}}{2e^2}$ (b) $2e^2\,e^{\pi/2}$

(c) $\frac{2}{e^2}e^{\pi/2}$ (d) 0

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. The value of $\int_0^1 \frac{2x^2+3x+3}{(x+1)(x^2+2x+2)}\,dx$ is :

(a) $\frac{\pi}{4} + 2\ \ell n\,2 - \tan^{-1}2$

(b) $\frac{\pi}{4} + 2\ \ell n\,2 - \tan^{-1}\frac{1}{3}$

(c) $2\ \ell n\,2 - \cot^{-1}3$

(d) $-\frac{\pi}{4} + \ell n\,4 + \cot^{-1}2$

2. The value of integral $\int_0^{\pi} x f(\sin x)\,dx$ is

(a) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

(b) $\pi\int_0^{\pi/2} f(\sin x)\,dx$

(c) 0

(d) None of these

3. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)}\,dx$

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{2}$

(c) is same as $\int_0^{\pi} \frac{dx}{(1+x)(1+x^2)}$

(d) cannot be evaluated

4. The value of integral $\int_a^b \frac{|x|}{x}\,dx$, $a < b$ is

(a) $b - a$ if $a > 0$ (b) $a - b$ if $b < 0$

(c) $b + a$ if $a < 0 < b$ (d) $|b| - |a|$

5. If $f(x) = \int_0^x (\cos^4 t + \sin^4 t)\,dt$, $f(x + \pi)$ will be equal to

(a) $f(x) + f(\pi)$ (b) $f(x) + 2(\pi)$

(c) $f(x) + f\left(\frac{\pi}{2}\right)$ (d) $f(x) + 2f\left(\frac{\pi}{2}\right)$

6. A function $f(x)$ which satisfies, $f'(\sin^2 x) = \cos^2 x$ for all real x & $f(1) = 1$ is :

(a) $f(x) = x - \frac{x^3}{3} + \frac{1}{3}$

(b) $f(x) = x^2 - \frac{x}{2} + \frac{1}{2}$

(c) a polynomial of degree two

(d) $f(0) = 1/2$

7. If $f(x) = \int_0^x (2\cos^2 3t + 3\sin^2 3t)\,dt$, $f(x + \pi)$ is equal to

(a) $f(x) + f(\pi)$ (b) $f(x) + 2f\left(\frac{\pi}{2}\right)$

(c) $f(x) + 4f\left(\frac{\pi}{4}\right)$ (d) None of these

8. If $I = \int_0^{2\pi} \sin^2 x\,dx$, then

(a) $I = 2\int_0^{\pi} \sin^2 x\,dx$

(b) $I = 4\int_0^{\pi/2} \sin^2 x\,dx$

(c) $I = \int_0^{2\pi} \cos^2 x\,dx$

(d) $I = 8\int_0^{\pi/4} \sin^2 x\,dx$

9. Given f is an odd function defined everywhere, periodic with period 2 and integrable on every interval. Let $g(x) = \int_0^x f(t)\,dt$. Then:

(a) $g(2n) = 0$ for every integer n

(b) $g(x)$ is an even function

(c) $g(x)$ and $f(x)$ have the same period

(d) none of these

10. If $f(x) = 2^{\{x\}}$, where $\{x\}$ denotes the fractioal aprt of x. Then which of the following is true?

(a) f is periodic (b) $\int_0^1 2^{\{x\}}\,dx = \frac{1}{\ell n\, 2}$

(c) $\int_0^1 2^{\{x\}}\,dx = \log_2 e$ (d) $\int_0^{100} 2^{\{x\}}\,dx = 100 \log_2 e$

11. If $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$; $n \in N$, then which of the following statements hold good?

(a) $2n\, I_{n+1} = 2^{-n} + (2n - 1)\, I_n$

(b) $I_2 = \frac{\pi}{8} + \frac{1}{4}$

(c) $I_2 = \frac{\pi}{8} - \frac{1}{4}$

(d) $I_3 = \frac{\pi}{16} - \frac{5}{48}$

12. If f(x) is integrable over [1, 2], then $\int_1^2 f(x)\,dx$ is equal to:

(a) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ (b) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$

(c) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)$ (d) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right)$

Comprehension – 1 (No. 13 to 15)

If $y = \int_{u(x)}^{v(x)} f(t)\,dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x)\,f^2(v(x)) - u'(x)\,f^2(u(x))$ and the equation of the tangent at (a, b) as $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$

13. If $y = \int_x^{x^2} t^2\,dt$, then equation of tangent at x = 1 is

(a) y = x + 1 (b) x + y = 1

(c) y = x – 1 (d) y = x

14. If $F(x) = \int_1^x e^{t^2/2}\,(1 - t^2)\,dt$, then $\frac{d}{dx} F(x)$ at x = 1 is

(a) 0 (b) 1

(c) 2 (d) –1

15. If $y = \int_{x^3}^{x^4} \ell nt\,dt$, then $\lim_{x\to 0^+} \frac{dy}{dx}$ is

(a) 0 (b) 1

(c) 2 (d) –1

Comprehension – 2 (No. 16 to 18)

If function f(x) is continuous in the interval (a, b) and having same definition between a and b, then we can find $\int_a^b f(x)dx$ if f(x) is discontiuous and not same definition between a and b, then we must break the interval such that f(x) becomes continuous and having same definition in the breaking intervals.

Now, if f(x) is discontinuous at x = c (a < c < b), then $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ and also if f(x) is discontinuous at x = a in (0, 2a), then we can write

$\int_0^{2a} f(x)\,dx = \int_0^a \{f(a - x) + f(a + x)\}\,dx$

On the basis of above information, answere the following questions :0

16. $\int_0^{10} \left[\frac{x^2 + 2}{x^2 + 1}\right] dx$ (where [.] denotes greatest integer function) is equal to

(a) 0 (b) 2

(c) 5 (d) None of these

17. $\int_0^1 \sin([x] + [2x])dx$ (where [.] denotes the greatest integer function) is equalt to

(a) sin 1 (b) $\sin\left(\frac{3}{2}\right)$

(c) $\frac{\sin 1}{2}$ (d) $\frac{\sin 2}{3}$

18. $\int_{-1}^{1} [|x|]d\left(\frac{1}{1 + e^{-1/x}}\right)$ (where [.] denotes the greates integer functions) is equal to

(a) –3 (b) –2

(c) –1 (d) None of these

Comprehension – 3 (No. 19 to 21)

Let $g(t) = \int_{x_1}^{x_2} f(t, x)\,dx$. Then $g'(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial t} (f(t, x))\,dx$.

Consider $f(x) = \int_0^{\pi} \frac{\ell n\,(1 + x\cos\theta)}{\cos\theta}\,d\theta$.

19. Range of f(x) is

(a) $(0, \pi)$ (b) $(0, \pi^2)$

(c) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left(\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right)$

20. The number of critical points of f(x), in the interior of its domain, is

(a) 0 (b) 1

(c) 2 (d) infinitely many

21. f(x) is

(a) discontinuous at x = 0

(b) continuous but not differentiable at x = 1

(c) continuous at x = 0

(d) differentiable at x = 1

22. **Match the column:**

Column – I	**Column – II**
(a) $\int_5^{10} \left[\frac{x-5}{5}\right] dx =$	(p) 1
(b) $\int_{-\tan 1}^{0} \left[-\tan^{-1} x\right] dx =$	(q) 2
(c) $\frac{6}{\pi} \int_{\pi/6}^{\pi/3} [2\sin x]\,dx =$	(r) 0
(d) $\int_{-1}^{1} \frac{\cot^{-1} x}{\pi}\,dx =$	(s) 3

23. Match the column:

Column–I	Column–II
(a) $\int_0^{\pi/2} \ell n\,(\tan x + \cot x)\,dx =$	(p) $\frac{\pi^2}{4}$
(b) $\int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2}\,dx =$	(q) $\pi\,\ell n\,2$
(c) $\int_0^{2\pi} x\,(\sin^2 x \cos^2 x)\,dx =$	(r) 0
(d) $\int_0^{\pi/2} (2\,\ell n \sin x - \ell n \sin 2x)\,dx =$	(s) $-\frac{\pi}{2}\,\ell n\,2$

NUMERICAL VALUE BASED

24. $\int_0^{100} \{x\}dx$, where $\{x\}$ denotes the fraction part of x.

25. $\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3} =$

26. $\int_0^{\pi/2} \sqrt{\sin x}\cos^5 x\,dx =$

27. $\lim_{n\to\infty} \frac{1}{n}\left(1+\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+.....+\sqrt{\frac{n}{4n-3}}\right)$ is equal to

28. $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x\cos x}dx =$

29. If $f(\pi) = 2$ and $\int_0^{\pi}(f(x)+f''(x))\sin x\,dx = 5$ then $f(0)$ is equal to

30. Find the value of $\int_0^9 \left[\sqrt{x}+2\right]dx$, where [.] is the greatest integer function

31. Value of $\int_1^5 \left(\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2(x-1)}\right)dx$ is

32. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is.....

33. Let f be a positive function. let, $I_1 = \int_{1-k}^{k} xf[x(1-x)]\,dx$.

$I_2 = \int_{1-k}^{k} f[x(1-x)]dx$, where $2k-1>0$. Then $\frac{I_1}{I_2}$ is:

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)...3n}{n^{2n}}\right)^{1/n}$ is equal to:

[JEE Main-2016]

(a) $\frac{18}{e^4}$

(b) $\frac{27}{e^2}$

(c) $\frac{9}{e^2}$

(d) $3\log 3 - 2$

2. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to:

[JEE Main-2017]

(a) -1 (b) -2

(c) 2 (d) 4

3. The value of $\int_0^{\pi} |\cos x|^3\,dx$ is :

[JEE Main-2019 (January)]

(a) 0 (b) $\frac{4}{3}$

(c) $\frac{2}{3}$ (d) $-\frac{4}{3}$

4. Let f be a differentiable function R to R such that $|f(x)-f(y)| \le 2|x-y|^{\frac{3}{2}}$, for all $x, y \in R$. If $f(0)=1$ then $\int_0^1 f^2(x)dx$ is equal to

[JEE Main-2019 (January)]

(a) 0 (b) $\frac{1}{2}$

(c) 2 (d) 1

5. If $\int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta = 1-\frac{1}{\sqrt{2}}, (k>0)$, then the value of k is:

[JEE Main-2019 (January)]

(a) 2 (b) $\frac{1}{2}$

(c) 4 (d) 1

6. If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$, then $f'\left(\frac{1}{2}\right)$ is:

[JEE Main-2019 (January)]

(a) $\frac{24}{25}$ (b) $\frac{18}{25}$

(c) $\frac{4}{5}$ (d) $\frac{6}{25}$

7. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest integer less than or equal to t, is

[JEE Main-2019 (January)]

(a) $\frac{1}{12}(7\pi+5)$ (b) $\frac{1}{12}(7\pi-5)$

(c) $\frac{3}{20}(4\pi-3)$ (d) $\frac{3}{10}(4\pi-3)$

8. The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x\left(\tan^5 x+\cot^5 x\right)}$ equals:

[JEE Main-2019 (January)]

(a) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$

(b) $\frac{1}{10}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$

(c) $\frac{\pi}{40}$

(d) $\frac{1}{5}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$

9. The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$ (where [x] denotes the greatest integer less than or equal to x) is

[JEE Main-2019 (January)]

(a) 0 (b) $\sin 4$

(c) 4 (d) $4-\sin 4$

10. Let f and g be continuous functions on [0,a] such that $f(x) = f(a-x)$ and $g(x)+g(a-x)=4$, then $\int_0^a f(x)g(x)dx$ is equal to:

[JEE Main-2019 (January)]

(a) $4\int_0^a f(x)dx$ (b) $\int_0^a f(x)dx$

(c) $2\int_0^a f(x)dx$ (d) $-3\int_0^a f(x)dx$

11. The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^{x}\right\}\log_e x\,dx$ is equal to:

[JEE Main-2019 (January)]

(a) $\frac{1}{2}-e-\frac{1}{e^2}$

(b) $-\frac{1}{2}+\frac{1}{e}-\frac{1}{2e^2}$

(c) $\frac{3}{2}-\frac{1}{e}-\frac{1}{2e^2}$

(d) $\frac{3}{2}-e-\frac{1}{2e^2}$

12. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+....+\frac{1}{5n}\right)$ is equal to

[JEE Main-2019 (January)]

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}(3)$

(c) $\frac{\pi}{2}$ (d) $\tan^{-1}(2)$

13. If $f(x)=\frac{2-x\cos x}{2+x\cos x}$ and $g(x)=\log_e x, (x>0)$ then the value of integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(f(x))dx$ is:

[JEE Main-2019 (April)]

(a) $\log_e 3$ (b) $\log_e 2$

(c) $\log_e e$ (d) $\log_e 1$

14. Let $f(x) = \int_0^x g(t)\,dt$, where g is a non-zero even function. If $f(x+5) = g(x)$, then $\int_0^x f(t)\,dt$ equals:

[JEE Main-2019 (April)]

(a) $\int_{x+5}^{5} g(t)\,dt$ (b) $5\int_{x+5}^{5} g(t)\,dt$

(c) $\int_{5}^{x+5} g(t)\,dt$ (d) $2\int_{5}^{x+5} g(t)\,dt$

15. The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}dx$ is

[JEE Main-2019 (April)]

(a) $\frac{\pi-2}{4}$ (b) $\frac{\pi-2}{8}$

(c) $\frac{\pi-1}{4}$ (d) $\frac{\pi-1}{2}$

16. If $f: R \to R$ is a differentiable function and $f(2) = 6$, then $\lim_{x\to 2} \frac{\int_6^{f(x)} 2t\,dt}{x-2}$ is:

[JEE Main-2019 (April)]

(a) 0 (b) $2f'(2)$

(c) $12f'(2)$ (d) $24f'(2)$

17. The value of the integral $\int_0^1 x\cot^{-1}(1-x^2+x^4)dx$ is:

[JEE Main-2019 (April)]

(a) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$ (b) $\frac{\pi}{2} - \log_e 2$

(c) $\frac{\pi}{2} - \frac{1}{2}\log_e 2$ (d) $\frac{\pi}{4} - \log_e 2$

18. The value of $\int_0^{2\pi} [\sin 2x(1+\cos 3x)]dx$, where [t] denotes the greatest integer function, is:

[JEE Main 2019]

(a) -2π (b) π

(c) $-\pi$ (d) 2π

19. The integrala $\int_{\pi/6}^{\pi/3} \sec^{2/3} x\,\text{cosec}^{4/3} x\,dx$ equal to:

[JEE Main-2019 (April)]

(a) $3^{7/6} - 3^{5/6}$ (b) $3^{5/3} - 3^{1/3}$

(c) $3^{4/3} - 3^{1/3}$ (d) $3^{5/6} - 3^{2/3}$

20. $\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \ldots + \frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to:

[JEE Main-2019 (April)]

(a) $\frac{4}{3}(2)^{4/3}$ (b) $\frac{3}{4}(2)^{4/3} - \frac{4}{3}$

(c) $\frac{3}{4}(2)^{4/3} - \frac{3}{4}$ (d) $\frac{4}{3}(2)^{3/4}$

21. Let $f: R \to R$ be a continuously differentiable function which that $f(2) = 6$ and $f'(2) = \frac{1}{48}$. If $\int_6^{f(x)} 4t^3 dt = (x-2)g(x)$, then $\lim_{x\to 2} f(x)$ is equal to:

[JEE Main-2019 (April)]

(a) 24 (b) 36

(c) 12 (d) 18

22. A value of α such that $\int_\alpha^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right)$ is:

[JEE Main-2019 (April)]

(a) $\frac{1}{2}$ (b) 2

(c) $-\frac{1}{2}$ (d) -2

23. If $f(a+b+1-x) = f(x)$, for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x)+f(x+1))\,dx$ is equal to

[JEE Main-2020 (January)]

(a) $\int_{a+1}^{b+1} f(x+1)\,dx$ (b) $\int_{a-1}^{b-1} f(x+1)\,dx$

(c) $\int_{a+1}^{b+1} f(x)\,dx$ (d) $\int_{a-1}^{b-1} f(x)\,dx$

24. If θ_1 and θ_2 be be respectively the smallest and the largest values of θ in $(0, 2\pi) - (\pi)$ which satisfy the equation, $2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$, then $\int_{\theta_1}^{\theta_2} \cos^2 3\theta\,d\theta$ is equal to:

[JEE Main-2020 (January)]

(a) $\frac{\pi}{9}$ (b) $\frac{\pi}{3} + \frac{1}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

25. The value of α for which $4\alpha\int_{-1}^{2} e^{-\alpha|x|}dx = 5$ is:

[JEE Main-2020 (January)]

(a) $\log_e\left(\frac{3}{2}\right)$ (b) $\log_e\left(\frac{4}{3}\right)$

(c) $\log_e\sqrt{2}$ (d) $\log_e 2$

26. If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

[JEE Main-2020 (January)]

(a) $\frac{1}{6} < I^2 < \frac{1}{2}$ (b) $\frac{1}{16} < I^2 < \frac{1}{9}$

(c) $\frac{1}{8} < I^2 < \frac{1}{4}$ (d) $\frac{1}{9} < I^2 < \frac{1}{8}$

27. $\lim_{x \to 0} \frac{\int_0^x t \sin(10t)dt}{x}$ is equal to

[JEE Main-2020 (January)]

(a) $-\frac{1}{10}$ (b) $\frac{1}{10}$

(c) $-\frac{1}{5}$ (d) 0

28. If for all real triplets (a, b, c), $f(x) = a + bx + cx^2$; then $\int_0^1 f(x)dx$ is equal to : [JEE Main-2020 (January)]

(a) $\frac{1}{3}\left\{f(0) + f\left(\frac{1}{2}\right)\right\}$

(b) $\frac{1}{3}\left\{f(1) + 3f\left(\frac{1}{2}\right)\right\}$

(c) $\frac{1}{6}\left\{f(0) + f(1) + 4f\left(\frac{1}{2}\right)\right\}$

(d) $2\left\{3f(1) + 2f\left(\frac{1}{2}\right)\right\}$

29. The value of $\int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx$ is equal to:

[JEE Main-2020 (January)]

(a) 2π (b) π^2

(c) 4π (d) $2\pi^2$

30. Let [t] denote the greatest integer less than or equal to t. Then the value of $\int_1^2 |2x - [3x]| dx$ is _____.

[JEE Main-2020 (September)]

31. The integral $\int_0^2 ||x-1| - x| dx$ is equal to _____.

[JEE Main-2020 (September)]

32. If the value of the integral $\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx$ is $\frac{k}{6}$, then k is equal to:

[JEE Main-2020 (September)]

(a) $2\sqrt{3} + \pi$ (b) $3\sqrt{2} - \pi$

(c) $3\sqrt{2} + \pi$ (d) $2\sqrt{3} - \pi$

33. $\int_{-\pi}^{\pi} |\pi - |x|| dx$ is equal to :

[JEE Main-2020 (September)]

(a) π^2 (a) $2\pi^2$

(c) $\sqrt{2}\pi^2$ (d) $\frac{\pi^2}{2}$

34. Let {x} and [x] denote the fractional part of x and the greatest integer $\leq$ x respectively of a real number x. If $\int_0^n \{x\}dx, \int_0^n [x]dx$ and $10(n^2 - n), (n \in N, n > 1)$ are three consecutive terms of a G.P., then n is equal to _____.

[JEE Main-2020 (September)]

35. The Integral

$$\int_{\pi/6}^{\pi/3} \tan^3 x . \sin^2 3x (2\sec^2 x . \sin^2 3x + 3\tan x . \sin 6x)dx$$

is equal to:

[JEE Main-2020 (September)]

(a) $\frac{7}{18}$ (b) $\frac{9}{2}$

(c) $-\frac{1}{18}$ (d) $-\frac{1}{9}$

36. The value of $\int_{-\pi/2}^{\pi/2} \frac{1}{1 + e^{\sin x}} dx$ is:

[JEE Main-2020 (September)]

(a) $\frac{\pi}{4}$ (b) π

(c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{2}$

37. The integral $\int_1^2 e^x . x^x (2 + \log_e x)dx$ equals:

[JEE Main-2020 (September)]

(a) $e(2e-1)$ (b) $e(4e-1)$

(c) $4e^2 - 1$ (d) $e(4e+1)$

38. If $I_1 = \int_0^1 (1 - x^{50})^{100} dx$ and $I_2 = \int_0^1 (1 - x^{50})^{101}$ such that $I_2 = \alpha I_1$ then α equals to :

[JEE Main-2020 (September)]

(a) $\frac{5051}{5050}$ (b) $\frac{5050}{5051}$

(c) $\frac{5050}{5049}$ (d) $\frac{5049}{5050}$

39. The value of the integral, $\int_1^3 \left[x^2 - 2x - 2\right] dx$, where $[\cdot]$ denotes the greatest integer less than or equal to x is:

[*JEE Main-2021 (February)*]

(*a*) -5 (*b*) $-\sqrt{2} - \sqrt{3} + 1$

(*c*) -4 (*d*) $-\sqrt{2} - \sqrt{3} - 1$

40. If a curve $y = f(x)$ passes through the point $(1, 2)$ and satisfies $x\frac{dy}{dx} + y = bx^4$, then for what value of b, $\int_1^2 f(x)dx = \frac{62}{5}$?

[*JEE Main-2021 (February)*]

(*a*) $\frac{31}{5}$ (*b*) 10

(*c*) 5 (*d*) $\frac{62}{5}$

41. $\lim\limits_{x \to 0^+} \frac{\int_0^{x^2} (\sin\sqrt{t})\, dt}{x^3}$ is equal to:

[*JEE Main-2021 (February)*]

(*a*) $\frac{1}{15}$ (*b*) 0

(*c*) $\frac{2}{3}$ (*d*) $\frac{3}{2}$

JEE ADVANCED PREVIOUS YEARS

Comprehension – 1 (No. 1 & 2)

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in IR$ and let

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right) f(t)\, dt \text{ for all } x \in (1, \infty).$$

1. Which of the following is true? [*IIT JEE-2012*]

(*a*) g is increasing on $(1, \infty)$

(*b*) g is decreasing on $(1, \infty)$

(*c*) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(*d*) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

2. Consider the statements:

P : There exists some $x \in IR$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in IR$ such that $2f(x) + 1 = 2x(1 + x)$

Then

(*a*) both P and Q are true

(*b*) P is true and Q is false

(*c*) P is false and Q is true

(*d*) both P and Q are false

3. If $f(x) = \int_0^x e^{t^2} (t-2)(t-3)\, dt$ for all $x \in (0, \infty)$, then

[*IIT JEE-2012*]

(*a*) f has a local maximum at $x = 2$

(*b*) f is decreasing on $(2, 3)$

(*c*) there exists some $c \in (0, \infty)$ such that $f'(c) = 0$

(*d*) f has a local minimum at $x = 3$

4. Let $f: \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(*a*) $(2e - 1, 2e)$ (*b*) $(e - 1, 2e - 1)$

(*c*) $\left(\frac{e-1}{2}, e - 1\right)$ (*d*) $\left(0, \frac{e-1}{2}\right)$

[*IIT JEE-2012*]

5. For $a \in R$ (the set of all real numbers), $a \neq -1$,

$$\lim_{n \to \infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1) + (na+2) + \ldots + (na+n)]} = \frac{1}{60}$$

Then $a =$

[*JEE Advanced-2013*]

(*a*) 5 (*b*) 7

(*c*) $\frac{-15}{2}$ (*d*) $\frac{-17}{2}$

6. Let $f:[a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a, \\ \int_a^x f(t)dt & \text{if } a \le x \le b, \\ \int_a^b f(t)dt & \text{if } x > b. \end{cases}$$, Then

[*JEE Advanced-2014*]

(*a*) g(x) is continuous but not differentiable at a

(*b*) g (x) is differentiable on R

(*c*) g(x) is continuous but not differentiable at b

(*d*) g(x) is continuous and differentiable at either a or b but not both

7. Let f: $(0, \infty) \to R$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then

[*JEE Advanced-2014*]

(*a*) f(x) is monotonically increasing on $[1, \infty)$

(*b*) f(x) is monotonically decreasing on (0, 1)

(*c*) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(*d*) $f(2^x)$ is an odd function of x on R

8. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$ is

[*JEE Advanced-2014*]

9. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec} x)^{17} dx$ is equal to

[*JEE Advanced-2014*]

(*a*) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ (*b*) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$

(*c*) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$ (*d*) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

Comprehension – 2 (No. 10 & 11)

Given that for each $a \in (0, 1)$

$$\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\, dt$$

exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0, 1).

$$\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\, dt$$

[*JEE Advanced-2014*]

10. The value of $g\left(\frac{1}{2}\right)$ is

(*a*) π (*b*) 2π

(*c*) $\frac{\pi}{2}$ (*d*) $\frac{\pi}{4}$

11. The value of $g'\left(\frac{1}{2}\right)$ is

(*a*) $\frac{\pi}{2}$ (*b*) π

(*c*) $-\frac{\pi}{2}$ (*d*) 0

12.

List-I	List-II
P. The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	**1.** 8
Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.** 2
R. $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	**3.** 4
S. $\frac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	**4.** 0

[*JEE Advanced-2014*]

Codes:

	P	Q	R	S
(*a*)	3	2	4	1
(*b*)	2	3	4	1
(*c*)	3	2	1	4
(*d*)	2	3	1	4

13. Let f: R → R be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$ where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)} dx$, then the value of (4I – 1) is

[*JEE Advanced-2015*]

14. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ [*JEE Advanced-2015*]

15. Let $f: R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is.

[*JEE Advanced-2015*]

16. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

[*JEE Advanced-2015*]

(*a*) $\int_0^{\pi/4} xf(x)\,dx = \frac{1}{12}$

(*b*) $\int_0^{\pi/4} f(x)\,dx = 0$

(*c*) $\int_0^{\pi/4} xf(x)\,dx = \frac{1}{6}$

(*d*) $\int_0^{\pi/4} f(x)\,dx = 1$

17. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\,dx \le M$, then the possible values of m and M are

[*JEE Advanced-2015*]

(*a*) $m = 13, M = 24$

(*b*) $m = \frac{1}{4}, M = \frac{1}{2}$

(*c*) $m = -11, M = 0$

(*d*) $m = 1, M = 12$

18. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\,?$$

[*JEE Advanced-2015*]

(*a*) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (*b*) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(*c*) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (*d*) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

Comprehension – 2 (No. 19 & 20)

Let $F: R \to R$ be a thrice differentiable function. Supose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$.

[*JEE Advanced-2015*]

19. The correct statement(s) is(are)

(*a*) $f'(1) < 0$

(*b*) $f(2) < 0$

(*c*) $f'(x) \neq 0$ for any $x \in (1, 3)$

(*d*) $f'(x) = 0$ for some $x \in (1, 3)$

20. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is(are)

(*a*) $9f'(3) + f'(1) - 32 = 0$

(*b*) $\int_1^3 f(x)dx = 12$

(*c*) $9f'(3) - f'(1) + 32 = 0$

(*d*) $\int_1^3 f(x)dx = -12$

21. The total number of distinct $x \in [0,1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is [*JEE-Advanced-2016*]

22. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ is equal to

[*JEE Advanced-2016*]

(*a*) $\frac{\pi^2}{4} - 2$ (*b*) $\frac{\pi^2}{4} + 2$

(*c*) $\pi^2 - e^{\frac{\pi}{2}}$ (*d*) $\pi^2 + e^{\frac{\pi}{2}}$

23. Let $f(x)=\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x>0$. Then

[JEE Advanced-2016]

(a) $f\left(\frac{1}{2}\right)\geq f(1)$ (b) $f\left(\frac{1}{3}\right)\leq f\left(\frac{2}{3}\right)$

(c) $f'(2)\leq 0$ (d) $\frac{f'(3)}{f(3)}\geq\frac{f'(2)}{f(2)}$

24. Let $f:R\to R$ be a differentiable function such that $f(0)=0, f\left(\frac{\pi}{2}\right)=3$ and $f'(0)=1$.

If $g(x)=\int_x^{\frac{\pi}{2}}\left[f'(t)\operatorname{cosec}t-\cot t\operatorname{cosec}tf(t)\right]dt$ for $x\in\left(0,\frac{\pi}{2}\right]$, then $\lim_{x\to 0}g(x)=$

[JEE Advanced-2017]

25. If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then

[JEE Advanced-2017]

(a) $I>\log_e 99$ (b) $I<\log_e 99$

(c) $I<\frac{49}{50}$ (d) $I>\frac{49}{50}$

26. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$, *then*

[JEE Advanced-2017]

(a) $g'\left(-\frac{\pi}{2}\right)=2\pi$ (b) $g'\left(-\frac{\pi}{2}\right)=-2\pi$

(c) $g'\left(\frac{\pi}{2}\right)=2\pi$ (d) $g'\left(\frac{\pi}{2}\right)=-2\pi$

27. For each positive integer n, let

$y_n=\left\{\frac{1}{n}(n+1)(n+2)\ldots\ldots(n+n)\right\}^{1/n}$

For $x\in\mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty}y_n=L$, then the value of [L] is _____

[JEE Advanced-2018]

28. The value of the integral $\int_0^{\frac{1}{2}}\frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}dx$ is:

[JEE Advanced-2018]

29. If $I=\frac{2}{\pi}\int_{\pi/4}^{\pi/4}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals:

[JEE Advanced-2019]

30. For $a\in R$, $|a|>1$, let

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+\ldots.+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+\ldots..+\frac{1}{(an+n)^2}\right)}\right)=54.$$

Then the possible value (s) of a is/are :

[JEE Advanced-2019]

(a) 8 (b) –9

(c) –6 (d) 7

31. The value of the integral $\int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$ equals:

[JEE Advanced-2019]

32. Which of the following inequalities is/are TRUE?

[JEE Advanced-2020]

(a) $\int_0^1 x\cos x\,dx\geq\frac{3}{8}$ (b) $\int_0^1 x\sin x\,dx\geq\frac{3}{10}$

(c) $\int_0^1 x^2\cos x\,dx\geq\frac{1}{2}$ (d) $\int_0^1 x^2\sin x\,dx\geq\frac{2}{9}$

33. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi)=-6$. If $F:[0,\pi]\to\mathbb{R}$ is defined by $F(x)=\int_0^x f(t)dt$, and if

$\int_0^{\pi}\left(f'(x)+F(x)\right)\cos x\,dx=2$, then the value of $f(0)$ is:

[JEE Advanced-2020]

ANSWER KEY

Topicwise Questions

1. (a)	2. (b)	3. (a)	4. (b)	5. (a)	6. (a)	7. (a)	8. (a)	9. (b)	10. (c)
11. (b)	12. (d)	13. (a)	14. (d)	15. (a)	16. (b)	17. (c)	18. (d)	19. (b)	20. (c)
21. (c)	22. (a)	23. (a)	24. (c)	25. (c)	26. (a)	27. (b)	28. (b)	29. (d)	

Learning Plus

1. (a)	2. (c)	3. (c)	4. (c)	5. (d)	6. (a)	7. (c)	8. (a)	9. (b)	10. (a)
11. (a)	12. (a)	13. (d)	14. (b)	15. (d)	16. (b)	17. (c)	18. (a)	19. (c)	20. (c)
21. (b)	22. (a)	23. (b)	24. (c)	25. (a)	26. (d)	27. (c)	28. (d)	29. (b)	30. (b)
31. (b)	32. (d)	33. (c)	34. (b)	35. (c)					

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (a, c)	2. (a, b)	3. (a, c)	4. (a, b, c, d)	5. (a, d)	6. (c, d)	7. (a, b)	8. (a, b, c)	9. (a, b, c)	
10. (a, b, c, d)	11. (a, b)	12. (b, c)	13. (c)	14. (a)	15. (a)	16. (d)	17. (c)	18. (d)	19. (d)
20. (a)	21. (c)	22. (a) → (r), (b) → (r), (c) → (p), (d) → (p)				23. (a) → (q), (b) → (r), (c) → (p), (d) → (s)			
24. (50)	25. (0.67)	26. (0.277)	27. (b)	28. (0)	29. (3)	30. (31)	31. (11.33)	32. (0.5)	33. (0.5)

JEE Mains & Advanced Past Years Questions

JEE MAIN PREVIOUS YEARS

1. (b)	2. (c)	3. (b)	4. (d)	5. (a)	6. (a)	7. (c)	8. (b)	9. (a)	10. (c)
11. (d)	12. (d)	13. (d)	14. (a)	15. (c)	16. (c)	17. (a)	18. (c)	19. (a)	20. (c)
21. (d)	22. (d)	23. (b)	24. (d)	25. (d)	26. (d)	27. (d)	28. (c)	29. (b)	30. (1)
31. (1.5)	32. (d)	33. (a)	34. (21)	35. (c)	36. (d)	37. (b)	38. (b)	39. (d)	40. (b)
41. (c)									

JEE-ADVANCED PREVIOUS YEARS

1. (b)	2. (c)	3. (a, b, c, d)	4. (d)	5. (b, d)	6. (a, c)	7. (a, c, d)	8. (2)	9. (a)	10. (a)
11. (d)	12. (d)	13. (0)	14. (9)	15. (7)	16. (a, b)	17. (d)	18. (a, c)	19. (a, b, c)	20. (c, d)
21. (0001)	22. (a)	23. (b, c)	24. (b)	25. (b, d)	26. (Bonus)	27. (1)	28. (2)	29. (4.00)	30. (a,b)
31. (0.50)	32. (a, b, d)	33. (4.00)							

t.me/realprayasians2

24 INDEFINITE INTEGRATION

INTRODUCTION

If f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. and says that $\int_a^b f(x)\,dx$ can be found by evaluating F(b) – F(a), where F is an antiderivative of f. We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)dx$ is traditionally used for antiderivative of f and is called and **indefinite integral**. Thus

$$\int f(x)dx = F(x) + C \text{ means } F'(x) = f(x)$$

For example, we can write

$$\int x^2 dx = \frac{x^3}{3} + C \text{ because } \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire family of function (one antiderivative for each value of the constant C).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is family of functions. The connection between them is given by part 2 of the Fundamental Theorem. If f is continuous on [a, b] then

$$\int_a^b f(x)\,dx = \int f(x)dx\Big|_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions. We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval. Thus, we write

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$$

With the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This true despite the fact that the general antiderivative of the function $f(x) = 1/x^2, x \neq 0$, is

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x < 0 \\ -\frac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Note : If $\phi_1(x)$ & $\phi_2(x)$ are two antiderivatives of a function f(x) on [a, b] the difference between them is constant

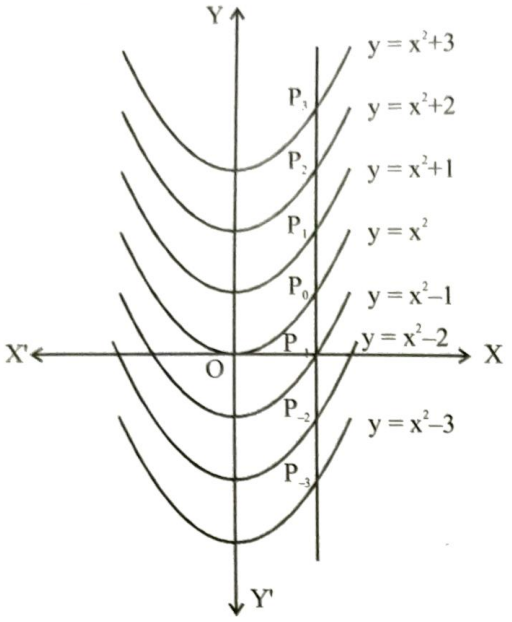
GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRATION

$\int f(x)dx = F(x) + C = y$(say), represents a family of curves, The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral. This is the geometrical interpretation of indefinite integral. Let f(x) = 2x. Then $\int f(x)dx = x^2 + C$. For different values of C, we get different integral. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different valeus to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

If the line x = a intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these point equals 2a. This indicates theat the tangents to the curves at these points are parallel.

Thus, $\int 2x\,dx = x^2 + C = f(x) + C$, implies that the tangents to all the curves $f(x) + C$, $C \in R$, at the points of intersection of the curves by the line $x = a$, $(a \in R)$, are parallel.



ELEMENTARY INTEGRALS

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\,\ell n\,|ax+b| + c$

(iii) $\int e^{ax+b}dx = \frac{1}{a}\,e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{a^{px+q}}{p\,\ell n\,a}\ (a>0) + c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\,\ell n\sec|ax+b| + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\,\ell n\sin|ax+b| + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b).\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \operatorname{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

OR $\ln\tan\left|\frac{\pi}{4}+\frac{x}{2}\right| + c$ OR $-\ln|\sec x - \tan x| + c$

(xiv) $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} - \cot x| + c$

OR $\ln\left|\tan\frac{x}{2}\right| + c$ OR $-\ln|\operatorname{cosec} x + \cot x|$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right| + C$

OR $\sinh^{-1}\frac{x}{a} + C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right| + C$

OR $\cosh^{-1}\frac{x}{a} + C$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(xxii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(xxiii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ell n\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right| + C$

(xxiv) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ell n\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C$

(xxv) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$

(xxvi) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$

PROPERTIES OF INTEGRATION

P-1: $\int (f_1(x) + f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx$

P-2: $\int af(x)dx = a\int f(x)dx$

P-3: If $\int f(x)dx = \phi(x) + C$

$\Rightarrow \int f(ax + b)dx = \frac{1}{a}\phi(ax + b) + C$

P-4: $\frac{d}{dx}(\int f(x)dx) = f(x)$

P-5: $\int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C$

TRAIN YOUR BRAIN

Example 1: Evaluate : $\int\left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

Sol. $\int\left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

$= \int x^3\, dx + \int 5x^2\, dx - \int 4dx + \int \frac{7}{x}dx + \int \frac{2}{\sqrt{x}}dx$

$= \int x^3\, dx + 5.\int x^2\, dx - 4.\int 1.\, dx + 7.\int \frac{1}{x}dx + 2.\int x^{-1/2}dx$

$= \frac{x^4}{4} + 5.\frac{x^3}{3} - 4x + 7\,\ell n|x| + 2\left(\frac{x^{1/2}}{1/2}\right) + C$

$= \frac{x^4}{4} + \frac{5}{3}x^3 - 4x + 7\,\ell n|x| + 4\sqrt{x} + C$

Example 2: Evaluate : $\int\left(e^{x\ell na} + e^{a\ell nx} + e^{a\ell na}\right)dx$, $a > 0$

Sol. We have,

$\int (e^{x\ell na} + e^{a\ell nx} + e^{a\ell na})\, dx$

$= \int (e^{\ell na^x} + e^{\ell nx^a} + e^{\ell na^a})\, dx$

$= \int (a^x + x^a + a^a)\, dx$

$= \int a^x\, dx + \int x^a\, dx + \int a^a\, dx = \frac{a^x}{\ell na} + \frac{x^{a+1}}{a+1} + a^a.x + C.$

Example 3 : Evaluate : $\int \frac{2^x + 3^x}{5^x}dx$

Sol. $\int \frac{2^x + 3^x}{5^x}dx = \int\left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right)dx$

$= \int\left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right]dx = \frac{(2/5)^x}{\ell n\frac{2}{5}} + \frac{(3/5)^x}{\ell n\frac{3}{5}} + C$

Example 4 : Integrate $(1 + \sin x)/(1 - \cos x)$.

Sol. Here

$I = \int \frac{1 + \sin x}{1 - \cos x}dx = \int \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$

$= \frac{1}{2}\int \operatorname{cosec}^2\frac{x}{2}dx + \int \cot\frac{x}{2}dx$

$= -\cot\frac{x}{2} + 2\log\left(\sin\frac{x}{2}\right) + c$

Example 5 : Evaluate : $\int \sin^3 x \cos^3 x \, dx$

Sol. $\int \sin^3 x \cos^3 x \, dx = \frac{1}{8} \int (2 \sin x \cos x)^3 \, dx = \frac{1}{8} \int \sin^3 2x \, dx = \frac{1}{8} \int \frac{3\sin 2x - \sin 6x}{4} \, dx = \frac{1}{32} \int (3\sin 2x - \sin 6x) \, dx$

$= \frac{1}{32} \left[-\frac{3}{2} \cos 2x + \frac{1}{6} \cos 6x \right] + C$

Example 6 : Evaluate : $\int \frac{x^4}{x^2+1} \, dx$

Sol. $\int \frac{x^4}{x^2+1} dx = \int \frac{x^4 - 1 + 1}{x^2+1} dx$

$= \int \left(\frac{x^4 - 1}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$= \int (x^2 - 1) \, dx + \int \frac{1}{x^2+1} \, dx$

$= \frac{x^3}{3} - x + \tan^{-1} x + C$

Example 7 : Evaluate : $\int \frac{1}{4+9x^2} \, dx$

Sol. We have

$\int \frac{1}{4+9x^2} \, dx = \frac{1}{9} \int \frac{1}{\frac{4}{9} + x^2} dx$

$= \frac{1}{9} \int \frac{1}{(2/3)^2 + x^2} \, dx$

$= \frac{1}{9} \cdot \frac{1}{(2/3)} \tan^{-1} \left(\frac{x}{2/3} \right) + C$

$= \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right) + C$

INTEGRATION BY TRANSFORMATION

$\int \cos ax \cos bx \, dx$, $\int \cos ax \cos bx \, dx$,

$\int \cos ax \cos bx \, dx$, in which $a \neq b$.

We can use these addition formulae to change products to sums or differences, and the later can be integrated easily. We next consider integrals of the type $\int \cos^m x \sin^n x \, dx$, in which at least one of the exponents m and n is an odd positive integer (the other exponent need only be a real number). Suppose that $m = 2k + 1$, where k is a non-negative integer. Then $\cos^m x \sin^n x = \cos^{2k+1} x \sin^n x = (\cos^2 x)^k \sin^n x \cos x$. Using the identity $\cos^2 x = 1 - \sin^2 x$, we obtain $\int \cos^m x \sin^n x \, dx = \int (1 - \sin^2 x)^k \sin^n x \cos x \, dx$. The factor $(1 - \sin^2 x)^k$ can be expanded by the Binomial Theorem, and the result is that $\int \cos^m x \sin^n x \, dx$ can be written as a sum of constant multiples of integrals of the form $\int \sin^q x \cos x \, dx$. Since

$$\int \sin^q x \cos x \, dx = \begin{cases} \frac{1}{q+1} \sin^{q+1} x + c & \text{if } q \neq -1, \\ \ell n |\sin x| + c & \text{if } q = -1, \end{cases}$$

If follows that $\int \cos^m x \sin^n x \, dx$ can be readily evaluated, An entirely analogous argument follows if the exponent n is an odd positive integer. The third type of integral we consider consists of those of the form $\int \cos^m x \sin^n x \, dx$, in which both m and n are even non-negative integers. These function are not so simple to integrate as those containing an odd power. We first consider the special case in which either m = 0 or n = 0. The simplest non-trivial examples are the two integrals $\int \cos^2 x \, dx$ and $\int \sin^2 x \, dx$, which can be integrated by means of the identities $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ Evaluation of the two integrals is now a simple matter. We get $\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{x}{2} + \frac{1}{4} \sin 2x + c$. Going on to the higher powers, consider the integral $\int \cos^{2i} x \, dx$, where i is an arbitrary positive integer. We write $\cos^{2i} x = (\cos^2 x)^i = \left[\frac{1}{2}(1 + \cos 2x) \right]^i = \frac{1}{2^i}(1 + \cos 2x)^i$ We expand using binomial theorem and integrate the terms using previous methods.

TRAIN YOUR BRAIN

Example 8 : Integrate $\int \sin 8x \sin 3x \, dx$

Sol. $\sin 8x \sin 3x = \frac{1}{2} (\cos 5x - \cos 11x)$,

and so $\int \sin 8x \sin 3x \, dx$

$= \frac{1}{2} \int (\cos 5x - \cos 11x) \, dx = \frac{1}{10} \sin 5x - \frac{1}{22} \sin 11x + c$,

Example 9 : Integrate

(a) $\int \cos^3 4x\, dx$,

(b) $\int \sin^5 x \cos^4 x\, dx$

Sol. In (a) illustrates that the method just described is applicable to odd positive integer powers of the sine or cosine (i.e., either m or n may be zero). We obtain

$$\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx$$

$$= \int (1-\sin^2 4x)\cos 4x\, dx$$

$$= \int \cos 4x\, dx - \int \sin^2 4x \cos 4x\, dx$$

$$= \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + c.$$

In **(b)** it is the exponent of the sine which is an odd positive integer.

$$\int \sin^5 x \cos^4 x\, dx = \int (\sin^2 x)^2 \cos^4 x \sin x\, dx$$

$$= \int (1-\cos^2 x)^2 \cos^4 x \sin x\, dx$$

$$= \int (1-2\cos^2 x+\cos^4 x)\cos^4 x \sin x\, dx$$

$$= \int \cos^4 x \sin x\, dx - 2$$

$$\int \cos^6 x \sin x\, dx + \int \cos^8 x \sin x\, dx$$

$$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + c.$$

INTEGRATION BY SUBSTITUTION

Let g be a function whose range is an interval ℓ, and let f be a function that is continuous on ℓ. If g is differentiable on its domain and F is an antiderivative of f on ℓ, then

$\int f(g(x))g'(x)\, dx = F(g(x)) + C$. If $u = g(x)$, then $du = g'(x)$ and

$\int f(u)\, du = F(u) + C$.

Guidelines for Making A Change of Variable

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Evaluate the resulting integral in terms of u.
5. Replace u by g(x) to obtain an antiderivative in terms of x.

The general power rule for integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C,\ n \neq -1$$

Rationalizing substitutions

Some irrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Some standard substitutions

(i) $\int \frac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$, take x^n common & put

$1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1+x^{-n}=t$.

(iv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\, dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$;

put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(v) $\int \sqrt{\frac{x-\alpha}{x-\beta}}\, dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$;

put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

(vi) For terms of the form $x^2 + a^2$ or $\sqrt{x^2+a^2}$,

put $x = a\tan\theta$ or $a\cot\theta$

(vii) For terms of the form $x^2 - a^2$ or $\sqrt{x^2-a^2}$,

put $x = a\sec\theta$ or $a\operatorname{cosec}\theta$

(viii) For terms of the form $a^2 - x^2$ or $\sqrt{a^2-x^2}$,

put $x = a\sin\theta$ or $a\cos\theta$

(ix) If both $\sqrt{a+x}, \sqrt{a-x}$ are present, then put $x = a\cos 2\theta$

TRAIN YOUR BRAIN

Example 10 : Evaluate $\int (x^2+1)^2\,(2x)\,dx$.

Sol. Assuming $g(x) = x^2 + 1$, we obtain $g'(x) = 2x$ and $f(g(x)) = [g(x)]^2$.

From this, we can recognize that the integrand and follows the $f(g(x))\ g'(x)$ pattern. Thus, we can write

$$\int \overbrace{(x^2+1)^2}^{[g(x)]^2} \overbrace{(2x)}^{g'(x)}\,dx$$

$$= \frac{1}{3}(x^2+1)^3 + C.$$

Example 11 : Evaluate $\int \frac{-4x}{(1-2x^2)^2}\,dx$

Sol. $$\int \overbrace{(1-2x^2)^{-2}}^{u^{-2}} \overbrace{(-4x)\,dx}^{du} = \frac{\overbrace{(1-2x^2)^{-1}}^{u^{-1}(-1)}}{-1} + C$$

Example 12 : Evaluate $\int x^3 \cos(x^4+2)\,dx$.

Sol. Let $u = x^4 + 2 \Rightarrow du = 4x^3\,dx$

$$\int x^3 \cos(x^4+2)\,dx$$

$$= \int \cos u \cdot \frac{1}{4}\,du = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}$$

$$\sin(x^4+2) + C$$

Example 13 : Evaluate $\int \frac{x^2dx}{(x^3-2)^5}$.

Sol. Let $u = x^3 - 2$. Then $du = 3x^2\,dx$. so by substitution :

$$\int \frac{x^2dx}{(x^3-2)^5} = \int \frac{du/3}{u^5} = \frac{1}{3}\int u^{-5}du$$

$$= \frac{1}{3}\frac{u^{-4}}{-4} + C = -\frac{1}{12}(x^3-2)^{-4} + C.$$

Example 14 : Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$

Sol. Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$.

Therefore $\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du$

$$= 2\int \frac{u^2}{u^2-4}\,du = 2\int \left(1 + \frac{4}{u^2-4}\right)du$$

$$= 2\int du + 8\int \frac{du}{u^2-4} = 2u + 8.\frac{1}{2\cdot 2}\,\ell n\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\,\ell n\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

Example 15 : Evaluate : $\int x^3 \sin x^4\,dx$

Sol. We have

$$I = \int x^3 \sin x^4\,dx$$

Let $x^4 = t \Rightarrow d(x^4) = dt \Rightarrow 4x^3\,dx = dt \Rightarrow dx = \frac{1}{4x^3}\,dt$

$$I = \frac{1}{4}\int \sin t\,dt = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos x^4 + C$$

Example 16 : Evaluate : $\int \frac{(\ell n\,x)^2}{x}\,dx$

Sol. Let $I = \int \frac{(\ell n\,x)^2}{x}\,dx$ Put $\ell nx = t$

$$\Rightarrow \frac{1}{x}\,dx = dt$$

$$\Rightarrow I = \int t^2 dt = \frac{t^3}{3} + c = \frac{(\ell n\,x)^3}{3} + C$$

Example 17 : Evaluate : $\int (1+\sin^2 x)\cos x\,dx$

Sol. Let $I = \int (1+\sin^2 x)\cos x\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\Rightarrow I = \int (1+t^2)\,dt = t + \frac{t^3}{3} + c = \sin x + \frac{\sin^3 x}{3} + C$$

Example 18: Evaluate $\int \frac{x}{x^4+x^2+1}\,dx$

Sol. We have,

$$I=\int \frac{x}{x^4+x^2+1}\,dx=\int \frac{x}{(x^2)^2+x^2+1}\,dx$$

{Put $x^2=t \Rightarrow x.dx=\frac{dt}{2}$}

$$\Rightarrow I=\frac{1}{2}\int \frac{1}{t^2+t+1}\,dt=\frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$=\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

Example 19: Evaluate $\int \frac{dx}{1+e^x}$

Sol. Rewrite the integrand as follows : $\frac{1}{1+e^x}=\frac{e^{-x}}{e^{-x}}\left(\frac{1}{1+e^x}\right)=\frac{e^{-x}}{e^{-x}+1}$ $(u=e^{-x}+1; du=-e^{-x}dx)$

$$\int \frac{dx}{1+e^x}=\int \frac{e^{-x}dx}{e^{-x}+1}=\int \frac{-du}{u}=-\ell n|u|+C=-\ell n(e^{-x}+1)+c \quad (\because e^{-x}+1>0)$$

Example 20 : Evaluate $\int \cos x\sqrt{(4-\sin^2 x)}\,dx$

Sol. Put $\sin x=t$ so that $\cos x\,dx=dt$. Then the given integral

$$=\int \sqrt{(4-t^2)}\,dt=\int \sqrt{(2^2-t^2)}\,dt$$

$$=\frac{1}{2}t\sqrt{(2^2-t^2)}+\frac{2^2}{2}\sin^{-1}(t/2)+c$$

$$=\frac{1}{2}\sin x.\sqrt{(4-\sin^2 x)}+2\sin^{-1}(1/2\sin x)+c$$

Example 21: Integrate

(i) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$,

(ii) $\frac{10x^9+10^x\log_e 10}{10^x+x^{10}}$

Sol. (i) Let $I=\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$. Now putting $e^x+e^{-x}=t$, so that $(e^x-e^{-x})\,dx=dt$,

we have $I=\int (1/t)\,dt=\log t=\log(e^x+e^{-x})$.

(ii) Here $I=\int \frac{10x^9+10^x.\log_e 10}{10^x+x^{10}}\,dx$. Now putting $10^x+x^{10}=t$, and $(10^x\log_e 10+10x^9)\,dx=dt$, we have

$$I=\int (1/t)\,dt=\log t=\log(10^x+x^{10})+c$$

Example 22 : Integrate

(i) $\frac{1}{x\cos^2(1+\log x)}$,

(ii) $\frac{1}{x(1+\log x)^m}$

Sol. (i) Here $I=\int dx/\{x\cos^2(1+\log x)\}$. Putting $1+\log x=t$, so that $(1/x)\,dx=dt$, we have

$$I=\int dt/\cos^2 t=\int \sec^2 t\,dt=\tan t=\tan(1+\log x)+c.$$

(ii) Here $I=\int dx/\{x(1+\log x)^m\}$. Putting $1+\log x=t$, so that $(1/x)dx=dt$, we have

$$I=\int \frac{dt}{t^m}=\frac{t^{-m+1}}{-m+1}=\frac{(1+\log x)^{-m+1}}{(1-m)}$$

$$=\frac{1}{(1-m)}(1+\log x)^{1-m}+c.$$

Example 23 : Integrate

(i) $\frac{\cot x}{\log(\sin x)}$

(ii) $\frac{\tan x}{(\log(\sec x)}$

Sol.

(i) Here $\frac{d}{dx}(\log \sin x) = \frac{1}{\sin x}\cos x = \cot x$.

$\therefore\ I = \int \frac{\cot x\, dx}{\log \sin x} = \int \frac{\cot x}{t} \times \frac{dt}{\cot x} = \log|\log(\sin x)| + c$

(ii) We have $\int \frac{\tan x\, dx}{\log \sec x} = \log|\log \sec x| + c$

Example 24 : Integrate $\cos^5 x$.

Sol. $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx$

$= \int (1-\sin^2 x)^2 \cos x\, dx$

$= \int (1-t^2)^2\, dt$, [put $\sin x = t$

$\Rightarrow \cos x\, dx = dt$]

$= \int (1-2t^2+t^4)\, dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + c$

$= \frac{\sin^5 x}{5} - \frac{2}{x}\sin^3 x + \sin x + c$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

Consider the integral $\int \sqrt{a^2 - x^2}\, dx$

If we change the variable from x to θ by the substitution x = a sinθ, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the roots sign because

$$\int \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = a|\cos\theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) are the substitution x = a sin θ (the old variable is a function of the new one).

In general we can make a substitution of the form x = g(t) by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

This kind of substitution is called inverse substitution. We can make the inverse substitution x = a sin θ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

Some Standard Integrals

(i) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(ii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(iii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left[x + \sqrt{x^2 + a^2}\right]$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left[x + \sqrt{x^2 - a^2}\right]$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(vii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(viii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(ix) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

(x) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

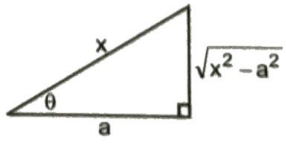
TRAIN YOUR BRAIN

Example 25: Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$

Sol. We let $x = a\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{a^2\tan^2\theta} = a|\tan\theta| = a\tan\theta$$



Therefore $\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta$

$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c$

The triangle in figure gives $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$,

so we have $\frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c$

$= \ln|x + \sqrt{x^2-a^2}| - \ln a + C = \ln|x + \sqrt{x^2-a^2}| + C, \sec\theta = \frac{x}{a}$

Example 26 : Integrate $1/(2x^2+x-1)$.

Sol. We have $\int \frac{dx}{(2x^2+x-1)} = \frac{1}{2}\int \frac{dx}{\left(x^2+\frac{x}{2}-\frac{1}{2}\right)}$

$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2 - \frac{1}{2} - \frac{1}{16}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2 - \frac{9}{16}}$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{3}\log\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}} + C$

$= \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + c = \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right| - \frac{1}{3}\log 2$

$+ c = \frac{1}{3}\log(2x-1)/(x-1)| + C_1$

Example 27: Integrate $(3x+1)/(2x^2-2x+3)$.

Sol. Here $(d/dx)(2x^2-2x+3) = 4x-2$.

$\therefore\ I = \int \frac{3x+1}{2x^2-2x+3}\, dx = \int \frac{\frac{3}{4}(4x-2)+1+\frac{3}{2}}{(2x^2-2x+3)} dx$

$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\, dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\, dx$

$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{2.2}\int \frac{dx}{x^2-x+(3/2)}$

$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right) - \left(\frac{1}{4}\right)}$

$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + (\sqrt{5/2})^2}$

$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\cdot\frac{1}{(\sqrt{5/2})}$

$\left(\tan^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{5/2})}\right\}\right) + c$

$= \frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{10}}\right) + c$

Example 28: Integrate $\frac{1}{\sqrt{(4+3x-2x^2)}}$

Sol. We have $\int \frac{dx}{\sqrt{(4+3x-2x^2)}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left\{2+\frac{9}{16}-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)\right\}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\{(41/16)-(x-3/4)^2\}}}$

$= \frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{x-\frac{3}{4}}{\left(\frac{\sqrt{41}}{4}\right)}\right\} + c$

$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right) + c$

Example 29: Evaluate $\int \sqrt{(x-1)(2-x)}\, dx$.

Sol. We have $\int \sqrt{(x-1)(2-x)}\, dx$

$= \int \sqrt{(-x^2+3x-2)}dx$

$= \int \sqrt{\left\{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}\right\}}\, dx \int \sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}}\, dx,$

[from $\int \sqrt{(a^2-x^2)}\, dx$]

$$= \frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}} + \frac{1}{2}\cdot\frac{1}{4}\sin^{-1}$$

$$\left\{\left(x-\frac{3}{2}\right)/(1/2)\right\}+c$$

$$= \frac{1}{4}(2x-3)\sqrt{(3x-x^2-2)}+\frac{1}{8}\sin^{-1}(2x-3)+c$$

Example 30: Integrate $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}}$

Sol. We have $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}} = \int \frac{x(x^2+1)-x+3}{\sqrt{(x^2+1)}}\,dx =$

$$\int \frac{x(x^2+1)}{\sqrt{(x^2+1)}}dx - \int \frac{xdx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$$

$$= \frac{1}{2}\int (2x)\sqrt{(x^2+1)}\;dx - \frac{1}{2}\int \frac{2x\,dx}{\sqrt{(x^2+1)}}$$

$$+3\int \frac{dx}{\sqrt{(x^2+1)}}$$

$$= \frac{1}{2}\left[\frac{2}{3}(x^2+1)^{3/2}\right] - \frac{1}{2}\left[2\sqrt{(x^2+1)}\right] + 3\,\ell n$$

$$(x+\sqrt{(x^2+1)})+c$$

$$= \frac{1}{3}(x^2+1)^{2/3} - \sqrt{(x^2+1)} + 3\,\ell n\,(x+\sqrt{(x^2+1)})+c$$

INTEGRATION BY PARTS

$\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & **(b)** $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ is simple to integrate.

Note:-

(i) If there is no other function, then unity is taken as the second function e.g. in the itegration of $\int \tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.

(ii) This is generally obtained, by keeping the order of u & v as per the order of the letter in **ILATE**, where

I – Inverse function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

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Example 31: Integrate $x^n \log x$

Sol. We have $\int x^n \log x\,dx = \int (\log x).x^n dx$

$$= (\log x)\cdot\frac{x^{n+1}}{n+1} - \int\frac{1}{x}.\frac{x^{n+1}}{n+1}\,dx = (\log x)\cdot\frac{x^{n+1}}{n+1} -$$

$$\int\frac{x^n}{n+1}dx = (\log x)\cdot\frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}+C$$

Example 32: Evaluate : $\int x\tan^{-1}x\,dx$

Sol. Let $I = \int x\tan^{-1}x\,dx$

$$= (\tan^{-1}x)\frac{x^2}{2} - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}[x-\tan^{-1}x]+C.$$

Example 33: Evaluate : $\int x\,\ell n(1+x)\,dx$

Sol. Let $I = \int x\,\ell n(1+x)\,dx$

$$= \ell n\,(x+1)\cdot\frac{x^2}{2} - \int\frac{1}{x+1}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\ell n\,(x+1) - \frac{1}{2}\int\frac{x^2}{x+1}\,dx$$

$$\frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\,dx$$

$$\frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int\left(\frac{x^2-1}{x+1}+\frac{1}{x+1}\right)dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int\left((x-1)+\frac{1}{x+1}\right)dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\frac{x^2}{2}-x+\ln|x+1|\right]+C$$

Example 34: Evaluate $\int \frac{\log(\sec^{-1}x)dx}{x\sqrt{(x^2-1)}}\,dx]$

Sol. Put $\sec^{-1}x = t$ so that $\frac{1}{x\sqrt{(x^2-1)}}\,dx = dt$.

Then the given integral $= \int \log t\,dt = \int (\log t).1\,dt =$

$(\log t).t - \int \frac{1}{t}t\,dt = t\log t - t + c$

$= t(\log t - \log e) + c = \sec^{-1}x(\log(\sec^{-1}x) - 1) + c$

$= \sec^{-1}x\left(\log\left(\frac{\sec^{-1}x}{e}\right)\right) + c$

Example 35: Evaluate $\int x^2\tan^{-1}x\,dx$.

Sol. We have $\int x^2\tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \int \frac{x^3}{3}\cdot\frac{1}{1+x^2}$,

integrating by parts taking x^2 as the second function

$\frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x(x^2+1)-x}{1+x^2}\,dx$

$[\because x^3 = x(x^2+1)-x]$

$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx + \frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}\,dx = \frac{x^3}{3}$

$\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log(1+x^2) + c$

Example 36: Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$.

Sol. $I = \int \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\}dx$

$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$

$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\,d\theta = \frac{3}{2}\int\theta\sec^2\theta\,d\theta$

(put, $2x+2 = 3\tan\theta \Rightarrow 2\,dx = 3\sec^2\theta\,d\theta$)

$= \frac{3}{2}(\theta\tan\theta - \int\tan\theta\,d\theta) = \frac{3}{2}\{\theta\tan\theta - \log(\sec\theta)\} +$

c

$I = \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$

$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\frac{\sqrt{4x^2+8x+13}}{3}\right\} + c$

$\Rightarrow I = (x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \frac{1}{4}\log(4x^2+8x+13) + c$

Remember

(i) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(ii) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Evaluate $I = \int e^{ax}\sin bx\,dx$

Integrating by parts taking sin bx as the second function,

We get $I = -\frac{e^{ax}\cos bx}{b} - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx$

$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx$

Again integrating by parts taking cos bx as the second function, we get

$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax}\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right]$

or $I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$

or $I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx) - \frac{a^2}{b^2}I$. $[\because e^{ax}\sin bx\,dx = I]$

Transposing the term $-\frac{a^2}{b^2}I$ to the left hand side we get

$$I - \frac{a^2}{b^2}I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

or $\frac{I}{b^2}(a^2+b^2) = \frac{1}{b^2}e^{ax}(a\sin bx - b\cos bx)$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Thus, $\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$

Standard Integrals :

(ii) $\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$

(iii) $\int[f(x)+xf'(x)]dx = xf(x) + c$

Example 37: Evaluate $\int \frac{xe^x}{(x-1)^2}dx$

Sol. We have $\int \frac{xe^x}{(x-1)^2}dx = \int xe^x\frac{1}{(x-1)^2}dx$

$$\int \frac{xe^x}{(x-1)^2}dx = (xe^x)\left(-\frac{1}{x-1}\right) - \int(e^x+xe^x)\left(-\frac{1}{x-1}\right)dx,$$

[Note that the integral of $\frac{1}{(x-1)^2}$ is $-\frac{1}{x+1}$]

$$= -\frac{xe^x}{(x+1)^2} - \int e^x(x+1)\frac{1}{x+1}dx$$

$$= -\frac{xe^x}{x+1} + \int e^x dx = -\frac{xe^x}{x+1} + e^x + c$$

$$= e^x\left[1-\frac{x}{x+1}\right] + c = e^x\frac{x+1-x}{x+1} + c = \frac{e^x}{x+1} + c$$

Alternative solution

We have $\int \frac{xe^x}{(x+1)^2}dx = \int e^x\frac{(x+1)-1}{(x+1)}dx$

$$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$$

$$= \int e^x[f(x)+f'(x)]dx, \text{ where } f(x) = \frac{1}{x+1} = e^x\frac{1}{x+1} + c$$

Example 38: Evaluate $\int e^x\frac{2+\sin 2x}{1+\cos 2x}dx$

Sol. We have $\int e^x\frac{2+\sin 2x}{1+\cos 2x}dx$

$$= \int e^x\left(\frac{2}{2\cos^2 x} + \frac{2\sin x\cos x}{2\cos^2 x}\right)dx$$

$$= \int e^x[\sec^2 x + \tan x]dx$$

$$= \int e^x[f(x)+f'(x)]dx, \text{ where } f(x) = \tan x = e^x f(x) + c$$

$$= e^x\tan x + c$$

Example 39 : Evaluate : $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Sol. Given integral $= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$$= \int e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx = -e^x\cot\frac{x}{2} + C$$

Example 40 : Evaluate : $\int\left[\ell n(\ell nx) + \frac{1}{(\ell nx)^2}\right]dx$

Sol. Let $I = \int\left[\ell n(\ell nx) + \frac{1}{(\ell nx)^2}\right]dx$

{put $x = e^t \Rightarrow e^t\,dt$}

$$\therefore I = \int e^t\left(\ell nt + \frac{1}{t^2}\right)dt = \int e^t\left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt$$

$$= e^t\left(\ell nt - \frac{1}{t}\right) + C = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + C$$

FORCE INTEGRATION

Example 41: Integrate $(x-2)/(2x^2+4x+3)^2$.

Sol. Here $(d/dx)(2x^2+4x+3) = 4x+4$.

$$\therefore \int\frac{(x+2)dx}{(2x^2+4x+3)^2} = \int\frac{\frac{1}{4}(4x+4)+2-1}{(2x^2+4x+3)^2}dx$$

$$= \frac{1}{4}\int\frac{(4x+4)dx}{(2x^2+4x+3)^2} + \int\frac{(2-1)dx}{\left(x^2+2x+\frac{3}{2}\right)^2}$$

$$= \frac{1}{4}\int(2x^2+4x+3)^{-2}(4x+4)dx + \frac{1}{4}\int\frac{dx}{\left(x^2+2x+\frac{3}{2}\right)^2}$$

$$= -\frac{1}{4(2x^2+4x+3)} + \frac{1}{4}\int \frac{dx}{\left\{(x+1)^2+\frac{1}{2}\right\}^2}$$

Now put $x+1=t$ and then applying the reduction formula, we get

$$I = \frac{-1}{4(2x^2+4x+3)} + \frac{1}{4}\left[\frac{(x+1)}{(x+1)^2+\frac{1}{2}} + \sqrt{2}\tan^{-1}\{\sqrt{2}(x+1)\}\right] + c$$

Example 42: Integrate $(2x+3)/(x^2+2x+3)^2$.

Sol. Here $(d/dx)(x^2+2x+3) = 2x+2$

$$\therefore I = \int \frac{(2x+3)}{(x^2+2x+3)^2} = \int \frac{(2x+2+1)\,dx}{(x^2+2x+3)^2}$$

$$= \int \frac{(2x+2)dx}{(x^2+2x+3)^2} + \int \frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{(x^2+2x+3)} + \int \frac{dx}{(x^2+2x+3)^2} \quad \text{....(i)}$$

Now let $I_1 = \int \frac{dx}{[(x^2+1)^2+2]^2}$ (Put $x+1=\sqrt{2}\tan t$, so that $dx = \sqrt{2}\sec^2 t\,dt$)

$$\therefore I_1 = \int \frac{\sqrt{2}\sec^2 t\,dt}{(2\tan^2 t+2)^2} = \frac{\sqrt{2}}{4}\int \cos^2 t\,dt$$

$$= \frac{\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{8}[t+\frac{1}{2}\sin 2t] + \frac{\sqrt{2}}{8}[t+\sin t\cos t] + c$$

Now $\tan t = \frac{x+1}{\sqrt{2}}$. Therefore $\sin t = \frac{x+1}{\sqrt{\{(x+1)^2+2\}}}$

$$= \frac{x+1}{\sqrt{(x^2+2x+3)}}, \text{ and } \cos t = \frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$$

Also $t = \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$. Hence $I_1 = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

$$+ \frac{\sqrt{2}}{8}\cdot\frac{x+1}{\sqrt{\{(x^2+2x+3)\}}}\cdot\frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}.$$

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{4}\frac{x+1}{(x^2+2x+3)}$$

$$\therefore I = -\frac{1}{x^2+2x+3} + \frac{1}{4}\frac{x+1}{x^2+2x+3} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c, \quad \text{from (i)}$$

$$= \frac{x+1-4}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$= \frac{x-3}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called partial fractions, that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x-1)$ and $1/(x+2)$ to a common denominator we obtain $\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation $\int \frac{x+5}{x^2+x-2}\,dx$

$$= \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx$$

$$= 2\ell n|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function $f(x) = \frac{P(x)}{Q(x)}$

Where P and Q are polynomials. It's possible to express f as sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called proper. Recall that if

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

where $a_n \neq 0$, then the degree of P is n and we write deg (P) = n.

If f is improper, that is, deg(P) ≥ deg (Q), then we must take the preliminary step of dividing Q into P (by division) until a remainder R(x) is obtained such that deg (R) < deg(Q). The division statement is

(1) $f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ where S and R are also polynomials.

The next step is to factor the denominator Q(x) as far as possible . It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as $Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

The third step is to express the proper rational function $R(x)/Q(x)$ (from equation 1) as a sum of partial fractions of the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^j}$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Case I : The Denominator Q(x) is a product of distinct linear factors.

This means that we can write $Q(x) = (a_1x + b_1)(a_2x + b_2) \ldots (a_kx + b_k)$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that.

$$(2)\quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_k}{a_kx + b_k}$$

Case II : Q(x) is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times, that is, $(a_1x + b_1)^r$ occurs in the factorization of Q(x). Then instead of the single term $A_1/(a_1x + b_1)$ in equation 2, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \ldots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write $\frac{x^3 - x + 1}{x^2(x-1)^3}$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

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Example 43: Resolve $\frac{3x + 2}{x^3 - 6x^2 + 11x - 6}$ into partial fractions.

Sol. We have,

$$\frac{3x + 2}{x^3 - 6x^2 + 11x - 6} = \frac{3x + 2}{(x-1)(x-2)(x-3)}$$

Let $\frac{3x + 2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$.

Then,

$$\Rightarrow \frac{3x + 2}{(x-1)(x-2)(x-3)}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow 3x+2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \ldots\ldots (i)$$

Putting $x - 1 = 0$ or $x = 1$ in (i), we get

$$5 = A(1-2)(1-3) \Rightarrow A = \frac{5}{2},$$

Putting $x - 2 = 0$ or, $x = 2$ in (i), we obtain

$$8 = B(2-1)(2-3) \Rightarrow B = -8.$$

Putting $x - 3 = 0$ or, $x = 3$ in (i), we obtain

$$11 = C(3-1)(3-2) \Rightarrow C = \frac{11}{2}.$$

$$\therefore \frac{3x + 2}{x^3 - 6x^2 + 11x - 6} = \frac{3x + 2}{(x-1)(x-2)(x-3)}$$

$$= \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$$

Example 44: Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$.

Sol. Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as $2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x-1)(x+2)$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form.

$$\frac{x^2 + 2x - 1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To denominator the values of A, B and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Expanding the right side of equation 4 and writing it in the standard form for polynomials, we get

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side-namely, 1. Likewise. The coefficients of x are equal and the constant terms are equal. This gives the following system of equation for A, B and C.

$2A+B+2C=1, 3A+2B-C=2, \quad -2A=-1$

Solving, we get $A=\frac{1}{2}$, $B=\frac{1}{5}$, and $C=-\frac{1}{10}$, and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

$$=\int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}\ell n|x|+\frac{1}{10}\ell n|2x-1|-\frac{1}{10}\ell n|x+2|+K$$

In integrating the middle term we have made the mental substitution u = 2x – 1, which gives du = 2dx and dx = du/2.

CASE: III When some of the factors of denominator g(x) are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}$

The following example illustrates the procedure

Example 45: Evaluate $\int \frac{dx}{\cos x+\cos ecx}$

Sol. $I=\int\frac{dx}{\cos x+\frac{1}{\sin x}}=\int\frac{\sin x\,dx}{\cos x.\sin x+1}$

$$=\int\frac{2\sin x\,dx}{2+2\sin x\cos x}dx=\int\frac{2\sin x}{2+\sin 2x}dx$$

$$=\int\frac{[(\sin x+\cos x)+(\sin x-\cos x)]dx}{2+\sin 2x}$$

$$=\int\frac{\sin x+\cos x}{2+\sin 2x}dx+\int\frac{\sin x-\cos x}{2+\sin 2x}dx$$

$$=\int\frac{\sin x+\cos x}{3-(1-\sin 2x)}dx+\int\frac{\sin x-\cos x}{1+(1+\sin 2x)}$$

$$=\int\frac{\sin x+\cos x}{3-(\sin x-\cos x)^2}.dx+\int\frac{\sin x-\cos x}{1+(\sin x+\cos x)^2}.dx$$

put $\sin x-\cos x=s$ and $\sin x+\cos x=t$

$\Rightarrow (\cos x+\sin x)\,dx=ds$ and $(\cos x-\sin x)\,dx=dt$

$$I=\int\frac{ds}{3-s^2}-\int\frac{dt}{1+t^2}$$

$$=\int\frac{ds}{(\sqrt{3})^2-(s)^2}-\int\frac{dt}{1+t^2}$$

$$=\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+s}{\sqrt{3}-s}\right|-\tan^{-1}t+c$$

$$=\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|-\tan^{-1}(\sin x+\cos x)+c$$

Example 46: Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\frac{2x-3}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$.

Then,

$2x-3=A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1)$ .....(i)

Putting x = 1 in (i), we get $-1=A(1+1)^2$

$\Rightarrow A=-\frac{1}{4}$

Comparing coefficients of like powers of x on both side of (i), we have

$A+B=0, C-B=0, 2A+B-C+D=0,$

$C+E-B-D=2$ and $A-C-E=-3$.

Putting $A=-\frac{1}{4}$ and solving these equations, we get

$B=\frac{1}{4}=C, D=\frac{1}{2}$ and $E=\frac{5}{2}$

$$\therefore \frac{2x-3}{(x-1)(x^2+1)^2}=\frac{-1}{4(x-1)}+\frac{x+1}{4(x^2+1)}+\frac{x+5}{2(x^2+1)^2}$$

SOME ADVANCED TYPE INTEGRATION

Type-1 $\int\frac{dx}{ax^2+bx+c}, \int\frac{dx}{\sqrt{ax^2+bx+c}}, \int\sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

TRAIN YOUR BRAIN

Example 47: Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Sol. We have,

$$\int \sqrt{x^2+2x+5}$$

$$= \int \sqrt{x^2+2x+1+4}\ dx = \int \sqrt{(x+1)^2+2^2}$$

$$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}.(2)^2 \ell n|(x+1)+\sqrt{(x+1)^2+2^2}\,| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\ell n\,|(x+1)+\sqrt{x^2+2x+5}\,| + C$$

Example 48: Evaluate : $\int \frac{1}{x^2-x+1}\ dx$

Sol. $\int \frac{1}{x^2-x+1}\ dx = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}\ dx$

$$= \int \frac{1}{(x-1/2)^2+3/4}\ dx = \int \frac{1}{(x-1/2)^2+\left(\sqrt{3}/2\right)^2}\ dx$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Example 49: Evaluate : $\int \frac{1}{\sqrt{9+8x-x^2}}\ dx$

Sol. $\int \frac{1}{\sqrt{9+8x-x^2}}\ dx = \int \frac{1}{\sqrt{-\{x^2-8x-9\}}}\ dx$

$$= \int \frac{1}{\sqrt{-\{x^2-8x+16-25\}}}\ dx$$

$$= \int \frac{1}{-\{(x-4)^2-5^2\}}\ dx = \int \frac{1}{\sqrt{5^2-(x-4)^2}}\ dx$$

$$= \sin^{-1}\left(\frac{x-4}{5}\right) + C$$

Type-2 Integration of type

$$\int \frac{px+q}{ax^2+bx+c}\ dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}\ dx,$$

$$\int (px+q)\sqrt{ax^2+bx+c}\ dx$$

Express $px + q = A$ (differential co–efficient of denominator) + B.

TRAIN YOUR BRAIN

Example 50: Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\ dx$

Sol. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\ dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\ dx$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+1}}\ dx - \int \frac{1}{\sqrt{x^2+4x+1}}\ dx$$

$$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}}\ dx,$$

where $t = (x^2 + 4x + 1)$ for Ist integral

$$= 2\sqrt{t} - \ell n\,|(x+2)+\sqrt{x^2+4x+1}\,| + C$$

$$= 2\sqrt{x^2+4x+1} - \ell n\,|x+2+\sqrt{x^2+4x+1}\,| + C$$

Example 51: Evaluate : $\int (x-5)\sqrt{x^2+x}\ dx$

Sol. Let $(x-5) = \lambda \cdot \frac{d}{dx}(x^2+x) + \mu$. Then,

$x - 5 = \lambda(2x + 1) + \mu$.

Comparing coefficients of like powers of x, we get

$$1 = 2\lambda \text{ and } \lambda + \mu = -5 \Rightarrow \lambda = \frac{1}{2} \text{ and } \mu = -\frac{11}{2}$$

Hence, $\int (x-5)\sqrt{x^2+x}\ dx$

$$= \int \left(\frac{1}{2}(2x+1) - \frac{11}{2}\right)\sqrt{x^2+x}\ dx$$

$$= \int \frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx \quad \text{(where } t = x^2+x \text{ for first integral)}$$

$$= \frac{1}{2}\cdot\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.\right.$$

$$\left.\left. - \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 \ell n\left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}\right] + C$$

Type-3 Integration of trigonometric functions

(i) $\int \frac{dx}{a+b\sin^2 x}$ **OR** $\int \frac{dx}{a+b\cos^2 x}$

OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put $\tan x = t$.

(ii) $\int \frac{dx}{a+b\sin x}$ **OR** $\int \frac{dx}{a+b\cos x}$

OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}\,dx.$

Express Nr $\equiv A(Dr) + B\frac{d}{dx}(Dr) + C$ & proceed.

TRAIN YOUR BRAIN

Example 52: Evaluate : $\int \frac{1}{1+\sin x+\cos x}\,dx$

Sol. $I = \int \frac{1}{1+\sin x+\cos x}\,dx$

$$= \int \frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx$$

$$= \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}\,dx$$

$$= \int \frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$, we get

$$I = \int \frac{1}{t+1}\,dt = \ell n\,|t+1| + C = \ell n\left|\tan\frac{x}{2}+1\right| + C$$

Example 53: Evaluate : $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

SoL. $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Let $3\sin x + 2\cos x = \lambda.(3\cos x + 2\sin x) + \mu\frac{d}{dx}(3\cos x + 2\sin x)$

$\Rightarrow 3\sin x + 2\cos x = \lambda(3\cos x + 2\sin x) + \mu(-3\sin x + 2\cos x)$

Comparing the coefficients of sin x and cos x on both sides, we get

$$\lambda = \frac{12}{13} \text{ and } \mu = -\frac{5}{13}$$

$$\therefore I = \int \frac{\lambda(3\cos x + 2\sin x) + \mu(-3\sin x + 2\cos x)}{3\cos x + \sin x}\,dx$$

$$= \lambda\int 1.\,dx + \mu\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$$

$$= \lambda x + \mu\int \frac{dt}{t}, \text{ where } t = 3\cos x + 2\sin x$$

$$= \lambda x + \mu\ell n\,|t| + C = \frac{12}{13}x - \frac{5}{13}\ell n\,|3\cos x + 2\sin x| + C$$

Type-4 Integration of type

$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1}\,dx$ where K is any constant.

Divide Nr & Dr by x^2 & put $x \mp \frac{1}{x} = t$.

TRAIN YOUR BRAIN

Example 54: Evaluate : $\int \frac{1-x^2}{1+x^2+x^4}\,dx$

Sol. Let $I = \int \frac{1-x^2}{1+x^2+x^4}\,dx = -\int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$

$\{\text{put } x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt\}$

$\therefore I = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\ell n\left|\frac{t-1}{t+1}\right| + C$

$= -\frac{1}{2}\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

TRAIN YOUR BRAIN

Example 55: Evaluate $\int \frac{1}{(e^x-1)^2}dx$.

Sol. We have $\int \frac{1}{(e^x-1)^2}\,dx + \int \frac{e^x}{e^x(e^x-1)^2}dx$,

[multiplying the Nr. and Dr. by e^x]

$= \int \frac{dt}{t(t-1)^2}$, putting $e^x = t$ so that $e^x\,dx = dt$.

Now $\frac{1}{t(t-1)} \equiv \frac{A}{t}+\frac{B}{t-1}+\frac{C}{(t-1)^2} \Rightarrow 1 \equiv A(t-1)^2 +$

$Bt(t-1)+Ct$ ....(1) (on resolving into partial fractions)

To find A, putting $t = 0$ on both sides of (1), we get $A = 1$.

To find C, put $t = 1$ and we get $C = 1$. Thus $1 \equiv (t-1)^2 + Bt(t-1) + t$

Comparing the coefficients of t^2 on both sides, we get

$0 = 1 + B$ or $B = -1$

$\therefore \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}$

Hence $\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2} =$

$\log t - \log(t-1) - \{1/(t-1)\} + C$

$= \log e^x - \log(e^x-1) - \{1/(e^x-1)\} + c = x - \log(e^x-1) - \{1/(e^x-1)\} + c$

INTEGRATION OF IRRATIONAL FUNCTIONS

Certain types of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by a appropriate change of the variable. Such transformation of an integral is called its rationalization.

(i) If the integrand is a rational function of fractional powers of an independent variable x, i.e. the function R $\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x = t^m$, where m is the least common multiple of the numbers $q_1, q_2, \ldots, q_k$.

(ii) If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d} = t^m$ where m has the same sense as above.

Some important substitutions

1. $\int \frac{dx}{(ax+b)\sqrt{px+q}}$; Put $px+q = t^2$
2. $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$; Put $ax+b = \frac{1}{t}$
3. $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; Put $px+q = t^2$
4. $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ then put $x = \frac{1}{t}$

or the trigonomtric substitutions are also helpful.

TRAIN YOUR BRAIN

Example 56: Evaluate : $\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

Sol. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}}\,dx$ {Put $x+1=t^2$

$\Rightarrow dx = 2t\,dt$}

$\therefore I = \int \frac{1}{(t^2-1-3)}\frac{2t}{\sqrt{t^2}}\,dt$

$\Rightarrow I = 2\int \frac{dt}{t^2-2^2} = 2\cdot\frac{1}{2(2)}\,\ell n\left|\frac{t-2}{t+2}\right| + C$

$\Rightarrow I = \frac{1}{2}\,\ell n\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C.$

Example 57: Evaluate : $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Sol. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Putting $x+1=t^2$, and $dx = 2t\,dt$, we get

$I = \int \frac{(t^2+1)\,2t\,dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$

$\Rightarrow 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$ {put $t-\frac{1}{t}=u$}

$= 2\int \frac{du}{u^2+\left(\sqrt{3}\right)^{2'}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\}+C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\}+C$

Example 58: Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$.

Then, $2x-3 = (x^2+1)^2\cdot A + (Bx+c)(x-1)$ .....(i)

$(x^2+1)+(Dx+E)\cdot(x-1)$

Putting $x=1$ in (i), we get $-1 = A(1+1)^2 \Rightarrow A = -\frac{1}{4}$

Comparing coefficients of like powers of x on both side of (i), we have

$A+B=0$, $C-B=0$, $2A+B-C+D=0$,

$C+E-B-D=2$ and $A-C-E=-3$.

Putting $A=-\frac{1}{4}$ and solving these equations,

we get $B=\frac{1}{4}=C$, $D=\frac{1}{2}$ and $E=\frac{5}{2}$

$\therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)}+\frac{x+1}{4(x^2+1)}+\frac{x+5}{2(x^2+1)^2}$

Example 59: Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$. Then,

$2x = A(x^2+x+1)+(Bx+C)(x-1)$ .....(i)

Putting $x-1=0$ or, $x=1$ in (i), we get $2=3A$

$\Rightarrow A = \frac{2}{3}$

Putting $x=0$ in (i), we get $A-C=0 \Rightarrow C=A=\frac{2}{3}$

Putting $x=-1$ in (i), we get $-2=A+2B-2C$.

$\Rightarrow -2 = \frac{2}{3}+2B-\frac{4}{3} \Rightarrow B=-\frac{2}{3}$

$\therefore \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1}+\frac{(-2/3)x+2/3}{x^2+x+1}$ or $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1}+\frac{2}{3}\frac{1-x}{x^2+x+1}$

INTEGRATION OF A BINOMIAL DIFFERENTIAL

The integral $\int x^m(a+bx^n)^p\,dx$, where m, n, p are rational numbers, is expressed through elementary functions only in the following three cases :

Case I : p is an integer. Then, if $p>0$, the integrand is expanded by the formula of the binomial; but if $p<0$, then we put $x=t^k$, where k is the common denominator of the fractions and n.

Case II : $\frac{m+1}{n}$ is an integer. We put $a+bx^n=t^\alpha$, where α is the denominator of the fraction p.

Case III : $\frac{m+1}{n}+p$ is an integer we put $a+bx^n=t^\alpha x^n$, where a is the denominator of the fraction p.

TRAIN YOUR BRAIN

Example 60: Evaluate $I = \int \sqrt[3]{x}\,(2+\sqrt{x})^2\,dx$.

Sol. $I = \int x^{\frac{1}{3}}(2+x^{\frac{1}{2}})^2\,dx$. Here p = 2, i.e. an integer, hence we have case I.

$$I = \int x^{\frac{1}{3}}(x+4x^{\frac{1}{2}}+4)\,dx$$

$$= \int (x^{\frac{4}{3}}+4x^{\frac{5}{6}}+4x^{\frac{1}{3}})\,dx$$

$$= \frac{3}{7}x^{\frac{7}{3}}+\frac{24}{11}x^{\frac{11}{6}}+3x^{\frac{4}{3}}+C.$$

Example 61: Evaluate $I = \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$.

Sol. $I = \int x^{-\frac{2}{3}}(1+x^{\frac{1}{3}})^{\frac{1}{2}}dx$. Here $m = -\frac{2}{3}; n = \frac{1}{3}; p = \frac{1}{2}$;

$$\frac{m+1}{n} = \frac{\left(-\frac{2}{3}+1\right)}{\frac{1}{3}} = 1, \text{ i.e. an integer.}$$

we have case II. Let us make the substitution. Hence,

$$I = 6\int t^2 dt = 2t^3 + C = 2(1+x^{\frac{1}{3}})^{\frac{3}{2}} + C.$$

REDUCTION FORMULA

Reduction formula of $\int \tan^n x\,dx$, $\int \cot^n x\,dx$, $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$

1. $I_n = \int \tan^n x\,dx = \int \tan^2 x \tan^{n-2} x\,dx$

$$= \int (\sec^2 x - 1)\tan^{n-2} x\,dx$$

$$\Rightarrow I_n = \int \sec^2 x \tan^{n-2} x\,dx - I_{n-2}$$

$$\Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2},\ n \geq 2$$

2. $I_n = \int \cot^n x\,dx = \int \cot^2 x \cdot \cot^{n-2} x\,dx$

$$= \int (\operatorname{cosec}^2 x - 1)\cot^{n-2} x\,dx$$

$$\Rightarrow I_n = -\int \operatorname{cosec}^2 x \cot^{n-2} x\,dx - I_{n-2}$$

$$\Rightarrow I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2},\ n \geq 2$$

3. $I_n = \int \sec^n x\,dx = \int \sec^2 x \sec^{n-2} x\,dx$

$$\Rightarrow I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2)\sec^{n-1} x \cdot \sec x \tan x\,dx.$$

$$\Rightarrow I_n = \tan x \sec^{n-2} x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\,dx$$

$$\Rightarrow (n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

4. $I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^2 x \operatorname{cosec}^{n-2} x\,dx$

$$\Rightarrow I_n = -\cot x \operatorname{cosec}^{n-2} x + \int (\cot x)(n-2)(-\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot x)\,dx$$

$$\Rightarrow -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \cot^2 x \operatorname{cosec}^{n-2} x\,dx$$

$$\Rightarrow I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int (\operatorname{cosec}^2 x - 1)\operatorname{cosec}^{n-2} x\,dx$$

$$\Rightarrow (n-1) I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{\cot x \operatorname{cosec}^{n-2} x}{-(n-1)} + \frac{n-2}{n-1} I_{n-2}$$

TRAIN YOUR BRAIN

Example 62: Obtain reducation formula for $I_n = \int \sin^n x\,dx$.

Hence evaluate $\int \sin^4 x\,dx$

Sol. $I_n = \int (\sin x)(\sin x)^{n-1}\,dx$

$$= -\cos x (\sin x)^{n-1} + (n-1)\int (\sin x)^{n-2}\cos^2 x\,dx$$

$$= -\cos x (\sin x)^{n-1} + (n-1)\int (\sin x)^{n-2}(1-\sin^2 x)\,dx$$

$$I_n = -\cos x (\sin x)^{n-1} + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n = -\frac{\cos x (\sin x)^{n-1}}{n} + \frac{(n-1)}{n} I_{n-2}\ (n \geq 2)$$

Hence $I_4 = -\frac{\cos x (\sin x)^3}{4} + \frac{3}{4}\left(-\frac{\cos x (\sin x)}{2} + \frac{1}{2}x\right) + C$

Topicwise Questions

Standard Integrals

1. $\int \left(x+\frac{1}{x}\right)^3 dx =$

(a) $\frac{1}{4}\left(x+\frac{1}{x}\right)^4 + c$

(b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

(c) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x+\frac{1}{x^2}+c$

(d) $\frac{x^4}{4}+\frac{3x^2}{2}-3\log x-\frac{1}{2x^2}+c$

2. $\int \sqrt{1+\sin x}\, dx =$

(a) $\frac{1}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+c$ (b) $\frac{1}{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$

(c) $2\sqrt{1+\sin x}+c$ (d) $-2\sqrt{1-\sin x}+c$

Integration by substitution

3. $\int \frac{dx}{1+e^x} =$

(a) $\log(1+e^x)+C$ (b) $-\log(1+e^{-x})+C$

(c) $-\log(1-e^{-x})+C$ (d) $\log(e^{-x}+e^{-2x})+C$

4. $\int \frac{dx}{x+x\log x} =$

(a) $\log(1+\log x)+C$ (b) $\log\log(1+\log x)+C$

(c) $\log x+\log(\log x)+C$ (d) $\log(1-\log x)+C$

5. $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx =$

(a) $\log(1+x^2)+c$ (b) $\log e^{\tan^{-1}x}+c$

(c) $e^{\tan^{-1}x}+c$ (d) $\tan^{-1}e^{\tan^{-1}x}+c$

6. $\int \frac{1}{\cos^2 x(1-\tan x)^2}dx =$

(a) $\frac{1}{\tan x-1}+c$ (b) $\frac{1}{1-\tan x}+c$

(c) $-\frac{1}{3}\frac{1}{(1-\tan x)^3}+c$ (d) $\frac{1}{1+\tan x}+c$

7. $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\, dx =$

(a) $-\frac{1}{2}\cos\sqrt{x}+c$ (b) $-2\cos\sqrt{x}+c$

(c) $\frac{1}{2}\cos\sqrt{x}+c$ (d) $2\cos\sqrt{x}+c$

8. $\int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x+4}} =$

(a) $\log\left[\tan x+\sqrt{\tan^2 x+4}\right]+c$

(b) $\frac{1}{2}\log\left[\tan x+\sqrt{\tan^2 x+4}\right]+c$

(c) $\log\left[\frac{1}{2}\tan x+\frac{1}{2}\sqrt{\tan^2 x+4}\right]+c$

(d) $\frac{1}{2}\log\left[\tan x+\frac{1}{2}\sqrt{\tan^2 x+4}\right]+c$

9. $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}$ dx is equal to

(a) $\frac{a^{\sqrt{x}}}{\sqrt{x}}+c$ (b) $\frac{2a^{\sqrt{x}}}{\log a}+c$

(c) $2a^{\sqrt{x}}.\ell n\, a+c$ (d) None of these

10. $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$ is equal to

(a) $2\sqrt{\tan x}+c$ (b) $2\sqrt{\cot x}+c$

(c) $\frac{\sqrt{\tan x}}{2}+c$ (d) None of these

Integration by parts

11. $\int x\tan^{-1}x\,dx =$

(a) $\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x+c$

(b) $\frac{1}{2}(x^2-1)\tan^{-1}x-\frac{1}{2}x+c$

(c) $\frac{1}{2}(x^2+1)\tan^{-1}x+\frac{1}{2}x+c$

(d) $\frac{1}{2}(x^2+1)\tan^{-1}x-x+c$

12. $\int(x-1)e^{-x}\,dx$ is equal to

(a) $-xe^{x}+C$ (b) $xe^{x}+C$

(c) $-xe^{-x}+C$ (d) $xe^{-x}+C$

13. $\int e^{2x}(-\sin x+2\cos x)\,dx =$

(a) $e^{2x}\sin x+c$ (b) $-e^{2x}\sin x+c$

(c) $-e^{2x}\cos x+c$ (d) $e^{2x}\cos x+c$

14. The value of $\int\frac{\log x}{(x+1)^2}dx$ is

(a) $\frac{-\log x}{x+1}+\log x-\log(x+1)+C$

(b) $\frac{\log x}{(x+1)}+\log x-\log(x+1)+C$

(c) $\frac{\log x}{x+1}-\log x-\log(x+1)+C$

(d) $\frac{-\log x}{x+1}-\log x-\log(x+1)+C$

15. $\int e^{x}(1-\cot x+\cot^2 x)\,dx$ equals

(a) $e^{x}\cot x+c$ (b) $e^{x}\operatorname{cosec} x+c$

(c) $-e^{x}\cot x+c$ (d) $-e^{x}\operatorname{cosec} x+c$

16. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to

(a) $xe^{\tan^{-1}x}+c$ (b) $x^2e^{\tan^{-1}x}+c$

(c) $\frac{1}{x}e^{\tan^{-1}x}+c$ (d) $\frac{1}{x^2}e^{\tan^{-1}x}+c$

Partial fraction and Integration of Rational function

17. $\int\frac{dx}{(x-x^2)}=$

(a) $\log x-\log(1-x)+c$ (b) $\log(1-x^2)+c$

(c) $-\log x+\log(1-x)+c$ (d) $\log(x-x^2)+c$

18. $\int\frac{dx}{x^2+x+1}$ is equal to

(a) $\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

(b) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

(c) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

(d) None of these

19. $\int\frac{dx}{1+x+x^2+x^3}=$

(a) $\log\sqrt{1+x}-\frac{1}{2}\log\sqrt{1+x^2}+\frac{1}{2}\tan^{-1}x+c$

(b) $\log\sqrt{1+x}-\log\sqrt{1+x^2}+\tan^{-1}x+c$

(c) $\log\sqrt{1+x^2}-\log\sqrt{1+x}+\frac{1}{2}\tan^{-1}x+c$

(d) $\log\sqrt{1+x}+\tan^{-1}x+\log\sqrt{1+x^2}+c$

20. Correct evaluation of $\int \frac{x}{(x-2)(x-1)}dx$ is

(where p is an arbitrary constant)

(a) $\log_e \frac{(x-2)^2}{(x-1)} + p$ (b) $\log_e \frac{(x-1)}{(x-2)} + p$

(c) $\frac{x-1}{x-2} + p$ (d) $2\log_e\left(\frac{x-2}{x-1}\right) + p$

21. $\int \frac{1}{(x-1)(x^2+1)}dx =$

(a) $\frac{1}{2}\log(x-1) - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$

(b) $\frac{1}{2}\log(x-1) + \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$

(c) $\frac{1}{2}\log(x-1) - \frac{1}{2}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$

(d) $\frac{1}{2}\log(x+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c$

22. $\int \frac{x^2+x-1}{x^2+x-6}dx =$

(a) $x + \log(x+3) + \log(x-2) + c$

(b) $x - \log(x+3) + \log(x-2) + c$

(c) $x - \log(x+3) - \log(x-2) + c$

(d) $x + \log(x+3) - \log(x-2) + c$

23. $\int \frac{dx}{(x^2+1)(x^2+4)} =$

(a) $\frac{1}{3}\tan^{-1}x - \frac{1}{3}\tan^{-1}\frac{x}{2} + c$

(b) $\frac{1}{3}\tan^{-1}x + \frac{1}{3}\tan^{-1}\frac{x}{2} + c$

(c) $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + c$

(d) $\tan^{-1}x - 2\tan^{-1}\frac{x}{2} + c$

24. $\int \frac{1}{x-x^3}dx =$

(a) $\frac{1}{2}\log\frac{(1-x^2)}{x^2} + c$ (b) $\log\frac{(1-x)}{x(1+x)} + c$

(c) $\log x(1-x^2) + c$ (d) $\frac{1}{2}\log\frac{x^2}{(1-x^2)} + c$

Integration of irrational function

25. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} =$

(a) $\frac{2}{3(b-a)}[(x+a)^{3/2} - (x+b)^{3/2}] + c$

(b) $\frac{2}{3(a-b)}[(x+a)^{3/2} - (x+b)^{3/2}] + c$

(c) $\frac{2}{3(a-b)}[(x+a)^{3/2} + (x+b)^{3/2}] + c$

(d) $\frac{2}{3(b-a)}[(x+a)^{3/2} + (x+b)^{3/2}] + c$

Integration of Trigonometric function

26. $\int \frac{1}{\cos x(1+\cos x)}dx =$

(a) $\log(\sec x + \tan x) + 2\tan\frac{x}{2} + c$

(b) $\log(\sec x + \tan x) - 2\tan\frac{x}{2} + c$

(c) $\log(\sec x + \tan x) + \tan\frac{x}{2} + c$

(d) $\log(\sec x + \tan x) - \tan\frac{x}{2} + c$

27. $\int \sin^3 x \cos^2 x\, dx =$

(a) $\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$ (b) $\frac{\cos^5 x}{5} + \frac{\cos^3 x}{3} + c$

(c) $\frac{\sin^5 x}{5} - \frac{\sin^3 x}{3} + c$ (d) $\frac{\sin^5 x}{5} - \frac{\sin^3 x}{3} + c$

28. $\int \sin 2x \cos 3x \, dx =$

(a) $\frac{1}{2}\left(\cos x + \frac{1}{5}\cos 5x\right) + c$

(b) $\frac{1}{2}\left(\cos x - \frac{1}{5}\cos 5x\right) + c$

(c) $\cos x + \frac{1}{5}\cos 5x + c$

(d) $\cos x - \frac{1}{5}\cos 5x + c$

29. $\int \frac{e^x}{(1+e^x)(2+e^x)} dx =$

(a) $\log[(1+e^x)(2+e^x)] + c$

(b) $\log\left[\frac{1+e^x}{2+e^x}\right] + c$

(c) $\log[(1+e^x)\sqrt{2+e^x}] + c$

(d) $\log\left[\frac{(1+e^x)}{\sqrt{2+e^x}}\right] + c$

30. The value of $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

(a) $\frac{-1}{\sin x + \cos x} + C$

(b) $\ell n |\sin x + \cos x| + C$

(c) $\ell n |\sin x - \cos x| + C$

(d) $\ell n (\sin x + \cos x)^2 + C$

31. $\int \frac{dx}{5 + 4\cos x} =$

(a) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right) + c$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right) + c$

(c) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$ (d) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$

32. $\int \sin\sqrt{x} \, dx =$

(a) $2[\sin\sqrt{x} - \cos\sqrt{x}] + c$

(b) $2[\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}] + c$

(c) $2[\sin\sqrt{x} + \cos\sqrt{x}] + c$

(d) $2[\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x}] + c$

33. $\int \sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x \, dx$ is equal to

(a) $\frac{\sin 16x}{1024} + c$ (b) $-\frac{\cos 32x}{1024} + c$

(c) $\frac{\cos 32x}{1096} + c$ (d) $-\frac{\cos 32x}{1096} + c$

Special integrals

34. $\int \frac{x^2+1}{x^4+1} dx =$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{2x}\right) + c$

(b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2x}}\right) + c$

(3) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{2\sqrt{x}}\right) + c$

(d) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c$

35. $\int \frac{x^2-1}{x^4+x^2+1} dx =$

(a) $\frac{1}{2}\log\left(\frac{x^2+x+1}{x^2-x+1}\right) + c$ (b) $\frac{1}{2}\log\left(\frac{x^2-x-1}{x^2+x+1}\right) + c$

(c) $\log\left(\frac{x^2-x+1}{x^2+x+1}\right) + c$ (d) $\frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right) + c$

1. If $\int \frac{1}{1+\sin x}\,dx = \tan\left(\frac{x}{2}+a\right)+b$, then

(a) $a=-\frac{\pi}{4}, b\in R$ (b) $a=\frac{\pi}{4}, b\in R$

(c) $a=\frac{5\pi}{4}, b\in R$ (d) none of these

2. The value of $\int \frac{\cos 2x}{\cos x}\,dx$ is equal to

(a) $2\sin x - \ell n|\sec x+\tan x|+C$
(b) $2\sin x - \ell n|\sec x-\tan x|+C$
(c) $2\sin x + \ell n|\sec x+\tan x|+C$
(d) None of these

3. $\int[f(x)g''(x)-f''(x)g(x)]\,dx$ is equal to

(a) $\frac{f(x)}{g'(x)}+c$

(b) $f'(x)\,g(x)-f(x)\,g'(x)+c$

(c) $f(x)\,g'(x)-f'(x)\,g(x)+c$
(d) $f(x)\,g'(x)+f'(x)\,g'(x)+c$

4. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ is equal to

(a) $\frac{1}{2}\sin 2x+c$ (b) $-\frac{1}{2}\sin 2x+c$

(c) $-\frac{1}{2}\sin x+c$ (d) $-\sin^2 x+c$

5. If $\int \frac{2^x}{\sqrt{1-4^x}}\,dx = K\sin^{-1}(2^x)+C$, then K is equal to

(a) $\ln 2$ (b) $\frac{1}{2}\ln 2$

(c) $\frac{1}{2}$ (d) $\frac{1}{\ell n 2}$

6. $\int \tan^3 2x\sec 2x\,dx$ is equal to

(a) $\frac{1}{3}\sec^3 2x-\frac{1}{2}\sec 2x+c$

(b) $-\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+c$

(c) $\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+c$

(d) $\frac{1}{3}\sec^3 2x+\frac{1}{2}\sec 2x+c$

7. $\int\sqrt{\frac{e^x-1}{e^x+1}}\,dx$ is equal to

(a) $\ell n\,(e^x+\sqrt{e^{2x}-1})-\sec^{-1}(e^x)+C$

(b) $\ell n\,(e^x+\sqrt{e^{2x}-1})+\sec^{-1}(e^x)+C$

(c) $\ell n\,(e^x-\sqrt{e^{2x}-1})-\sec^{-1}(e^x)+C$

(d) None of these

8. $\int\sqrt{\sec x-1}\,dx$ is equal to

(a) $2\,\ell n\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(b) $2\,\ell n\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(c) $-2\,\ell n\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(d) None of these

9. The value of $\int\frac{dx}{x\sqrt{1-x^3}}$ is equal to

(a) $\frac{1}{3}\ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C$

(b) $\frac{1}{3}\ell n\left|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right|+C$

(c) $\frac{1}{3}\ell n\left|\frac{1}{\sqrt{1-x^3}}\right|+C$

(d) $\frac{1}{3}\ell n|1-x^3|+C$

10. Antiderivative of $\frac{\sin^2 x}{1+\sin^2 x}$ w.r.t. x is :

(*a*) $x - \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\tan x) + c$

(*b*) $x + \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + c$

(*c*) $x - \sqrt{2} \arctan(\sqrt{2}\tan x) + c$

(*d*) $x - \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + c$

11. $\int\left\{\frac{(\log x - 1)}{1+(\log x)^2}\right\}^2 dx$ is equal to

(*a*) $\frac{x}{(\log x)^2+1} + c$ (*b*) $\frac{xe^x}{1+x^2} + c$

(*c*) $\frac{x}{x^2+1} + c$ (*d*) $\frac{\log x}{(\log x)^2+1} + c$

12. If $\int \frac{\sin x}{\sin(x-a)} dx = Ax + B\log\sin(x-a) + c$, then value of (A, B) is

(*a*) (sin a, cos a) (*b*) (cos a, sin a)

(*c*) (–sin a, cos a) (*d*) (–cos a, sin a)

13. $\int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}} dx$ is equal to

(*a*) $\frac{2}{3}\sqrt{1+\ell n|x|}(\ell n|x|-2) + c$

(*b*) $\frac{2}{3}\sqrt{1+\ell n|x|}(\ell n|x|+2) + c$

(*c*) $\frac{1}{3}\sqrt{1+\ell n|x|}(\ell n|x|-2) + c$

(*d*) $\frac{1}{3}\sqrt{1+\ell n|x|}(3\ell n|x|+2) + c$

14. $\int \{1+2\tan x(\tan x+\sec x)\}^{1/2} dx$ is equal to

(*a*) $\ell n \sec x(\sec x - \tan x) + c$

(*b*) $\ell n \operatorname{cosec} x(\sec x + \tan x) + c$

(*c*) $\ell n \sec x(\sec x + \tan x) + c$

(*d*) $\ell n(\sec x + \tan x) + c$

15. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}(x + \sqrt{x}) dx$ is equal to

(*a*) $2e^{\sqrt{x}}[\sqrt{x} - x + 1] + c$

(*b*) $2e^{\sqrt{x}}[x - 2\sqrt{x} + 1] + c$

(*c*) $2e^{\sqrt{x}}[x - \sqrt{x} + 1] + c$

(*d*) $2e^{\sqrt{x}}[x + \sqrt{x} + 1] + c$

16. $\int e^{\tan\theta}(\sec\theta - \sin\theta) d\theta$ is equal to

(*a*) $-e^{\tan\theta}\sin\theta + c$ (*b*) $e^{\tan\theta}\sin\theta + c$

(*c*) $e^{\tan\theta}\sec\theta + c$ (*d*) $e^{\tan\theta}\cos\theta + c$

17. If $\int e^{3x}\cos 4x\, dx = e^{3x}(A\sin 4x + B\cos 4x) + C$ then :

(*a*) 4A = 3B (*b*) 2A = 3B

(*c*) 3A = 4B (*d*) None of these

18. $\int \frac{x^3-1}{x^3+x} dx$ is equal to

(*a*) $x - \ell n x + \ell n(x^2+1) - \tan^{-1} x + c$

(*b*) $x - \ell n x + \frac{1}{2}\ell n(x^2+1) - \tan^{-1} x + c$

(*c*) $x + \ell n x + \frac{1}{2}\ell n(x^2+1) + \tan^{-1} x + c$

(*d*) None of these

19. If $y = \int \frac{dx}{(1+x^2)^{3/2}}$ and y = 0 when x = 0, then value of y when x = 1, is:

(*a*) $\sqrt{\frac{2}{3}}$ (*b*) $\sqrt{2}$

(*c*) $3\sqrt{2}$ (*d*) $\frac{1}{\sqrt{2}}$

20. $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$ is equal to

(*a*) $\left(\sqrt{\cos x} + \frac{1}{5}\tan^{5/2} x\right) + C$

(*b*) $\sqrt{2}\left(\sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x\right) + C$

(*c*) $\sqrt{2}\left(\sqrt{\tan x} - \frac{1}{5}\tan^{5/2} x\right) + C$

(*d*) None of these

21. If $\int \frac{\cos 4x+1}{\cot x-\tan x}\,dx = A\cos 4x + B$ where A & B are constants, then

(a) A = –1/4 & B may have any value
(b) A = – 1/8 & B may have any value
(c) A = –1/2 & B = – 1/4
(d) None of these

22. $\int 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}\,dx$ is equal to

(a) $\cos x-\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+c$

(b) $\cos x-\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+c$

(c) $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+c$

(d) $\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+c$

23. $\int\frac{1}{\cos^6 x+\sin^6 x}\,dx$ is equal to

(a) $\tan^{-1}(\tan x+\cot x)+c$
(b) $-\tan^{-1}(\tan x+\cot x)+c$
(c) $\tan^{-1}(\tan x-\cot x)+c$
(d) $-\tan^{-1}(\tan x-\cot x)+c$

24. If $\int\frac{dx}{\sqrt{\sin^3 x\cos^5 x}} = a\sqrt{\cot x}+b\sqrt{\tan^3 x}+c$ where c is an arbitrary constant of integration then the values of 'a' and 'b' are respectively:

(a) $-2\ \&\ \frac{2}{3}$ (b) $2\&-\frac{2}{3}$

(c) $2\ \&\ \frac{2}{3}$ (d) None of these

25. $\int\frac{dx}{\cos x-\sin x}$ is equal to

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{3\pi}{8}\right)\right|+c$

(b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right|+c$

(c) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}-\frac{3\pi}{8}\right)\right|+c$

(d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right|+c$

26. $\int\frac{1}{\sqrt{\sin^3 x\cos x}}\,dx$ is equal to

(a) $\frac{-2}{\sqrt{\tan x}}+c$ (b) $2\sqrt{\tan x}+c$

(c) $\frac{2}{\sqrt{\tan x}}+c$ (d) $-2\sqrt{\tan x}-c$

27. $\int\frac{1}{x(x^n+1)}\,dx$ is equal to

(a) $\frac{1}{n}\ell n\left(\frac{x^n}{x^n+1}\right)+c$ (b) $\frac{1}{n}\ell n\left(\frac{x^n+1}{x^n}\right)+c$

(c) $\ell n\left(\frac{x^n}{x^n+1}\right)+c$ (d) None of these

28. The value of $\int\frac{1}{x^2(x^4+1)^{3/4}}\,dx$ is equal to

(a) $\left(1+\frac{1}{x^4}\right)^{1/4}+C$ (b) $(x^4+1)^{1/4}+C$

(c) $\left(1-\frac{1}{x^4}\right)^{1/4}+C$ (d) $-\left(1+\frac{1}{x^4}\right)^{1/4}+C$

29. $\int\frac{1-x^7}{x(1+x^7)}\,dx$ is equal to

(a) $\ell n x+\frac{2}{7}\ell n(1+x^7)+c$

(b) $\ell n x-\frac{2}{7}\ell n(1-x^7)+c$

(c) $\ell n x-\frac{2}{7}\ell n(1+x^7)+c$

(d) $\ell n x+\frac{2}{7}\ell n(1-x^7)+c$

30. Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is

(a) $\frac{x}{x^4+x+1}+c$ (b) $-\frac{x}{x^4+x+1}+c$

(c) $\frac{x+1}{x^4+x+1}+c$ (d) $-\frac{x+1}{x^4+x+1}+c$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. The value of $\int 2^{mx}.3^{nx}\,dx$ (when $m, n \in N$) is equal to:

(a) $\dfrac{2^{mx}+3^{nx}}{m\ell n2+n\ell n3}+C$

(b) $\dfrac{e^{(m\ell n2+n\ell n3)x}}{m\ell n2+n\ell n3}+C$

(c) $\dfrac{2^{mx}.3^{nx}}{\ell n\left(2^m.3^n\right)}+C$

(d) $\dfrac{(mn).2^x.3^x}{m\ell n2+n\ell n3}+C$

2. The value of $\int \dfrac{x^2+\cos^2 x}{1+x^2}\,\text{cosec}^2 x\,dx$ is equal to:

(a) $\cot x-\cot^{-1}x+C$

(b) $-\cot x+\cot^{-1}x+C$

(c) $-\tan^{-1}x-\dfrac{\text{cosec}\,x}{\sec x}+C$

(d) $-e^{\ell n\,\tan^{-1}x}-\cot x+C$

3. The value of $\int \dfrac{\ell n\left(\frac{x-1}{x+1}\right)}{x^2-1}\,dx$ is equal to

(a) $\dfrac{1}{2}\ell n^2\dfrac{x-1}{x+1}+C$ (b) $\dfrac{1}{4}\ell n^2\dfrac{x-1}{x+1}+C$

(c) $\dfrac{1}{2}\ell n^2\dfrac{x+1}{x-1}+C$ (d) $\dfrac{1}{4}\ell n^2\dfrac{x+1}{x-1}+C$

4. The value of $\int \dfrac{\ell n\,(\tan x)}{\sin x\,\cos x}\,dx$ is equal to

(a) $\dfrac{1}{2}\ell n^2(\cot x)+C$

(b) $\dfrac{1}{2}\ell n^2(\sec x)+C$

(c) $\dfrac{1}{2}\ell n^2(\sin x\sec x)+C$

(d) $\dfrac{1}{2}\ell n^2(\cos x\,\text{cosec}\,x)+C$

5. The value of the integral $\int \dfrac{\ell n(x+1)-\ell n\,x}{x(x+1)}\,dx$ is

(a) $-\dfrac{1}{2}[\ell n\,(x+1)]^2-\dfrac{1}{2}(\ell n\,x)^2+\ell n(x+1)\,\ell n\,x+c$

(b) $-[\{\ell n\,(x+1)\}]^2-(\ell n\,x)^2]+\ell n(x+1).\ell n\,x+c$

(c) $-\dfrac{1}{2}[\ell n\,(1+1/x)]^2+c$

(d) None of these

6. If $\int e^{3x}\cos 4x\,dx=e^{3x}(A\sin 4x+B\cos 4x)+c$ then

(a) 4A=3B (b) 2A=3B

(c) 3A=4B (d) 4A+3B=1

7. The value of $\int \dfrac{dx}{\sqrt{x-x^2}}$ is equal to

(a) $2\sin^{-1}\sqrt{x}+C$ (b) $\sin^{-1}(2x-1)+C$

(c) $-2\cos^{-1}(2x-1)+C$ (d) $\cos^{-1}2\sqrt{x-x^2}+C$

8. $\int \dfrac{(x-1)\,dx}{x^2\sqrt{2x^2-2x+1}}$ is equal to $\dfrac{\sqrt{f(x)}}{g(x)}+c$ ther.

(a) $(x)=2x^2-2x+1$ (b) $g(x)=x+1$

(c) $g(x)=x$ (d) $f(x)=\sqrt{2x^2-2x}$

9. If $\int \dfrac{dx}{5+4\cos x}=I\tan^{-1}\left(m\tan\dfrac{x}{2}\right)+C$ then :

(a) I=2/3 (b) m=1/3

(c) I=1/3 (d) m=2/3

10. The value of $\int \dfrac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$ is equal to :

(a) $\cot^{-1}(\cot^2 x)+C$ (b) $-\cot^{-1}(\tan^2 x)+C$

(c) $\tan^{-1}(\tan^2 x)+C$ (d) $-\tan^{-1}(\cos 2x)+C$

11. Let $f'(x)=3x^2\sin\dfrac{1}{x}-x\cos\dfrac{1}{x}$, if $x\neq 0$; $f(0)=0$ and $f(1/\pi)=0$ then :

(a) f(x) is continuous at x = 0

(b) f(x) is non derivable at x = 0

(c) f'(x) is continuous at x = 0

(d) f'(x) is non derivable at x = 0

12. If $I_n = \int \cot^n x\, dx$ and $I_0 + I_1 + 2(I_2 + \ldots + I_8) + I_9 + I_{10} = A\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right) + C$, where $u = \cot x$ and C is an arbitrary constant, then

(a) A is constant (b) $A = -1$

(c) $A = 1$ (d) A is dependent on

Comprehension – 1 (No. 13 to 15)

A curve is represented parametrically by the equations $x = e^t \cos t$ and $y = e^t \sin t$ where t is a parameter. Then

13. The relation between the parameter 't' and the angle α between the tangent to the given curve and the x-axis is given by, 't' equals

(a) $\frac{\pi}{2} - \alpha$ (b) $\frac{\pi}{4} + \alpha$

(c) $\alpha - \frac{\pi}{4}$ (d) $\frac{\pi}{4} - \alpha$

14. The value of $\frac{d^2y}{dx^2}$ at the point where $t = 0$ is

(a) 1 (b) 2

(c) -2 (d) 3

15. If $F(t) = \int (x + y)dt$ then the value of $F\left(\frac{\pi}{2}\right) - F(0)$ is

(a) 1 (b) -1

(c) $e^{\pi/2}$ (d) 0

Comprehension – 2 (No. 16 to 18)

Let $f(x)$ be a polynomial function of degree 2 satisfying

$$\int \frac{f(x)}{x^3 - 1} dx = \ln\left|\frac{x^2 + x + 1}{x - 1}\right| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C,$$

where C is indefinite integration constant.

16. The value of $f(1)$ is equal to

(a) 1 (b) 2

(c) -1 (d) -3

17. Let $\int \frac{1 - 6\,\text{cosec}\, x}{6 + f(\sin x)} d(\sin x) = g(x) + K$, where $g(x)$ contains no constant term. Then $\lim_{t \to \frac{\pi}{2}} g(t)$ is equal to

(where K is indefinite integration constant.)

(a) $\ln 1$ (b) $\ln 2$

(c) $\ln 3$ (d) $\ln 4$

18. Let $\int \frac{5 + f(\sin x) + f(\cos x)}{\sin x + \cos x} dx = h(x) + \lambda$, where $h(1) = -1$. The value of $\tan^{-1}(h(2)) + \tan^{-1}(h(3))$ is equal to (where λ is indefinite integration constant.)

(a) $\frac{\pi}{4}$ (b) $\frac{-\pi}{4}$

(c) $\frac{3\pi}{4}$ (d) $\frac{-3\pi}{4}$

19. **Match the column:**

Column – I

(a) If $F(x) = \int \frac{x + \sin x}{1 + \cos x} dx$ and $F(0) = 0$, then the value of $F(\pi/2)$ is

(b) Let $F(x) = \int e^{\sin^{-1} x}\left(1 - \frac{x}{\sqrt{1 - x^2}}\right) dx$ and $F(0) = 1$, If $F(1/2) = \frac{k\sqrt{3}\, e^{\pi/6}}{\pi}$, then the value of k is

(c) Let $F(x) = \int \frac{dx}{(x^2 + 1)(x^2 + 9)}$ and $F(0) = 0$, if $F(\sqrt{3}) = \frac{5}{36} k$, then the value of k is

(d) Let $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$ and $F(0) = 0$ if $F(\pi/4) = \frac{2k}{\pi}$, then the value of k is

Column – II

(p) $\frac{\pi}{2}$ (q) $\frac{\pi}{3}$

(r) $\frac{\pi}{4}$ (s) π

20. If $I = \int \frac{dx}{a + b\cos x}$, where $a, b > 0$ and $a + b = u$, $a - b = v$, then match the following column

Column – I	**Column – II**
(a) $v = 0$	(p) $I = \frac{1}{\sqrt{uv}} \ell n \left\lvert \frac{\sqrt{u} + \sqrt{v}\, \tan\frac{x}{2}}{\sqrt{u} - \sqrt{v}\, \tan\frac{x}{2}} \right\rvert + C$
(b) $v > 0$	(q) $I = \frac{2}{\sqrt{uv}} \tan^{-1}\left(\sqrt{\frac{v}{u}} \tan\frac{x}{2}\right) + C$
(c) $v < 0$	(r) $I = \frac{1}{\sqrt{-uv}} \ell n \left\lvert \frac{\sqrt{u} + \sqrt{-v}\, \tan\frac{x}{2}}{\sqrt{u} - \sqrt{-v}\, \tan\frac{x}{2}} \right\rvert + C$
	(s) $\frac{2}{u} \tan\frac{x}{2} + C$

NUMERICAL VALUE BASED

21. If $\int \frac{(3x^2+2x)\,dx}{x^n+2x^5+x^4+2x^3+2x^2+5} = \frac{1}{A}\tan^{-1}\left(\frac{x^n+x^2+C}{D}\right)+K$

then the value of A + B + C + D is?

22. If $\int \cos^4 x\,dx = Ax + B\sin 2x + C\sin 4x + D$, then the value of 8A + 4B + 32 C is?

23. If $\int x^{13/2}(1+x^{5/2})^{1/2}\,dx$

$= \frac{A}{35}(1+x^{5/2})^{7/2} - \frac{8B}{25}(1+x^{5/2})^{5/2} + \frac{C}{15}(1+x^{5/2})^{3/2} + K$

then the value of A + B + C is?

24. If $\int \frac{dx}{5+4\sin x} = A\tan^{-1}\left\{\frac{5}{3}\tan\left(\frac{x}{2}\right)+\frac{4}{3}\right\}+c$, then find value of A

25. If $\int \frac{\cos 4x+1}{\cot x - \tan x}dx = -k\cos 4x + c$ then find value of k

26. If $\int \frac{\left(\sqrt{x}\right)^5 dx}{\left(\sqrt{x}\right)^7+x^6} = \lambda\log\left(\frac{x^a}{x^a+1}\right)+c$, then $a+\lambda$ is greater then

27. If $\int \frac{\cos x + \sin 2x}{(2-\cos^2 x)(\sin x)}dx =$

$\int \frac{A}{\sin x}dx + B\int \frac{\sin x}{1+\sin^2 x}dx + C\int \frac{dx}{1+\sin^2 x}$, then find value of A + B + C

28. If $\int \tan^4 x\,dx = K\tan^3 x - \tan x + x + c$, then K is equals

29. $\int \frac{dx}{x^2\left(x^4+1\right)^{3/4}} = A\left(\frac{1+x^4}{x^4}\right)^{1/4} + C$ then absolute value of A equals

30. If $\int \frac{\log_x e.\log_{ex} e.\log_{e^2 x} e}{x}dx$

$= A\log_e\log_e x + B\log(1+\log_e x) + C\log(2+\log_e x) + \lambda$ then absolute value A + B equals

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The integral $\int \frac{2x^{12}+5x^9}{\left(x^5+x^3+1\right)^3}dx$ is equal to: where C is an arbitrary constant.

[JEE Main - 2016]

(a) $\frac{-x^5}{\left(x^5+x^3+1\right)^2}+C$ (b) $\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+C$

(c) $\frac{x^5}{2\left(x^5+x^3+1\right)^2}+C$ (d) $\frac{-x^{10}}{2\left(x^5+x^3+1\right)^2}+C$

2. Let $I_n = \int \tan^n x\,dx, (n>1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to : [JEE Main - 2017]

(a) $\left(-\frac{1}{5},0\right)$ (b) $\left(-\frac{1}{5},1\right)$

(c) $\left(\frac{1}{5},0\right)$ (d) $\left(\frac{1}{5},-1\right)$

3. The integral (where C is a constant of integration)

$\int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x\right)^2}dx$

is equal to [JEE Main - 2018]

(a) $\frac{-1}{3\left(1+\tan^3 x\right)}+C$ (b) $\frac{1}{1+\cot^3 x}+C$

(c) $\frac{-1}{1+\cot^3 x}+C$ (d) $\frac{1}{3\left(1+\tan^3 x\right)}+C$

4. For $x^2 \neq n\pi+1, n\in N$ (the set of natural numbers), the integral $\int x\cdot\sqrt{\frac{2\sin\left(x^2-1\right)-\sin 2\left(x^2-1\right)}{2\sin\left(x^2-1\right)+\sin 2\left(x^2-1\right)}}dx$ is

[JEE Main - 2019 (January)]

(a) $\log_e\left|\frac{1}{2}\sec^2\left(x^2-1\right)\right|+c$

(b) $\frac{1}{2}\log_e\left|\frac{1}{2}\sec^2\left(x^2-1\right)\right|+c$

(c) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right|+c$

(d) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right|+c$

5. If $f(x)=\int\frac{5x^8+7x^6}{(x^2+1+2x^7)^2}\,dx\ (x\geq 0)$ and $f(0)=0$, then the valuen of $f(1)$ is: **[JEE Main - 2019 (January)]**

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) $-\frac{1}{4}$ (d) $\frac{1}{4}$

6. Let $n\geq 2$ be a natural number and $0<\theta<\pi/2$. Then $\int\frac{\left(\sin^n\theta-\sin\theta\right)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}d\theta$ is equal to:
(where C is a constant of integration)
[JEE Main - 2019 (January)]

(a) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(b) $\frac{n}{n^2+1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(c) $\frac{n}{n^2-1}\left(1+\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(d) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}}+C$

7. If $\int x^5e^{-4x^3}dx=\frac{1}{48}e^{-4x^3}f(x)+C$, is a constant of integration, then $f(x)$ is equal to:
[JEE Main - 2019 (January)]

(a) $-2x^3-1$ (b) $-4x^3-1$

(c) $-2x^3+1$ (d) $4x^3+1$

8. If $\int\frac{x+1}{\sqrt{2x-1}}dx=f(x)\sqrt{2x-1}+C$ where C is a constant of integration of integration, then $f(x)$ is equal to.
[JEE Main - 2019 (January)]

(a) $\frac{1}{3}(x+1)$ (b) $\frac{2}{3}(x+2)$

(c) $\frac{2}{3}(x-4)$ (d) $\frac{1}{3}(x+4)$

9. If $\int\frac{\sqrt{1-x^2}}{x^4}dx=A(x)\left(\sqrt{1-x^2}\right)^m+C$, for a suitable chosen integer m and a function $A(x)$, where C is a constant of integration, then $(A(x))^m$ equals :-
[JEE Main - 2019 (January)]

(a) $\frac{-1}{27x^9}$ (b) $\frac{-1}{3x^1}$

(c) $\frac{1}{27x^6}$ (d) $\frac{1}{9x^4}$

10. The integral $\int\cos(\log_e x)\,dx$ is equal to : (where C is a constant of integration) **[JEE Main - 2019 (January)]**

(a) $\frac{x}{2}\left[\sin\left(\log_e x\right)-\cos\left(\log_e x\right)\right]+C$

(b) $x\left[\cos\left(\log_e x\right)+\sin\left(\log_e x\right)\right]+C$

(c) $\frac{x}{2}\left[\cos\left(\log_e x\right)+\sin\left(\log_e x\right)\right]+C$

(d) $x\left[\cos\left(\log_e x\right)-\sin\left(\log_e x\right)\right]+C$

11. Let f be a differentiable function such that $f(1)=2$ and $f'(x)=f(x)$ for all $x\in R$. If $h(x)=f(f(x))$, then $h'(1)$ is equal to **[JEE Main - 2019 (January)]**

(a) $2e^2$ (b) $4e$

(c) $2e$ (d) $4e^2$

12. The integral $\int\frac{3x^{13}+2x^{11}}{\left(2x^4+3x^2+1\right)^4}dx$ is equal to (where C is a constant of integration)
[JEE Main - 2019 (January)]

(a) $\frac{x^4}{6\left(2x^4+3x^2+1\right)^3}+C$ (b) $\frac{x^{12}}{6\left(2x^4+3x^2+1\right)^3}+C$

(c) $\frac{x^4}{\left(2x^4+3x^2+1\right)^3}+C$ (d) $\frac{x^{12}}{\left(2x^4+3x^2+1\right)^3}+C$

13. If $\int\frac{dx}{x^3\left(1+x^6\right)^{2/3}}=xf(x)(1+x^6)^{1/3}+C$ where C is a constant of integration, then the function $f(x)$ is equal to :-
[JEE Main - 2019(April)]

(a) $-\frac{1}{6x^3}$ (b) $\frac{3}{x^2}$

(c) $-\frac{1}{2x^2}$ (d) $-\frac{1}{2x^3}$

14. The integral $\int\sec^{2/3}x\,\mathrm{cosec}^{4/3}x\,dx$ is equal to (Hence C is a constant of integration) **[JEE Main - 2019(April)]**

(a) $3\tan^{1/3}x+C$ (b) $-\frac{3}{4}\tan^{-4/3}x+C$

(c) $-3\cot^{1/3}x+C$ (d) $-3\tan^{-1/3}x+C$

15. If $\int e^{\sec x}(\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x))dx = e^{\sec x} f(x) + C$, then a possible choice of $f(x)$ is :-

[JEE Main - 2019(April)]

(a) $\sec x \tan x - \frac{1}{2}$ (b) $x \sec x + \tan x + \frac{1}{2}$

(c) $\sec x + x \tan x - \frac{1}{2}$ (d) $\sec x + \tan x + \frac{1}{2}$

16. If $\int x^9 e^{x^2} dx = g(x) e^{x^2} + c$, where c is a constant of integration, then $g(-1)$ is equal to :

[JEE Main - 2019(April)]

(a) $-\frac{5}{2}$ (b) 1

(c) $-\frac{1}{2}$ (d) -1

17. If $\int \frac{dx}{(x^2-2x+10)^2}$

$= A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2-2x+10}\right) + C$ where C is a constant of integration, then :

[JEE Main - 2019(April)]

(a) $A = \frac{1}{27}$ and $f(x) = 9(x-1)$

(b) $A = \frac{1}{81}$ and $f(x) = 3(x-1)$

(c) $A = \frac{1}{54}$ and $f(x) = 9(x-1)^2$

(d) $A = \frac{1}{54}$ and $f(x) = 3(x-1)$

18. The integral $\int \frac{2x^3-1}{x^4+x} dx$ is equal to:

(Here C is a constant of intergration)

[JEE Main - 2019(April)]

(a) $\log_e \left|\frac{x^3+1}{x}\right| + C$ (b) $\frac{1}{2}\log_e \frac{(x^3+1)^2}{|x^3|}$

(c) $\frac{1}{2}\log_e \frac{|x^3+1|}{x^2} + C$ (d) $\log_e \frac{|x^3+1|}{x^2} + C$

19. Let $\alpha \in (0, \pi/2)$ be fixed. It the integral $\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha} dx = A(x)\cos 2\alpha + B(x)\sin 2\alpha + C$, where C is a constant of integration, then the functions A(x) and B(x) are respectively :

[JEE Main - 2019(April)]

(a) $x - \alpha$ and $\log_e|\cos(x-\alpha)|$

(b) $x + \alpha$ and $\log_e|\sin(x-\alpha)|$

(c) $x - \alpha$ and $\log_e|\sin(x-\alpha)|$

(d) $x + \alpha$ and $\log_e|\sin(x+\alpha)|$

20. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}} dx$ is equal to :

[JEE Main-2019(April)]

(a) $2x + \sin x + 2\sin 2x + c$

(b) $x + 2\sin x + 2\sin 2x + c$

(c) $x + 2\sin x + \sin 2x + c$

(d) $2x + \sin x + \sin 2x + c$

21. If $\int \frac{\cos x}{\sin^3 x (1+\sin^6 x)^{2/3}} dx = f(x)(1+\sin^6 x)^{1/\lambda} + c$, where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal to

[JEE Main-2020 (January)]

(a) 2 (b) $\frac{-9}{8}$

(c) $\frac{9}{8}$ (d) -2

22. Let $f(x) = \left(\sin(\tan^{-1} x) + \sin(\cot^{-1} x)\right)^2 - 1, |x| > 1$. If $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\sin^{-1}(f(x))\right)$ and $y\left(\sqrt{3}\right) = \frac{\pi}{6}$, then $y\left(-\sqrt{3}\right)$ is equal to

[JEE Main-2020 (January)]

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{-\pi}{6}$ (d) $\frac{5\pi}{6}$

23. The integral $\int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$ is equal to :

[JEE Main-2020 (January)]

(where C is a constant of integration)

(a) $\left(\frac{x-3}{x+4}\right)^{1/7} + C$ (b) $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-13/7} + C$

(c) $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{3/7} + C$ (d) $-\left(\frac{x-3}{x+4}\right)^{-1/7} + C$

24. If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)} = \lambda\tan\theta + 2\log_e|f(\theta)| + C$ is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to:

[JEE Main-2020 (January)]

(a) $(1, 1+\tan\theta)$ (b) $(-1, 1-\tan\theta)$

(c) $(-1, 1+\tan\theta)$ (d) $(1, 1-\tan\theta)$

25. If $x^2dy + xy\,dx = x^2dy + 2y\,dx$; $y(2) = e$ and $x > 1$, then $y(4)$ is equal to: [JEE Main-2020 (September)]

(a) $\frac{\sqrt{e}}{2}$ (b) $\frac{1}{2}+\sqrt{e}$

(c) $\frac{3}{2}+\sqrt{e}$ (d) $\frac{3}{2}\sqrt{e}$

26. If $\int \sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)dx = A(x)\tan^{-1}(\sqrt{x}) + B(x) + C$, where C is a constant of integration, t hen the ordered pair $(A(x), B(x))$ can be [JEE Main-2020 (September)]

(a) $(x+1, -\sqrt{x})$ (b) $(x+1, \sqrt{x})$

(c) $(x-1, -\sqrt{x})$ (d) $(x-1, \sqrt{x})$

27. Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2}dx (x \geq 0)$. Then $f(3) - f(1)$ is equal to: [JEE Main-2020 (September)]

(a) $\frac{\pi}{12}+\frac{1}{2}-\frac{\sqrt{3}}{4}$ (b) $-\frac{\pi}{6}+\frac{1}{2}+\frac{\sqrt{3}}{4}$

(c) $-\frac{\pi}{12}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6}+\frac{1}{2}-\frac{\sqrt{3}}{4}$

28. The integral $\int \left(\frac{x}{x\sin x + \cos x}\right)^2 dx$ is equal to (where C is a constant of integration): [JEE Main-2020 (September)]

(a) $\tan x - \frac{x\sec x}{x\sin x + \cos x} + C$

(b) $\sec x - \frac{x\tan x}{x\sin x + \cos x} + C$

(c) $\tan x + \frac{x\sec x}{x\sin x + \cos x} + C$

(d) $\sec x + \frac{x\tan x}{x\sin x + \cos x} + C$

29. If $\int \frac{\cos\theta}{5 + 7\sin\theta - 2\cos^2\theta}d\theta = A\log_e|B(\theta)| + C$, where C is a constant of intergation, then $\frac{B(\theta)}{A}$ can be : [JEE Main-2020 (September)]

(a) $\frac{2\sin\theta+1}{5(\sin\theta+3)}$ (b) $\frac{2\sin\theta+1}{\sin\theta+3}$

(c) $\frac{5(2\sin\theta+1)}{\sin\theta+3}$ (d) $\frac{5(\sin\theta+3)}{2\sin\theta+1}$

30. If $\int (e^{2x} + 2e^x - e^{-x} - 1)e^{(e^x+e^{-x})}dx = g(x)e^{(e^x+e^{-x})} + c$, where c is a constant of integration, then $g(0)$ is equal to: [JEE Main-2020 (September)]

(a) e^2 (b) 1

(c) 2 (d) e

31. $f(x) = \int \frac{5x^8 + 7x^6}{(x^7 + 1 + 2x^7)^2}dx, (x \geq 0), f(0) = 0$ and $f(1) = \frac{1}{K}$ then the value of K is. [JEE Main-2021 (March)]

32. For real numbers α, β, γ and δ, if [JEE Main-2021 (March)]

$$\int \frac{\left(x^2-1\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx =$$

$$\alpha\log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right) + \beta\tan^{-1}\left(\frac{\gamma\left(x^2+1\right)}{x}\right) + \delta\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where, C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to. [JEE Main-2021 (March)]

JEE-ADVANCED PREVIOUS YEARS

1. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}dx$ equals (for some arbitrary constant K) [JIT JEE-2012]

(a) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

2. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_p. If PY_p has length 1 for each point P on Γ, then which of the following options is/are correct ? [JEE Advanced -2019]

(a) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(b) $xy' - \sqrt{1-x^2} = 0$

(c) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(d) $xy' + \sqrt{1-x^2} = 0$

ANSWER KEY

Topicwise Questions

1. (*b*)	**2.** (*d*)	**3.** (*b*)	**4.** (*a*)	**5.** (*c*)	**6.** (*b*)	**7.** (*b*)	**8.** (*a*)	**9.** (*b*)	**10.** (*a*)
11. (*a*)	**12.** (*c*)	**13.** (*d*)	**14.** (*a*)	**15.** (*c*)	**16.** (*a*)	**17.** (*a*)	**18.** (*b*)	**19.** (*a*)	**20.** (*a*)
21. (*a*)	**22.** (*b*)	**23.** (*c*)	**24.** (*d*)	**25.** (*b*)	**26.** (*d*)	**27.** (*a*)	**28.** (*b*)	**29.** (*b*)	**30.** (*b*)
31. (*c*)	**32.** (*b*)	**33.** (*b*)	**34.** (*d*)	**35.** (*d*)					

Learning Plus

1. (*a*)	**2.** (*a*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)	**6.** (*c*)	**7.** (*a*)	**8.** (*c*)	**9.** (*a*)	**10.** (*a*)
11. (*a*)	**12.** (*b*)	**13.** (*a*)	**14.** (*c*)	**15.** (*c*)	**16.** (*d*)	**17.** (*c*)	**18.** (*b*)	**19.** (*d*)	**20.** (*b*)
21. (*b*)	**22.** (*b*)	**23.** (*c*)	**24.** (*a*)	**25.** (*a*)	**26.** (*a*)	**27.** (*a*)	**28.** (*d*)	**29.** (*c*)	**30.** (*b*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*bc*)	**2.** (*bcd*)	**3.** (*bd*)	**4.** (*acd*)	**5.** (*ac*)	**6.** (*cd*)	**7.** (*abd*)	**8.** (*ac*)	**9.** (*ab*)	**10.** (*abcd*)
11. (*acd*)	**12.** (*ab*)	**13.** (*c*)	**14.** (*b*)	**15.** (*c*)	**16.** (*d*)	**17.** (*c*)	**18.** (*d*)		

19. (*a*) → (p), (*b*) → (p), (*c*) → (r), (*d*) → (s) **20.** (*a*) → (s) ; (*b*) → (q) ; (*c*) → (r) **21.** (8) **22.** (5) **23.** (9)

24. (0.66) **25.** (0.125) **26.** (2) **27.** (2) **28.** (0.33) **29.** (1) **30.** (0.5)

JEE Mains & Advanced Past Years Questions

JEE-MAIN

PREVIOUS YEARS

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*d*)	**5.** (*d*)	**6.** (*a*)	**7.** (*b*)	**8.** (*d*)	**9.** (*a*)	**10.** (*c*)
11. (*b*)	**12.** (*b*)	**13.** (*d*)	**14.** (*d*)	**15.** (*d*)	**16.** (*a*)	**17.** (*d*)	**18.** (*a*)	**19.** (*c*)	**20.** (*c*)
21. (*d*)	**22.** (*d*)	**23.** (*a*)	**24.** (*c*)	**25.** (*d*)	**26.** (*a*)	**27.** (*a*)	**28.** (*a*)	**29.** (*c*)	**30.** (*c*)
31. (4)	**32.** (6)								

JEE-ADVANCED

PREVIOUS YEARS

1. (*c*) **2** (*a,d*)

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26

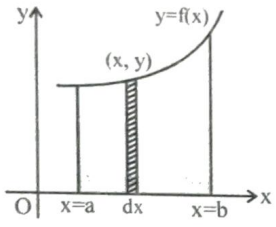
APPLICATION OF INTEGRALS

AREA BY VERTICAL STRIPS

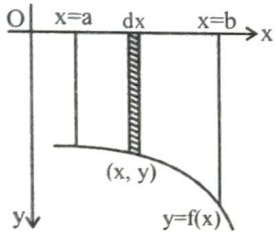
To determine area bounded by curve $y = f(x)$, the x-axis and the ordinates at $x = a$ & $x = b$ is

Case-I : If $y = f(x)$ lies completely above the x-axis i.e.,

$$A = \int_a^b f(x)dx = \int_a^b ydx$$



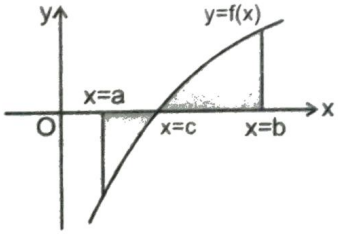
Case-II : If $y = f(x)$ lies completely below the x-axis then A is negative. The convention is to consider the magnitude only



i.e. $A = \left|\int_a^b f(x)dx\right| = \left|\int_a^b ydx\right|$

Case-III : If $y = f(x)$ cuts the x-axis at $x = c \in (a, b)$

$$A = \left|\int_a^c f(x)\,dx\right| + \int_c^b f(x)\,dx$$



TRAIN YOUR BRAIN

Example 1: Find the area bounded by $y = \sec^2 x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ & x-axis

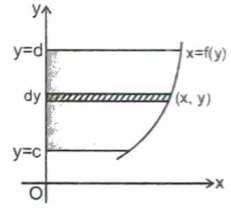
Sol. Area bounded

$$= \int_{\pi/6}^{\pi/3} ydx = \int_{\pi/6}^{\pi/3} \sec^2 dx = [\tan x]_{\pi/6}^{\pi/3}$$

$$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3}\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \text{ sq. units}$$

AREA BY HORIZONTAL STRIPS

To determine area bounded by the curve $x = f(y)$, the y-axis and abscissa at $y = c$ & $y = d$ is



i.e. $A = \int_c^d f(y)dy = \int_c^d xdy$

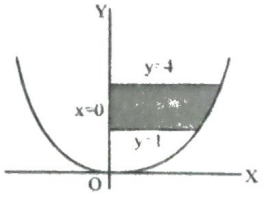
TRAIN YOUR BRAIN

Example 2: Find the area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

Sol. The required area

$$= \int_1^4 xdy = \int_1^4 \frac{\sqrt{y}}{2}dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4$$

$$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1] = \frac{7}{3} = 2\frac{1}{3} \text{ sq. units}$$

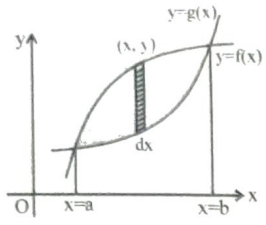


AREA ENCLOSED BETWEEN TWO CURVES

Case-I: (By vertical strips)

Area between the curves $y = f(x)$ & $y = g(x)$ between the ordinates at $x = a$ & $x = b$ is, such that $f(x) > g(x)$

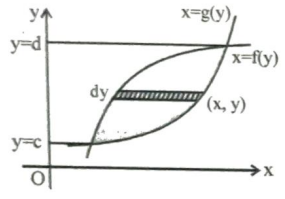
$$\text{i.e. } A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$



Case-II: (By horizontal strips)

Area between the curves $x = f(y)$ & $x = g(y)$ between the ordinates at $y = c$ & $y = d$ is

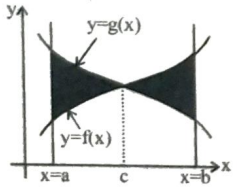
$$\text{i.e. } A = \int_c^d f(y)dy - \int_c^d g(y)dy = \int_c^d [f(y) - g(y)]dy$$



Case-III:

If the curves $y = f(x)$ and $y = g(x)$ intersect at $x = c$, then required area

$$A = \int_a^c (g(x) - f(x)dx) + \int_c^b (f(x) - g(x))dx = \int_a^b |f(x) - g(x)|dx$$



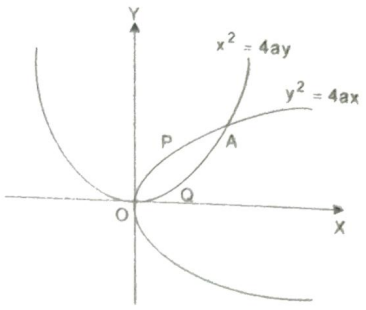
Note: Required area must have all the boundaries indicated in the problem.

TRAIN YOUR BRAIN

Example 3: Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol. The two parabolas intersect at O(0,0) and $A(4a, 4a)$. The area included between the two curves = area $OQAP$.

$$= \int_{x=0}^{x=4a} (y_1 - y_2)dx$$



$$= \int_0^{4a} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right)dx = \left[2\sqrt{a}\frac{2}{3}x^{3/2} - \frac{x^3}{12a}\right]^{4a}$$

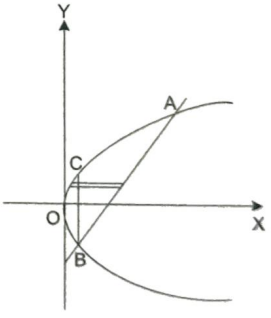
$$= 4\frac{\sqrt{a}}{3}8.a^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{16a^2}{3} \text{ sq.units.}$$

Note: Sometimes it is better to use the formula $\int_c^d x\,dy$ instead of $\int_a^b y\,dx$ in the computation of area to simplify calculations, as the following illustration shows.

Example 4: Find the area of the segment cut off from the parabola $y^2 = 2x$ by the line $y = 4x - 1$

Sol. The line $y = 4x - 1$ intersects the parabola $y^2 = 2x$ at A and B



$$2x = (4x-1)^2 \Rightarrow 16x^2 - 10x + 1 = 0$$

$$\Rightarrow (8x-1)(2x-1) = 0$$

$$\therefore A = \left(\frac{1}{2}, 1\right)$$

$$\text{and } B = \left(\frac{1}{8}, -\frac{1}{2}\right)$$

If the formula $\int y\,dx$ is to be used then the area will have to be split up as OBC and CBA. Instead the problem can be done directly by using the formula $\int (x_2 - x_1)\,dy$.

Area required $= \int_{y=1/2}^{1} (x_2 - x_1)\,dy$

$= \int_{1/2}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy = \left[\frac{y^2}{8} + \frac{y}{4} - \frac{y^3}{6}\right]_{1/2}^{1}$

$= \left(\frac{1}{8} + \frac{1}{4} - \frac{1}{6}\right) - \left(\frac{1}{32} - \frac{1}{8} + \frac{1}{48}\right)$

$= \frac{(3+6-4)}{24} - \frac{(3-12+2)}{96} = \frac{5}{24} + \frac{7}{96}$

$= \frac{27}{96} = \frac{9}{32}$ sq. units

USEFUL RESULTS

(*a*) Whole area of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is π ab sq. units

(*b*) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $\frac{16ab}{3}$ sq. units

(*c*) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $\frac{8a^2}{3m^3}$ sq. units

AVERAGE VALUE OF A FUNCTION

$y = f(x)$ w.r.t x over an interval $a \le x \le b$ is defined as : y(avg)

$= \frac{1}{b-a}\int_a^b f(x)dx$

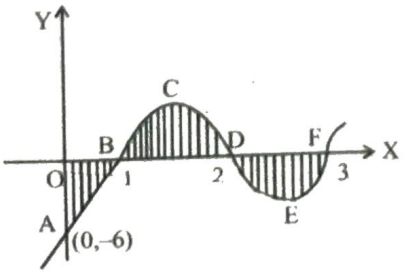
TRAIN YOUR BRAIN

Example 5: Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$ and x-axis.

Sol. To determine the sign, we follow the usual rule as of change of sign.

$y = +ve$ for $x > 3$

$y = -ve$ for $2 < x < 3$



$y = +ve$ for $1 < x < 2$

$y = -ve$ for $x < 1$

$\int_0^3 |y|dx = \int_0^1 |y|dx + \int_1^2 |y|dx + \int_2^3 |y|dx$

$= \int_0^1 ydx + \int_1^2 ydx + \int_2^3 ydx$

Now let $F(x) = \int (x-1)(x-2)(x-3)dx$

$= \int (x^3 - 6x^2 + 11x - 6)dx$

$= \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$

$\therefore\ F(0) = 0,\ F(1) = -\frac{9}{4}, F(2) = -2, F(3) = -\frac{9}{4}$

Hence required Area $= -[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq. units.

Example 6: The curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area enclosed by the curve, the axis of x and the line $x = 4$ is 8 square units. Determine a, b, where a and b are positive.

Sol. The curve passes through (0, 0). Hence the limits of x are 0 to 4.

$A = \int_0^4 ydx = \int_0^4 (a\sqrt{x} + bx)dx$

or $8 = \left[a.\frac{2}{3}x^{3/2} + b\frac{x^2}{2}\right]_0^4$ or $8 = \frac{16a}{3} + 8b$ ...(i)

Again the curve passes through (1, 2)

$\therefore\ 2 = a + b$ ...(ii)

Solving (i) and (ii), we get $a = 3, b = -1$.

IMPORTANT NOTES

Shifting of Origin

Since area remains invariant even if the coordinate axes are shifted, hence shifting of origin in many cases prove to be very convenient in computing the areas.

Enclosed Area for Curve Reprsented Parametrically

If the equation of the curve be in parametric form then $A = \int_{t=\alpha}^{t=\beta} y.\frac{dx}{dt}.dt$ or $A = \int_{t=\gamma}^{t=\delta} x\frac{dy}{dt}dt$, where α & β are values corresponding to values of x and γ & δ are values corresponding to values of y.

TRAIN YOUR BRAIN

Example 7: Find the smaller of the area bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.

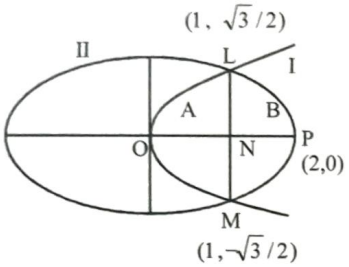
Sol. C_1 is $4(y^2 - 2y) = 3x - 7$ or $4(y-1)^2 = 3x - 3 = 3(x-1)$ ...(i)

above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $(x-1)^2 + 4(y-1)^2 = -1 + 1 + 4$

or $\frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$ ...(ii)



Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are $4Y^2 = 3X$ and $\frac{X^2}{2^2} + \frac{Y^2}{1} = 1$

They meet at $\left(1, \pm\frac{\sqrt{3}}{2}\right)$

Required area $= 2(A + B)$

$= 2[\int Y_1 dX + \int Y_2 dX]$

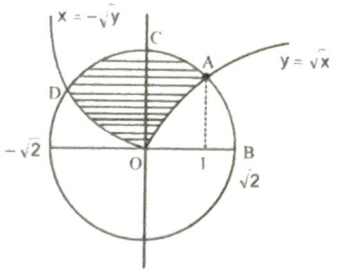
$$= 2\left[\frac{\sqrt{3}}{2}\int_0^1 \sqrt{X}dX + \int_1^2 \frac{\sqrt{4-X^2}}{2}dX\right] = \left[\frac{\sqrt{3}}{6} + \frac{2\pi}{3}\right]$$

Example 8: Find the area bounded by the curve $y \geq \sqrt{x}$ & $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$

Sol. Common region is given by the diagram

If area of region OAB = λ then area of OCD = λ

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$ will bound same area with x & y axis respectively.



$y = \sqrt{x}$ $\Rightarrow y^2 = x$

$x = -\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are symmetric with respect to line $y = x$

Area of first quadrant OBC

$= \frac{\pi r^2}{4} = \frac{\pi}{2}$ $(\because r = \sqrt{2})$

area of region OCA $= \frac{\pi}{2} - \lambda$

area of shaded region $= (\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq. units

Example 9: Find the area bounded by the curves $x^2 + y^2 = 4$ & $x^2 = -\sqrt{2}y$ and the line $x = y$, below x-axis.

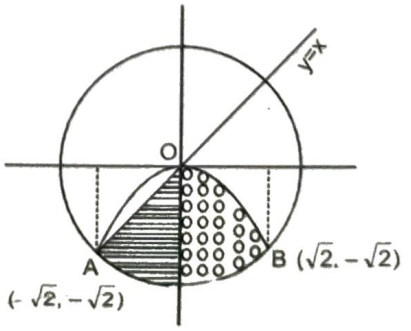
Sol. Let C is $x^2 + y^2 = 4$, P is $y = -\frac{x^2}{\sqrt{2}}$ and L is $y = x$.

We have above three curves. Solving P and C we get the points

$A(-\sqrt{2}, -\sqrt{2})$, $B(\sqrt{2}, -\sqrt{2})$

Also the line $y = x$ passes through $A(-\sqrt{2}, -\sqrt{2})$

$\therefore$ Required area = shaded + dotted



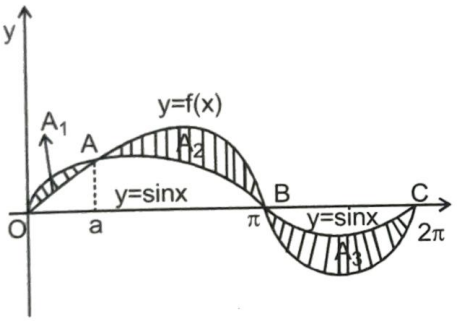
$$= \int_{\sqrt{2}}^{0} (y_1 - y_1)dx + \int_{0}^{\sqrt{2}} (y_2 - y_1)dx$$

$$= \int_{\sqrt{2}}^{0} x\,dx + \int_{0}^{\sqrt{2}} \frac{x^2}{\sqrt{2}}dx - \int_{\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_{\sqrt{2}}^{0} - \frac{1}{\sqrt{2}}\left[\frac{x^3}{3}\right]_{0}^{\sqrt{2}} - \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{2}}^{\sqrt{2}}$$

$$\therefore \quad |A| = \frac{3\pi + 16}{6}$$

Example 10: In the adjacent graphs of two functions $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at A(a f(a)); B $(\pi, 0)$ and C$(2\pi, 0)$. $A_i (i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$ between $x = 0$ and $x = a$, $(i = 1)$, between $x = a$ and $x = \pi$; $(i = 2)$, between $x = \pi$ and $x = 2\pi$; $(i = 3)$. If $A_1 = 1 - \sin a + (a - 1)\cos a$, determine the function f(x). Hence determine 'a' and A_1. Also calculate A_2 and A_3.



Sol. From the figure it is clear that

$$\int_{0}^{a} (\sin x - f(x))dx = 1 - \sin a + (a - 1)\cos a \quad \text{differentiate}$$

w.r.t. a

$$\sin a - f(a) = -\cos a + \cos a - (a - 1)\sin a$$

$$\Rightarrow \sin a - f(a) = -a \sin a + \sin a \Rightarrow f(a) = a \sin a$$

$$\Rightarrow f(x) = x \sin x$$

The points where f(x) & sinx intersect are $x \sin x = \sin x$

$\Rightarrow \sin x = 0$ or $x = 1$. We can say that $a = 1$

$$A_1 = \int_{0}^{1} (\sin x - x \sin x)dx = (1 - \sin 1) \text{ sq. units;}$$

$$A_2 = \int_{1}^{\pi} (f(x) - \sin x)dx = \int_{1}^{\pi} (x \sin x - \sin x)dx$$

$$= (\pi - 1 - \sin 1) \text{ sq. units}$$

$$A_3 = \left|\int_{\pi}^{2\pi} (\sin x - x \sin x)dx\right| = (3\pi - 2) \text{ sq. units}$$

Area bounded by Region, Volume and Surface Area of Solids of Revolution

1. Area bounded by the curve $y = \log x$, x – axis and the ordinates $x = 1, x = 2$ is
   (a) $\log 4$ *sq. unit* (b) $(\log 4 + 1)$ *sq. uni t*
   (c) $(\log 4 - 1)$ *sq. unit* (d) $\log 2$ sq. unit

2. Area bounded by the parabola $y = 4x^2$, y – axis and the lines $y = 1, y = 4$ is
   (a) 3 sq. unit (b) $\frac{7}{5}$ sq. unit
   (c) $\frac{7}{3}$ sq. unit (d) $\frac{2}{3}$ sq. unit

3. Area enclosed between curve $y = \sin 2x + \cos 2x$ and x-axis from $x = 0$ and $x = \frac{\pi}{4}$, is
   (a) 2 sq. unit (b) 1 sq. unit
   (c) 3 sq. unit (d) 4 sq. unit

4. Area under the curve $y = \sqrt{3x+4}$ between $x = 0$ and $x = 4$, is
   (a) $\frac{56}{9}$ sq.unit (b) $\frac{64}{9}$ sq. unit
   (c) 8 sq. unit (d) $\frac{112}{9}$ sq. unit

5. The area bounded by the parabola $y^2 = 4ax$, its axis and two ordinates $x = 4, x = 9$ is
   (a) $4a^2$ (b) $4a^2.4$
   (c) $4\sqrt{a}\left(\frac{84}{3}\right)$ (d) $\frac{152\sqrt{a}}{3}$

6. The area bounded by curve $y^2 = x$, line $y = 4$ and y-axis is
   (a) $\frac{16}{3}$ (b) $\frac{64}{3}$
   (c) $7\sqrt{2}$ (d) $16\sqrt{2}$

7. The measurement of the area bounded by the co-ordinate axes and the curve $y = \log_e x$ is
   (a) 1 (b) 2
   (c) 3 (d) ∞

8. The area of smaller part between the circle $x^2 + y^2 = 4$ and the line $x = 1$ is
   (a) $\frac{4\pi}{3} - \sqrt{3}$ (b) $\frac{8\pi}{3} - \sqrt{3}$
   (c) $\frac{4\pi}{3} + \sqrt{3}$ (d) $\frac{5\pi}{3} + \sqrt{3}$

9. The area between the curve $y = \sin^2 x$, x – axis and the ordinates $x = 0$ and $x = \frac{\pi}{2}$ is
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
   (c) $\frac{\pi}{8}$ (d) π

10. The area bounded by the circle $x^2 + y^2 = 4$, line $x = \sqrt{3}y$ and x– axis lying in the first quadrant, is
    (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
    (c) $\frac{\pi}{3}$ (d) π

11. The area formed by triangular shaped region bounded by the curves $y = \sin x, y = \cos x$ and $x = 0$ is
    (a) $\sqrt{2} - 1$ (b) 1
    (c) $\sqrt{2}$ (d) $1 + \sqrt{2}$

12. The area bounded by the curve $y = 4x - x^2$ and the x – axis, is
    (a) $\frac{30}{7}$ sq. unit (b) $\frac{31}{7}$ sq. unit
    (c) $\frac{32}{3}$ sq. unit (d) $\frac{34}{3}$ sq. unit

13. Area of the region bounded by the curve $y = \tan x$, and line $x = \frac{\pi}{4}$ and x-axis is
    (a) $\frac{1}{4}$ (b) $\log\sqrt{2} + \frac{1}{4}$
    (c) $\log\sqrt{2} - \frac{1}{4}$ (d) $\log\sqrt{2}$

14. The area between the curve $y = 4 + 3x - x^2$ and x-axis is
    (a) 125/6 (b) 125/3
    (c) 125/2 (d) 125/7

15. The area bounded by the curve $y = x$, x-axis and ordinates $x = -1$ to $x = 2$ is
    (a) 0 (b) 1/2
    (c) 3/2 (d) 5/2

16. Area inside the parabola $y^2 = 4ax$, between the lines $x = a$ and $x = 4a$ is equal to
    (a) $4a^2$ (b) $8a^2$
    (c) $\frac{56a^2}{3}$ (d) $35\frac{a^2}{3}$

17. The area bounded by $y = -x^2 + 2x + 3$ and $y = 0$ is

(a) 32 (b) $\frac{32}{3}$

(c) $\frac{1}{32}$ (d) $\frac{1}{3}$

18. The area enclosed between the curves $y = x^3$ and $y = \sqrt{x}$ is, (in square units)

(a) $\frac{5}{3}$ (b) $\frac{5}{4}$

(c) $\frac{5}{12}$ (d) $\frac{12}{5}$

19. Area enclosed by the parabola $ay = 3(a^2 - x^2)$ and x-axis is

(a) $4a^2$ sq. unit (b) $12a^2$ sq. unit

(c) $4a^3$ sq. unit (d) $24a^2$ sq. unit

20. Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) πab sq. unit (b) $\frac{1}{2}\pi ab$ sq. unit

(c) $\frac{1}{4}\pi ab$ sq. unit (d) $\frac{\pi}{3}ab$ sq. unit

21. The area of the region bounded by $y = |x - 1|$ and $y = 1$ is

(a) 2 (b) 1

(c) $\frac{1}{2}$ (d) 1/4

22. The area enclosed by the parabolas $y = x^2 - 1$ and $y = 1 - x^2$ is

(a) 1/3 (b) 2/3

(c) 4/3 (d) 8/3

23. The area bounded by the curves $y^2 = 8x$ and $y = x$ is

(a) $\frac{128}{3}$ sq. unit (b) $\frac{32}{3}$ sq. unit

(c) $\frac{64}{3}$ sq. unit (d) 32 sq. unit

24. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is

(a) 3 e (b) e - 3

(c) $\frac{1}{2}(3 - e)$ (d) $\frac{1}{2}(e - 3)$

25. The area of the region bounded by the curves $y = x^2$ and $y = |x|$ is

(a) 1/6 (b) 1/3

(c) 5/6 (d) 5/3

26. The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$ is

(a) $\frac{(\pi^3 - 8)}{4}$ (b) $\frac{\pi^3}{4}$

(c) $\frac{(\pi^3 - 16)}{4}$ (d) $\frac{(\pi^3 - 8)}{2}$

27. Area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(a) $\frac{8}{9}$ sq. unit (b) $\frac{9}{8}$ sq. unit

(c) $\frac{4}{3}$ sq. unit (d) $\frac{3}{4}$ sq. unit

28. If the area bounded by $y = ax^2$ and $x = ay^2$ $a > 0$, is 1, then a =

(a) 1 (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{2}}$

29. The area enclosed between the curve $y = \log_e(x + e)$ and the co-ordinate axes is

(a) 3 (b) 4

(c) 1 (d) 2

Learning Plus

1. The area bounded by the x-axis and the curve $y = 4x - x^2 - 3$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{8}{3}$

2. The area of the region bounded by the curve $y = \sin x$ and the x-axis in $[-\pi, \pi]$ is

(a) 4 (b) 8

(c) 12 (d) 2

3. The area bounded by the curve $y = \sin ax$ with x-axis in between any two successive points of inflection is

(a) $\frac{4}{a}$ (b) $\frac{2}{a}$

(c) $\frac{1}{a}$ (d) 2a

4. The area bounded by the curve $y = \cos x$ and x-axis between $x = 0$ and $x = \pi$

(a) 2 (b) 4

(c) 6 (d) 3

5. Area bounded by curve $y^3 - 9y + x = 0$ and y-axis is

(a) $\frac{9}{2}$ (b) 9

(c) $\frac{81}{2}$ (d) 81

6. The area bounded by the curve $y^2 = 4x$ and the line $x = 2$ is

(a) $\frac{8}{3}$ (b) $\frac{16}{3}$

(c) $\frac{8\sqrt{2}}{3}$ (d) $\frac{16\sqrt{2}}{3}$

7. The area of the figure bounded by right of the line $y = x + 1$, $y = \cos x$ and x–axis is:

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{5}{6}$ (d) $\frac{3}{2}$

8. The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is

(a) 3 (b) 2

(c) 1 (d) 4

9. The area bounded by the parabola $y = 4x^2$, $x = 0$ and $y = 1$, $y = 4$ in first quardrant is

(a) 7 (b) $\frac{7}{2}$

(c) $\frac{7}{3}$ (d) $\frac{7}{4}$

10. The area bounded by the curve $y = e^x$ and the lines $y = |x - 1|$, $x = 2$ is given by:

(a) $e^2 + 1$ (b) $e^2 - 1$

(c) $e^2 - 2$ (d) $e^2 + 2$

11. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is:

(a) $\frac{4 + 3\ell n 3}{2}$ (b) $\frac{4 - 3\ell n 3}{2}$

(c) $\frac{3}{2} + \ell n 3$ (d) $\frac{1}{2} + \ell n 3$

12. A continuous function f(x) satisfying

(i) $x^4 - 4x^2 \le f(x) \le 2x^2 - x^3$,

(ii) area bounded by $y = f(x)$, $y = x^4 - 4x^2$, y - axis and line $x = t$, $(0 \le t \le 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y – axis and line $x = t$, $(0 \le t \le 2)$, is given as

(a) $f(x) = \frac{x^4 - kx^3 + (2k - 4)x^2}{1 + k}$

(b) $f(x) = \frac{x^4 + x^3 + kx^2}{k + 1}$

(c) $f(x) = \frac{kx^4 - x^3 - 2kx^2}{k + 1}$

(d) $f(x) = \frac{k}{k + 1}(x^4 - x^2)$

13. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals $\frac{9}{2}$?

(a) -4 & 2 (b) -2 & 4

(c) 2 & 4 (d) -2 & -4

14. The area bounded by the curve $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{5}{3}$

15. The area bounded in the first quadrant between the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the line $3x + 4y = 12$ is:

(a) $6(\pi - 1)$ (b) $3(\pi - 2)$

(c) $3(\pi - 1)$ (d) $6(\pi - 2)$

16. The area of the region (s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is

(a) 1/3 (b) 2/3

(c) 1/6 (d) 1

17. The area of the closed figure bounded by $y = x$, $y = -x$ & the tangent to the curve $y = \sqrt{x^2 - 5}$ at the point (3, 2) is

(a) 5 (b) $2\sqrt{5}$

(c) 10 (d) $\frac{5}{2}$

18. The area of the triangle formed by the positive x-axis and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$

(c) $2\sqrt{3}$ (d) $\frac{\sqrt{3}}{2}$

19. The area of the region bounded by $x = 0$, $y = 0$, $x = 2$, $y = 2$, $y \le e^x$ and $y \ge \ell n\, x$, is

(a) $6 - 4\ell n 2$ (b) $4\ell n 2 - 2$

(c) $2\ell n 2 - 4$ (d) $6 - 2\ell n 2$

20. Let $f(x) = x^3 + 3x + 2$ and g(x) is the inverse of it. Then the area bounded by g(x), the x-axis and the ordinate at $x = -2$ and $x = 6$ is

(a) $\frac{1}{4}$ (b) $\frac{4}{3}$

(c) $\frac{9}{2}$ (d) $\frac{7}{3}$

21. The area bounded by the curve $y = xe^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(*a*) $1-3e^{-2}$ (*b*) $1-2e^{-2}$

(*c*) $1-e^{-2}$ (*d*) 1

22. The line $y = mx$ bisects the area enclosed by the curve $y = 1 + 4x - x^2$ & the line $x = 0$, $x = \frac{3}{2}$ & $y = 0$. Then the value of m is

(*a*) $\frac{13}{6}$ (*b*) $\frac{6}{13}$

(*c*) $\frac{3}{2}$ (*d*) 4

23. If $(a, 0)$; $a > 0$ is the point where the curve $y = \sin 2x - \sqrt{3}\sin x$ cuts the x-axis first, A is the area bounded by this part of the curve, the origin and the positive x-axis, then

(*a*) $4A + 8\cos a = 7$

(*b*) $4A + 8\sin a = 7$

(*c*) $4A - 8\sin a = 7$

(*d*) $4A - 8\cos a = 7$

24. The area bounded by the curve $y = \frac{1}{x^2}$ and its asymptote from $x = 1$ to $x = 3$ is

(*a*) $\frac{1}{3}$ (*b*) $\frac{2}{3}$

(*c*) $\frac{1}{2}$ (*d*) $\frac{1}{6}$

25. The area between two arms of the curve $|y| = x^3$ from $x = 0$ to $x = 2$ is

(*a*) 2 (*b*) 4

(*c*) 8 (*d*) 16

26. The area of the region for which $0 < y < 3 - 2x - x^2$ & $x > 0$ is

(*a*) $\int_1^3 (3-2x-x^2)dx$

(*b*) $\int_0^3 (3-2x-x^2)dx$

(*c*) $\int_0^1 (3-2x-x^2)dx$

(*d*) $\int_1^3 (3-2x+x^2)dx$

27. The curve $f(x) = Ax^2 + Bx + C$ passes through the point $(1, 3)$ and line $4x + y = 8$ is tangent to it at the point $(2, 0)$. The area enclosed by $y = f(x)$, the tangent line and the y-axis is

(*a*) $\frac{4}{3}$ (*b*) $\frac{8}{3}$

(*c*) $\frac{16}{3}$ (*d*) $\frac{32}{3}$

28. A circle and a square have the same perimeter. Then

(*a*) area of the square is larger

(*b*) area of circle is π times the area of square

(*c*) their areas are equal

(*d*) area of the circle is larger

29. The area bounded by the curve $y = f(x)$, the x-axis & the ordinates $x = 1$ & $x = b$ is $(b-1)\sin(3b+4)$. Then $f(x)$ is

(*a*) $(x-1)\cos(3x+4)$

(*b*) $\sin(3x+4)$

(*c*) $\sin(3x+4) + 3(x-1).\cos(3x+4)$

(*d*) none

30. The area enclosed by the curve $y^2 + x^4 = x^2$ is

(*a*) $\frac{2}{3}$ (*b*) $\frac{4}{3}$

(*c*) $\frac{8}{3}$ (*d*) $\frac{10}{3}$

31. The average value of $f(x) = \sec^2 x$ from $x = 0$ to $x = \frac{\pi}{4}$ is

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{2}$

(*c*) $\frac{2}{\pi}$ (*d*) $\frac{4}{\pi}$

32. Let s(t) be the area of the ΔOAB with O(0, 0, 0), A (2, 2, 1) and B(t, 1, t + 1) and define $f(x) = \int_0^x (s(t))^2\, dt$.

The area bounded by $y = f(x)$, x-axis between $x = 0$ and $x = 6$ is

(*a*) 162 (*b*) $\frac{567}{2}$

(*c*) $\frac{345}{2}$ (*d*) 263

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/ NUMERICAL

1. Area of the region bounded by the curve $y = \tan x$ and lines $y = 0$ and $x = 1$ is equal to

   (*a*) $\int_0^1 \tan(1-x)\,dx$ (*b*) $\int_0^1 \tan(1-x)\,dx$

   (*c*) $\int_0^1 \tan^{-1} x\,dx$ (*d*) $\tan 1 - \int_0^{\tan 1} \tan^{-1} y\,dy$

2. Let S be the area bounded by the curve $y = \sin x$ $(0 \le x \le \pi)$ and the x-axis and T be the area bounded by the curve $y = \sin x \left(0 \le x \le \frac{\pi}{2}\right)$, $y = a\cos x \left(0 \le x \le \frac{\pi}{2}\right)$ and the x-axis (where $a \in R^+$).

   If $S : T = 1 : \frac{1}{3}$ then which of the following is(are) correct?

   (*a*) The value of S is equal to 1.

   (*b*) The value of T is equal to $\frac{2}{3}$.

   (*c*) The value of a is $\frac{2}{3}$.

   (*d*) The value of a is $\frac{4}{3}$.

3. Area of the region bounded by the curve $y = e^{x^2}$ and the line $y = e$ is

   (*a*) $2\int_1^e \sqrt{(\ln y)}\,dy$ (*b*) $2e - \int_{-1}^1 e^{x^2}\,dx$

   (*c*) $\int_{-1}^1 \left(e - e^{x^2}\right)dx$ (*d*) $2\int_0^1 \sqrt{x}\,e^x\,dx$

4. If $C_1 \equiv y = \frac{1}{1+x^2}$ and $C_2 \equiv y = \frac{x^2}{2}$ be two curve lying in XY plane. Then

   (*a*) area bounded by curve C_1 and $y = 0$ is π

   (*b*) area bounded by C_1 and C_2 is $\frac{\pi}{2} - \frac{1}{3}$

   (*c*) area bounded by C_1 and C_2 is $1 - \frac{\pi}{2}$

   (*d*) area bounded by curve C_1 and x-axis is $\frac{\pi}{2}$

5. Let T be the triangle with vertices $(0, 0)$, $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then

   (*a*) Area $(R) = \frac{c^3}{6}$ (*b*) Area of $R = \frac{c^3}{3}$

   (*c*) $\lim_{c\to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = 3$ (*d*) $\lim_{c\to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = \frac{3}{2}$

6. Let $f(x) = \begin{cases} -1 & ; \ x < 0 \\ 0 & ; \ x = 0 \\ 1 & ; \ x > 1 \end{cases}$, $g(x) = x(1 - x^2)$ and $h''(x) = 6x - 4$. If $h(x)$ has local minima value 5 at $x = 1$, then which of the following are (is) correct?

   (*a*) Tangent to $y = h(x)$ at $(2, 7)$ is $5x - y = 3$.

   (*b*) Area bounded by $y = h(x)$, $y = g(f(x))$ between $x = 0$ and $x = 2$ is $\frac{32}{3}$.

   (*c*) Area bounded by ordinates of local maxima, local minima, $y = h(x)$ and x-axis can be expressed as $\int_{\frac{1}{3}}^1 \left(\left(\frac{4}{3} - x\right)^3 - 2\left(\frac{4}{3} - x\right)^2 - x + \frac{19}{3}\right)dx$.

   (*d*) Area bounded by $y = h(x)$, its tangent at $(2, 7)$ and y-axis is $\frac{20}{3}$.

7. Consider the following regions in the plane $R_1 = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ and $R_2 = \{(x,y)\} : x^2 + y^2 \le 4/3\}$. The area of the region $R_1 \cap R_2$ can be expressed as $\frac{a\sqrt{3} + b\pi}{9}$, where a and b are integers, then

   (*a*) $a = 3$ (*b*) $a = 1$

   (*c*) $b = 1$ (*d*) $b = 3$

8. The area of the region of the plane bounded by $(|x|, |y|) \le 1$ & $xy \le \frac{1}{2}$ is

   (*a*) less than $4\ell n\, 3$ (*b*) $\frac{15}{4}$

   (*c*) $2 + 2\,\ell n\, 2$ (*d*) $1 - \ell n\, 2$

9. Consider the functions $f(x)$ and $g(x)$, both defined from $R \to R$ and are defined as $f(x) = 2x - x^2$ and $g(x) = x^n$ where $n \in N$. If the area between $f(x)$ and $g(x)$ in first quadrant is 1/2 then *n* is a divisor of

   (*a*) 12 (*b*) 15

   (*c*) 20 (*d*) 30

Comprehension–1 (No. 10 to 12)

Let f(x) be a differentiable function, satisfying f (0) = 2, f′(0) = 3 and f″(x) = f(x)

10. Graph of y = f(x) cuts x -axis at

(*a*) $x = -\frac{1}{2}\ell n5$ (*b*) $x = \frac{1}{2}\ell n5$

(*c*) $x = -\ell n5$ (*d*) $x = \ell n5$

11. Area enclosed by y = f(x) in the second quadrant is

(*a*) $3 + \frac{1}{2}\ell n\sqrt{5}$ (*b*) $2 + \frac{1}{2}\ell n5$

(*c*) $3 - \sqrt{5}$ (*d*) 3

12. Area enclosed by y = f(x), $y = f^{-1}(x)$, x + y = 2 and $x + y = -\frac{1}{2}\ell n5$ is

(*a*) $8 + \frac{1}{8}(\ell n5)^2$

(*b*) $8 - 2\sqrt{5} + \frac{1}{8}(\ell n5)^2$

(*c*) $2\sqrt{5} - \frac{1}{8}(\ell n5)^2$

(*d*) $8 + 2\sqrt{5} - \frac{1}{8}(\ell n5)^2$

Comprehension–2 (No. 13 to 15)

Area enclosed by curve y = f(x) and $y = x^2 + 2$ between the abscissa x = 2 and $x = \alpha$, $\alpha > 2$ is given as $(\alpha^3 - 4\alpha^2 + 8)$ sq. unit. It is known that curve y = f(x) lies below the parabola $y = x^2 + 2$.

13. Area enclosed by curve y = f(x) with x axis, x = 0, x = 1 is

(*a*) $\frac{8}{3}$ (*b*) $\frac{16}{3}$

(*c*) $\frac{16}{7}$ (*d*) $\frac{4}{3}$

14. f(x) lies above x-axis in $x \in (p, q)$, then (q + p) is equal to

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 8

15. Value of area bounded by line y = x + 2 and y = f(x), x = 2 and x = 4 is

(*a*) $\frac{36}{5}$ (*b*) $\frac{7}{5}$

(*c*) $\frac{123}{13}$ (*d*) $\frac{79}{12}$

16. Match the column:

Column – I	Column – II
(*a*) Area bounded by $y = x^3$ with x-axis between x = 1 and x = 2 is	(p) 1/3
(*b*) Area bounded by $y \le 4x - x^2$ in I quadrant is	(q) 15/4
(*c*) Area of region $\{(x, y) : x^2 \le y \le |x|\}$ is	(r) 3/2
(*d*) Area of portion cut off by $y = x^3$, $y = x^2$ and ordinates x = 0, x = 2 is a	(s) 32/3

17. Match the column:

Column – I	Column – II
(*a*) Area bounded by region $0 \le y \le 4x - x^2 - 3$ is	(p) 32/3
(*b*) Area of the region enclosed by $y^2 = 8x$ and y = 2x is	(q) 1/2
(*c*) The area bounded by $|x| + |y| \le 1$ and $|x| \ge 1/2$ is	(r) 8/3
(*d*) Area bounded by $x \le 4 - y^2$ and $x \ge 0$ is	(s) 4/3

NUMERICAL VALUE BASED®

18. The area bounded by loop of $|y| = \sin x$ for $0 < x < \pi$, is

19. Area enclosed by the quadrilateral formed by the lines y = 2x + 3, y = 0, x = 2 and x = 4 is

20. Area of the region bounded by the curves y = |x–1| and y = 3 – |x| is

21. The area bounded by the curves $y = \sqrt{x}$, 2y + 3 = x and x - axis in the 1st quadrant is

22. The area of the triangle ABC, coordinates of whose vertices are A(2, 0), B(4, 5) and C(6, 3) is

23. The area bounded by $y = 2 - x^2$ and x + y = 0 is a/b then find value of a+b

24. Find the value of m (m > 0) for which the area bounded by the line y = mx + 2 and $x = 2y - y^2$ is 9/2 square units.

25. Find the area of the region bounded by curves $f(x) = (x - 4)^2$, $g(x) = 16 - x^2$ and the x-axis.

26. If the area bounded by the curve $y = 2x - x^2$ & the straight line $y = -x$ is A then 2A is equal to.....sq. units.

27. The area bounded by the curves $y = x(x - 3)^2$ and y = x is (in sq. units)

JEE-MAIN PREVIOUS YEARS

1. The are (in sq. units) of the region : $\{(x, y\} : x \geq 0, x + y \leq 3, x^2 \leq 4y$ and $y \leq 1 + \sqrt{x}\}$ is :

[JEE Main- 2017]

(a) $\frac{5}{2}$ (b) $\frac{59}{12}$

(c) $\frac{3}{2}$ (d) $\frac{7}{3}$

2. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curvey = (gof) (x) and the lines $x = \alpha$, $x = \beta$ and $y = 0$ is -

[JEE Main-2018]

(a) $\frac{1}{2}\left(\sqrt{3}+1\right)$ (b) $\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$

(c) $\frac{1}{2}\left(\sqrt{2}-1\right)$ (d) $\frac{1}{2}\left(\sqrt{3}-1\right)$

3. The area (in sq. units) bounded by the parabolay = x^2–1, the tangent at the point (2, 3) to it and the y–axis is L

[JEE Main - 2019 (January)]

(a) $\frac{8}{3}$ (b) $\frac{32}{3}$

(c) $\frac{53}{3}$ (d) $\frac{14}{3}$

4. The area of the region A$\{(x, y) : 0 \leq y \leq x\,|x| + 1$ and $-1 \leq x \leq 1\}$ in sq. units, is : [JEE Main - 2019 (January)]

(a) 2/3 (b) 1/3

(c) 2 (d) 4/3

5. Let $I = \int_a^b \left(x^4 - 2x^2\right)dx$. If I is minimum then the ordered pair (a, b) is: [JEE Main - 2019 (January)]

(a) $\left(0, \sqrt{2}\right)$ (b) $\left(-\sqrt{2}, 0\right)$

(c) $\left(\sqrt{2}, -\sqrt{2}\right)$ (d) $\left(-\sqrt{2}, \sqrt{2}\right)$

6. If the area enclosed between the curves $y = kx^2$ and $x = ky^2$, $(k > 0)$, is 1 square unit. Then k is:

[JEE Main - 2019 (January)]

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\sqrt{3}$ (d) $\frac{2}{\sqrt{3}}$

7. The area (in sq. untis) in the first quadrant bounded by the parabola, $y = x^2 + 1$, then tangent to it at the point (2, 5) and the coordinate axes is :

[JEE Main - 2019 (January)]

(a) $\frac{8}{3}$ (b) $\frac{37}{24}$

(c) $\frac{187}{24}$ (d) $\frac{14}{3}$

8. The area (in sq. units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is:

[JEE Main - 2019 (January)]

(a) 5/4 (b) 9/8

(c) 7/8 (d) 3/4

9. The area (in sq. units) of the region bounded by the parabola, $y = x^2 + 2$ and the lines, $y = x + 1$, $x = 0$ and $x = 3$, is: [JEE Main - 2019 (January)]

(a) $\frac{15}{4}$ (b) $\frac{21}{2}$

(c) $\frac{17}{4}$ (d) $\frac{15}{2}$

10. Let S be the set of all points in $(-\pi, \pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is non-differentiable. Then S is a subset of which of the following?

[JEE Main - 2019 (January)]

(a) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$ (b) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

(c) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ (d) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

11. The area (in sq. units) of the region $A = \{(x, y) \in R \times R \,|\, 0 \leq x \leq 3, 0 \leq y \leq 4, y \leq x^2 + 3x\}$ is :

[JEE Main - 2019(April)]

(a) $\frac{53}{6}$ (b) $\frac{59}{6}$

(c) 8 (d) $\frac{26}{3}$

12. Let $S(\alpha) = \{(x,y) : y^2 \leq x, 0 \leq x \leq \alpha\}$ and $A(\alpha)$ is area of the region $S(\alpha)$. If for a λ, $0 < \lambda < 4$, $A(\lambda) : A(4) = 2 : 5$, then λ equals [JEE Main - 2019(April)]

(a) $2\left(\frac{4}{25}\right)^{\frac{1}{3}}$ (b) $4\left(\frac{4}{25}\right)^{\frac{1}{3}}$

(c) $2\left(\frac{2}{5}\right)^{\frac{1}{3}}$ (d) $4\left(\frac{2}{5}\right)^{\frac{1}{3}}$

13. The area (in sq. units) of the region $A = \{(x, y) : x^2 \le y \le x + 2\}$ is

[JEE Main - 2019(April)]

(a) $\frac{10}{3}$ (b) $\frac{9}{2}$

(c) $\frac{31}{6}$ (d) $\frac{13}{6}$

14. The area (in sq. units) of the region $A = \{(x, y) : \frac{y^2}{2} \le x \le y + 4\}$ is :-

[JEE Main - 2019(April)]

(a) $\frac{53}{3}$ (b) 18

(c) 30 (d) 16

15. The area (in sq. units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is :

[JEE Main - 2019(April)]

(a) $\frac{3}{2} - \frac{1}{\log_e 2}$ (b) $\frac{1}{2}$

(c) $\log_e 2 + \frac{3}{2}$ (d) $\frac{3}{2}$

16. if the area (in sq. sunits) of the region $\{(x, y) : y^2 \le 4x, x + y \le 1, x \ge 0, y \ge 0\}$ is $a\sqrt{2} + b$, then $a - b$ is equal to :

[JEE Main - 2019(April)]

(a) $\frac{3}{8}$ (b) $\frac{10}{3}$

(c) 6 (d) $-\frac{2}{3}$

17. If the area (in sq. units) bounded by the parabola $y^2 = 4\lambda x$ and line $y = \lambda x, \lambda > 0$, is $\frac{1}{9}$, then λ is equal to :

[JEE Main - 2019(April)]

(a) 24 (b) 48

(c) $4\sqrt{3}$ (d) $2\sqrt{6}$

18. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is

[JEE Main-2020 (January)]

(a) $\frac{1}{3}(6\pi - 1)$ (b) $\frac{1}{3}(12\pi - 1)$

(c) $\frac{1}{6}(12\pi - 1)$ (d) $\frac{1}{6}(24\pi - 1)$

19. The area (in units) of the region $\{(x, y) \in R^2 | 4x^2 \le y \le 8x + 12\}$ is :

[JEE Main-2020 (January)]

(a) $\frac{127}{3}$ (b) $\frac{125}{3}$

(c) $\frac{128}{3}$ (d) $\frac{124}{3}$

20. For $a > 0$, let the curves $C_1 : y^2 = ax$ and $C_2 = x^2 = ay$ intersect at origin O and a point P. Let the line $x = b (0 < b < a)$ intersect the chord OP and the x-axis at points Q and R, respectively. If the line $x = b$ bisects the area bounded by the curves, C_1 and C_2, and the area of $\Delta OQR = \frac{1}{2}$, then 'a' satisfies the equation

[JEE Main-2020 (January)]

(a) $x^6 - 12x^3 - 4 = 0$ (b) $x^6 - 6x^3 + 4 = 0$

(c) $x^6 - 12x^3 + 4 = 0$ (d) $x^6 + 6x^3 - 4 = 0$

21. The area (in sq. units) of the region $(x,y) \in R^2 ; x^2 \le y \le 3 - 2x\}$, is

[JEE Main-2020 (January)]

(a) $\frac{29}{3}$ (b) $\frac{34}{3}$

(c) $\frac{32}{3}$ (d) $\frac{31}{3}$

22. Given $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \le 1 \end{cases}$ and $g(x) = \left(x - \frac{1}{2}\right)^2$

$x \in R$. Then the area (in sq. units) of the region bounded by the curves, $y = f(x)$ and $y = g(x)$ between the lines, $2x = 1$ and $2x = \sqrt{3}$, is :

[JEE Main-2020 (January)]

(a) $\frac{1}{2} - \frac{\sqrt{3}}{4}$ (b) $\frac{1}{3} + \frac{\sqrt{3}}{4}$

(c) $\frac{1}{2} + \frac{\sqrt{3}}{4}$ (d) $\frac{\sqrt{3}}{4} - \frac{1}{3}$

23. Consider a region $R = \{(x, y) \in R^2 : x^2 \le y \le 2x\}$. if a line $y = \alpha$ divides the area of region R into two equal parts, then which of the following is true?

[JEE Main-2020 (September)]

(a) $3\alpha^2 - 8\alpha + 8 = 0$ (b) $\alpha^3 - 6\alpha^{3/2} - 16 = 0$

(c) $3\alpha^2 - 8\alpha^{3/2} + 8 = 0$ (d) $\alpha^3 - 6\alpha^2 + 16 = 0$

24. Area (in sq. units) of the region outside $\frac{|x|}{2} + \frac{|y|}{3} = 1$ and inside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is :

[JEE Main-2020 (September)]

(a) $3(4 - \pi)$ (b) $6(4 - \pi)$

(c) $6(\pi - 2)$ (d) $3(\pi - 2)$

25. The area (in sq. units) of the region $\left\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, \frac{1}{2} \le x \le 2\right\}$ is

[JEE Main-2020 (September)]

(a) $\frac{79}{16}$ (b) $\frac{23}{6}$

(c) $\frac{79}{24}$ (d) $\frac{23}{16}$

26. Let $f(x)=|x-2|$ and $g(x)=f(f(x)), x \in [0,4]$.

Then $\int_0^3 (g(x)-f(x))dx$ is equal to :

[JEE Main-2020 (September)]

(a) 0 (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ (d) 1

27. The area (in sq. units) of the region $A=\{(x, y):(x-1)[x] \le y \le 2\sqrt{x}, 0 \le x \le 2\}$, where [t] denotes the greatest integer function, is : [JEE Main-2020 (September)]

(a) $\frac{8}{3}\sqrt{2}-1$ (b) $\frac{4}{3}\sqrt{2}+1$

(c) $\frac{8}{3}\sqrt{2}-\frac{1}{2}$ (d) $\frac{4}{3}\sqrt{2}-\frac{1}{2}$

28. The area (in sq. units) of the region enclosed by the curves $y=x^2-1$ and $y=1-x^2$ is equal to :

[JEE Main-2020 (September)]

(a) $\frac{7}{2}$ (b) $\frac{4}{3}$

(c) $\frac{8}{3}$ (d) $\frac{16}{3}$

29. The area (in sq. units) of the region $A=\{(x, y):|x|+|y| \le 1, 2y^2 \ge |x|\}$ is :

[JEE Main-2020 (September)]

(a) $\frac{1}{6}$ (b) $\frac{7}{6}$

(c) $\frac{5}{6}$ (d) $\frac{1}{3}$

30. The area, enclosed by the curves $y=\sin x+\cos x$ and $y=|\cos x-\sin x|$ and the lines $x=0, x=\frac{\pi}{2}$, is :

[JEE Main-2021]

(a) $2\sqrt{2}(\sqrt{2}-1)$ (b) $2(\sqrt{2}+1)$

(c) $4(\sqrt{2}-1)$ (d) $2\sqrt{2}(\sqrt{2}+1)$

31. If the area of the bounded region

$R=\left\{(x, y): \max\{0, \log_e x\} \le y \le 2^x, \frac{1}{2} \le x \le 2\right\}$ is,

$\alpha(\log_e 2)^{-1}+\beta(\log_e 2)+\gamma_1$ then the value of

$(\alpha+\beta-2\lambda)^2$ is equal to: [JEE Main-2021]

(a) 8 (b) 2

(c) 4 (d) 1

JEE-ADVANCED PREVIOUS YEARS

1. If S be the area of the region enclosed by $y=e^{-x^2}$, $y=0, x=0$, and $x=1$. Then [IIT-JEE 2012]

(a) $S \ge \frac{1}{e}$ (b) $\left(\frac{1}{5}\right)$

(c) $S \le \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ (d) $S \le \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

2. The area enclosed by the curves $y=\sin x+\cos x$ and $y=|\cos x-\sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE (Advanced) 2013]

(a) $4(\sqrt{2}-1)$ (b) $2\sqrt{2}(\sqrt{2}-1)$

(c) $2(\sqrt{2}+1)$ (d) $2\sqrt{2}(\sqrt{2}+1)$

3. If the line $x=\alpha$ divides the area o region $R=\{(x, y) \in R^2: x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then

[JEE (Advanced)-2017]

(a) $2\alpha^4-4\alpha^2+1=0$ (b) $\alpha^4+4\alpha^2-1=0$

(c) $\frac{1}{2}<\alpha<1$ (d) $0<\alpha \le \frac{1}{2}$

4. Let $f:[0, \infty) \to \mathbb{R}$ be a containuous function such that $f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$ for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE?

[JEE Advanced - 2018]

(a) The curve $y=f(x)$ passes through the point (1, 2)

(b) The curve $y=f(x)$ passes through the point (2, –1)

(c) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(d) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

5. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y=x^n (n>1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is

[JEE Advanced - 2018]

6. The area of the region $\{(x,y) : xy \le 8, 1 \le y \le x^2\}$ is

[JEE Advanced -2019]

(a) $8\log_e 2 - \frac{14}{3}$

(b) $16\log_e 2 - \frac{14}{3}$

(c) $16\log_e 2 - 6$

(d) $8\log_e 2 - \frac{7}{3}$

7. Let the functions $f: R \to R$ and $g: R \to R$ be defined by

$f(x) = e^{x-1} - e^{|x-1|}$ and $g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

[JEE(Advanced) - 2020]

(a) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$ (b) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$

(c) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$ (d) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

ANSWER KEY

Topicwise Questions

1. (c)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (a)
21. (b)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (c)	

Learning Plus

1. (c)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (c)
11. (b)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (c)
21. (a)	**22.** (a)	**23.** (a)	**24.** (b)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (b)
31. (d)	**32.** (b)								

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (a,d)	**2.** (b,d)	**3.** (a,b,c,d)	**4.** (a,b)	**5.** (a,c)	**6.** (a,b,c,d)	**7.** (a,c)	**8.** (a,d)	**9.** (b,c,d)	**10.** (a)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (a) → (q), (b) → (s), (c) → (p), (d) → (r)				
17. (a) → (s), (b) → (s), (c) → (q), (d) → (p)				**18.** [0004]	**19.** [0018]	**20.** [0004]	**21.** [0009]	**22.** [0007]	**23.** [0011]
24. [1]	**25.** [64]	**26.** [9]	**27.** [8]						

JEE Mains & Advanced Past Years Questions

JEE MAIN PREVIOUS YEARS

1. (a)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (b)
11. (b)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (c)
21. (c)	**22.** (d)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (c)	**29.** (c)	**30.** (a)
31. (b)									

JEE ADVANCED PREVIOUS YEARS

1. (a,b,d)	**2.** (b)	**3.** (a,c)	**4.** (b,c)	**5.** (4)	**6.** (b)	**7.** (a)

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27 DIFFERENTIAL EQUATIONS

INTRODUCTION

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a differential equation. There are two kinds of differential equation:

Ordinary Differential Equation : If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for example $\frac{dy}{dx} + \frac{dz}{dx} = y + z,$

$$\frac{dy}{dx} + xy = \sin x, \quad \frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x,$$

$$k\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2},$$

$$y = x\frac{dy}{dx} + k\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

Partial differential equation : If the dependent variables depend on two or more independent variables, then it is known as partial differential equation

for example $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y} = ax, \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : Order is the highest differential appearing in a differential equation.

Degree : The power of the highest order derivative occurring in a differential equation is called the degree of the differential equation, for this purpose the differential equation is made free from radicals and fractions of derivatives.

Note:

(i) In the differential equation, all the derivatives should be expressed in the polynomial form

$$f_1(x, y)\left[\frac{d^m y}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1} y}{dx^{m-1}}\right]^{n_2} + \ldots\ldots$$

$$+ f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

e.g. $\sin x\left(\frac{d^3y}{dx^3}\right) + \cos x\left(\frac{dy}{dx}\right)^{100} = 0$;

order 3, degree 1

(ii) If differential equation cannot be written as polynomial in derivatives, then degree is not defined.

e.g. $e^{\frac{d^2y}{dx^2}} = \frac{dy}{dx}$

TRAIN YOUR BRAIN

Example 1: Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$

(ii) $y = e^{\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)}$

(iii) $\sin\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = y$

(iv) $e^{y'''} - xy'' + y = 0$

Sol. (i) $\left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^6$

$\therefore$ order = 2, degree = 4

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \ell ny$

$\therefore$ order = 2, degree = 1

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin^{-1} y$

$\therefore$ order = 2, degree = 1

(iv) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

$\because$ equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

Example 2: Differential equation Order of D.E. and Degree of D.E.

- $\frac{dy}{dx} + 4y = \sin x$; (order -1, degree-1)
- $\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$; (order -2, degree-4)
- $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$; (order -2, degree-1)
- $\frac{dy}{dx} = \frac{x^4 - y^4}{xy(x^2 + y^2)}$; (order -1, degree-1)
- $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$

  $\Leftrightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$;

  (order -1, degree-2)
- $\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}$

  $\Leftrightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$;

  (order -2, degree-2)

FORMATION OF DIFFERENTIAL EQUATION

To form a differential equation, we differentiate the given family of curves and eliminate the unknown constants.

Note :

(i) The arbitrary constants in a differential equation are said to be independent, when it is impossible to deduce an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^B \cdot e^x = Ce^x$.

Similarly the relation $y = A\sin x + B\cos(x + C)$ appears to contain three arbitrary constatns, but they are realy equivalent to two only. Similarly

$y = (C_1 + C_2)\cos(x + C_3) - C_4e^{x-C_5}$ has only 3 independent arbitrary constants hence will be of order 3.

(ii) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

e.g.Consider the equation $y = ax$. This represents the Cartesian equation to a family of straight lines through the origin.

Differentiating $y = ax$, we get $\frac{dy}{dx} = \alpha$. Eliminating α, we get the differential equation $y = \frac{dy}{dx}.x$.

Hence $y = x\frac{dy}{dx}$ is the differential equation of all straight lines passing through the origin.

e.g.Consider another example, the equation $x^2 + y^2 = a^2$.This, for various a, represents a family of concentric circles with centre at origin.

Differentiating the relation we get

$2x + 2y\frac{dy}{dx} = 0$ (a is eliminated)

i.e. $x + y\frac{dy}{dx} = 0$

which may be said to be the differential equation to a family of concentric circles.

e.g. Now consider another equation representing a family of curves in the form

$f(x, y, a, b) = 0$.... (i)

containing two arbitrary constants. In this case, since there are two constants, it becomes necessary to differentiate equation twice so that the result contains

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and can be expressed in the form

$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$ ....(ii)

This equation is said to represent a differential equation of the family of curves represented by equation (i). Thus

in the case of $y = ax + b$ $\frac{dy}{dx} = a$

$\frac{d^2y}{dx^2} = 0$ which is the differential equation of the family of all straight lines.

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Example 3 : Form a differential equation of family of straight lines passing through origin.

Sol. Family of straight lines passing through origin is $y = mx$ where 'm' is a parameter.

Differentiating w.r.t. x

$$\frac{dy}{dx} = m$$

Eliminating 'm' from both equations, we obtain

$\frac{dy}{dx} = \frac{y}{x}$ which is the required differential equation.

Example 4: Form a differential equation of family of circles touching x-axis at the origin

Sol. Equation of family of circles touching x-axis at the origin is

$x^2 + y^2 + \lambda y = 0$ .....(i)

where λ is a parameter

$$2x + 2y\frac{dy}{dx} + \lambda\frac{dy}{dx} = 0 \text{.....(ii)}$$

Eliminating 'λ' from (i) and (ii)

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Which is required differential equation.

Example 5: Form the differential equation of the following relation:

(i) $x^2 + y^2 = 2ax$

(ii) $x^2 + y^2 = 2ax + b$

(iii) $y = ae^x + be^{2x}$

Sol. (i) Consider the relation $x^2 + y^2 = 2ax$

Differentiating, $2x + 2y\frac{dy}{dx} = 2a$

Eliminating a, $x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$

$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

In this case the relation contains only one constant and hence the differential equation contains only $\frac{dy}{dx}$.

(ii) Consider the relation $x^2 + y^2 = 2ax + b$

Differentiating $2x + 2y\frac{dy}{dx} = 2a$

Differentiating once again, $1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

which is the differential equation to the given equation and since there are two constants a and b, the differential equation contains (the second order) derivative $\frac{d^2y}{dx^2}$.

(iii) Consider the relation $y = ae^x + be^{2x}$

$$\frac{dy}{dx} = ae^x + 2be^{2x}$$

$$\frac{d^2y}{dx^2} = ae^x + 4be^{2x}$$

Consider

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = ae^x + 4be^{2x} - 3(ae^x + 2be^{2x}) + 2(ae^x + be^{2x})$$

$\therefore$ For the relation $y = ae^x + be^{2x}$, we get the (second order) differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

SOLUTION OF A DIFFERENTIAL EQUATION

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation

NOTE : The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be two types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive) :** A relation in x and y satisfying a given differential equation and involving exactly same number of arbitrary constants as order of differential equation.

(ii) **Particular Solution :** A solution obtained by assigning values to one or more than one arbitrary constant of general solution.

Differential Equation of First Order and First Degree :

A differential equation of first order and first degree is of the type $\frac{dy}{dx} + f(x, y) = 0$, which can also be written as : $Mdx + Ndy = 0$, where M and N are functions of x and y.

METHODS TO SOLVE DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Variables separable : If the differential equation can be put in the form, $f(x)\,dx = \phi(y)\,dy$ we say that variables are separable and solution can be obtained by integrating each side separately. A general solution of this will be $\int f(x)\,dx = \int \phi(y)\,dy + c$, where c is an arbitrary constant

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Example 6: Solve the differential equation $(1 + x)\, y\, dx = (y - 1)\, x\, dy$

Sol. The equation can be written as -

$$\left(\frac{1+x}{x}\right) dx = \left(\frac{y-1}{y}\right) dy$$

$$\int\left(\frac{1}{x}+1\right) dx = \int\left(1-\frac{1}{y}\right) dy$$

$$\ell n\, x + x = y - \ell ny + c$$

$$\ell ny + \ell nx = y - x + c$$

$$xy = ce^{y-x}$$

Example 7: Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol. Separating the variables

$$\frac{dy}{dx} = e^{-y} \times (e^x + x^2)$$

$e^y dy = (e^x + x^2)dx$, integrating the solution is

$$e^y = e^x + \frac{x^3}{3} + A$$

$\Rightarrow 3(e^y - e^x) = x^3 + C$($C$ is an arbitrary constant)

Example 8: Solve : $\frac{dy}{dx} = (e^x + 1)(1 + y^2)$

Sol. The equation can be written as

$$\frac{dy}{1+y^2} = (e^x + 1)dx$$

Integrating both sides,

$\tan^{-1} y = e^x + x + c$.

Polar coordinates transformations : Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

(a) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$

(ii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

(iii) $x\,dy - y\,dx = r^2 d\theta$

(b) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$

(ii) $x\,dy - y\,dx = r^2 \sec\theta\, d\theta$.

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Example 9: Solve the differential equation $xdx + ydy = x(xdy - ydx)$

Sol. Taking $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$2x\,dx + 2ydy = 2rdr$

$xdx + ydy = rdr$ .....(i)

$$\frac{y}{x} = \tan\theta$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$xdy - y\,dx = x^2 \sec^2\theta \cdot d\theta$

$xdy - ydx = r^2\, d\theta$ .....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\,dr = r\cos\theta \cdot r^2\, d\theta$

$$\frac{dr}{r^2} = \cos\theta\, d\theta \Rightarrow -\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}} = c \text{ where } -\lambda' = c$$

$(y + 1)^2 = c(x^2 + y^2)$

Equations Reducible to the Variables Separable form : If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is $\frac{dy}{dx} = f(ax + by + c)$ $a, b \neq 0$. To solve this, put $ax + by + c = t$.

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Example 10: Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{t^2+4} = dx \text{(Variables are separated)}$$

Integrating both sides,

$$\int \frac{dt}{4+t^2} = \int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Homogeneous Differential Equations : A differential equations of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equation.

It can also be written in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, by dividing both the functions by x^n where n is the degree of function. To solve this equation, substitute $\frac{y}{x} = t$ or $y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$ Then the equation reduces to $t + x\frac{dt}{dx} = f(t)$ which can be easily reduced to variable separable as $\frac{dt}{f(t)-t} = \frac{dx}{x}$.

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Example 11: Solve the differential equation $\frac{dy}{dx} = \frac{x-y}{x+y}$ (Note: x – y, x + y are homogeneous in x and y of degree one)

Sol. Taking $y = vx$,

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

Substituting in the given equation

$$v + x\frac{dv}{dx} = \frac{x-vx}{x+vx} = \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v}$$

Now, separating the variables

$$\int \frac{1+v}{1-2v-v^2}dv = \int \frac{dx}{x} + A$$

$$-\frac{1}{2}\log(1-2v-v^2) = \log x + A$$

$\log[(1-2v-v^2)x^2] = \text{constant}$; $(1-2v-v^2)x^2 = C$.

$x^2 - 2xy - y^2 = C$ is therefore the solution where C is an arbitrary constant.

Example 12: Solve : $(x^2 - y^2)\,dx + 2xydy = 0$ given that $y = 1$ when $x = 1$

Sol. $\frac{dy}{dx} = -\frac{x^2-y^2}{2xy}$

$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = -\frac{1-v^2}{2v}$$

$$\int \frac{2v}{1+v^2}dv = -\int \frac{dx}{x}$$

$\ell n(1+v^2) = -\ell nx + c$

at $x = 1, y = 1$

$\therefore v = 1$

$\ell n\,2 = c$

$$\therefore \ell n\left\{\left(1+\frac{y^2}{x^2}\right).x\right\} = \ell n2$$

$x^2 + y^2 = 2x$

Equations Reducible to the Homogeneous form

Equations of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C} \quad \text{.....(1)}$$

can be made homogeneous (in new variables X and Y) by substituting $x = X + h$ and $y = Y + k$, where h and k are constants to obtain,

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)} \quad \text{.....(2)}$$

These constants are chosen such that $ah + bk + c = 0$, and $Ah + Bk + C = 0$. Thus we obtain the following differential equation

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}.$$

The differential equation can now be solved by substituting $Y = vX$.

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Example 13: Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

Sol. Let $x = X + h$, $y = Y + k$

$$\frac{dy}{dX} = \frac{d}{dX}(Y+k)$$

$$\frac{dy}{dX} = \frac{dY}{dX} \quad \text{.....(i)}$$

$$\frac{dx}{dX} = 1 + 0 \quad \text{.....(ii)}$$

on dividing (i) by (ii) $\frac{dy}{dx} = \frac{dY}{dX}$

$$\therefore \frac{dY}{dX} = \frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4}$$

$$= \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$$

h & k are such that $h + 2k - 5 = 0$ & $2h + k - 4 = 0$ $h = 1, k = 2$

$$\therefore \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

which is homogeneous differential equation.

Now, substituting $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX} \Rightarrow \therefore X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X}$$

$$\int \left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right) dv = \ell n\, X + c$$

$$\frac{1}{2}\ell n(v+1) - \frac{3}{2}\ell n(1-v) = \ell n\, X + c$$

$$\ell n\left|\frac{v+1}{(1-v)^3}\right| = \ell n X^2 + 2c$$

$$\frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2} = e^{2c}$$

$X + Y = c'(X-Y)^3$ where $e^{2c} = c'$

$x - 1 + y - 2 = c'(x - 1 - y + 2)^3$

$x + y - 3 = c'(x - y + 1)^3$

Special case :

Case 1: In equation (1) if $\frac{a}{A} = \frac{b}{B}$, then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Case 2: In equation (1), if $b + A = 0$, then by a simple cross multiplication equation (1) becomes an **exact differential equation**.

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Example 14: Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Sol. Putting $u = 2x + 3y$

$$\frac{du}{dx} = 2 + 3\cdot\frac{dy}{dx}$$

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5}$$

$$\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$$

$$\int \frac{2u-5}{7u-13}\,dx = \int dx$$

$$\Rightarrow \frac{2}{7}\int 1.du - \frac{9}{7}\int \frac{1}{7u-13}.du = x + c$$

$$\Rightarrow \frac{2}{7}u - \frac{9}{7}\cdot\frac{1}{7}\ell n(7u-13) = x + c$$

$$\Rightarrow 4x + 6y - \frac{9}{7}\ell n(14x+21y-13) = 7x + 7c$$

$$\Rightarrow -3x + 6y - \frac{9}{7}\ell n(14x+21y-13) = c'$$

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Example 15: Solve $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Sol. Cross multiplying,

$2xdy + y\,dy - dy = xdx - 2ydx + 5dx$

$2(xdy + y\,dx) + ydy - dy = xdx + 5\,dx$

$2\,d(xy) + y\,dy - dy = xdx + 5dx$

On integrating,

$$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$\Rightarrow x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

Note: If the homogeneous equation is of the form : $yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Exact Differential Equation : These are equations of the form $\sum f(\phi(x,y))d\phi = 0$. Sometimes given differential equations may be expressed as different derivatives of functions, and then integrating them gives the result. Following results may be helpful in such problems:

- $d(xy) = xdy + ydx$
- $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
- $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$
- $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$
- $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$
- $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$
- $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$
- $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$
- $d[\ln(xy)] = \frac{xdy + ydx}{xy}$
- $d\left(\ln\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$
- $d\left[\frac{1}{2}\ln(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$
- $d\left[\ln\left(\frac{y}{x}\right)\right] = \frac{xdy - ydy}{xy}$
- $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$
- $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$
- $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$
- $d(x^m y^n) = x^{m-1}y^{n-1}(mydx + nxdy)$

Example 16: Solve : $y\,dx + x\,dy = \frac{xdy - ydx}{x^2 + y^2}$

Sol. $ydx + xdy = \frac{xdy - ydx}{x^2 + y^2}$

$d(xy) = d(\tan^{-1} y/x)$

Integrating both sides

$xy = \tan^{-1} y/x + c$

Example 17: Solve : $(2x\,\ell ny)\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

Sol. The given equation can be written as

$$\ell ny(2x)\,dx + x^2\left(\frac{dy}{y}\right) + 3y^2\,dy = 0$$

$\Rightarrow \ell ny\,d(x^2) + x^2\,d(\ell ny) + d(y^3) = 0$

$\Rightarrow d(x^2\,\ell ny) + d(y^3) = 0$

Now integrating each term, we get

$x^2\,\ell ny + y^3 = c$

Linear Differential Equation : A linear differential equation has the following characteristics :

(i) Dependent variable and its derivative in first degree only and are not multiplied together

(ii) All the derivatives should be in a polynomial form

(iii) The order may be more than one

Note: The mth order linear differential equation is of the form.

$P_0(x)\frac{d^m y}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots\ldots + P_{m-1}(x)\frac{dy}{dx} + P_m(x)\,y = \phi(x)$, where $P_0(x), P_1(x) \ldots\ldots\ldots\ldots P_m(x)$ are called the coefficients of the differential equation. $\frac{dy}{dx} + y^2\sin x = \ln x$ is not a Linear differential equation.

Linear differential equations of first order: The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y. (Where P and Q are functions of x only).

Integrating Factor (I.F.) : It is an expression which multiplied to a differential equation converts it into an exact form. I.F for linear differential equation $= e^{\int Pdx}$ (constant of integration will not be considered)

$\therefore$ after multiplying above equation by I.F it becomes;

$$\frac{dy}{dx}\cdot e^{\int Pdx} + Py\cdot e^{\int Pdx} = Q\cdot e^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}(y\cdot e^{\int Pdx}) = Q\cdot e^{\int Pdx}$$

$$\Rightarrow y\cdot e^{\int Pdx} = \int Qe^{\int Pdx}dx + C$$

Note: Some times differential equation becomes linear, if x is taken as the dependent variable, and y as independent variable. The differential equation has then the following

form : $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are functions of y.

The I.F. now is $e^{\int P_1 dy}$

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Example 18: Solve $\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$

Sol. $\frac{dy}{dx} + Py = Q$

$P = \frac{3x^2}{1+x^3}$

$IF = e^{\int P.dx} = e^{\int \frac{3x^2}{1+x^3} dx} = e^{\ell n(1+x^3)} = 1+x^3$

$\therefore$ General solution is

$y(I.F.) = \int Q(I.F.).dx + c$

$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3} (1+x^3) dx + c$

$y(1+x^3) = \int \frac{1-\cos 2x}{2} dx + c$

$y(1+x^3) = \frac{1}{2} x - \frac{\sin 2x}{4} + c$

Example 19: Solve : $x \ell nx \frac{dy}{dx} + y = 2 \ell n x$

Sol. $\frac{dy}{dx} + \frac{1}{x \ell nx} y = \frac{2}{x}$

$P = \frac{1}{x \ell nx}, Q = \frac{2}{x}$

$IF = e^{\int P.dx} = e^{\int \frac{1}{x \ell nx} dx} = e^{\ell n(\ell nx)} = \ell n x$

$\therefore$ General solution is

$y.(\ell n x) = \int \frac{2}{x} . \ell nx . dx + c$

$y(\ell n x) = (\ell n x)^2 + c$

Example 20: Solve the differential equation $x \frac{dy}{dx} = y - \cos\left(\frac{1}{x}\right)$.

Sol. Here, $x \frac{dy}{dx} - y = -\cos\left(\frac{1}{x}\right)$

$\therefore \frac{dy}{dx} - \frac{1}{x} . y = -\frac{1}{x} \cos\left(\frac{1}{x}\right)$; this is in the linear form.

Integrating factor $e^{\int \frac{(-1)}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$.

Multiplying by the integrating factor,

$\frac{1}{x} . \frac{dy}{dx} - \frac{1}{x^2} y = -\frac{1}{x^2} \cos\left(\frac{1}{x}\right)$

or $\frac{d}{dx}\left\{\frac{1}{x} . y\right\} = -\frac{1}{x^2} \cos\left(\frac{1}{x}\right)$

or $d\left\{\frac{y}{x}\right\} = -\frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$

or $\int d\left(\frac{y}{x}\right) = \int -\frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$

$\therefore \frac{y}{x} = \int \cos\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right)$

or $\frac{y}{x} = \sin\left(\frac{1}{x}\right) + c$, where c is an arbitrary constant.

$\therefore y - x \sin\left(\frac{1}{x}\right) = cx$.

Equations reducible to the linear differential equation By change of variable.

If equation is of the form.

$R(y)\frac{dy}{dx} + P(x)S(y) = Q(x)$

such that $\frac{dS}{dy} = R$, then put $S(y) = t$

$\Rightarrow \frac{dt}{dx} = \frac{dS}{dx} = \frac{dS}{dy} . \frac{dy}{dx} = \frac{Rdy}{dx}$

Thus differential equation reduces to

$\frac{dt}{dx} + P(x)t = Q(x)$

which is linear differential equation.

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Example 21: Solve the differential equation $\cos 2y \frac{dy}{dx} + \frac{1}{x} \sin 2y = e^x$.

Sol. Put $\sin 2y = t$

$\Rightarrow 2\cos 2y \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \cos 2y \frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

so given equation reduces to $\frac{1}{2}\frac{dt}{dx} + \frac{1}{x} t = e^x$

$\Rightarrow \frac{dt}{dx} + \frac{2}{x} t = 2e^x$

$I.F. = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\therefore$ Solution is $tx^2 = 2\int x^2 e^x dx$

$\Rightarrow (\sin y)x^2 = 2(x^2e^x - 2xe^x + 2e^x) + c$, where 'c' is an arbitrary constant.

Example 22: Solve : $y \sin x \frac{dy}{dx} = \cos x (\sin x - y^2)$

Sol. The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$2y \frac{dy}{dx} = \frac{dz}{dx}$

differential equation becomes

$\frac{\sin x}{2}\frac{dz}{dx} + \cos x . z = \sin x \cos x$

$\frac{dz}{dx} + 2 \cot x . z = 2 \cos x$ which is linear in $\frac{dz}{dx}$

$IF = e^{\int 2 \cot x dx} = e^{2 \ln \sin x} = \sin^2 x$

$\therefore$ General solution is -

$z . \sin^2 x = \int 2 \cos x . \sin^2 x . dx + c$

$y^2 \sin^2 x = \frac{2}{3} \sin^3 x + c$

Bernoulli's equation : Differential equation of the form

$\frac{dy}{dx} + Py = Qy^n$;

P, Q are functions of x is called Bernoulli's equation.

To solve this, divide the equation by y^n, then

$\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$

Put $\frac{1}{y^{n-1}} = t$

$\Rightarrow -\frac{(n-1)}{y^n}\frac{dy}{dx} = \frac{dt}{dx}$

Differential equation reduces to

$\frac{1}{(1-n)}\frac{dt}{dx} + P.t = Q$

e.g. : $2 \sin x \frac{dy}{dx} - y \cos x = xy^3 e^x$.

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Example 23: Solve : $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$ (Bernoulli's equation)

Sol. Dividing both sides by y^2

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2}$ ..... (1)

Putting $\frac{1}{y} = t$

$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Differential equation (1) becomes,

$-\frac{dt}{dx} - \frac{t}{x} = \frac{1}{x^2}$

$\frac{dt}{dx} + \frac{t}{x} = -\frac{1}{x^2}$

Which is linear differential equation in $\frac{dt}{dx}$

$IF = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$\therefore$ General solution is -

$$t.x = \int -\frac{1}{x^2} . x\, dx + c$$

$$tx = -\ell nx + c$$

$$\frac{x}{y} = -\ell nx + c$$

Example 24: Solve the differential equation $\frac{dy}{dx} = x^3y^3 - xy$.

Sol. Here, $\frac{dy}{dx} + xy = x^3y^3$

$$\therefore \frac{1}{y^3}\frac{dy}{dx} + x.\frac{1}{y^2} = x^3 \qquad \ldots(i)$$

Put $\frac{1}{y^2} = z$; then $\frac{-2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes $-\frac{1}{2}\frac{dz}{dx} + xz = x^3$

or $\frac{dz}{dx} - 2xz = -2x^3$

It is in the linear form, So, integrating factor

$e^{\int -2x dx} = e^{-x^2}$.

Multiplying by it,

$$e^{-x^2}\frac{dz}{dx} - 2xe^{-x^2}.z = -2x^3.e^{-x^2}$$

or $\frac{d}{dx}\left(ze^{-x^2}\right) = -2x^3e^{-x^2}$

$\therefore d\left(ze^{-x^2}\right) = -2x^3e^{-x^2}dx$

or $\int d\left(ze^{-x^2}\right) = \int -2x^3.e^{-x^2}dx$

or $ze^{-x^2} = -\int te^{-t}dt$ (putting $x^2 = t$)

$$= -\left[t.\frac{e^{-t}}{-1} - \int \frac{e^{-t}}{-1}dt\right] = te^{-t} - \int e^{-t}dt$$

or $\frac{1}{y^2}e^{-x^2} = x^2.e^{-x^2} + e^{-x^2} + c$

$\therefore (x^2 + 1 + ce^{x^2})y^2 = 1$

Where 'c' is an arbitrary constant.

Note : Sometimes the differential equation can be written as linear differential equation in x as dependent variable and y as independent variable.

Example 25: Solve the differential equation $\frac{dy}{dx} = \frac{y}{y\sin y - x}$.

Sol. $\frac{dy}{dx} = \frac{y}{y\sin y - x}$

$$\Rightarrow \frac{dx}{dy} = \frac{y\sin y - x}{y} = \sin y - \frac{x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y}x = \sin y$$

which is a linear differential equation in x and y.

It's integrating factor is I.F. $= e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Thus solution is $xy = \int y\sin y\, dy$

$\Rightarrow xy = -y\cos y + \sin y + c$, where 'c' is an arbitrary constant.

Clairaut's Equation :

The differential equation

$y = mx + f(m)$, ...(1)

where $m = \frac{dy}{dx}$

is known as Clairaut's Equation.

To solve (1), differentiate it w.r.t. x, which gives

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx}$$

$$x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$.....(2)

or $x + f'(m) = 0$ .....(3)

Note :

(i) If m is eliminated between (1) and (2), the solution obtained is a general solution of (1)

(ii) If m is eliminated between (1) and (3), then solution obtained does not contain any arbitrary constant and is not particular solution of (1). This solution is called singular solution of (1).

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Example 26: Solve : $y = mx + m - m^3$ where, $m = \frac{dy}{dx}$

Sol. $y = mx + m - m^3$ ..... (i)

The given equation is in clairaut's form.

Now, differentiating wrt. x -

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x + 1 - 3m^2) = 0 \Rightarrow \frac{dm}{dx} = 0 \Rightarrow m = c \text{ ..... (ii)}$$

$$\text{or } x + 1 - 3m^2 = 0 \Rightarrow m^2 = \frac{x+1}{3} \text{ ..... (iii)}$$

Eliminating 'm' between (i) & (ii) is called the general solution of the given equation.

$y = cx + c - c^3$where, 'c' is an arbitrary constant.

Again, eliminating 'm' between (i) & (iii) is called singular solution of the given equation.

$$y = m(x + 1 - m^2)$$

$$y = \pm\left(\frac{x+1}{3}\right)^{1/2}\left(x + 1 - \frac{x+1}{3}\right)$$

$$y = \pm\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1) \Rightarrow y = \pm 2\left(\frac{x+1}{3}\right)^{3/2}$$

$$y^2 = \frac{4}{27}(x+1)^3$$

$$27y^2 = 4(x+1)^3$$

ORTHOGONAL TRAJECTORY

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of a given family of curve at right angle.

Steps to find orthogonal trajectory :

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation obtained in (iii).

Hence solution obtained in (iv) is the required orthogonal trajectory.

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Example 27: Find the orthogonal trajectory of family of straight lines passing through the origin.]

Sol. Family of straight lines passing through the origin is-

$y = mx$ ..... (i)

where 'm' is an arbitrary constant.

Differentiating wrt x

$$\frac{dy}{dx} = m \text{ ..... (ii)}$$

Eliminate 'm' from (i) & (ii)

$$y = \frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$y = -\frac{dx}{dy}x \Rightarrow x\,dx + y\,dy = 0$$

Integrating each term,

$$\frac{x^2}{2} + \frac{y^2}{2} = c \Rightarrow x^2 + y^2 = 2c$$

which is the required orthogonal trajectory.

Example 28: Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

Sol. $y^2 = 4ax$ ..... (i)

$$2y\frac{dy}{dx} = 4a \text{ ..... (ii)}$$

Eliminating 'a' from (i) & (ii)

$$y^2 = 2y\frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$y = 2\left(-\frac{dx}{dy}\right)x$$

$2x\,dx + y\,dy = 0$

Integrating each term,

$$x^2 + \frac{y^2}{2} = c \Rightarrow 2x^2 + y^2 = 2c$$

which is the required orthogonal trajectories.

GEOMETRICAL APPLICATION OF DIFFERENTIAL EQUATION

Form a differential equation from a given geometrical problem. Often following formulae are useful to remember

(i) Length of tangent $(L_T) = \left|\frac{y\sqrt{1+m^2}}{m}\right|$

(ii) Length of normal $(L_N) = \left|y\sqrt{1+m^2}\right|$

(iii) Length of sub-tangent $(L_{ST}) = \left|\frac{y}{m}\right|$

(iv) Length of subnormal $(L_{SN}) = |my|$

where y is the ordinate of the point, m is the slope of the tangent $\left(\frac{dy}{dx}\right)$

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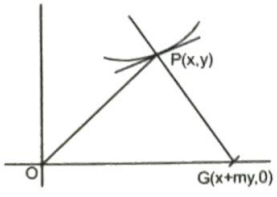
Example 29: Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Sol. Let the equation of the curve be $y = f(x)$. P(x, y) be any point on the curve.

Slope of the tanget at P(x, y) is $\frac{dy}{dx} = m$

$\therefore$ Slope of the normal at P is

$$m' = -\frac{1}{m}$$



Equation of the normal at 'P'

$$Y - y = -\frac{1}{m}(X - x)$$

Co-ordinates of G (x + my, 0)

Now, $OP^2 = PG^2$

$x^2 + y^2 = m^2y^2 + y^2$

$$m = \pm\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Taking + as the sign

$$\frac{dy}{dx} = \frac{x}{y}$$

$y\,dy = x\,dx$

$$\frac{y^2}{2} = \frac{x^2}{2} + \lambda$$

$x^2 - y^2 = 2\lambda$

$x^2 - y^2 = c$ (Rectangular hyperbola)

Again taking as –ve sign

$$\frac{dy}{dx} = -\frac{x}{y}$$

$y\,dy = -x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \lambda'$$

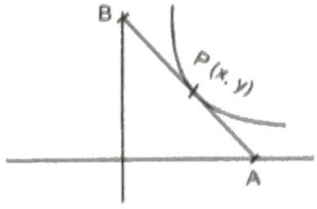
$x^2 + y^2 = 2\lambda'$

$x^2 + y^2 = c'$ (circle)

Example 30: Find the curves for which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact.

Sol. Let P (x, y) be any point on the curve

Equation of tangent at P (x, y) is -



$Y - y = m(X - x)$ where $m = \frac{dy}{dx}$ is slope of the tangent at P(x, y).

Co-ordinates of $A\left(\frac{mx - y}{m}, 0\right)$ & B (0, y – mx)

P is the middle point of A & B

$\therefore \quad \frac{mx - y}{m} = 2x$

$\Rightarrow mx - y = 2mx \Rightarrow mx = -y$

$\Rightarrow \frac{dy}{dx} x = -y \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \ell nx + \ell ny = \ell nc$

$\therefore xy = c$

Example 31: Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$ where a is a parameter.

Sol. Here, $x^2 + y^2 - ay = 0$; differentiating w.r.t. x,

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\therefore \quad 2x + 2y\frac{dy}{dx} - \frac{x^2 + y^2}{y}\frac{dy}{dx} = 0$$

$$\left(\because x^2 + y^2 - ay = 0 \Rightarrow a = \frac{x^2 + y^2}{y}\right)$$

$$\text{or} \quad 2x + \frac{y^2 - x^2}{y}\frac{dy}{dx} = 0.$$

This is the differential equation of the circles. The equation of orthogonal trajectories is

$$2x + \frac{y^2 - x^2}{y}.\left(-\frac{dx}{dy}\right) = 0,$$

$$\left(\text{putting } \frac{-dx}{dy} \text{ in place of } \frac{dy}{dx}\right)$$

or $2xy\,dy + (x^2 - y^2)\,dx = 0.$

It is a homogenous equation.

Put $y = vx$; then $\frac{dy}{dx} = v + x\frac{dy}{dx}$

$$\therefore \quad 2x \cdot vx \cdot \left(v + x\frac{dv}{dx}\right) + x^2 - v^2x^2 = 0$$

$$\text{or} \quad 2v\left(v + x\frac{dv}{dx}\right) + 1 - v^2 = 0$$

$$\therefore \quad 1 + v^2 + 2vx\frac{dv}{dx} = 0$$

$$\text{or} \quad \frac{dx}{x} + \frac{2v}{1 + v^2}dv = 0 \text{ ; integrating,}$$

we get $\log x + \log(1 + v^2) = \log c$

$\therefore \quad x(1 + v^2) = c$

$$\text{or} \quad x\left(1 + \frac{y^2}{x^2}\right) = c \text{ , i.e., } x^2 + y^2 = cx.$$

OTHER APPLICATIONS

Differential equations, their formation and solution is of practical importance. Following are some examples of application of differential equations in problems involving rate of change of temperature and rate of change of flow of fluids etc.

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Example 32: The rate of cooling of a substance in moving air is proportional to the difference of temperatures of the substance and the air. A substance cools from 36°C to 34°C in 15 minutes. Find when the substance will have the temperature 32°C, it being known that the constant temperature of air is 30°C.

Sol. Let the temperature of the substance at time t minutes be T. The rate of cooling of the substance.

$= k(T - 30°)$°C/minute (from the question).

But the rate of cooling = rate of decrease of temperature

$$= -\frac{dT}{dt}$$

$\therefore$ From the question,

$$-\frac{dT}{dt} = k(T - 30) \text{ or } -\frac{dT}{T - 30} = kdt$$

$$\therefore \quad \int_{36}^{34} -\frac{dT}{T - 30} = \int_0^{15} kdt \text{ or } [-\log(T - 30)]_{36}^{34} = k[t]_0^{15}$$

$$\therefore \quad \log\frac{6}{4} = k.15 \text{ ; } \quad \therefore \quad k = \frac{1}{15}\log\frac{3}{2}$$

Again, $\int_{36}^{32} -\frac{dT}{T - 30} = \int_0^t kdt$,

where t minutes is the required time.

$$\therefore \quad [-\log(T - 30)]_{36}^{32} = \frac{1}{15}\log\frac{3}{2}.[t]_0^t$$

$$\therefore \quad \log 3 = \frac{1}{15}\log\frac{3}{2}.t$$

$$\therefore \quad t = \frac{15\log 3}{\log 3/2} \text{ minutes.}$$

Topicwise Questions

Order and degree of differential equations

1. The order and degree of the differential equation $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$ are

(a) 1, 2 (b) 2, 1

(c) 1, 1 (d) 2, 2

2. The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/4} = \left(\frac{d^2y}{dx^2}\right)^{1/3}$ is

(a) $\frac{1}{3}$ (b) 4

(c) 9 (d) $\frac{3}{4}$

Formation of differential equations

3. $y = 4\sin 3x$ is a solution of the differential equation

(a) $\frac{dy}{dx} + 8y = 0$ (b) $\frac{dy}{dx} - 8y = 0$

(c) $\frac{d^2y}{dx^2} + 9y = 0$ (d) $\frac{d^2y}{dx^2} - 9y = 0$

4. The differential equation whose solution is $y = A\sin x + B\cos x$, is

(a) $\frac{d^2y}{dx^2} + y = 0$ (b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{dy}{dx} + y = 0$ (d) $\frac{d^2y}{dx} + 2y = 0$

5. The differential equation of the family of curves $y = a\cos(x + b)$ is

(a) $\frac{d^2y}{dx^2} - y = 0$ (b) $\frac{d^2y}{dx^2} + y = 0$

(c) $\frac{d^2y}{dx^2} + 2y = 0$ (d) $\frac{d^2y}{dx^2} - 2y = 0$

6. The differential equation of the family of curves represented by the equation $x^2y = a$, is

(a) $\frac{dy}{dx} + \frac{2y}{x} = 0$ (b) $\frac{dy}{dx} + \frac{2x}{y} = 0$

(c) $\frac{dy}{dx} - \frac{2y}{x} = 0$ (d) $\frac{dy}{dx} - \frac{2x}{y} = 0$

Variable separable type differential equations

7. The solution of the differential equation $3e^x \tan y dx + (1 - e^x)\sec^2 y dy = 0$ is

(a) $\tan y = c(1 - e^x)^3$ (b) $(1 - e^x)^3 \tan y = c$

(c) $\tan y = c(1 - e^x)$ (d) $(1 - e^x)\tan y = c$

8. The solution of the differential equation $(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0$ is

(a) $e^x(\sin x + \cos x) + c = 0$

(b) $e^y(\sin x + \cos x) = c$

(c) $e^y(\cos x - \sin x) = c$

(d) $e^x(\sin x - \cos x) = c$

9. The solution of the differential equation $\frac{dy}{dx} = (1 + x)(1 + y^2)$ is

(a) $y = \tan(x^2 + x + c)$ (b) $y = \tan(2x^2 + x + c)$

(c) $y = \tan(x^2 - x + c)$ (d) $y = \tan\left(\frac{x^2}{2} + x + c\right)$

10. The solution of the differential equation $x\sec y\frac{dy}{dx} = 1$ is

(a) $x\sec y\tan y = c$ (b) $cx = \sec y + \tan y$

(c) $cy = \sec x\tan x$ (d) $cy = \sec x + \tan x$

11. If $\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}=0$, then

(a) $y+\sin^{-1}x=c$ (b) $y^2+2\sin^{-1}x+c=0$

(c) $x+\sin^{-1}y=0$ (d) $x^2+2\sin^{-1}y=1$

12. The solution of the differential equation $\frac{dy}{dx}=\frac{x-y+3}{2(x-y)+5}$ is

(a) $2(x-y)+\log(x-y)=x+c$

(b) $2(x-y)-\log(x-y+2)=x+c$

(c) $2(x-y)+\log(x-y+2)=x+c$

(d) $2(x-y)-\log(x+y+2)=x+c$

13. The solution of $(x\sqrt{1+y^2})dx+(y\sqrt{1+x^2})dy=0$ is

(a) $\sqrt{1+x^2}+\sqrt{1+y^2}=c$

(b) $\sqrt{1+x^2}-\sqrt{1+y^2}=c$

(c) $(1+x^2)^{3/2}+(1+y^2)^{3/2}=c$

(d) $(1+x^2)^{3/2}-(1+y^2)^{3/2}=c$

14. The solution of the equation $\frac{dy}{dx}=(x+y)^2$ is

(a) $x+y+\tan(x+c)=0$

(b) $x-y+\tan(x+c)=0$

(c) $x+y-\tan(x+c)=0$

(d) $x+y+\cot(x+c)=0$

15. Solution of the differential equation $\sin\frac{dy}{dx}=a$ with $y(0)=1$ is

(a) $\sin^{-1}\frac{(y-1)}{x}=a$ (b) $\sin\frac{(y-1)}{x}=a$

(c) $\sin\frac{(1-y)}{(1+x)}=a$ (d) $\sin\frac{y}{(x+1)}=a$

16. The solution of the differential equation $(x+y)^2\frac{dy}{dx}=a^2$ is

(a) $(x+y)^2=\frac{a^2}{2}x+c$ (b) $(x+y)^2=a^2x+c$

(c) $(x+y)^2=2a^2x+c$ (d) $y=a\tan^{-1}\left(\frac{x+y}{a}\right)+c$

17. Solution of $\frac{dy}{dx}=\frac{x\log x^2+x}{\sin y+y\cos y}$ is

(a) $y\sin y=x^2\log x+c$

(b) $y\sin y=x^2+c$

(c) $y\sin y=x^2+\log x+c$

(d) $y\sin y=x\log x+c$

18. Solution of the differential equation $\frac{dy}{dx}\tan y=\sin(x+y)+\sin(x-y)$ is

(a) $\sec y+2\cos x=c$ (b) $\sec y-2\cos x=c$

(c) $\cos y-2\sin x=c$ (d) $\tan y-2\sec y=c$

Homogeneous differential equations

19. Solution of differential equation $2xy\frac{dy}{dx}=x^2+3y^2$ is (where p is a constant)

(a) $x^3+y^2=px^2$ (b) $\frac{x^2}{2}+\frac{y^3}{x}=y^2+p$

(c) $x^2+y^3=px^2$ (d) $x^2+y^2=px^3$

20. The solution of the equation $\frac{dy}{dx}=\frac{x}{2y-x}$ is

(a) $(x-y)(x+2y)^2=c$ (b) $y=x+c$

(c) $y=(2y-x)+c$ (d) $y=\frac{x}{2y-x}+c$

21. The solution of the equation $\frac{dy}{dx}=\frac{y}{x}\left(\log\frac{y}{x}+1\right)$ is

(a) $\log\left(\frac{y}{x}\right)=cx$ (b) $\frac{y}{x}=\log y+c$

(c) $y=\log y+1$ (d) $y=xy+c$

Exact differential equations

22. The solution of $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$ is

(a) $\frac{x}{y} + e^{x^3} = c$ (b) $\frac{x}{y} - e^{x^3} = 0$

(c) $\frac{-x}{y} + e^{x^3} = 0$ (d) $\frac{x}{y} + e^{x^2} = 0$

23. The solution of $(x - y^3)dx + 3xy^2dy = 0$ is

(a) $\log x + \frac{x}{y}$ (b) $\log x + \frac{y^3}{x} = k$

(c) $\log x - \frac{x}{y^3} = k$ (d) $\log xy - y^3 = k$

24. The solution of $ye^{-x/y}dx - (xe^{-x/y} + y^3)dy = 0$ is

(a) $\frac{y^2}{2} + e^{-x/y} = k$ (b) $\frac{x^2}{2} + e^{-x/y} = k$

(c) $\frac{x^2}{2} + e^{x/y} = k$ (d) $\frac{y^2}{2} + e^{x/y} = k$

Linear differential equations

25. An integrating factor for the differential equation $(1+y^2)dx - (\tan^{-1}y - x)dy = 0$

(a) $\tan^{-1}y$ (b) $e^{\tan^{-1}y}$

(c) $\frac{1}{1+y^2}$ (d) $\frac{1}{x(1+y^2)}$

26. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is

(a) $4xy = x^4 + c$ (b) $xy = x^4 + c$

(c) $\frac{1}{4}xy = x^4 + c$ (d) $xy = 4x^4 + c$

27. The solution of the equation $x\frac{dy}{dx} + 3y = x$ is

(a) $x^3y + \frac{x^4}{4} + c = 0$ (b) $x^3y = \frac{x^4}{4} + c$

(c) $x^3y + \frac{x^4}{4} = 0$ (d) $x^2y = \frac{x^4}{2} + c$

28. $y + x^2 = \frac{dy}{dx}$ has the solution

(a) $y + x^2 + 2x + 2 = ce^x$ (b) $y + x + x^2 + 2 = ce^{2x}$

(c) $y + x + 2x^2 + 2 = ce^x$ (d) $y^2 + x + x^2 + 2 = ce^x$

Application of differential equations

29. The equation of the curve that passes through the point (1, 2) and satisfies the differential equation $\frac{dy}{dx} = \frac{-2xy}{(x^2+1)}$ is

(a) $y(x^2+1) = 4$ (b) $y(x^2+1) + 4 = 0$

(c) $y(x^2-1) = 4$ (d) $y(x^2-1) + 4 = 0$

30. The equation of the curve through the point (1,0) and whose slope is $\frac{y-1}{x^2+x}$ is

(a) $(y-1)(x+1) + 2x = 0$ (b) $2x(y-1) + x + 1 = 0$

(c) $x(y-1)(x+1) + 2 = 0$ (d) $x(y+1)(x+1) + 2 = 0$

Learning Plus

1. The order and degree of the differential equation

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$ are respectively

(a) 2, 2 (b) 2, 3

(c) 2, 1 (d) 1, 2

2. The order of the differential equation whose general solution is given by

$y = (C_1 + C_2)\sin(x + C_3) - C_4\,e^{x+C_5}$ is

(a) 5 (b) 4

(c) 2 (d) 3

3. The order and degree of differential equation of all tangent lines to parabola $x^2 = 4y$ is

(a) 1, 2 (b) 2, 2

(c) 3, 1 (d) 4, 1

4. If p and q are order and degree of differential equation

$$y^2\left(\frac{d^2y}{dx^2}\right)^2 + 3x\left(\frac{dy}{dx}\right)^{1/3} + x^2y^2 = \sin x, \text{ then :}$$

(a) $p > q$ (b) $\frac{p}{q} = \frac{1}{2}$

(c) $p = q$ (d) $p < q$

5. Family $y = Ax + A^3$ of curve represented by the differential equation of degree

(a) three (b) two

(c) one (d) degree is not defined

6. The order and degree of the differential equation

$$\left(1+3\frac{dy}{dx}\right)^{\frac{2}{3}} = 4\frac{d^3y}{dx^3} \text{ are}$$

(a) $1, \frac{2}{3}$ (b) 3, 1

(c) 1, 2 (d) 3, 3

7. The degree and order of the differential equation of the family of all parabolas whose axis is x-axis are respectively

(a) 2, 1 (b) 1, 2

(c) 3, 2 (d) 2, 3

8. The order and degree of the differential equation

$$\sqrt[3]{\frac{dy}{dx}} - 4\frac{d^2y}{dx^2} - 7x = 0 \text{ are a and b, then a + b is}$$

(a) 3 (b) 4

(c) 5 (d) 6

9. The differential equations of all conics whose centre lie at the origin is of order :

(a) 2 (b) 3

(c) 4 (d) 1

10. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants is of-

(a) first order and first degree

(b) second order and first degree

(c) second order and second degree

(d) first order and second degree

11. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation

$D^3(y) - 3D^2(y) - 4D(y) + 12y = 0$ is

(Where $D^n(y) = \frac{d^n(y)}{dx^n}$)

(a) 0 (b) 1

(c) 2 (d) more than 2

12. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation

$D^2(y) - 3D(y) - 4y = -4x$

(Where $D^n(y) = \frac{d^n(y)}{dx^n}$)

(a) $m = -1, c = 3/4$ (b) $m = 1, c = 3/4$

(c) no such real m, c (d) $m = 1, c = -3/4$

13. If $\frac{dy}{dx} = e^{2y}$ and $y = 0$ when $x = 5$, the value of x for $y = 3$ is

(a) e^5 (b) $e^6 + 1$

(c) $\frac{e^6 + 9}{2}$ (d) $\log_e 6$

14. The differential equation for all the straight lines which are at a unit distance from the origin is

(a) $\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

(b) $\left(y + x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(c) $\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(d) $\left(y + x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

15. Integral curve satisfying $y' = \frac{x^2 + y^2}{x^2 - y^2}$, $y(1) = 2$, has the slope at the point (1, 2) of the curve, equal to

(a) $-\frac{5}{3}$ (b) -1

(c) 1 (d) $\frac{5}{3}$

16. The differential equation of the family of curves represented by $y = a + bx + ce^{-x}$ (where a, b, c are arbitrary constants) is

(a) $y''' = y'$ (b) $y''' + y' = 0$

(c) $y''' - y'' + y' = 0$ (d) $y''' + y'' - y' = 0$

17. The differential equation whose solution is $(x-h)^2 + (y-k)^2 = a^2$ is (where a is a constant)

(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(b) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\frac{d^2y}{dx^2}$

(c) $\left[1+\left(\frac{dy}{dx}\right)\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(d) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = a^2\frac{d^2y}{dx^2}$

18. If $y = e^{(K+1)x}$ is a solution of differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, then k equals

(a) -1 (b) 0

(c) 1 (d) 2

19. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is

(a) $(x^2 - y^2)y' = 2xy$ (b) $2(x^2 + y^2)y' = xy$

(c) $2(x^2 - y^2)y' = xy$ (d) $(x^2 + y^2)y' = 2xy$

20. The solution of the differential equation

$\frac{dy}{dx} - ky = 0, y(0) = 1$, approaches zero when $x \to \infty$, if

(a) $k = 0$ (b) $k > 0$

(c) $k < 0$ (d) $k \neq 0$

21. The equation of the curve whose subnormal is constant a is

(a) $y = ax + b$ (b) $y^2 = 2ax + b$

(c) $ay^2 - x^2 = a$ (d) $x^2 + y^2 = a^2$

22. If $\frac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then function y is

(a) $e^{(1-x)^2/2}$ (b) $e^{(1+x)^2/2} - 1$

(c) $\log_e(1 + x) - 1$ (d) $1 + x$

23. Solution of differential equation $xdy - y\,dx = 0$ represents :

(a) rectangular hyperbola

(b) straight line passing through origin

(c) parabola whose vertex is at origin

(d) circle whose centre is at origin

24. The solution to the differential equation $y\ell ny + xy' = 0$, where $y(1) = e$, is

(a) $x(\ell ny) = 1$ (b) $xy(\ell ny) = 1$

(c) $(\ell ny)^2 = 2$ (d) $\ell ny + \left(\frac{x^2}{2}\right) y = 1$

25. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(a) $\log\left(\frac{x}{y}\right) = cy$ (b) $\log\left(\frac{y}{x}\right) = cx$

(c) $x\log\left(\frac{y}{x}\right) = cy$ (d) $y\log\left(\frac{x}{y}\right) = cx$

26. The solution of the differential equation

$ydx + (x + x^2y)\,dy = 0$ is

(a) $\frac{1}{xy} + \log y = c$ (b) $\log y = cx$

(c) $\frac{1}{xy} = c$ (d) $-\frac{1}{xy} + \log y = c$

27. The solution of the differential equation

$(2x - 10y^3)\frac{dy}{dx} + y = 0$ is

(a) $x + y = ce^{2x}$ (b) $y^2 = 2x^3 + c$

(c) $xy^2 = 2y^5 + c$ (d) $x(y^2 + xy) = 0$

28. The general solution of the differential equation. $y' + y\varphi'(x) - \varphi(x).\varphi'(x) = 0$ where $\varphi(x)$ is a known function is

(a) $y = ce^{-\varphi(x)} + \varphi(x) - 1$

(b) $y = ce^{\varphi(x)} + \varphi(x) + K$

(c) $y = ce^{-\varphi(x)} - \varphi(x) + 1$

(d) $y = ce^{-\varphi(x)} + \varphi(x) + K$

29. Solution of differential equation

$(1 + y^2)dx + (x - e^{\tan^{-1}y})\,dy = 0$ is

(a) $ye^{\tan^{-1}x} = \tan^{-1}x + c$

(b) $xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + c$

(c) $2x = e^{\tan^{-1}y} + c$

(d) $y = xe^{-\tan^{-1}x} + c$

30. The solution of $\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$ is

(a) $y = x\cot(c - x)$

(b) $\cos^{-1}y/x = -x + c$

(c) $y = x\tan(c - x)$

(d) $y^2/x^2 = x\tan(c - x)$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/ NUMERICAL

1. The differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y + x^2 = 0$ is of the following type
   (a) linear (b) homogeneous
   (c) order two (d) degree one
2. The solution of the differential equation
   $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}(e^x + e^{-x}) + 1 = 0$ is
   (a) $y + e^{-x} = c$ (b) $y - e^{-x} = c$
   (c) $y + e^x = c$ (d) $y - e^x = c$
3. Let $y = (A + Bx)e^{3x}$ be a solution of the differential equation $\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = 0$, m, n $\in$ I, then
   (a) $m + n = 3$ (b) $n^2 - m^2 = 64$
   (c) $m = -6$ (d) $n = 9$
4. The differential equation $\frac{d^2y}{dx^2} + y + \cot^2 x = 0$ must be satisfied by
   (a) $y = 2 + c_1 \cos x + \sqrt{c_2} \sin x$
   (b) $y = \cos x \cdot \ell n\left(\tan\frac{x}{2}\right) + 2$
   (c) $y = 2 + c_1 \cos x + c_2 \sin x + \cos x \log\left(\tan\frac{x}{2}\right)$
   (d) all the above
5. If $y = e^{-x} \cos x$ and $y_n + k_n y = 0$, where $y_n = \frac{d^n y}{dx^n}$ and k_n, $n \in N$ are constants.
   (a) $k_4 = 4$ (b) $k_8 = -16$
   (c) $k_{12} = 20$ (d) $k_{16} = -24$
6. The solution of $x^2 y_1^2 + xy y_1 - 6y^2 = 0$ are
   (Where $y_1 = \frac{dy}{dx}$)
   (a) $y = Cx^2$ (b) $x^2 y = C$
   (c) $\frac{1}{2}\ell n\, y = C + \ell n x$ (d) $x^3 y = C$
7. Solution of the differential equation
   $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$ is
   (a) $\tan^{-1} y + \sin^{-1} x = c$
   (b) $\tan^{-1} x + \sin^{-1} y = c$
   (c) $\tan^{-1} y \cdot \sin^{-1} x = c$
   (d) $\cot^{-1}\frac{1}{y} + \cos^{-1}\sqrt{1-x^2} = c$
8. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$ is
   (a) $f(x) = c \cdot e^{(2-\sqrt{3})x}$ (b) $f(x) = c \cdot e^{(2+\sqrt{3})x}$
   (c) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$ (d) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$
9. The equation of the curve passing through (3, 4) & satisfying the differential equation,
   $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ can be
   (a) $x - y + 1 = 0$ (b) $x^2 + y^2 = 25$
   (c) $x^2 + y^2 - 5x - 10 = 0$ (d) $x + y - 7 = 0$
10. Solution of differential equation
    $f(x)\frac{dy}{dx} = f^2(x) + f(x)\, y + f'(x)\, y$ is
    (a) $y = f(x) + ce^x$ (b) $y = -f(x) + ce^x$
    (c) $y = -f(x) + ce^x f(x)$ (d) $y = cf(x) + e^x$
11. The solution of $(x + y + 1)\, dy = dx$ are
    (a) $x + y + 2 = Ce^y$
    (b) $x + y + 4 = = C \log y$
    (c) $\log(x + y + 2) = Cy$
    (d) $\log(x + y + 2) = C + y$
12. The graph of the function $y = f(x)$ passing through the point (0, 1) and satisfying the differential equation
    $\frac{dy}{dx} + y \cos x = \cos x$ is such that
    (a) it is a constant function
    (b) it is periodic
    (c) it is neither an even nor an odd function
    (d) it is continuous & differentiable for all x.
13. The orthogonal trajectories of the system of curves
    $\left(\frac{dy}{dx}\right)^2 = a/x$ are
    (a) $9\, a(y + c)^2 = 4x^3$ (b) $y + C = \frac{-2}{3\sqrt{a}} x^{3/2}$
    (c) $y + C = \frac{2}{3\sqrt{a}} x^{3/2}$ (d) none of these

14. The orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2 = \frac{4}{x}$ are

(a) $9(y + c)^2 = x^3$ (b) $y + c = \frac{-x^{3/2}}{3}$

(c) $y + c = \frac{x^{3/2}}{3}$ (d) all of these

Comprehension – 1(no. 15 to 17)

Differential equations are solved by reducing them to the exact differential of an expression in x & y i.e., they are reduced to the form d(f(x, y)) = 0

15. The general solution of
$(2x^3 - xy^2)\,dx + (2y^3 - x^2y)\,dy = 0$ is
(a) $x^4 + x^2y^2 - y^4 = c$ (b) $x^4 - x^2y^2 + y^4 = c$
(c) $x^4 - x^2y^2 - y^4 = c$ (d) $x^4 + x^2y^2 + y^4 = c$

16. General solution of the differential equation $\frac{xdy}{x^2+y^2} + \left(1 - \frac{y}{x^2+y^2}\right) dx = 0$ is

(a) $x + \tan^{-1}\left(\frac{y}{x}\right) = c$ (b) $x + \tan^{-1}\frac{x}{y} = c$

(c) $x - \tan^{-1}\left(\frac{y}{x}\right) = c$ (d) none of these

17. General solution of the differential equation $e^y\,dx + (xe^y - 2y)\,dy = 0$ is
(a) $xe^y - y^2 = c$ (b) $ye^x - x^2 = c$
(c) $ye^y + x = c$ (d) $xe^y - 1 = cy^2$

18. Match the following

Column-I

(a) Solution of $y - \frac{xdy}{dx} = y^2 + \frac{dy}{dx}$ is

(b) Solution of $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is

(c) Solution of $\sec^2 y\,dy + \tan y\,dx = dx$ is

(d) Solution of $\sin y \frac{dy}{dx} = \cos y\,(1 - x\cos y)$ is

Column-II

(p) $xy^2 = 2y^5 + c$
(q) $\sec y = x + 1 + ce^x$
(r) $(x + 1)(1 - y) = cy$
(s) $\tan y = 1 + ce^x$

19. Match the following

Column-I

(a) $xdy = y(dx + ydy)$, $y(1) = 1$ and $y(x_0) = -3$, then $x_0 =$

(b) If y(t) is solution of $(t + 1)\frac{dy}{dt} - ty = 1$, $y(0) = -1$, then $y(1) =$

(c) $(x^2 + y^2)\,dy = xydx$ and $y(1) = 1$ and $y(x_0) = e$, then $x_0 =$

(d) $\frac{dy}{dx} + \frac{2y}{x} = 0$, $y(1) = 1$, then $y(2) =$

Column-II

(p) $\frac{1}{4}$ (q) -15

(r) $-\frac{1}{2}$ (s) $\sqrt{3}\,e$

NUMERICAL VALUE BASED

20. The order of the differential equation whose general solution is given by
$y = c_1\cos(2x + c_2) - (c_3 + c_4)a^{x+c_5} + c_6\sin(x - c_7)$
is

21. The real value of m for which the substitution $y = u^m$ will transform the differential equation $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is

22. The degree of the differential equation of all tangent lines to the parabola $y^2 = 4ax$ is

23. The equation of the family of curves which intersect the hyperbola $xy = 2$ orthogonally is $y = \frac{x^A}{B} + c$, then find value of A/B

24. If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\frac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = k$ then the value of k is

25. If $y = c_1e^{2x} + c_2e^x + c_3e^{-x}$ satisfies the differential equation
$\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, then absolute value of $\frac{a^3 + b^3 + c^3}{abc}$ is equal to

26. The curves which intersect the hyperbola xy=4 at an angle $\frac{\pi}{2}$ is $y = \frac{x^3}{k} + \frac{c}{4}$ then $\frac{k}{3} =$ ---

27. The curve represented by the differential equation $xdy - ydx = ydy$ intersects the y axis at A(0, 1) and the line y = e at (a, b) then find a + b

28. If soluton of $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x$ is given by $y^2 = \frac{x^3}{\lambda} + c_1x + c_2$ then the numerical quantity λ should be equal to.

JEE-MAIN PREVIOUS YEARS

1. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1+xy)dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to: [JEE Main- 2016]

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$

(c) $\frac{2}{5}$ (d) $\frac{4}{5}$

2. If $(2 + \sin x)\frac{dy}{dx} + (y+1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to : [JEE (Main) 2017]

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$

(c) $-\frac{2}{3}$ (d) $-\frac{1}{3}$

3. Let $y = y(x)$ be the solution of the differential equation $\sin x \frac{dy}{dx} + y\cos x = 4x, x \in (0,\pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : [JEE Main - 2018]

(a) $\frac{-8}{9\sqrt{3}}\pi^2$ (b) $-\frac{8}{9}\pi^2$

(c) $-\frac{4}{9}\pi^2$ (d) $\frac{4}{9\sqrt{3}}\pi^2$

4. If $y = y(x)$ is solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1)=1$, then $y\left(\frac{1}{2}\right)$ is equal to [JEE Main - 2019 (January)]

(a) $\frac{7}{64}$ (b) $\frac{1}{4}$

(c) $\frac{49}{16}$ (d) $\frac{13}{16}$

5. Let $f : [0, 1] \to R$ be such that $f(xy) = f(x)\,f(y)$ for all $x,y \in [0, 1]$, and $f(0) \neq 0$. If $y = y(x)$ satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with $y(0) = 1$, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to :- [JEE Main - 2019 (January)]

(a) 4 (b) 3

(c) 5 (d) 2

6. If $\frac{dy}{dx} + \frac{3}{\cos^2 x}y = \frac{1}{\cos^2 x}, x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals: [JEE Main - 2019 (January)]

(a) $\frac{1}{3} + e^6$ (b) $\frac{1}{3}$

(c) $-\frac{4}{3}$ (d) $\frac{1}{3} + e^3$

7. The curve amongst the family of curves represented by the differential equation, $(x^2 - y^2)\,dx + 2xy\,dy = 0$ which passes through (1, 1), is : [JEE Main - 2019 (January)]

(a) a circle with centre on the x-axis
(b) an ellipse with major axis along the y-axis
(c) a circle with centre on the y-axis
(d) a hyperbola with transverse axis along the x-axis

8. Let f be a differential function such that $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}, (x > 0)$ and $f(1) \neq 4$. Then $\lim_{x\to 0} x f\left(\frac{1}{x}\right)$: [JEE Main - 2019 (January)]

(a) exists and equals $\frac{4}{7}$ (b) exists and equals 4.

(c) does not exist. (d) exists and equls 0.

9. The solution of the differential equation $\frac{dy}{dx} = (x-y)^2$ when $y(1) = 1$, is : [JEE Main - 2019 (January)]

(a) $\log_e\left|\frac{2-x}{2-y}\right| = x - y$

(b) $-\log_e\left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

(c) $-\log_e\left|\frac{1+x-y}{1-x+y}\right| = x + y - 2$

(d) $\log_e\left|\frac{2-y}{2-x}\right| = 2(y-1)$

10. If $y(x)$ is the solution of the differential equation $\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}, x>0$ where $y(1)=\frac{1}{2}e^{-2}$, then

[JEE Main - 2019 (January)]

(a) $y(\log_e 2)=\log_e 4$

(b) $y(\log_e 2)=\frac{\log_e 2}{4}$

(c) $y(x)$ is decreasing in $\left(\frac{1}{2},1\right)$

(d) $y(x)$ is decreasing in $(0, 1)$

11. Let $y=y(x)$ be the solution of the differential equation, $x\frac{dy}{dx}+y=x\log_e x, (x>1)$. If $2y(2)=\log_e 4-1$, then $y(e)$ is equal to :

[JEE Main - 2019 (January)]

(a) $-\frac{e}{2}$ (b) $-\frac{e^2}{2}$

(c) $\frac{e}{4}$ (d) $\frac{e^2}{4}$

12. If a curve passes through the point $(1, -2)$ and has slope of the tangent at any point (x, y) on it as $\frac{x^2-2y}{x}$, then the curve also passes through the point:

[JEE Main - 2019 (January)]

(a) $(3, 0)$ (b) $\left(\sqrt{3},0\right)$

(c) $(-1, 2)$ (d) $\left(-\sqrt{2},1\right)$

13. Let $y=y(x)$ be the solution of the differential equation, $\left(x^2+1\right)^2\frac{dy}{dx}+2x(x^2+1)y=1$ such that $y(0)=0$. If $\sqrt{a}y(1)=\frac{\pi}{32}$, then the value of 'a' is :

[JEE Main - 2019(April)]

(a) $\frac{1}{2}$ (b) $\frac{1}{16}$

(c) $\frac{1}{4}$ (d) 1

14. Given that the slope of the tangent to a curve $y=y(x)$ at any point (x, y) is $\frac{2y}{x^2}$. If the curve passes through the centre of the circle $x^2+y^2-2x-2y=0$, then its equation is:

[JEE Main - 2019(April)]

(a) $x\log_e|y|=2(x-1)$ (b) $x\log_e|y|=2(x-1)$

(c) $x^2\log_e|y|=-2(x-1)$ (d) $x\log_e|y|=-2(x-1)$

15. The solution of the differential equation $x\frac{dy}{dx}+2y=x^2$ $(x\neq 0)$; with $y(1)=1$, is

[JEE Main - 2019(April)]

(a) $y=\frac{x^3}{5}+\frac{1}{5x^2}$ (b) $y=\frac{4}{5}x^3+\frac{1}{5x^2}$

(c) $y=\frac{3}{4}x^2+\frac{1}{4x^2}$ (d) $y=\frac{x^2}{4}+\frac{3}{4x^2}$

16. If $\cos x\frac{dy}{dx}-y\sin x=6x, \left(0<x<\frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{3}\right)=0$, then $y\left(\frac{\pi}{6}\right)$ is equal to :-

[JEE Main - 2019(April)]

(a) $-\frac{\pi^2}{4\sqrt{3}}$ (b) $-\frac{\pi^2}{2}$

(c) $-\frac{\pi^2}{2\sqrt{3}}$ (d) $\frac{\pi^2}{2\sqrt{3}}$

17. If $y=y(x)$ is the solution of the differential equation $\frac{dy}{dx}=(\tan x-y)\sec^2 x, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that $y(0)=0$, then $y\left(-\frac{\pi}{4}\right)$ is equal to :

[JEE Main - 2019(April)]

(a) $2+\frac{1}{e}$ (b) $\frac{1}{2}-e$

(c) $e-2$ (d) $\frac{1}{2}-e$

18. Let $y=y(x)$ be the solution of the differential equation, $\frac{dy}{dx}+y\tan x=2x+x^2\tan x$, [JEE Main - 2019(April)]

$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, such that $y(0)=1$, then :

(a) $y'\left(\frac{\pi}{4}\right)+y'\left(\frac{-\pi}{4}\right)=-\sqrt{2}$

(b) $y'\left(\frac{\pi}{4}\right)-y'\left(\frac{-\pi}{4}\right)=\pi-\sqrt{2}$

(c) $y\left(\frac{\pi}{4}\right)-y\left(-\frac{\pi}{4}\right)=\sqrt{2}$

(d) $y\left(\frac{\pi}{4}\right)+y\left(-\frac{\pi}{4}\right)=\frac{\pi^2}{2}+2$

19. If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at $x = 0$ is equal to : [JEE Main - 2019(April)]

(a) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$ (b) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$

(c) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (d) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

20. Consider the differential equation, $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If value of y is 1 when $x = 1$, then the value of x for which $y = 2$, is : [JEE Main - 2019(April)]

(a) $\frac{1}{2} + \frac{1}{\sqrt{e}}$ (b) $\frac{3}{2} - \sqrt{e}$

(c) $\frac{5}{2} + \frac{1}{\sqrt{e}}$ (d) $\frac{3}{2} - \frac{1}{\sqrt{e}}$

21. The general solution of the differential equation [JEE Main - 2019(April)]

$(y^2 - x^3)dx - xydy = 0 (x \neq 0)$ is : (where c is a constant of integration)

(a) $y^2 + 2x^3 + cx^2 = 0$ (b) $y^2 + 2x^2 + cx^3 = 0$

(c) $y^2 - 2x^3 + cx^2 = 0$ (d) $y^2 - 2x^2 + cx^3 = 0$

22. Let $y = f(x)$ is the solution of the differential equation $e^y\left(\frac{dy}{dx} - 1\right) = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to [JEE Main-2020 (January)]

(a) 2e (b) $1 + \log_e 2$

(c) $\log_e 2$ (d) $2 + \log_e 2$

23. Let $y = y(x)$ be the solution curve of the differential equation, $(y^2 - x)\frac{dy}{dx} = 1$, satisfying $y(0) = 1$. This curve intersects the x-axis at a point whose abscissa is: [JEE Main-2020 (January)]

(a) $2 - e$ (b) $2 + e$

(c) $-e$ (d) 2

24. Let $y = y(x)$ be a solution of the differential equation, $\sqrt{1-x^2}\frac{dy}{dx} + \sqrt{1-y^2} = 0, |x| < 1$. If $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$, then $y\left(\frac{-1}{\sqrt{2}}\right)$ is equal to [JEE Main-2020 (January)]

(a) $\frac{-\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-1}{\sqrt{2}}$

25. The differential equation of the family of curves, $x^2 = 4b(y + b), b \in R$, is [JEE Main-2020 (January)]

(a) $x(y')^2 = x + 2yy'$ (b) $xy'' = y'$

(c) $x(y')^2 = x + 2yy'$ (d) $x(y')^2 = 2yy' + x$

26. If for $x > 0$, $y = y(x)$ is the solution of the differential equation, $(x + 1)dy = \{(x + 1)^2 + y - 3\}dx$, $y(2) = 0$, then $y(3)$ is equal to : [JEE Main-2020 (January)]

27. If $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$; $y(1) = 1$; then a value of x satisfying $y(x) = e$ is : [JEE Main-2020 (January)]

(a) $\sqrt{3}e$ (b) $\frac{e}{\sqrt{2}}$

(c) $\sqrt{2}e$ (d) $\frac{1}{2}\sqrt{3}e$

28. If a curve $y = f(x)$, passing through the point $(1, 2)$, is the solution of the differential equation, $2x^2dy = (2xy + y^2)dx$, then $f\left(\frac{1}{2}\right)$ is equal to : [JEE Main-2020 (September)]

(a) $\frac{1}{1 + \log_e 2}$ (b) $\frac{-1}{1 + \log_e 2}$

(c) $1 + \log_e 2$ (d) $\frac{1}{1 - \log_e 2}$

29. Let $y = y(x)$ be the solution of the differential equation, $\frac{2 + \sin x}{y + 1}\cdot\frac{dy}{dx} = -\cos x, y > 0, y(0) = 1$. If $y(\pi) = a$ and $\frac{dy}{dx}$ at $x = \pi$ is b, then the ordered pair (a, b) is equal to: [JEE Main-2020 (September)]

(a) (2,1) (b) (1, - 1)

(c) $\left(2, \frac{3}{2}\right)$ (d) (1, 1)

30. The solution curve of the differential equation, $\left(1 + e^{-x}\right)(1 + y^2)\frac{dy}{dx} = y^2$, which passes through the point (0,1), is [JEE Main-2020 (September)]

(a) $y^2 + 1 = y\left(\log_e\left(\frac{1 + e^{-x}}{2}\right) + 2\right)$

(b) $y^2 + 1 = y\left(\log_e\left(\frac{1 + e^{x}}{2}\right) + 2\right)$

(c) $y^2 = 1 + y\log_e\left(\frac{1 + e^{-x}}{2}\right)$

(d) $y^2 = 1 + y\log_e\left(\frac{1 + e^{x}}{2}\right)$

31. The solution of the differential equation

[JEE Main-2020 (September)]

$\frac{dy}{dx} - \frac{y+3x}{\log_e(y+3x)} + 3 = 0$ is :

(where C is a constant of integration.)

(a) $x - \frac{1}{2}(\log_e(y+3x))^2 = C$

(b) $y + 3x - \frac{1}{2}(\log_e x)^2 = C$

(c) $x - 2\log_e(y+3x) = C$

(d) $x - \log_e(y+3x) = c$

32. Let $y = y(x)$ be the soluiton of the differential equation, $xy' - y = x^2(x\cos x + \sin x)$, $x > 0$. If $y(\pi) = \pi$, then $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right)$ is equal to :

[JEE Main-2020 (September)]

(a) $1 + \frac{\pi}{2}$ (b) $1 + \frac{\pi}{2} + \frac{x^2}{4}$

(c) $2 + \frac{\pi}{2}$ (d) $2 + \frac{\pi}{2} + \frac{\pi^2}{4}$

33. Let $y = y(x)$ be the solution of the differential equation $\cos x \frac{dy}{dx} + 2y\sin x = \sin 2x$, $x \in \left(0, \frac{\pi}{2}\right)$. If $y(\pi/3) = 0$, then $y(\pi/4)$ is equal to : [JEE Main-2020 (September)]

(a) $\frac{1}{\sqrt{2}} - 1$ (b) $\sqrt{2} - 2$

(c) $2 - \sqrt{2}$ (d) $2 + \sqrt{2}$

34. If $y = y(x)$ is the solution of the differential equation $\frac{5+e^x}{2+y} \cdot \frac{dy}{dx} + e^x = 0$ satisfying $y(0) = 1$, then a value of $y(\log_e 13)$ is : [JEE Main-2020 (September)]

(a) -1 (b) 2

(c) 0 (d) 1

35. If $y = \left(\frac{2}{\pi}x - 1\right)\operatorname{cosec} x$ is the solution of the differential equation, $\frac{dy}{dx} + p(x)y = \frac{2}{\pi}\operatorname{cosec} x$, $0 < x < \frac{\pi}{2}$, then the function $p(x)$ is equal to:

[JEE Main-2020 (September)]

(a) $\tan x$ (b) $\cot x$

(c) $\operatorname{cosec} x$ (d) $\sec x$

36. The general solution of the differential equation

[JEE Main-2020 (September)]

$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$ is :

(where C is a constant of integration)

(a) $\sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right) + C$

(b) $\sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + C$

(c) $\sqrt{1+y^2} - \sqrt{1+x^2} = \frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + C$

(d) $\sqrt{1+y^2} - \sqrt{1+x^2} = \frac{1}{2}\log_e\left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right) + C$

37. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over R is equal to :

(a) 8

(b) $\frac{1}{2}$

(c) $-\frac{15}{4}$

(d) $\frac{1}{8}$

38. The differential equation satisfied by the system of parabolas $y^2 = 4a(x+a)$ is : [JEE Main-2021 (March)]

(a) $y\left(\frac{dy}{dx}\right)^2 - 2x\left(\frac{dy}{dx}\right) - y = 0$

(b) $y\left(\frac{dy}{dx}\right)^2 - 2x\left(\frac{dy}{dx}\right) + y = 0$

(c) $y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$

(d) $y\left(\frac{dy}{dx}\right) + 2x\left(\frac{dy}{dx}\right) - y = 0$

JEE-ADVANCED
PREVIOUS YEARS

1. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then [IIT-JEE 2012]

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

2. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. Then the equation of the curve is [JEE Advanced-2013]

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

3. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4 + 2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then is [JEE Advanced- 2014]

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

4. Let $f : [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals [JEE Advanced- 2014]

(a) $e^2 - 1$ (b) $e^4 - 1$

(c) $e - 1$ (d) e^4

5. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true ? [JEE Advanced-2015]

(a) $y(-4) = 0$

(b) $y(-2) = 0$

(c) $y(x)$ has a critical point in the interval $(-1, 0)$

(d) $y(x)$ has no critical point in the interval $(-1, 0)$

6. Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' (here $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true? [JEE Advanced-2015]

(a) $P = y + x$

(b) $P = y - x$

(c) $P + Q = 1 - x + y + y' + (y')^2$

(d) $P - Q = x + y - y' - (y')^2$

7. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then [JEE Advanced-2016]

(a) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = 1$

(b) $\lim_{x \to 0+} x f\left(\frac{1}{x}\right) = 2$

(c) $\lim_{x \to 0+} x^2 f'(x) = 0$

(d) $|f(x)| \leq 2$ for all $x \in (0, 2)$

8. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point $(1, 3)$. Then the solution curve [JEE Advanced-2016]

(a) intersects $y = x + 2$ exactly at one point

(b) intersects $y = x + 2$ exactly at two points

(c) intersects $y = (x + 2)^2$

(d) does NOT intersect $y = (x + 3)^2$

9. If $y = y(x)$ satisfies the differential equation $8\sqrt{x}\left(\sqrt{9 + \sqrt{x}}\right)dy = \left(\sqrt{4 + \sqrt{9 + \sqrt{x}}}\right)^{-1} dx$, $x > 0$ and $y(0) = \sqrt{7}$, then $y(256) =$ [JEE (Advanced)-2017]

(a) 16 (b) 3

(c) 9 (d) 80

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$ then the value of $\lim_{x \to -\infty} f(x)$ is [JEE Advanced - 2018]

ANSWER KEY

Topicwise Questions

1. (*a*)	**2.** (*b*)	**3.** (*c*)	**4.** (*a*)	**5.** (*b*)	**6.** (*a*)	**7.** (*a*)	**8.** (*b*)	**9.** (*d*)	**10.** (*b*)
11. (*a*)	**12.** (*c*)	**13.** (*a*)	**14.** (*c*)	**15.** (*b*)	**16.** (*d*)	**17.** (*a*)	**18.** (*a*)	**19.** (*d*)	**20.** (*a*)
21. (*a*)	**22.** (*a*)	**23.** (*b*)	**24.** (*a*)	**25.** (*b*)	**26.** (*a*)	**27.** (*b*)	**28.** (*a*)	**29.** (*a*)	**30.** (*a*)

Learning Plus

1. (*a*)	**2.** (*d*)	**3.** (*a*)	**4.** (*d*)	**5.** (*a*)	**6.** (*d*)	**7.** (*b*)	**8.** (*c*)	**9.** (*b*)	**10.** (*b*)
11. (*c*)	**12.** (*d*)	**13.** (*c*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*a*)	**18.** (*c*)	**19.** (*a*)	**20.** (*c*)
21. (*b*)	**22.** (*b*)	**23.** (*b*)	**24.** (*a*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*a*)	**29.** (*b*)	**30.** (*c*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. (*cd*)	**2.** (*ad*)	**3.** (*acd*)	**4.** (*bc*)	**5.** (*ab*)	**6.** (*acd*)	**7.** (*ad*)	**8.** (*cd*)	**9.** (*ab*)	**10.** (*c*)
11. (*ad*)	**12.** (*abd*)	**13.** (*abc*)	**14.** (*abcd*)	**15.** (*b*)	**16.** (*a*)	**17.** (*a*)			
18. (*a*) – (r); (*b*) – (p); (*c*) – (s); (*d*) – (q)				**19.** (*a*) – (q), (*b*) – (r), (*c*) – (s), (*d*) – (p)				**20.** (5)	
21. (1.5)	**22.** (2)	**23.** (0.5)	**24.** (12)	**25.** (–0.25)	**26.** (4)	**27.** (0)	**28.** (3)		

JEE Mains & Advanced Past Years Questions

JEE MAIN PREVIOUS YEARS

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*b*)	**6.** (*a*)	**7.** (*a*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)
11. (*c*)	**12.** (*b*)	**13.** (*b*)	**14.** (*a*)	**15.** (*d*)	**16.** (*c*)	**17.** (*c*)	**18.** (*b*)	**19.** (*a*)	**20.** (*d*)
21. (*a*)	**22.** (*b*)	**23.** (*a*)	**24.** (*b*)	**25.** (*c*)	**26.** (3)	**27.** (*a*)	**28.** (*a*)	**29.** (*d*)	**30.** (*d*)
31. (*a*)	**32.** (*c*)	**33.** (*b*)	**34.** (*a*)	**35.** (*b*)	**36.** (*a*)	**37.** (*d*)	**38.** (c)		

JEE ADVANCED PREVIOUS YEARS

1. (*a,d*)	**2.** (*a*)	**3.** (*b*)	**4.** (*b*)	**5.** (*a,c*)	**6.** (*b,c*)	**7.** (*a*)	**8.** (*a,d*)	**9.** (*b*)	**10.** (0.4)

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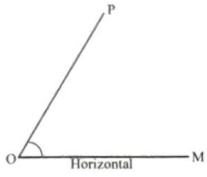
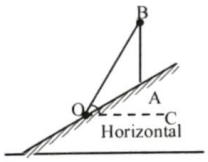
33 HEIGHTS AND DISTANCES

ANGLE OF ELEVATION

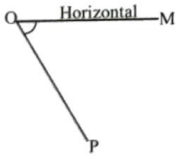
Consider a point P being observed from a point O (usually called observer) at a lower horizontal level. Then, OP is called the line of sight or observation and $\angle POM$ is called the angle of elevation of point P as seen from O.





ANGLE OF DEPRESSION

Consider a point P being observed from a point O at a higher horizontal level $\angle POM$ here is called the angle of depression of the point P as seen from O. This is also measured w.r.t the horizontal at O.



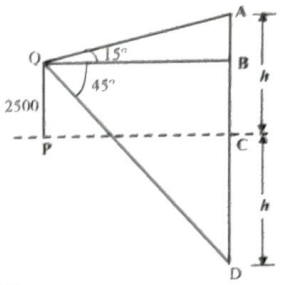
TRAIN YOUR BRAIN

Example 1: The angle of elevation of a stationary cloud from a point 2500 metres above a lake is 15° and the angle of depression of its reflection in the lake is 45°. What is the height of the cloud above the lake level?

Sol. $\tan 15° = \dfrac{AB}{BQ}$

$\Rightarrow \tan(45°-30°) = \dfrac{h-2500}{BQ}$

$\Rightarrow \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \dfrac{h-2500}{BQ}$



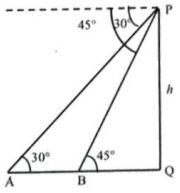
$\tan 45° = \dfrac{BD}{BQ}$

$\Rightarrow 1 = \dfrac{h+2500}{BQ}$ (2)

$(1) \div (2) \Rightarrow \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{h-2500}{h+2500} \Rightarrow h = 2500\sqrt{3}$ m

Example 2: The shadow of a tower standing on a level plane is found to be 60 m longer when the sun's elevation is 30° then when it is 45°. Find the height of tower.

Sol.



$AB = 60\,m = AQ - BQ$

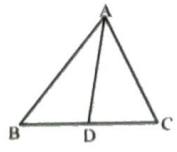
$= h\cot 30° - h\cot 45°$

$\Rightarrow h = \dfrac{60}{\sqrt{3}-1}\text{ m} = \dfrac{60(\sqrt{3}+1)}{3-1} = 81.96\text{ m}$

APOLLONIUS THEOREM

If AD is the median of a ΔABC, then

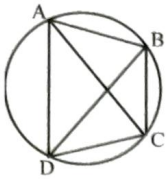
$AB^2 + AC^2 = 2(AD^2 + BD^2)$ or $2(AD^2 + DC^2)$



PTOLEMY'S THEOREM

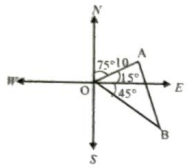
In a cyclic quadrilateral $ABCD$

$$AC.BD = AB.CD + AD.BC$$



Example 3: Two cars leave a place at the same time. One travels 10 km in the direction $N75°E$ and the other 20 km in the south-east direction. What is the final distance between the cars?

Sol.



$$\cos\angle AOB = \frac{OA^2 + OB^2 - AB^2}{2(OA)(OB)}$$

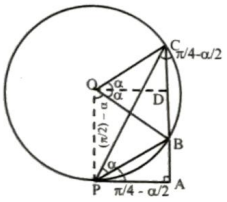
$$\Rightarrow \quad AB^2 = (10)^2 + (20)^2 - 2(10)(20)\cos 60°$$

$$= 100 + 400 - 200 = 300$$

$$\Rightarrow \quad AB = 10\sqrt{3}\ km$$

Example 4: A man is moving towards a building on which a flagstaff is mounted. The flagstaff subtends its maximum angle α at the man's eye when he is at a distance 'c' from the base of building. Obtain the height of building and flagstaff in terms of c and α.

Sol.



Let AB be the building and BC be the flagstaff.

Since angle subtended by BC is maximum at P, a circle drawn through P, B and C touches PA at P. The perpendicular bisector of chord BC and a line perpendicular to PA at P intersect at the center O of the circle. Consider chord BC.

$\because \quad \angle BPC = \alpha \Rightarrow \angle BOC = 2\alpha$

$\Rightarrow \quad \angle COD = \alpha$

$\therefore$ Height of flagstaff $= BC = 2CD$

$= 2OD \tan \angle COD$

$= 2PA \tan \alpha = 2c \tan \alpha$

Now consider chord PB.

$\because \quad \angle POB = \pi/2 - \alpha$

$\Rightarrow \quad \angle PCB = \pi/4 - \alpha/2$

$\Rightarrow \quad \angle BPA = \pi/4 - \alpha/2$ (alternate segment theorem)

$\therefore$ Height of building $= AB = AP \tan \angle BPA = C\tan(\pi/4 - \alpha/2)$

Example 5: A tower is observed from two stations A and B, where B is east of A at a distance 100 metres. The tower is due north of and due north west of B. The angle of elevation of the tower from A and B are complementary. Find the height of the tower.

Sol. Let OP be the tower of height h. A is a point due south of the tower and the angle of elevation of the tower at B will be $(90° - \theta)$ (complementary of θ).

i.e. $\angle PAO = \theta$; $\angle PBO = 90° - \theta$

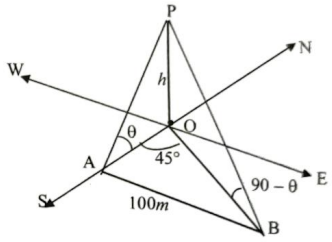
From right angled triangle POA.

$$\tan\theta = \frac{h}{OA} \qquad ...(1)$$

From right angle triangle POB,

$$\tan(90° - \theta) = \frac{h}{OB} \text{ or, } \tan\theta = \frac{OB}{h} \qquad (2)$$

From (1) and (2), we have



$$\frac{h}{OA} = \frac{OB}{h} \quad \therefore h = \sqrt{OA.OB} \qquad ...(3)$$

Since BO is north-west, OB will be south-east

$\therefore \quad \angle AOB = 45° = \angle OBA \qquad [\because \angle OAB = 90°]$

$\Rightarrow \quad OA = AB = 100\text{m}$

Now, From right angled triangle OAB,

$$OB^2 = OA^2 + AB^2 = 100^2 + 100^2$$

$\therefore \quad OB = 100\sqrt{2}\,\text{m} \qquad$ From (3)

$$h = \sqrt{100.100\sqrt{2}} = 100(2)^{1/4} \text{ metres}$$

Example 6: A spherical ball of diameter d subtends an angle α at a man's eye when the elevation of its centre is β. Prove that the height of the center of the ball is $1/2\ d \sin\beta \operatorname{cosec}(\alpha/2)$.

Sol. Let O be the center of the ball and E, the man's eye. EA and EB are the tangents covering the ball and hence

$$\angle AEB = \alpha$$

Draw $\quad OO' \perp EO'$

$$\angle OEO' = \beta$$

(From question)

Let $OO' = h$

From right angled triangle $OO'E$,

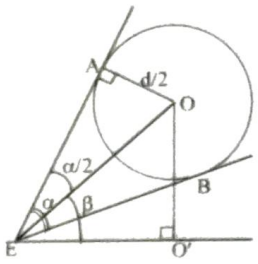
$$\sin\beta = \frac{h}{EO}$$

$\therefore$ $h = EO\sin\beta$ ...(1)

From geometry, it is clear that

$$\angle AEO = \angle BEO = \frac{\alpha}{2} \quad [\angle AEB = \alpha]$$

Now, From right angled triangle OAE,



$$\sin\frac{\alpha}{2} = \frac{d/2}{OE} \text{ or, } OE = \frac{1}{2}d\,\text{cosec}\frac{\alpha}{2} \quad ...(2)$$

From (1) and (2), $h = \frac{1}{2}d\ \sin\beta\ \text{cosec}\frac{\alpha}{2}$

Topicwise Questions

1. The angle of elevation of the top of a tower at point on the ground is 30°. If on walking 20 *metres* toward the tower, the angle of elevation become 60°, then the height of the tower is

(*a*) 10 meter (*b*) $\frac{10}{\sqrt{3}}$ meter

(*c*) $10\sqrt{3}$ meter (*d*) $100\sqrt{3}$ meter

2. The angle of elevation of the top of the tower observed from each of the three points A, B, C on the ground, forming a triangle is the same angle α. If R is the circum-radius of the triangle ABC, then the height of the tower is

(*a*) R sin α (*b*) R cos α

(*c*) R cot α (*d*) R tan α

3. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°. When he retires 40 meters from the bank, he finds the angle to be 30°. The breadth of the river is

(*a*) 20m (*b*) 40m

(*c*) 30m (*d*) 60m

4. A vertical pole consists of two parts, the lower part being one third of the whole. At a point in the horizontal plane through the base of the pole and distance 20 *meters* from it, the upper part of the pole subtends an angle whose tangent is $\frac{1}{2}$. The possible heights of the pole are

(*a*) 20 m and $20\sqrt{3}$ m (*b*) 20 m and 60 m

(*c*) 16 m and 48 m (*d*) 20 m and 16 mAns.

5. From a 60 meter high tower angles of depression of the top and bottom of a house are α and β respectively. If the height of the house is $\frac{60\sin(\beta-\alpha)}{x}$, then $x =$

(*a*) sin α sin β (*b*) cos α cos β

(*c*) sin α cos β (*d*) cos α sin β

6. An observer on the top of a tree, finds the angle of depression of a car moving towards the tree to be 30°. After 3 minutes this angle becomes 60°. After how much more time, the car will reach the tree

(*a*) 4min. (*b*) 4.5min.

(*c*) 1.5min. (*d*) 2min.

7. A house of height 100 *metres* subtends a right angle at the window of an opposite house. If the height of the window be 64 *metres*, then the distance between the two houses is

(*a*) 48m (*b*) 36m

(*c*) 54m (*d*) 72m

8. The length of the shadow of a pole inclined at 10° to the vertical towards the sun is 2.05 *metres*, when the elevation of the sun is 38°. The length of the pole is

(*a*) $\frac{2.05\sin 38^\circ}{\sin 42^\circ}$ (*b*) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$30m

(*c*) $\frac{2.05\cos 38^\circ}{\cos 42^\circ}$ (*d*) $\frac{2.05\cos 42^\circ}{\sin 38^\circ}$

9. From the top of a light house 60 *meters* high with its base at the sea level, the angle of depression of a boat is 15°. Thedistance of the boat from the foot of light house is

(*a*) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$60 m (*b*) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$60 m

(*c*) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$ m (*d*) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$30m

10. An observer in a boat finds that the angle of elevation of a tower standing on the top of a cliff is 60° and that of the bottom of cliff is 30°. If the height of the tower be 60 meters, then the height of the cliff is

(*a*) 30m (*b*) $40\sqrt{3}$ m

(*c*) $20\sqrt{3}$ m (*d*) $40\sqrt{3}$ m

11. From a point *a metre* above a lake the angle of elevation of a cloud is α and the angle of depression of its reflection is β. The height of the cloud is

(*a*) $\frac{a\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ metre (*b*) $\frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$ metre

(*c*) $\frac{a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}$ metre (*d*) $\frac{2a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}$

12. The shadow of a tower is found to be 60 *metre* shorter when the sun's altitude changes from 30° to 60°. The height of the tower from the ground is approximately equal to

(*a*) 62m (*b*) 301m

(*c*) 101m (*d*) 52m

13. If the angles of elevation of two towers from the middle point of the line joining their feet be 60° and 30° respectively, then the ratio of their heights is

(*a*) 2 : 1 (*b*) 1 : $\sqrt{2}$

(*c*) 3 : 1 (*d*) 1 : $\sqrt{3}$

14. Some portion of a 20 meters long tree is broken by the wind and the top struck the ground at an angle of 30°. The heightof the point where the tree is broken is

(*a*) 10m (*b*) $(2\sqrt{3}-3)20$ m

(*c*) $\frac{20}{3}$m (*d*) $10\sqrt{3}$m

15. The base of a cliff is circular. From the extremities of a diameter of the base the angles of elevation of the top of the cliff are 30° and 60°. If the height of the cliff be 500 *metres*, then the diameter of the base of the cliff is

(*a*) $1000\sqrt{3}$m (*b*) $2000/\sqrt{3}$m

(*c*) $1000/\sqrt{3}$m (*d*) $2000\sqrt{2}$m

16. The angle of elevation of the top of a tower from the top of a house is 60° and the angle of depression of its base is 30°. If the horizontal distance between the house and the tower be 12 *m*, then the height of the tower is

(*a*) $48\sqrt{3}$ m (*b*) $16\sqrt{3}$ m

(*c*) $24\sqrt{3}$ m (*d*) $16/\sqrt{3}$ m

17. A man whose eye level is 1.5 metres above the ground observes the angle of elevation of a tower to be 60°. If the distance of the man from the tower be 10 meters, the height of the tower is

(*a*) $(1.5+10\sqrt{3})$m (*b*) $10\sqrt{3}$m

(*c*) $\left(1.5+\frac{10}{\sqrt{3}}\right)$m (*d*) $100\sqrt{3}$m

18. A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point *h* vertically above the first, the depression of the foot of the tower is 60°. The height of the tower is

(*a*) *h*/3 (*b*) *h*/3*d*

(*c*) 3*h* (*d*) $\frac{3h}{d}$

19. A tree is broken by wind, its upper part touches the ground at a point 10 *metres* from the foot of the tree and makes an angle of 45° with the ground. The total length of tree is

(*a*) 15 metres (*b*) 20 metres

(*c*) $10(1+\sqrt{2})$ metres (*d*) $10\left(1+\frac{\sqrt{3}}{2}\right)$metres

20. The angle of elevation of a stationary cloud from a point 2500 *m* above a lake is 15° and the angle of depression of its reflection In the lake is 45°. The height of cloud above the lake level is

(*a*) $2500\sqrt{3}$ metres (*b*) 2500 metres

(*c*) $500\sqrt{3}$ metres (*d*) 5000 metres

21. From an aeroplane vertically over a straight horizontally road, the angles of depression of two consecutive mile stones on opposite sides of the aeroplane are observed to be α and β, then the height in miles of aeroplane above the road is

(*a*) $\frac{\tan\alpha.\tan\beta}{\cot\alpha+\cot\beta}$ (*b*) $\frac{\tan\alpha+\tan\beta}{\tan\alpha.\tan\beta}$

(*c*) $\frac{\cot\alpha+\cot\beta}{\tan\alpha.\tan\beta}$ (*d*) $\frac{\tan\alpha.\tan\beta}{\tan\alpha+\tan\beta}$

22. A flag-post 20 m high standing on the top of a house subtends an angle whose tangent is $\frac{1}{6}$ at a distance 70 m from the foot of the house. The height of the house is

(*a*) 30 m (*b*) 60 m

(*c*) 50m (*d*) 20m

23. A balloon is coming down at the rate of 4 *m/min.* and its angle of elevation is 45° from a point on the ground which has been reduced to 30° after 10 *minutes*. Balloon will be on the ground at a distance of how many meters from the observer

(*a*) $20\sqrt{3}$m (*b*) $20(3+\sqrt{3})$m

(*c*) $10(3+\sqrt{3})$m (*d*) $10\left(3-\sqrt{3}\right)$m

24. *AB* is a vertical pole resting at the end *A* on the level ground. *P* is a point on the level ground such that *AP* = 3 *AB*. If *C* is the mid- point of *AB* and *CB* subtends an angle β at *P*, the value of tan β is

(*a*) $\frac{18}{19}$ (*b*) $\frac{3}{19}$

(*c*) $\frac{1}{6}$ (*d*) $\frac{1}{64}$

25. Two straight roads intersect at an angle of 60°. A bus on one road is 2 km away from the intersection and a car on the otherroad is 3 km away from the intersection. Then the direct distance between the two vehicles is

(*a*) 1km (*b*) $\sqrt{2}$ *km*

(*c*) 4km (*d*) $\sqrt{7}$ *km*

26. The angle of elevation of a cliff at a point *A* on the ground and a point *B*, 100 *m* vertically at *A* are α and β respectively. The height of the cliff is

(*a*) $\frac{100\cot\alpha}{\cot\alpha-\cot\beta}$ (*b*) $\frac{100\cot\beta}{\cot\alpha-\cot\beta}$

(*c*) $\frac{100\cot\beta}{\cot\beta-\cot\alpha}$ (*d*) $\frac{100\cot\beta}{\cot\beta+\cot\alpha}$

27. The angular elevation of a tower CD at a point A due south of it is 60° and at a point B due west of A, the elevation is 30°. If $AB = 3\ km$, the height of the tower is

(a) $2\sqrt{3}$ km (b) $2\sqrt{6}$ km

(c) $\frac{3\sqrt{3}}{2}$ km (d) $\frac{3\sqrt{6}}{4}$ km

28. Two men are on the opposite side of a tower. They measure the angles of elevation of the top of the tower 45° and 30° respectively. If the height of the tower is 40 m, find the distance between the men

(a) 40m (b) $40\sqrt{3}$ m

(c) 68.280m (d) 109.28 m

29. The angles of elevation of the top of a tower (a) from the top (b) and bottom (d) at a building of height a are 30° and 45° respectively. If the tower and the building stand at the same level, then the height of the tower is

(a) $a\sqrt{3}$ (b) $\frac{a\sqrt{3}}{\sqrt{3}-1}$

(c) $\frac{a(3+\sqrt{3})}{2}$ (d) $a(\sqrt{3}-1)$

30. A ladder 5 *metre* long leans against a vertical wall. The bottom of the ladder is 3 *metre* from the wall. If the bottom of the ladder is pulled 1 *metre* farther from the wall, how much does the top of the ladder slide down the wall

(a) 1m (b) 7m

(c) 2m (d) 3m

31. The angle of elevation of the top of a pillar at any point A on the ground is $15°$. On walking 40 *metres* towards the pillar, the angle become $30°$. The height of the pillar is

(a) 40 *metres* (b) 20 *metres*

(c) $20\sqrt{3}$metres (d) $\frac{40}{3}\sqrt{3}$metres

32. The top of a hill observed from the top and bottom of a building of height h is at the angle of elevation p and q respectively. The height of the hills is

(a) $\frac{h\cot q}{\cot q - \cot p}$ (b) $\frac{h\cot p}{\cot p - \cot q}$

(c) $\frac{h\tan p}{\tan p - \tan q}$ (d) $\frac{h\cot p}{\cot q + \cot p}$

33. The shadow of a tower standing on a level ground is found to be 60 m longer when the sun's altitude is $30°$ than when it is $45°$. The height of the tower is

(a) 60m (b) 30m

(c) $60\sqrt{3}$m (d) $30(\sqrt{3}+1)$m

34. For a man, the angle of elevation of the highest point of the temple situated east of him is $60°$. On walking 240 *metres* to north, the angle of elevation is reduced to $30°$, then the height of the temple is

(a) $60\sqrt{6}$m (b) 60m

(c) $50\sqrt{3}$m (d) $30\sqrt{6}$m

35. A tower subtends angles $\alpha, 2\alpha, 3\alpha$ respectively at points A, B and C, all lying on a horizontal line through the foot of the tower. Then $AB/BC =$

(a) $\frac{\sin 3\alpha}{\sin 2\alpha}$ (b) $1+2\cos 2\alpha$

(c) $2+\cos 3\alpha$ (d) $\frac{\sin 2\alpha}{\sin \alpha}$

36. Two pillars of equal height stand on either side of a roadway which is 60 *metres* wide. At a point in the roadway between the pillars, the elevation of the top of pillars are $60°$ and $30°$. The height of the pillars is

(a) $15\sqrt{3}$m (b) $\frac{15}{\sqrt{3}}$m

(c) 15m (d) 20m

37. A ladder rests against a wall making an angle α with the horizontal. The foot of the ladder is pulled away from the wall through a distance x, so that it slides a distance y down the wall making an angle β with the horizontal. The correct relation is

(a) $x = y\tan\frac{\alpha+\beta}{2}$ (b) $y = x\tan\frac{\alpha+\beta}{2}$

(c) $x = y\tan(\alpha+\beta)$ (d) $y = x\tan(\alpha+\beta)$

Learning Plus

1. A pole 25m long stands on the top of a tower 225m high. If θ is the angle subtended by the pole at a point on the ground which is at a distance of 2.25 km from the foot of the tower, then $\tan\theta$ is equal to

(a) 1/90 (b) 1/91

(c) 1/10 (d) 1/9

2. The angle of elevation of the top of a vertical pole when observed from each vertex of a regular hexagon is $\pi/3$. If the area of the circle circumscribing the hexagon be A metre^2 then the area of the hexagon is

(a) $\frac{3\sqrt{3}}{8}A$ metre^2 (b) $\frac{\sqrt{3}}{\pi}A$ metre^2

(c) $\frac{3\sqrt{3}}{4\pi}A$ metre^2 (d) $\frac{3\sqrt{3}}{2\pi}A$ metre^2

3. A vertical pole PO is standing at the center O of a square ABCD. If AC subtends an angle 90° at the top P of the pole then the angle subtended by a side of the square at P is
(a) 45° (b) 30°
(c) 60° (d) 90°

4. The angles of elevation of the top of a tower standing on a horizontal plane, from two points on a line passing through its foot at distances a and b respectively, are complementary angles. If the line joining the two points subtends an angle θ at the top of the tower, then
(a) $\sin\theta = \frac{a-b}{a+b}$ (b) $\tan\theta = \frac{2\sqrt{ab}}{a-b}$
(c) $\sin\theta = \frac{a+b}{a-b}$ (d) $\cos\theta = \frac{2\sqrt{ab}}{a-b}$

5. A vertical lamp-post, 6 m high, stands at a distance of 2 m from a wall, 4 m high. A 1.5-m-tall man starts to walk away from the wall on the other side of the wall, in line with the lamp-post. The maximum distance to which the man can walk remaining in the shadow is
(a) $\frac{5}{2}m$ (b) $\frac{3}{2}m$
(c) $4m$ (d) $\frac{4}{3}$m

6. A circular ring of radius 3 cm is suspended horizontally from a point 4 cm vertically above the center by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings be θ then cos θ is
(a) 4/5 (b) 4/25
(c) 16/25 (d) 5/26

7. A flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and the distance of the man from the pillar, is
(a) $\sqrt{3}:1$ (b) 1 : 3
(c) $1:\sqrt{3}$ (d) $\sqrt{3}:2$

8. As seen from A, due west of a hill HL itself leaning east, the angle of elevation of top H of the hill is 60°; and after walking a distance of one kilometer along an incline of 30° to a point B, it was seen that the hill LH was printed at right angles to AB, the height LH of the hill is
(a) $\frac{1}{\sqrt{3}}$ km (b) $\sqrt{3}$
(c) $2\sqrt{3}$ km (d) $\frac{2}{\sqrt{3}}$ km

9. ABC is a triangular park with $AB = AC = 100$ metres. A clock tower is situated at the mid point of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1}3.2$ and $\text{cosec}^{-1}2.6$ respectively. The height of the tower is
(a) 16m (b) 25m
(c) 50m (d) 75m

10. The angles of elevation of the top of a tower from the top and bottom of a building of height 'a' are 30° and 45°, respectively. If the tower and the building stand at the same level, the height of the tower is
(a) $a\sqrt{3}$ (b) $\frac{3\sqrt{3}a}{2}$
(c) $\frac{\sqrt{3}a}{\sqrt{3}-1}$ (d) $a(\sqrt{3}+1)$

11. A flag staff of 5 mt high stands on a building of 25 mt high. At an observer at a height of 30 mt. the flag staff and the building subtend equal angles. The distance of the observer from the top of the flag staff is
(a) $\frac{5\sqrt{3}}{2}$ (b) $5\sqrt{\frac{3}{2}}$
(c) $5\sqrt{\frac{2}{3}}$ (d) $5\sqrt{\frac{2}{5}}$

12. If a flag-staff of 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is
(a) 60° (b) 30°
(c) 45° (d) 90°

13. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 mt. from its base is 45°. If the angle of elevation of the top of the complete pillar at the same point is to be 60°, then the height of the incomplete pillar is to be increased by
(a) $50\sqrt{2}$ mt (b) 100 mt
(c) $100(\sqrt{3}-1)$mt (d) $100(\sqrt{3}+1)$ mt

14. The angles of elevation of a cliff at a point A on the ground and a point B, 100 meters vertically at A are α and β respectively. The height of the cliff is
(a) $\frac{100\cot\alpha}{\cot\alpha-\cot\beta}$ (b) $\frac{100\cot\beta}{\cot\alpha-\cot\beta}$
(c) $\frac{100\cot\beta}{\cot\beta-\cot\alpha}$ (d) $\frac{100\cot\beta}{\cot\beta+\cot\alpha}$

15. The angle of elevation of a cloud from a point h mt. above is θ and the angle of depression of its reflection in the lake is ϕ. Then the height is
(a) $\frac{h\sin(\phi-\theta)}{\sin(\phi+\theta)}$ (b) $\frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$
(c) $\frac{h\sin(\theta+\phi)}{\sin(\theta-\phi)}$ (d) $\frac{h\sin(\phi-\theta)}{\sin(\theta-\phi)}$

16. On the level ground the angle of elevation of the top of a tower is 30°. On moving 20 m. nearer the tower, the angle of elevation is found to be 60°. The height of the tower is
(a) 10m (b) 20m
(c) $10\sqrt{3}$m (d) $100\sqrt{3}$m

17. Each side of a square subtends an angle of 60° at the top of a tower h metres high standing in the centre of the square. If a is the length of each side of the square, then

(a) $2a^2 = h^2$ (b) $2h^2 = a^2$

(c) $3a^2 = 2h^2$ (d) $3h^2 = 2a^2$.

18. From the top of light house 60 metres high with its base at the sea level, the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house is

(a) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.60 metres (b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.60 metres

(c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ metres (d) $\sqrt{3}+1$ metre

19. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°, when he retires 40 metres from the bank he finds the angle to be 30°. Then the breadth of the river is

(a) 40m (b) 60m

(c) 20m (d) 30m

20. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC subtends an angle β at P. If AP = n AB, then tan β =

(a) $\frac{n}{2n^2+1}$ (b) $\frac{n}{n^2-1}$

(c) $\frac{n}{n^2+1}$ (d) $\frac{n^2}{n+1}$

21. A tree is broken by wind, its upper part touches the ground at a point 10 metres from the foot of the tree and makes an angle of 45° with the ground. The entire length of the tree is

(a) 15 metres (b) 20 metres

(c) $10(1+\sqrt{2})$ metres (d) $\left(1+\frac{\sqrt{3}}{2}\right)$ metres

22. An aeroplane flying at a height of 300 metres above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. The height of the lower plane from the ground (in metres) is

(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$

(c) 50 (d) $150(\sqrt{3}+1)$

23. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point b ft. just above A is β. Then height of the tower is

(a) $b \tan\alpha \cot\beta$ (b) $b \cot\alpha \tan\beta$

(c) $b \tan\alpha \tan\beta$ (d) $b \cot\alpha \cot\beta$

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then, the time taken (in min) by him, from B to reach the pillar, is :

[*JEE Main-2016*]

(a) 6 (b) 10

(c) 20 (d) 5

2. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then tanβ is equal to :

[*JEE Main-2017*]

(a) $\frac{4}{9}$ (b) $\frac{6}{7}$

(c) $\frac{1}{4}$ (d) $\frac{2}{9}$

3. PQR is a triangular park with PQ = PR = 200m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is -

[*JEE Main-2018*]

(a) 50 (b) $100\sqrt{3}$

(c) $50\sqrt{2}$ (d) 100

4. Consider a triangular plot ABC with sides AB = 7 m, BC = 5m and CA = 6m. A vertical lamp-post at the mid point D of AC subtendes an angle 30° at B. The height (in m) of the lamp-post is: [*JEE Main-2019 (January)*]

(a) $\frac{3}{2}\sqrt{21}$ (b) $\frac{2}{3}\sqrt{21}$

(c) $2\sqrt{21}$ (d) $7\sqrt{21}$

5. If the angle of elevation of a cloud from apoint P which is 25 m above a lake be 30° and the angle of depressionof reflection of the cloud in the lake from P be 60°, then the height of the cloud (in meters) from the surface of the lake is [*JEE Main-2019 (January)*]

(a) 60 (b) 50

(c) 45 (d) 42

6. Two vertical poles of heights, 20m and 80m stand a part on a horizontal plane. The height (in meters) of the points of intersection of the lines joining the top of each pole to the foot of the other, from this horizontal plane is :

[*JEE Main-2019(April)*]

(*a*) 12 (*b*) 15

(*c*) 16 (*d*) 18

7. Two poles standing on a horizontal ground are of heights 5m and 10 m respectively. The line joining their tops makes an angle of 15° with ground. Then the distance **(in m)** between the poles, is :- [*JEE Main-2019 (April)*]

(*a*) $\frac{5}{2}(2+\sqrt{3})$ (*b*) $5(\sqrt{3}+1)$

(*c*) $5(2+\sqrt{3})$ (*d*) $10(\sqrt{3}-1)$

8. ABC is a triangular park with AB = AC = 100 metres. A vertical tower is situated at the mid point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}\left(3\sqrt{2}\right)$ and $\text{cosec}^{-1}\left(2\sqrt{2}\right)$ respectively, then the height of the tower (in metres) is :

[*JEE Main-2019 (April)*]

(*a*) $10\sqrt{5}$ (*b*) $\frac{100}{3\sqrt{3}}$

(*c*) 20 (*d*) 25

9. The angle of elevation of the top of vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point . If the angle of elevation of the top of the tower from B be 30° then the distance (in m) of the foot of the tower from the point A is: [*JEE Main-2019 (April)*]

(*a*) $15\left(3-\sqrt{3}\right)$ (*b*) $15\left(3+\sqrt{3}\right)$

(*c*) $15\left(1+\sqrt{3}\right)$ (*d*) $15\left(5-\sqrt{3}\right)$

10. The angle of elevation of a cloud C from a point P, 200 m above a still lake is 30°. If the angle of depression of the image of C in the lake from the point P is 60°, then PC (in m) is equal to: [*JEE Main-2020 (September)*]

(*a*) 400 (*b*) $400\sqrt{3}$

(*c*) 100 (*d*) $200\sqrt{3}$

11. Two vertical poles AB = 15 m and CD = 10 m are standing apart on a horizontal ground with points A and C on the ground. If P is the point of intersection of BC and AD, then the height of P (in m) above the line AC is :

[*JEE Main-2020 (September)*]

(*a*) 6 (*b*) 20/3

(*c*) 10/3 (*d*) 5

12. The angle of elevation of the summit of a mountain from a point on the ground is 45°. After climbing up one km towards the summit at an inclination of 30° from the ground, the angle of elevation of the summit is found to be 60°. Then the height (in km) of the summit from the ground is: [*JEE Main-2020 (September)*]

(*a*) $\frac{1}{\sqrt{3}+1}$ (*b*) $\frac{1}{\sqrt{3}-1}$

(*c*) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (*d*) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

13. The angle of elevation of the top of a hill from a point on the horizontal plane passing through the foot of the hill is found to be 45°. After walking a distance of 80 meters towards the top, up a slope inclined at an angle of 30° to the horizontal plane, the angle of elevation of the top of the hill becomes 75°. Then the height of the hill (in meters) is — [*JEE Main-2020 (September)*]

14. A vertical pole fixed to horizontal ground is divided in the ratio of 3 : 7 by a mark on it with lower part shorter than upper part. If the two parts subtend equal angles at a point on the ground 18m away from the base of other pole, then the height of the pole (in m) is:

[*JEE Mains-2021 (August)*]

(*a*) $12\sqrt{15}$ (*b*) $12\sqrt{10}$

(*c*) $8\sqrt{10}$ (*d*) $6\sqrt{10}$

15. Two poles, AB of length a metres and CD of length a + b metres (a ≠ b) are exected at same horizontal level with bases at B and D. If BD = x and $\tan\angle ACB=\frac{1}{2}$, then, [*JEE Mains-2021 (August)*]

(*a*) $x^2 + 2(a+2b)x - b(a+b) = 0$

(*b*) $x^2 + 2(a+2b)x + a(a+b) = 0$

(*c*) $x^2 - 2ax + b(a+b) = 0$

(*d*) $x^2 - 2ax + a(a+b) = 0$

16. The angle of elevation of jet plane from a point A on the ground is 60° After a flight of 20 seconds at the speed of 432 km/hr, the angle of elevation changes to 30°. If the jet plane is flying at a constant height, then it's height is:

[*JEE Mains-2021 (Febuary)*]

(*a*) $3600\sqrt{3}$ m (*b*) $1200\sqrt{3}$ m

(*c*) $1800\sqrt{3}$ m (*d*) $2400\sqrt{3}$ m

17. Two vertical poles are 150m apart and height of one is three times that of other. If from the middle point of the line joining their feet, an observer finds the angle of elevation of their tops to be complementary, then height of shorter pole (in m) is: [*JEE Mains-2021 (Febuary)*]

(*a*) 30 (*b*) 25

(*c*) $20\sqrt{3}$ (*d*) $25\sqrt{3}$

18. A man is observing from the top of tower a boat speeding towards the lower from a certain point A, with uniform speed. At that point, angle of depression of boat with man's eye is 30° (ignore man's height). After Sailing for 20 seconds, towards the base of tower (same level as of water), the boat has reached a point B, where angle of depression is 45°. Then time taken by boat (in seconds) from B to reach base of tower is:

[*JEE Mains-2021 (Febuary)*]

(*a*) $10\left(\sqrt{3}+1\right)$ (*b*) $10\left(\sqrt{3}-1\right)$

(*c*) 10 (*d*) $10\left(\sqrt{3}\right)$

ANSWER KEY

Topicwise Questions

1. (*c*) **2.** (*d*) **3.** (*a*) **4.** (*b*) **5.** (*d*) **6.** (*c*) **7.** (*a*) **8.** (*a*) **9.** (*b*) **10.** (*a*) **11.** (*b*) **12.** (*d*)
13. (*c*) **14.** (*c*) **15.** (*b*) **16.** (*b*) **17.** (*a*) **18.** (*a*) **19.** (*c*) **20.** (*a*) **21.** (*d*) **22.** (*c*) **23.** (*b*) **24.** (*b*)
25. (*d*) **26.** (*c*) **27.** (*d*) **28.** (*d*) **29.** (*c*) **30.** (*a*) **31.** (*b*) **32.** (*b*) **33.** (*d*) **34.** (*a*) **35.** (*b*) **36.** (*a*)
37. (*a*)

Learning Plus

1. (*b*) **2.** (*d*) **3.** (*c*) **4.** (*d*) **5.** (*a*) **6.** (*c*) **7.** (*c*) **8.** (*a*) **9.** (*b*) **10.** (*c*) **11.** (*b*) **12.** (*a*)
13. (*c*) **14.** (*c*) **15.** (*b*) **16.** (*c*) **17.** (*b*) **18.** (*b*) **19.** (*c*) **20.** (*a*) **21.** (*c*) **22.** (*a*) **23.** (*a*)

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

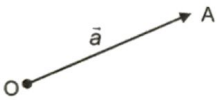
1. (*d*) **2.** (*d*) **3.** (*d*) **4.** (*b*) **5.** (*b*) **6.** (*c*) **7.** (*c*) **8.** (*c*) **9.** (*b*) **10.** (*a*) **11.** (*a*) **12.** (*b*)
13. (80.00) **14.** (*b*) **15.** (*c*) **16.** (*b*) **17.** (*d*) **18.** (*a*)

28 VECTOR ALGEBRA

VECTORS AND THEIR REPRESENTATION

Physical quantities, like displacements, velocity, acceleration etc. which have not only a value to signify how much (quantity wise) they are; – this aspect is called the magnitude, – but in addition have a directional element in them, are known as vector.

Any segment AB of a straight line which has distinct end points A and B is called a directed line segment. A and B are called respectively the initial point and end point of the line segment. A vector may be defined in terms of such a line segment and we write $\vec{a} = \overrightarrow{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.



Vectors are generally printed in bold faced type . When writing in the manuscript form the notation $\vec{a}, \overrightarrow{AB}$ may be used.

A vector, thus, is geometrically represented by a directed line segment. It is said that $\vec{a}$ is equivalently $\overrightarrow{OA}$ and they are vectorially indistinguishable.

GENERAL DEFINITION

Zero vector

A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O**. **The direction of zero vector is indeterminate.**

Unit vector

A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$, symbolically

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$

TRAIN YOUR BRAIN

Example 1: Find unit vector of $\hat{i} - 2\hat{j} + 3\hat{k}$

Sol. $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

then $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore \quad |\vec{a}| = \sqrt{14}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\,\hat{i} - \frac{2}{\sqrt{14}}\,\hat{j} + \frac{3}{\sqrt{14}}\,\hat{k}$$

Equal vectors

Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

Free vectors

Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as free vectors if the physical effect produced by them remains unaltered. e.g. displacement, velocity. (Generally IIT-JEE deals with free vectors only)

Localised vector

Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as located vectors if the physical effect produced by them are changed e.g. force in case of rotational motion (torque will be changed).

Note that :

(i) Number of distinct unit vectors in space perpendicular to a given plane is 2. (one upward and one downward).

(ii) Number of unit vectors in space parallel to a given plane is infinite.

(iii) Number of distinct unit vectors perpendicular to given line in space. (Infinitely many, think of the line as perpendicular to the xy plane. The unit vector might make any angle θ with the x-axis.)

(iv) Number of distinct unit vectors parallel to a line in space is 2.

(v) Two vectors are equal if they have equal component in an arbitrary direction.

Multiplication of vector by scalars

If $\vec{a}$ is a vector & m is a scalar, then $(m\vec{a})$ is a vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This multiplication is called **Scalar Multiplication.** If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

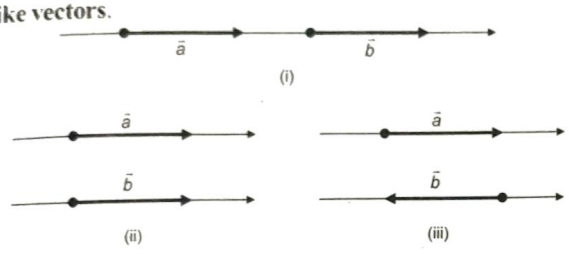
$$m(\vec{a}) = (\vec{a})m = m\vec{a},\ m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$

$$(m+n)\vec{a} = m\vec{a} + n\vec{a},\ m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Collinear vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors**.



Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$\Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3$$

$$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (=\lambda)$$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Result : If $\vec{a}$ and $\vec{b}$ are two non-zero non-collinear vectors then $x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0$ and $y = 0$.

Proof : Let $y \neq 0$

$$y\vec{b} = -x\vec{a} \Rightarrow \vec{b} = \left(-\frac{x}{y}\right)\vec{a} \Rightarrow \vec{b} = \lambda\vec{a}$$

$\Rightarrow \vec{a}$ and $\vec{b}$ are colinear, which is false.

Note : Two non-zero non collinear vectors are also called as **linearly independent vector or base vector.** If $\vec{a}$ and $\vec{b}$ are collinear then they are known as **linearly dependent vectors.**

TRAIN YOUR BRAIN

Example 2: Find values of x & y for which the vectors

$\vec{a} = (x+2)\hat{i} + (x-y)\hat{j} + \hat{k}$

$\vec{b} = (x-1)\hat{i} + (2x+y)\hat{j} + 2\hat{k}$ are parallel.

Sol. $\vec{a}$ and $\vec{b}$ are parallel if $\frac{x+2}{x-1} = \frac{y-x}{2x+y} = \frac{1}{2}$

$x = -5,\ y = -20$

Coplanar vectors

A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"two vectors are always coplanar"**.

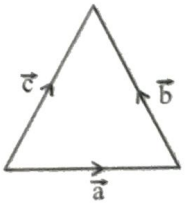
Self Practice Problems

1. Given a regular hexagon ABCDEF with centre O. show that
   (i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$
   (ii) $\overrightarrow{OD} + \overrightarrow{OA} = 2\overrightarrow{OB} + \overrightarrow{OF}$
   (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = 4\overrightarrow{AB}$
2. The vector $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$. Determine the unit vector along $\vec{c}$.
3. The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

Answers : (2) $-\frac{1}{3}\hat{i} + \frac{2}{15}\hat{j} - \frac{14}{15}\hat{k}$

ADDITION OF VECTORS

(i) Triangle law : If two vectors are represented in magnitude & direction by two sides of a triangle taken in same order then their sum is represented by the third side taken in reverse order.



Result : $\vec{a} + \vec{b} = \vec{c}$

(ii) Parallelogram law : If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

Properties of vector addition :

(i) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ **(commutative)**

(ii) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ **(associative)**

(iii) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$

(iv) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

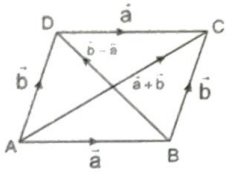
(v) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$

(vi) $|\vec{a}-\vec{b}|\ge||\vec{a}|-|\vec{b}||$

TRAIN YOUR BRAIN

Example 3: If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Sol. Let ABCD be a parallelogram such that $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}$.



Then, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC}=\vec{a}+\vec{b}=3\hat{i}+6\hat{j}-2\hat{k}$

$|\overrightarrow{AC}|=\sqrt{9+36+4}=7$

$\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$

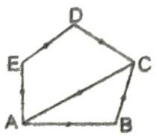
$\Rightarrow \overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}=\vec{b}-\vec{a}=\hat{i}+2\hat{j}-8\hat{k}$

$|\overrightarrow{BD}|=\sqrt{1+4+64}=\sqrt{69}$

$\therefore$ Unit vector along $\overrightarrow{AC}=\dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|}=\dfrac{1}{7}\left(3\hat{i}+6\hat{j}-2\hat{k}\right)$

and Unit vector along $\overrightarrow{BD}=\dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|}=\dfrac{1}{\sqrt{69}}\left(\hat{i}+2\hat{j}-8\hat{k}\right)$

Example 4: ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.



Sol. Let $\vec{R}$ be the resultant force

$\therefore \vec{R}=\overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$

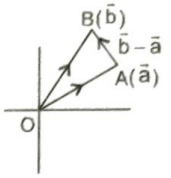
$\Rightarrow \vec{R}=(\overrightarrow{AB}+\overrightarrow{BC})+(\overrightarrow{AE}+\overrightarrow{ED}+\overrightarrow{DC})+\overrightarrow{AC}$

$\Rightarrow \vec{R}=\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}$

$\Rightarrow \vec{R}=3\overrightarrow{AC}$. Hence proved.

POSITION VECTOR OF A POINT

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then



$\overrightarrow{AB}=\vec{b}-\vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.

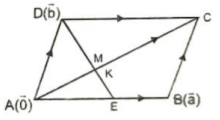
Example 5: If ABCD is a parallelogram and E is the mid point of AB. Show by vector method that DE trisect AC and is trisected by AC.

Sol. Let $\overrightarrow{AB}=\vec{a}$ and $AD=\vec{b}$

Then $\overrightarrow{BC}=\overrightarrow{AD}=\vec{b}$

and $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}=\vec{a}+\vec{b}$

Also let K be a point on AC, such that AK : AC = 1 : 3



$\Rightarrow AK=\dfrac{1}{3}AC$

$\Rightarrow \overrightarrow{AK}=\dfrac{1}{3}(\vec{a}+\vec{b})$ .....(i)

Again E being the mid point of AB, we have $\overrightarrow{AE}=\dfrac{1}{2}\vec{a}$

Let M be the point on DE such that DM : ME = 2 : 1

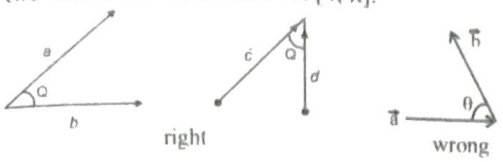
$\therefore \overrightarrow{AM}=\dfrac{\overrightarrow{AD}+2\overrightarrow{AE}}{1+2}=\dfrac{\vec{b}+\vec{a}}{3}$ .....(ii)

From (i) and (ii) we find that

$\overrightarrow{AK}=\dfrac{1}{3}(\vec{a}+\vec{b})=\overrightarrow{AM}$, and so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

ANGLE BETWEEN TWO VECTORS

It is defined as the smaller angle formed when the initial points or the terminal points of two vectors are brought together. Angle between two vectors lies in the interval $[0, \pi]$.



Example 6: If angle between equal vectors $\vec{a}$ and $\vec{b}$ is θ, then find the angle between $\vec{a}$ and $\vec{a}+\vec{b}$.

Sol. The angle between $\vec{a}$ and $\vec{a}+\vec{b}$ is $\theta/2$ as the diagonal of rhombus bisect the angle between two sides.

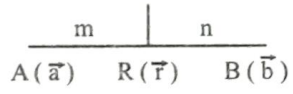
SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point R which divides AB internally in the ratio m : n is given by $\vec{r}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$

Proof : $\dfrac{AR}{RB}=\dfrac{m}{n}\Rightarrow n\overrightarrow{AR}=m\overrightarrow{RB}$

or $n(\vec{r}-\vec{a})=m(\vec{b}-\vec{r})$

$\therefore \vec{r}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$



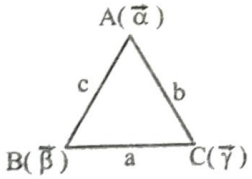
Similarly for external division $\vec{r}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$

Note : p.v. of mid point of $AB=\dfrac{\vec{a}+\vec{b}}{2}$.

Using section formula, we can prove that

(a) p.v. of the centroid of a triangle $ABC=\dfrac{\vec{\alpha}+\vec{\beta}+\vec{\gamma}}{3}$

(Concurrency of median)



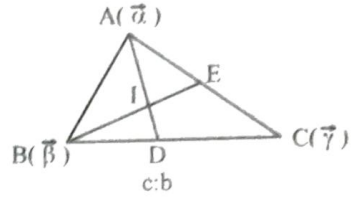
(b) incentre of the triangle $=\dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

$\dfrac{AI}{ID}=\dfrac{AB}{BD}=\dfrac{c}{\left(\frac{c}{b+c}\right)a}=\dfrac{b+c}{a}$

$\therefore D\left(\dfrac{b\vec{\beta}+c\vec{\gamma}}{b+c}\right)$

Excentres of the Δ are

$\dfrac{-a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{-a+b+c}$; $\dfrac{a\vec{\alpha}-b\vec{\beta}+c\vec{\gamma}}{a-b+c}$ and $\dfrac{a\vec{\alpha}+b\vec{\beta}-c\vec{\gamma}}{a+b-c}$

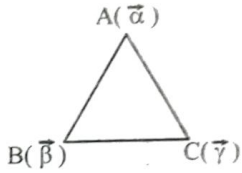


(c) Circumcentre of the

$\Delta=\dfrac{\vec{\alpha}\sin 2A+\vec{\beta}\sin 2B+\vec{\gamma}\sin 2C}{\sum\sin 2A}$

(d) Orthocentre of the

$\Delta=\dfrac{\vec{\alpha}\tan A+\vec{\beta}\tan B+\vec{\gamma}\tan C}{\sum\tan A}$



LINEAR COMBINATION OF VECTORS

The vector $r=\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\ldots+\alpha_n\vec{a}_n$ where $\alpha_1, \alpha_2, \ldots\alpha_n$ are any scalars (not all zero), is called linear combination of vectors $\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n$.

The following results are useful in determining coplanar and collinear vectors.

Fundamental theorem in two dimension

If $\vec{a}$ and $\vec{b}$ are non-collinear vectors then any vector $\vec{r}$, coplanar with $\vec{a}$ and $\vec{b}$, can be uniquely expressed as linear combination of $\vec{a}$ and $\vec{b}$, i.e., their exist unique $x, y \in R$ such that $\vec{r}=x\vec{a}+y\vec{b}$.

Also for any two non-collinear vector $\vec{a}$ and $\vec{b}$

$x\vec{a}+y\vec{b}=x'\vec{a}+y'\vec{b}$

$\Rightarrow x=x'$ and $y=y'$

Fundamental theorem in three dimension

If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors in space then any vector $\vec{r}$ can be uniquely expressed as linear combination of $\vec{a},\vec{b},\vec{c}$.

That is, there exists unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

Also for three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$,

$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c}$

$\Rightarrow \quad x = x', y = y'$ and $z = z'$

Linearly dependent and independent vectors

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly dependent if there exist a system of scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ (not all zero) such that

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \ldots + \lambda_n\vec{a}_n = \vec{0}$$

They are linearly independent if every relation of the type $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \ldots + \lambda_n\vec{a}_n = \vec{0}$ implies that $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_n = 0$.

Test for collinearity

Three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear if and only if there exist scalars x, y and z not all zero, such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$ where $x + y + z = 0$.

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if and only if there exist scalars x, y, z and w, not all zero, such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where $x + y + z + w = 0$.

Example 7: Show that the points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - \vec{c}, 4\vec{a} - 7\vec{b} + 7\vec{c}$ are collinear.

It is given that vectors $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

Sol. The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$\lambda_1(\vec{a} - 2\vec{b} + 3\vec{c}) + \lambda_2(-2\vec{a} + 3\vec{b} - \vec{c}) + \lambda_3(4\vec{a} - 7\vec{b} + 7\vec{c}) = 0$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Equating the coefficients $\vec{a}, \vec{b}$ and $\vec{c}$ separately to zero, we get

$\lambda_1 - 2\lambda_2 + 4\lambda_3 = 0, \quad -2\lambda_1 + 2\lambda_3 - 7\lambda_3 = 0$

and $3\lambda_1 - \lambda_2 + 7\lambda_3 = 0$.

We find that $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1$

so that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Hence the given vectors all collinear.

VECTOR EQUATION OF A LINE

Parametric vector equation of a line passing through two point $A(\vec{a})$ and $B(\vec{b})$ is given by $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where 't' is a parameter. If the line passes through the point $A(\vec{a})$ and is parallel to the vector $\vec{b}$, then its equation is $\vec{r} = \vec{a} + t\vec{b}$.

Note : $\vec{r}$ is the p.v. of the point on the line.

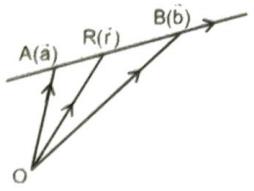


Fig. 1

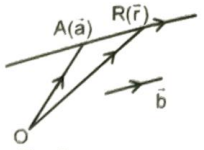


Fig. 2

A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} + \hat{b}\right)$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} - \hat{b}\right), \lambda \in R^+$.

Note that the equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are:

$$\vec{r} = \vec{a} + t\left(\hat{b} + \hat{c}\right) \text{ and } \vec{r} = \vec{a} + p\left(\hat{c} - \hat{b}\right).$$

Important note:

(i) Two lines in a plane are either intersecting or parallel, conversely two intersecting or parallel line must be in the same plane.

(ii) However in sapce we can have two neither parallel nor intersecting lines. Such non coplaner lines are known as skew lines.

(iii) If two lines are parallel and have a common point then they are coincident.

PRODUCT OF TWO VECTORS

To meet the needs of physical situations we have two kinds of products of two vectors:

(i) Resulting in scalar value and used in situations of work done by a force in a displacement; projection of a given length on another direction. This product is usually termed dot product or scalar product.

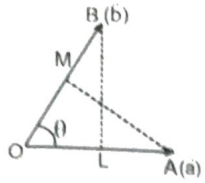
(ii) Resulting in a vector and used in situations of finding moment of a force about a point; or of finding the velocity of a particle having a rotating motion about a point with an angular velocity $\vec{\omega}$.

The two products with the related properties are stated:

Scalar product (Dot Product) of two vectors
Geometrical interpretation of scalar product

Let $\vec{a}$ and $\vec{b}$ be vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.

From ΔOBL and ΔOAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$.



Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|(|\vec{b}|\cos\theta)$

$= |\vec{a}|(OB\cos\theta) = |\vec{a}|(OL)$

= (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ...(i)

Again $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta) = |\vec{b}|(OM)$

= (magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ...(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

Properties of the Scalar product

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (Dot product is commutative)
2. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Dot product is distributive)
3. $\vec{a}\cdot\vec{b} = 0$ either when $|\vec{a}| = 0$ or when $|\vec{b}| = 0$ or when the vectors $\vec{a}$ and $\vec{b}$ are orthogonal. Thus for any two perpendicular vectors the dot product vanishes.
4. $\cos\theta = \left|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right|$ where θ is the acute angle made by $\vec{a}$ with $\vec{b}$.
5. $\vec{a}\cdot\vec{a} = \vec{a}^2 = |\vec{a}|^2$
6. For the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$

   $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

   $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{i}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{j}\cdot\hat{i} = \hat{i}\cdot\hat{j} = 0$
7. If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

   $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. so that

   $$\cos\theta = \left|\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right|$$
8. The projection of $\vec{a}$ on another direction represented by $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

   $\vec{a}\cdot\hat{i} = a_1\hat{i}\cdot\hat{i} = a_1$

   Similarly $\vec{a}\cdot\hat{j} = a_2$ and $\vec{a}\cdot\hat{k} = a_3$ so that

   $\vec{a} = (\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$
10. Work done by a force $\vec{F}$ in a displacement $\vec{AB}$ when the point of application of the force is displaced from A to B, is, $\vec{F}\cdot\vec{AB} = \vec{F}\cdot(\vec{OB} - \vec{OA}) = \vec{F}$. (position vector of B − position vector of A)

Note:

(a) Maximum value of $\vec{a}\cdot\vec{b}$ is $|\vec{a}||\vec{b}|$

(b) Minimum value of $\vec{a}\cdot\vec{b}$ is $-|\vec{a}||\vec{b}|$

(c) Any vector $\vec{a}$ can be written as

$\vec{a} = (\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$.

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Example 8: Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Sol. (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$

$\Rightarrow (3\hat{i} + 2\hat{j} + 9\hat{k}).(\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow p = -15$

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = \frac{2}{3}$$

Example 9: If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Sol. We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{c}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow 9 + 25 + 2(3)(5)\cos\theta = 49 \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$.

Example 10: Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\,\hat{k}$ is obtuse.

Sol. The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Now, θ is obtuse

$\Rightarrow \cos\theta < 0 \qquad \Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} < 0$

$\Rightarrow \vec{a}.\vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow 14x^2 - 8x + x < 0 \Rightarrow 7x(2x - 1) < 0$

$\Rightarrow x(2x - 1) < 0 \qquad \Rightarrow 0 < x < \frac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Example 11: Find the angle between $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$.

Sol. Angle is θ given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$= \frac{12 - 6 - 2}{\sqrt{9}\sqrt{49}} = \frac{4}{21}$$

Example 12: Show that

$\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 3\hat{i} - 62\hat{j} + 2\hat{k}$ & $\vec{c} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ taken two in pairs are perpendicular.

Sol. $\vec{a}\cdot\vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k}).(3\hat{i} - 62\hat{j} + 2\hat{k}) = 6 - 18 + 12 = 0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular. Similarly $\vec{b}$ and $\vec{c}$ and $\vec{c}$ and $\vec{a}$ are perpendicular.

Note: Though $\vec{a}\cdot\vec{b} = \vec{b}\vec{c} = \vec{c}\cdot\vec{a} = 0$; no one of $|\vec{a}|, |\vec{b}|, |\vec{c}|$ is zero.

Example 13: Given the two vectors $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$; find the projection of $\vec{a}$ on $\vec{b}$ and that of $\vec{b}$ on $\vec{a}$.

Sol. Projection of $\vec{a}$ and $\vec{b}$ $= (\vec{a}\cdot\vec{b})/|\vec{b}|$

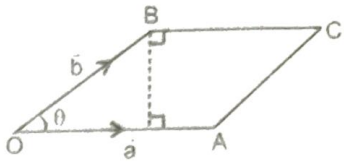
$$= \frac{(2 + 6 - 1)}{\sqrt{6}} = \frac{7}{\sqrt{6}}$$

Projection of $\vec{b}$ on $\vec{a}$ $= (\vec{a}\cdot\vec{b})/|\vec{a}| = \frac{7}{\sqrt{14}}$

Vector product (Cross Product) of two vectors

(i) If $\vec{a}$, $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$;

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

(iv) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(v) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(vi) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$ **(associative)**, where m is a scalar.

(vii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ **(distributive)**

(viii) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0$, $\vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(ix) Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(x) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\frac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(xi) If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

(xii) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \frac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$. The points A, B and C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

(xiii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

(xiv) **Lagrange's Identity:** For any two vectors $\vec{a}$ and $\vec{b}$; $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

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Example 14: Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

Sol. Let $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$. Then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&-1&3\\-2&1&-2\end{vmatrix}$$

$$=(2-3)\,\hat{i}-(-8+6)\,\hat{j}+(4-2)\,\hat{k}=-\hat{i}+2\hat{j}+2\hat{k}$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(-1)^2+2^2+2^2}=3$$

$$\therefore \text{ Required vector}=\pm 9\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)=\pm\frac{9}{3}\,(-\hat{i}+2\hat{j}+2\hat{k})$$

$$=\pm(-3\hat{i}+6\hat{j}+6\hat{k})$$

Example 15: Determine a unit vector perpendicular to the plane defined by the vectors

$$\vec{a}=2\hat{i}-6\hat{j}-3\hat{k} \text{ and } \vec{b}=4\hat{i}+3\hat{j}-3\hat{k}.$$

Sol. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$\vec{c}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-6&-3\\4&3&-1\end{vmatrix}\div|\vec{a}\times\vec{b}|$$

$$=\frac{\hat{i}(15)-\hat{j}(10)+\hat{k}(30)}{|\vec{a}\times\vec{b}|}=\frac{5(3\hat{i}-2\hat{j}+6\hat{k})}{5\sqrt{9+4+36}}$$

$$=3\hat{i}-2\hat{j}+6\hat{k}/7$$

2nd Method

Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{j}$ be a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\vec{c}\cdot\vec{a}=0$ i.e., $2x-6y-3z=0$

$\vec{c}\cdot\vec{b}=0$ i.e., $4x+3y-z=0$

Solve for x, y and z;

$$\frac{x}{15}=\frac{y}{-10}=\frac{z}{30} \text{ i.e., } \frac{x}{3}=\frac{y}{-2}=\frac{z}{6}$$

$$\frac{x}{3}=\frac{y}{-2}=\frac{z}{6}=\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{9+4+36}}=\frac{1}{7}$$

$$\therefore \vec{c}=(3\hat{i}-2\hat{j}+6\hat{k})/7$$

Example 16: For any three vectors $\vec{a},\vec{b},\vec{c}$, show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$.

Sol. We have, $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$$

[Using distributive law]

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}=\vec{0}$$

[$\because \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}$ etc.]

Example 17: For any vector $\vec{a}$, prove that

$$|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$$

Sol. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$$\vec{a}\times\hat{i}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i}$$

$$=a_1(\hat{i}\times\hat{i})+a_2(\hat{j}\times\hat{i})+a_3(\hat{k}\times\hat{i})=-a_2\hat{k}+a_3\hat{j}$$

$$\Rightarrow |\vec{a}\times\hat{i}|^2=a_2^2+a_3^2$$

$$\vec{a}\times\hat{j}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j}=a_1\hat{k}-a_3\hat{i}$$

$$\Rightarrow |\vec{a}\times\hat{j}|^2=a_1^2+a_3^2$$

$$\vec{a}\times\hat{k}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{k}=-a_1\hat{j}+a_2\hat{i}$$

$$\Rightarrow |\vec{a}\times\hat{k}|^2=a_1^2+a_2^2$$

$$\therefore |\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=a_2^2+a_3^2+a_1^2+a_3^2+a_1^2+a_2^2$$

$$=2(a_1^2+a_2^2+a_3^2)=2|\vec{a}|^2$$

Example 18: If the vector $\vec{x}$ is perpendicular to the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{x}.(2\hat{i}-\hat{j}+\hat{k})=-6$ find $\vec{x}$.

Sol. $\vec{x}=\lambda(\hat{i}-\hat{j}-\hat{k})$

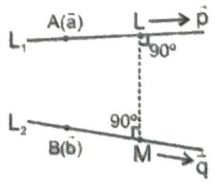
$$\Rightarrow \lambda(\hat{i}-\hat{j}-\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})=-6$$

$$\therefore \lambda=-3$$

The required $\vec{x}=-3\hat{i}-3\hat{j}+3\hat{k}$

SHORTEST DISTANCE BETWEEN TWO LINES

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect and are also not parallel are called **skew line**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines.



Let $\overline{LM}$ be the shortest distance vector between the lines L_1 and L_2. Then $\overline{LM}$ is perpendicular to both $\vec{p}$ and $\vec{q}$ i.e. $\overline{LM}$

is parallel to $\vec{p} \times \vec{q}$. Therefore the magnitude of the shortest distance vector (i.e. $|\overrightarrow{LM}|$) would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance.

$$\therefore \quad |\overrightarrow{LM}| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}\right|$$

$$= \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p}\times\vec{q}\right|$$

$$= \left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(i) The two lines directed along $\vec{p}$ and $\vec{q}$ will intersect only if shortest distance = 0

i.e. $(\vec{b}-\vec{a})\,.\,(\vec{p}\times\vec{q}) = 0$ i.e. $\left(\vec{b} - \vec{a}\right)$

lies in the plane containing $\vec{p}$ and $\vec{q} \Rightarrow (\vec{b}-\vec{a})\cdot\vec{p}\times\vec{q} = 0$.

(ii) If two parallel lines are given by

$\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$,

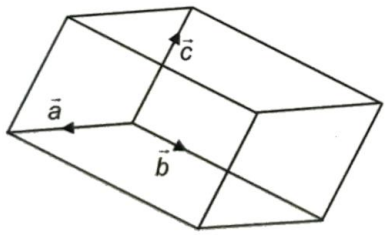
then distance (d) between them is given by

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

SCALAR TRIPLE PRODUCT (BOX PRODUCT) (S.T.P.)

Scalar triple product

For any two vectors $\vec{b}$ and $\vec{c}$, $\vec{b} \times \vec{c}$ is a vector. This can be scalarly multiplied with a third vector $\vec{a}$ to give the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$. This is a scalar whose value is the volume of a box having $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminous edges. Hence it is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and in this sense is called the box product.



General expression for $[\vec{a}\vec{b}\vec{c}]$: When $\vec{a},\vec{b},\vec{c}$ are expressend in terms of $\hat{i},\hat{j},\hat{k}$, then

$$\left[\vec{a}\vec{b}\vec{c}\right] = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}))$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note : Since three vectors line in a plane if $\left[\vec{a},\vec{b},\vec{c}\right] = 0$, Therefore, if 3 directed line segments are in the same plane the then the value of this determinant vanishes. This can be used to determine the equation of a plane through 3 non collinear points.

Properties of scalar triple product

1. $\vec{a}(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

   The value of the scalar triple product is unaltered for a cyclical change of the vectors in the product.

2. $\vec{a}(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

   The value of the scalar triple product is unaltered for a relative interchange of the positions of the dot and cross.

3. $\vec{a}(\vec{b}\times\vec{c}) = 0$ either when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar (or) when any one vector repeats in the triple product, like $\vec{a}\cdot(\vec{a}\times\vec{b})$, $\vec{a}\cdot(\vec{b}\times\vec{b})$ each one of these is zero.

4. For scalars, k, l and m

   $$k\vec{a}\cdot(l\vec{b}\times m\vec{c}) = klm\,\vec{a}\cdot(\vec{b}\times\vec{c})$$

5. In particular $\hat{i}\cdot(\hat{j}\times\hat{k}) = 1$

   (ix) $[\hat{i}\hat{j}\hat{k}] = 1$ (x) $[K\vec{a}\vec{b}\vec{c}] = K[\vec{a}\vec{b}\vec{c}]$

   (xi) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

   (xii) $\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right] = 0$ and

   $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

   (xiii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Tetrahedron and its properties

(a) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|\left[\vec{a}\ \vec{b}\ \vec{c}\ \right]\right|$

(b) If the position vectors of the vertices of tetrahedron are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

Note: This is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron

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Example 19: Prove that

$$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}].$$

Sol. $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$ -

$(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{c}]\ (\because \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{a}\cdot(\vec{b}\times\vec{c}))$

$= 2[\vec{a}\,\vec{b}\,\vec{c}]$

Example 20: Find λ if $\lambda\hat{i}+\hat{j}+2\hat{k}$; $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar.

Sol. The condition for coplanarity is $\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$

i.e., $\lambda(\lambda^2-1) - 1(\lambda+2) + 2(-1-2\lambda) = 0$

$\lambda^3 - 6\lambda - 4 = 0$

By inspection it is seen that $\lambda = -2$ is a root.

$\therefore \lambda^3 - 6\lambda - 4 = (\lambda+2)(\lambda^2-2\lambda-2)$

and $\lambda^2 - 2\lambda - 2 = 0$ for $\lambda = 1 \pm \sqrt{3}$

The required value of λ are

$\lambda_1 = -2\ ;\ \lambda_2 = 1+\sqrt{3}\ ;\ \lambda_3 = 1-\sqrt{3}$

Example 21: Find the volume of a parallelopiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$

Sol. Let $\vec{a} = -3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$.

Now $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$

$= -3(-21-15) - 7(15+21) + 5(25-49)$

$= 108 - 252 - 120 = -264$

So required volume of the parallelopiped

$= \left|[\vec{a}\ \vec{b}\ \vec{c}]\right| = |-264| = 264$ cubic units

Example 22: Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$.

Sol. Let four vertices be A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

$\therefore \vec{DA} = (\vec{a}-\vec{d})$

$\vec{DB} = (\vec{b}-\vec{d})$

$\vec{DC} = (\vec{c}-\vec{d})$

Hence volume $V = \frac{1}{6}[\vec{a}-\vec{d}\ \ \vec{b}-\vec{d}\ \ \vec{c}-\vec{d}]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[(\vec{b}-\vec{d})\times(\vec{c}-\vec{d})]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{d}+\vec{c}\times\vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{a}\ \ \vec{b}\ \ \vec{d}]+[\vec{a}\ \ \vec{c}\ \ \vec{d}]-[\vec{d}\ \ \vec{b}\ \ \vec{c}]\}$

$= \frac{1}{6}\{[\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{a}\ \ \vec{b}\ \ \vec{d}]+[\vec{a}\ \ \vec{c}\ \ \vec{d}]-[\vec{b}\ \ \vec{c}\ \ \vec{d}]\}$

Example 23: Show that the vectors

$\vec{a} = -2\hat{i}+4\hat{j}-2\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}-2\hat{k}$

and $\vec{c} = -2\hat{i}-2\hat{j}+4\hat{k}$ are coplanar.

Sol. $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix} = -2(-8-4) - 4(16-4) - 2(-8-4) = 24 - 48 + 24 = 0$

So vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

VECTOR TRIPLE PRODUCT

Vector triple product

For three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ a product of the form $\vec{a}\times(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\times\vec{c}$ is called a vector triple product.

This is a vector, and the value depends upon the placement of the brackets. In fact $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector in the plane of $\vec{b}$ and $\vec{c}$ (the two placed in the brackets).

In value

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$; and

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$

Note : Unit vector coplaner with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm\dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

Similarly unit vector coplaner with $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{a}$ is $\pm\dfrac{(\vec{a}\times\vec{b})\times\vec{a}}{|(\vec{a}\times\vec{b})\times\vec{a}|}$

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Example 24: For any vector $\vec{a}$, prove that

$$\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$$

Sol. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$.

Then $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\hat{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$= \{(\hat{i}.\hat{i})\vec{a}-(\hat{i}.\vec{a})\hat{i}\} + \{(\hat{j}.\hat{j})\vec{a}-(\hat{j}.\vec{a})\hat{j}\}$

$+ \{(\hat{k}.\hat{k})\vec{a}-(\hat{k}.\vec{a})\hat{k}\}$

$= \{(\vec{a}-(\hat{i}.\vec{a})\hat{i}\}+\{\vec{a}-(\hat{j}.\vec{a})\hat{j}\} + \{\vec{a}-(\hat{k}.\vec{a})\hat{k}\}$

$= 3\vec{a}-\{(\hat{i}.\vec{a})\hat{i}+(\hat{j}.\vec{a})\hat{j}+(\hat{k}.\vec{a})\hat{k}$

$= 3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = 3\vec{a}-\vec{a} = 2\vec{a}$

Example 25: Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = (\vec{b}.\vec{d})(\vec{a}\times\vec{c})-(\vec{b}.\vec{c})(\vec{a}\times\vec{d})$

Sol. We have, $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = \vec{a}\times\{(\vec{b}.\vec{d})\vec{c}-(\vec{b}.\vec{c})\vec{d}\}$

$= \vec{a}\times\{(\vec{b}.\vec{d})\vec{c}\}-\vec{a}\times\{(\vec{b}.\vec{c})\vec{d}\}$ [by dist. law]

$= (\vec{b}.\vec{d})(\vec{a}\times\vec{c})-(\vec{b}.\vec{c})(\vec{a}\times\vec{d})$

Example 26: Let $\vec{a} = \alpha\hat{i}+2\hat{j}-3\hat{k}$, $\vec{b} = \hat{i}+2\alpha\hat{j}-2\hat{k}$ and $\vec{c} = 2\hat{i}-\alpha\hat{j}+\hat{k}$. Find the value(s) of α, if any, such that $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = \vec{0}$.

Sol. $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{b}\times(\vec{c}\times\vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}]\{(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}\}$

which vanishes if

(i) $(\vec{a}.\vec{b})\,\vec{c} = (\vec{b}.\vec{c})\,\vec{a}$

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(i) $(\vec{a}.\vec{b})\,\vec{c} = (\vec{b}.\vec{c})\,\vec{a}$ leads to the equation $2\alpha^3+10\alpha+12=0$, $\alpha^2+6\alpha=0$ and $6\alpha-6=0$, which do not have a common solution.

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0 \Rightarrow 3\alpha = 2 \Rightarrow \alpha = \frac{2}{3}$$

Example 27: If $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}.\vec{a} = 1$ and $\vec{A}\times\vec{B} = \vec{b}$, then prove that $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$.

Sol. Given $\vec{A}+\vec{B} = \vec{a}$ .....(i)

$\Rightarrow \vec{a}.(\vec{A}+\vec{B}) = \vec{a}.\vec{a}$

$\Rightarrow 1+\vec{a}.\vec{B} = |\vec{a}|^2$

$\Rightarrow \vec{a}.\vec{B} = |\vec{a}|^2-1$ .....(ii)

Given $\vec{A}\times\vec{B} = \vec{b}$

$\Rightarrow \vec{a}\times(\vec{A}\times\vec{B}) = \vec{a}\times\vec{b}$

$\Rightarrow (\vec{a}.\vec{B})\,\vec{A}-(\vec{a}.\vec{A})\,\vec{B} = \vec{a}\times\vec{b}$

$\Rightarrow \left(|\vec{a}|^2-1\right)\vec{A}-\vec{B} = \vec{a}\times\vec{b}$ .....(iii)

[Using equation (ii)]

solving equation (i) and (iii) simultaneously, we get

$$\vec{A} = \frac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2} \text{ and } \vec{B} = \frac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$$

Example 28: Solve for $\vec{r}$ satisfying the simultaneous equations $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$, $\vec{r}.\vec{a} = 0$ provided $\vec{a}$ is not perpendicular to $\vec{b}$.

Sol. $(\vec{r}-\vec{c})\times\vec{b} = \vec{0} \Rightarrow \vec{r}-\vec{c}$ and $\vec{b}$ are collinear

$\therefore \vec{r}-\vec{c} = k\vec{b}$

$\Rightarrow \vec{r} = \vec{c}+k\vec{b}$ .....(i)

$\because \vec{r}.\vec{a} = 0$

$\Rightarrow (\vec{c}+k\vec{b})\,.\,\vec{a} = 0$

$\Rightarrow k = -\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}$ putting in (i) we get $\vec{r} = \vec{c}-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\,\vec{b}$

Example 29: If $\vec{x}\times\vec{a}+k\vec{x} = \vec{b}$, where k is a scalar and $\vec{a}, \vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a}, \vec{b}$ and k.

Sol. $\vec{x}\times\vec{a}+k\vec{x} = \vec{b}$ .....(i)

Premultiply the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a})+k\,(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$

$\Rightarrow (\vec{a}.\vec{a})\,\vec{x}-(\vec{a}.\vec{x})\,\vec{a}+k(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$ .....(ii)

Premultiply (i) scalarly by $\vec{a}$

$[\vec{a}\ \vec{x}\ \vec{a}] + k\,(\vec{a}.\vec{x}) = \vec{a}.\vec{b}$

$k(\vec{a}.\vec{x}) = \vec{a}.\vec{b}$ .....(iii)

Substituting $\vec{x}\times\vec{a}$ from (i) and $\vec{a}.\vec{x}$ from (iii) in (ii) we get $\vec{x} = \dfrac{1}{a^2+k^2}\left[k\vec{b}+(\vec{a}\times\vec{b})+\dfrac{(\vec{a}.\vec{b})}{k}\vec{a}\right]$

RECIPROCAL SYSTEM OF VECTORS

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$, then the two systems are called Reciprocal System of vectors.

Note : $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

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Example 30: If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system of vectors, prove that

(i) $\vec{a}.\vec{a}' + \vec{b}.\vec{b}' + \vec{c}.\vec{c}' = 3$

(ii) $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = \vec{0}$

Sol. (i) We have : $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

$\vec{a}.\vec{a}' + \vec{b}.\vec{b}' + \vec{c}.\vec{c}' = 1 + 1 + 1 = 3$

(ii) We have : $\vec{a}' = \lambda\ (\vec{b}\times\vec{c})$, $\vec{b}' = \lambda\ (\vec{c}\times\vec{a})$ and $\vec{c}' = \lambda\ (\vec{a}\times\vec{b})$, where $\lambda = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$\vec{a}\times\vec{a}' = \vec{a}\times\lambda(\vec{b}\times\vec{c}) = \lambda\{\vec{a}\times(\vec{b}\times\vec{c})\} = \lambda\{(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\}$

$\vec{b}\times\vec{b}' = \vec{b}\times\lambda(\vec{c}\times\vec{a}) = \lambda\{\vec{b}\times(\vec{c}\times\vec{a})\} = \lambda\{(\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a}\}$ and

$\vec{c}\times\vec{c}' = \vec{c}\times\lambda(\vec{a}\times\vec{b}) = \lambda\{\vec{c}\times(\vec{a}\times\vec{b})\} = \lambda\{(\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}\}$

$\therefore \vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}'$

$= \lambda\{(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\} + \lambda\{(\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a}\} + \lambda\{(\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}\}$

$= \lambda[(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} + (\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a} + (\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}]$

$= \lambda[(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} + (\vec{a}.\vec{b})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a} + (\vec{b}.\vec{c})\,\vec{a} - (\vec{a}.\vec{c})\,\vec{b}]$

$= \lambda\vec{0} = \vec{0}$

EQUATION OF A PLANE

(i) The equation $(\vec{r} - \vec{r}_0).\vec{n} = 0$ represents a plane containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.

The above equation can also be written as $\vec{r}.\vec{n} = d$, where

$$d = \vec{r}_0.\vec{n}$$

(ii) Angle between two planes is the angle between two normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

(iii) The length of perpendicular (p) from a point having position vector $\vec{a}$ to the plane $\vec{r}.\vec{n} = d$ is given by

$$p = \frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$$

(iv) If $(\vec{r} - \vec{a}).\vec{n}_1 = 0$ and $(\vec{r} - \vec{a}).\vec{n}_2 = 0$ are the equations of two planes, then the equation of line of intersection of these planes is given by $\vec{r} = \vec{a} + \lambda(\vec{n}_1 \times \vec{n}_2)$.

Necessary and sufficient condition for coplanarity

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that

$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$,

where $x + y + z + w = 0$.

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Example 31: Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors.

Sol. Let, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a} - \vec{b} + 3\vec{c} = x\left(\vec{a} + \vec{b} - 2\vec{c}\right) + y\left(\vec{a} + \vec{b} - 3\vec{c}\right)$, for some scalars x and y.

$\Rightarrow 2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\,\vec{a} + (x + y)\,\vec{b} + (-2x - 3y)\,\vec{c}$

$\Rightarrow 2 = x + y, -1 = x + y$ and $3 = -2x - 3y$.

Solving first and third of these equations, we get $x = 9$ and $y = -7$.

Clearly these values do not satisfy the second equation.

Hence the given vectors are not coplanar.

Example 32: Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Sol. Let the given four points be P, Q, R and S respectively.

These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\ \overrightarrow{PR} + y\ \overrightarrow{PS}$

$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = x\left(\vec{a} + \vec{b} - \vec{c}\right) + y\left(-\vec{a} - 9\vec{b} + 7\vec{c}\right)$

$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\,\vec{a} + (x - 9y)\,\vec{b} + (-x + 7y)\,\vec{c}$

$\Rightarrow x - y = -1, x - 9y = -5, -x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

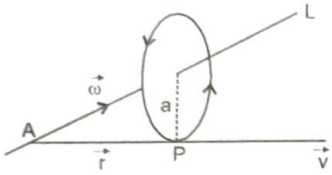
Solving the first two equations of these three equations, we get $x = -\frac{1}{2}, y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

APPLICATION OF VECTORS

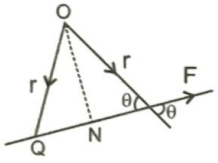
(*a*) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $W = \vec{F} \cdot \vec{s}$

(*b*) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector of the point P.



(*c*) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of P w.r.t. 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ & $\vec{M}$ form a right handed system.

(*d*) Moment of the couple $= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$, where $\vec{r}_1$ and $\vec{r}_2$ are position vectors of the point of the application of the forces $\vec{F}$ and $-\vec{F}$.



TRAIN YOUR BRAIN

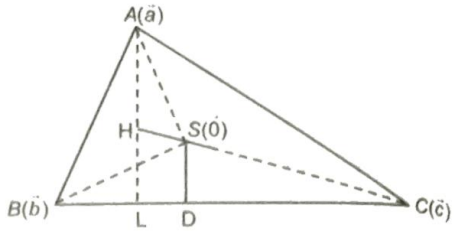
Example 33: In ΔABC, using vector method show that the distance between the circumcentre and the orthocentre is $R\sqrt{1-8\cos A\cos B\cos C}$ where R is the circumradius of the ΔABC.

Sol. In ΔABC, *S* is the circumcentre, *H* is the orthocentre. Here *S* is the origin of the system and position vectors of vertices be $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$.

$|\vec{a}| = |\vec{b}| = |\vec{c}| = R \Rightarrow \vec{a} + \vec{b} + \vec{c} = \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}$

$= \overrightarrow{SA} + 2\overrightarrow{SD}$ (D is the mid-point of BC)

$= \overrightarrow{SA} + \overrightarrow{AH} = \overrightarrow{SH}$ Now, $\overrightarrow{SH}^2 = (\vec{a} + \vec{b} + \vec{c})^2$



$= a^2 + b^2 + c^2 + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a}$

$= a^2 + b^2 + c^2 + 2ab\cos 2C + 2bc\cos 2A + 2ca\cos 2B$

$= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C)$

$= 3R^2 + 2R^2(-1 - 4\cos A\cos B\cos C)$

$= 3R^2 - 2R^2 - 8R^2\cos A\cos B\cos C$

$= R^2 - 8R^2\cos A\cos B\cos C$

$= SH = R\sqrt{1 - 8\cos A\cos B\cos C}$

Example 34: Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

Sol. We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

and, $\overrightarrow{CA}$ = Position vector of A – Position vector of C

$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$

Since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

$= (-\hat{i} - 2\hat{j} - 6\hat{k}) + (2\hat{i} - \hat{j} + \hat{k}) + (-\hat{i} + 3\hat{j} + 5\hat{k}) = \vec{0}$

So A, B and C are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k})$

$= -2 - 3 + 5 = 0$

$\Rightarrow \overrightarrow{BC} \perp \overrightarrow{CA} \Rightarrow \angle BCA = \frac{\pi}{2}$

Hence ABC is a right angled triangle.

Since A is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

$$= \frac{(-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (-6)^2}\sqrt{1^2 + (-3)^2 + (-5)^2}}$$

$$= \frac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}}$$

$$A = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\cos B = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$$= \frac{(\hat{i}+2\hat{j}+6\hat{k}).(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+2^2+6^2}\sqrt{2^2+(-1)^2+(1)^2}}$$

$$\Rightarrow \cos B = \frac{2-2+6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}} \Rightarrow \qquad B = \cos^{-1}\sqrt{\frac{6}{41}}$$

Example 35: If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Sol. Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore

$$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0 \qquad ...(i)$$

Now, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \vec{a}.\vec{a} + \vec{b}.\vec{b} + \vec{c}.\vec{c} + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$ [Using (i)]

$= 3\lambda^2$ [$\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$]

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda \qquad ...(ii)$

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}.(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}} \text{ [Using (ii)]}$$

$$\therefore \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \theta_1 = \theta_2 = \theta_3$.

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclineded with $\vec{a}, \vec{b}$ and $\vec{c}$

Example 36: Prove using vectors : If two medians of a triangle are equal, then it is isosceles.

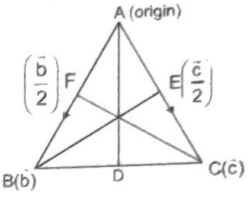
Sol. Let ABC be a triangle and let BE and CF be two equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

$$\text{P.V. of E} = \frac{1}{2}\vec{c} \text{ and P.V. of F} = \frac{1}{2}\vec{b}$$

$$\therefore \quad \overrightarrow{BE} = \frac{1}{2}(\vec{c}-2\vec{b}) \quad \overrightarrow{CF} = \frac{1}{2}(\vec{b}-2\vec{c})$$

Now, BE = CF $\Rightarrow |\overrightarrow{BE}| = |\overrightarrow{CF}|$

$$\Rightarrow |\overrightarrow{BE}|^2 = |\overrightarrow{CF}|^2 \Rightarrow \left|\frac{1}{2}(\vec{c}-2\vec{b})\right|^2 = \left|\frac{1}{2}(\vec{b}-2\vec{c})\right|^2$$



$$\Rightarrow \frac{1}{4}|\vec{c}-2\vec{b}|^2 = \frac{1}{4}|\vec{b}-2\vec{c}|^2$$

$$\Rightarrow |\vec{c}-2\vec{b}|^2 = |\vec{b}-2\vec{c}|^2$$

$$\Rightarrow (\vec{c}-2\vec{b}).(\vec{c}-2\vec{b}) = (\vec{b}-2\vec{c}).(\vec{b}-2\vec{c})$$

$$\Rightarrow \vec{c}.\vec{c} - 4\vec{b}.\vec{c} + 4\vec{b}.\vec{b} = \vec{b}.\vec{b} - 4\vec{b}.\vec{c} + 4\vec{c}.\vec{c}$$

$$\Rightarrow |\vec{c}|^2 - 4\vec{b}.\vec{c} + 4|\vec{b}|^2 = |\vec{b}|^2 - 4\vec{b}.\vec{c} + 4|\vec{c}|^2$$

$$\Rightarrow 3|\vec{b}|^2 = 3|\vec{c}|^2 \Rightarrow |\vec{b}|^2 = |\vec{c}|^2$$

$\Rightarrow$ AB = ACHence triangle ABC is an isosceles triangle.

Example 37: Using vectors : Prove that cos (A + B) = cos A cos B – sin A sin B

Sol. Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively. Let $\angle XOP = A$ and $\angle XOQ = B$. Drawn PL $\perp$ OX and QM $\perp$ OX.

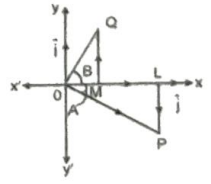
Clearly angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is A + B

In ΔOLP, OL = OP cos A and LP = OP sin A. Therefore

$\overrightarrow{OL} = (OP \cos A)\,\hat{i}$ and

$\overrightarrow{LP} = (OP \sin A)\left(-\hat{j}\right)$

Now, $\qquad \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OP}$



$$\Rightarrow \overrightarrow{OP} = OP[(\cos A)\,\hat{i} - (\sin A)\,\hat{j}] \qquad ...(i)$$

In ΔOMQ, $OM = OQ\cos B$ and $MQ = OQ\sin B$.

Therefore, $\overrightarrow{OM} = (OQ\cos B)\,\hat{i}$, $\overrightarrow{MQ} = (OQ\sin B)\,\hat{j}$

Now, $\overrightarrow{OM} + \overrightarrow{MQ} = \overrightarrow{OQ}$

$\Rightarrow \overrightarrow{OQ} = OQ[(\cos B)\hat{i} + (\sin B)\hat{j}]$ ...(ii)

From (i) and (ii), we get

$\overrightarrow{OP} \cdot \overrightarrow{OQ} = OP[(\cos A)\,\hat{i} \quad (\sin A)\,\hat{j}] \cdot OQ[(\cos B)\,\hat{i}$

$+ (\sin B)\,\hat{j}] = OP \cdot OQ[\cos A\cos B - \sin A\sin B]$

But, $\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}|\cos(A+B) = OP \cdot OQ\cos(A+B)$

$\therefore OP.OQ\cos(A+B) = OP.OQ[\cos A\cos B - \sin A\sin B]$

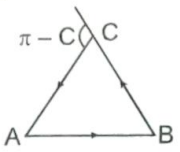
$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$

Example 38: Prove that in any triangle ABC

(i) $c^2 = a^2 + b^2 - 2ab\cos C$

(ii) $c = b\cos A + a\cos B$.

Sol. (i) In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$



$\Rightarrow \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$ ...(i)

Squaring both sides

$(\overrightarrow{BC})^2 + (\overrightarrow{CA})^2 + 2(\overrightarrow{BC}) \cdot \overrightarrow{CA} = (\overrightarrow{AB})^2$

$\Rightarrow a^2 + b^2 + 2(\overrightarrow{BC} \cdot \overrightarrow{CA}) = c^2$

$\Rightarrow c^2 = a^2 + b^2 + 2ab\cos(\pi - C)$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos C$

(ii) $(\overrightarrow{BC} + \overrightarrow{CA}) \cdot \overrightarrow{AB} = -\overrightarrow{AB} \cdot \overrightarrow{AB}$

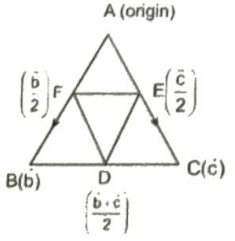
$\overrightarrow{BC} \cdot \overrightarrow{AB} + \overrightarrow{CA} \cdot \overrightarrow{AB} = -c^2$

$-ac\cos B - bc\cos A = -c^2$

$a\cos B + b\cos A = c$.

Example 39: If D, E, F are the mid-points of the sides of a triangle ABC, prove by vector method that area of $\Delta DEF = \frac{1}{4}$ (area of ΔABC)

Sol. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then the position vectors of D, E and F are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.



Now, $\overrightarrow{DE} = \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{b}}{2}$

and $\overrightarrow{DF} = \frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{c}}{2}$

$\therefore$ Vector area of $\Delta DEF = \frac{1}{2}(\overrightarrow{DE} \times \overrightarrow{DF})$

$= \frac{1}{2}\left(\frac{-\vec{b}}{2} \times \frac{-\vec{c}}{2}\right)$

$= \frac{1}{8}(\vec{b}\times\vec{c}) = \frac{1}{4}\left\{\frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC})\right\}$

$= \frac{1}{4}$(vector area of ΔABC)

Hence area of $\Delta DEF = \frac{1}{4}$ area of ΔABC.

Example 40: P, Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD. Show that $\Delta APD = \Delta CQB$.

Sol. Let $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AD} = \vec{d}$

Now DC is parallel to AB $\Rightarrow$ there exists a scalar t such that $\overrightarrow{DC} = t\,\overrightarrow{AB} = t\,\vec{b}$

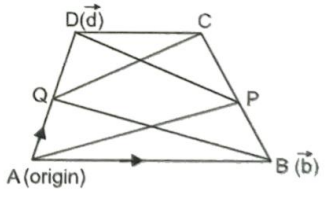
$\therefore \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d} + t\,\vec{b}$

The position vectors of P and Q are

$\frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b})$ and $\frac{1}{2}\vec{d}$ respectively.

Now $2\Delta\,\overrightarrow{APD} = \overrightarrow{AP} \times \overrightarrow{AD}$

$= \frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b}) \times \vec{d} = \frac{1}{2}(1+t)(\vec{b}\times\vec{d})$



Also $2\Delta\,\overrightarrow{CQB} = \overrightarrow{CQ} \times \overrightarrow{CB}$

$= \left[\frac{1}{2}\vec{d} - (\vec{d}+t\vec{b})\right] \times [\vec{b} - (\vec{d}+t\,\vec{b})]$

$= \left[-\frac{1}{2}\vec{d} - t\,\vec{b}\right] \times \left[-\vec{d} + (1-t)\,\vec{b}\right]$

$= -\frac{1}{2}(1-t)(\vec{d}\times\vec{b}) + t(\vec{b}\times\vec{d}) = \frac{1}{2}(1-t+2t)(\vec{b}\times\vec{d})$

$= \frac{1}{2}(1+t)(\vec{b}\times\vec{d}) = 2\Delta\overrightarrow{APD}$ Hence Prove.

Topicwise Questions

DEFINITION & TEST OF COLLINEARITY

1. The position vectors of A and B are $2\hat{i}-9\hat{j}-4\hat{k}$ and $6\hat{i}-3\hat{j}+8\hat{k}$ respectively, then the magnitude of $\overrightarrow{AB}$ is
(a) 11 (b) 12
(c) 13 (d) 14

2. The vectors $\hat{i}+2\hat{j}+3\hat{k}$, $\lambda\hat{i}+4\hat{j}+7\hat{k}$, $-3\hat{i}-2\hat{j}-5\hat{k}$ are collinear, if λ equals
(a) 3 (b) 4
(c) 5 (d) 6

Section formula and Important figures

3. The perimeter of the triangle whose vertices have the position vectors $(\hat{i}+\hat{j}+\hat{k}), (5\hat{i}+3\hat{j}-3\hat{k})$ and $(2\hat{i}+5\hat{j}+9\hat{k})$, is given by
(a) $15+\sqrt{157}$ (b) $15-\sqrt{157}$
(c) $\sqrt{15}-\sqrt{157}$ (d) $\sqrt{15}+\sqrt{157}$

4. If position vectors of a point A is $\vec{a}+2\vec{b}$ and $\vec{a}$ divides AB in the ratio 2 : 3 , then the position vector of B is
(a) 2a – b (b) b – 2a
(c) a – 3b (d) b

Dot product and Cross product

5. $(a.\hat{i})\hat{i}+(a.\hat{j})\hat{j}+(a.\hat{k})\hat{k}=$
(a) a (b) 2a
(c) 0 (d) None of these

6. If $\vec{r}.\hat{i}=\vec{r}.\hat{j}=\vec{r}.\hat{k}$ and $|\vec{r}|=3$, then $\vec{r}=$
(a) $\pm 3(\hat{i}+\hat{j}+\hat{k})$ (b) $\pm\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$
(c) $\pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ (d) $\pm\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$

7. The value of $\vec{b}$ such that scalar product of the vectors $(\hat{i}+\hat{j}+\hat{k})$ with the unit vector parallel to the sum of the vectors $(2\hat{i}+4\hat{j}-5\hat{k})$ and $(b\hat{i}+2\hat{j}+3\hat{k})$ is 1, is
(a) –2 (b) –1
(c) 0 (d) 1

8. If $|\vec{a}|=3, |\vec{b}|=1, |\vec{c}|=4$ and $\vec{a}+\vec{b}+\vec{c}=0$ then $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$
(a) –13 (b) –10
(c) 13 (d) 10

9. The angle between the vectors $3\hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}-2\hat{j}+4\hat{k}$ is
(a) $\cos^{-1}\frac{2}{\sqrt{7}}$ (b) $\sin^{-1}\frac{2}{\sqrt{7}}$
(c) $\cos^{-1}\frac{2}{\sqrt{5}}$ (d) $\sin^{-1}\frac{2}{\sqrt{5}}$

10. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is
(a) 45° (b) 60°
(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

11. If the scalar projection of the vectors $x\hat{i}-\hat{j}+\hat{k}$ on the vector $2\hat{i}-\hat{j}+5\hat{k}$ is $\frac{1}{\sqrt{30}}$ then value of x is equal to
(a) $\frac{-5}{2}$ (b) 6
(c) –6 (d) 3

12. A unit vector perpendicular to the plane determined by the points (1, –1, 2), (2, 0, –1) and (0, 2, 1) is
(a) $\pm\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{6}}(\hat{i}+2\hat{j}+\hat{k})$
(c) $\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+\hat{k})$ (d) $\frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$

13. $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=$
(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$
(c) $3|\vec{a}|^2$ (d) $4|\vec{a}|^2$

14. If $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|=$
(a) 16 (b) 8
(c) 3 (d) 12

15. If $|\vec{a}|=4$, $|\vec{b}|=2$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $(\vec{a}\times\vec{b})^2$ is equal to
(a) 48 (b) 16
(c) 8 (d) 40

16. Let a,b,c be three vectors such that $a \neq 0$, and $\vec{a} \times \vec{b} = 2\vec{a} \times \vec{c}$, $|\vec{a}| = |\vec{c}| = 1$, $|\vec{b}| = 4$ and $|\vec{b} \times \vec{c}| = 15$. If $\vec{b} - 2\vec{c} = \lambda\vec{a}$ then λ equals to

(a) 1 (b) ± 4

(c) 3 (d) -2

Triple Product (Scaler and vector)

17. If $\vec{a}, \vec{b}, \vec{c}$ are any three coplanar unit vectors, then

(a) $\vec{a}.(\vec{b} \times \vec{c}) = 1$ (b) $\vec{a}.(\vec{b} \times \vec{c}) = 3$

(c) $(\vec{a} \times \vec{b}).\vec{c} = 0$ (d) $(\vec{c} \times \vec{a}).\vec{b} = 1$

18. If the points whose position, vectors are $3\hat{i} - 2\hat{j} - \hat{k}$, $-\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$ lie on a plane, then $\lambda =$

(a) $-\frac{146}{17}$ (b) $\frac{146}{17}$

(c) $-\frac{17}{146}$ (d) $\frac{17}{146}$

19. If three vectors $\vec{a} = 12\hat{i} + 4\hat{j} + 3\hat{k}$, $\vec{b} = 8\hat{i} - 12\hat{j} - 9\hat{k}$ and $\vec{c} = 33\hat{i} - 4\hat{j} - 24\hat{k}$ represents a cube, then its volume will be

(a) 616 (b) 308

(c) 154 (d) 3696

20. $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{k} \times \hat{i}) + \hat{k}.(\hat{i} \times \hat{j}) =$

(a) 1 (b) 3

(c) –3 (d) 0

21. If the vectors $\hat{i} + 3\hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + 4\hat{k}$ and $3\hat{i} + 2\hat{j} + x\hat{k}$ are coplanar, then the value of x is

(a) –2 (b) 2

(c) 1 (d) 3

22. If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $[\vec{a}\,\vec{b}\,\vec{c}] = 4$, then $[\vec{a} \times \vec{b}\ \vec{b} \times \vec{c}\ \vec{c} \times \vec{a}] =$

(a) 16 (b) 64

(c) 4 (d) 8

23. The value of λ for which the four points $2\hat{i} + 3\hat{j} - \hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ $3\hat{i} + 4\hat{j} - 2\hat{k}$, $\hat{i} - \lambda\hat{j} + 6\hat{k}$ are coplanar

(a) 8 (b) 0

(c) –2 (d) 6

24. $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with

(a) $\vec{b}$ and $\vec{c}$ (b) $\vec{c}$ and $\vec{a}$

(c) $\vec{a}$ and $\vec{b}$ (d) $\vec{a}, \vec{b}$ and $\vec{c}$

25. If $\vec{u} = \hat{i} \times (a \times \hat{i}) + \hat{j} \times (a \times \hat{j}) + \hat{k} \times (a \times \hat{k})$ then

(a) $\vec{u} = 0$ (b) $\vec{u} = \hat{i} + \hat{j} + \hat{k}$

(c) $\vec{u} = 2a$ (d) $\vec{u} = a$

26. If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = i - 2j + 2k$ then $(\vec{a} \times \vec{b}) \times \vec{c}$ is equal to

(a) $24\hat{i} + 7\hat{j} - 5\hat{k}$ (b) $7\hat{i} - 24\hat{j} + 5\hat{k}$

(c) $12\hat{i} + 3\hat{j} - 5\hat{k}$ (d) $\hat{i} + \hat{j} - 7\hat{k}$

27. If $\vec{A} = \hat{i} - 2\hat{j} - 3\hat{k}, \vec{B} = 2\hat{i} + \hat{j} - \hat{k}, \vec{C} = \hat{i} + 3\hat{j} - 2\hat{k}$, then $(\vec{A} \times \vec{B}) \times \vec{C}$ is

(a) $5(-\hat{i} + 3\hat{j} + 4\hat{k})$ (b) $4(-\hat{i} + 3\hat{j} + 4\hat{k})$

(c) $5(-\hat{i} - 3\hat{j} - 4\hat{k})$ (d) $4(\hat{i} + 3\hat{j} + 4\hat{k})$

28. The distance between the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\vec{r}.(\hat{i} + 5\hat{j} + \hat{k}) = 5$ is

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$

(c) $\frac{10}{9}$ (d) $\frac{10}{3\sqrt{3}}$

29. The equation of the plane passing through the points $(-1,-2,0), (2,3,5)$ and parallel to the line $\vec{r} = -3\hat{j} + \hat{k} + (2\hat{i} + 5\hat{j} - \hat{k})$ is

(a) $\vec{r}.(-30\hat{i} + 13\hat{j} + 5\hat{k}) = 4$ (b) $\vec{r}.(30\hat{i} + 13\hat{j} + 5\hat{k}) = 4$

(c) $\vec{r}.(30\hat{i} + 13\hat{j} - 5\hat{k}) = 4$ (d) $\vec{r}.(30\hat{i} - 13\hat{j} - 5\hat{k}) = 4$

30. The line of intersection of the planes $\vec{r}.(\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r}.(2\hat{i} + 5\hat{j} - 3\hat{k}) = 2$ is parallel to the vector

(a) $-4\hat{i} + 5\hat{j} + 11\hat{k}$ (b) $4\hat{i} + 5\hat{j} + 11\hat{k}$

(c) $4\hat{i} - 5\hat{j} + 11\hat{k}$ (d) $4\hat{i} - 5\hat{j} - 11\hat{k}$

Straight line & Skew lines

31. The vector equation of a plane, which is at a distance of 8 unit from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$, is

(a) $\vec{r}.(2\hat{i} + \hat{j} + \hat{k}) = 24$ (b) $\vec{r}.(2\hat{i} + \hat{j} + 2\hat{k}) = 24$

(c) $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 24$ (d) $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 22$

32. The shortest distance between the lines $\vec{r} = (3\hat{i} - 2\hat{j} - 2\hat{k}) + it$ and $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + js$ (t and s being parameters) is

(a) $\sqrt{21}$ (b) $\sqrt{102}$

(c) 4 (d) 3

Linearly dependant & independnt vectors

33. The vector equation of the plane through the point $2\hat{i} - \hat{j} - 4\hat{k}$ and parallel to the plane $\vec{r}.(4\hat{i} - 12\hat{j} - 3\hat{k}) - 7 = 0$ is

(a) $\vec{r}.(4\hat{i} - 12\hat{j} - 3\hat{k}) = 0$ (b) $\vec{r}.(4\hat{i} - 12\hat{j} - 3\hat{k}) = 32$

(c) $\vec{r}.(4\hat{i} - 12\hat{j} - 3\hat{k}) = 12$ (d) $\vec{r}.(4\hat{i} - 12\hat{j} - 3\hat{k}) = 20$

34. The vector equation of the plane through the point (2, 1, 1) and passing through the line of intersection of the plane $\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) = 0$ and $\vec{r}.(\hat{j} + 2\hat{k}) = 0$ is

(a) $\vec{r}.(\hat{i} + 9\hat{j} + 11\hat{k}) = 0$ (b) $\vec{r}.(\hat{i} + 9\hat{j} + 11\hat{k}) = 6$

(c) $\vec{r}.(\hat{i} - 3\hat{j} - 13\hat{k}) = 0$ (d) $\vec{r}.(\hat{i} - 3\hat{j} - 13\hat{k}) = 10$

35. The position vector of a point at a distance of $3\sqrt{11}$ units from $\hat{i} - \hat{j} + 2\hat{k}$ on a line passing through the point $\hat{i} - \hat{j} + 2\hat{k}$ and parallel to vector $3\hat{i} + \hat{j} + \hat{k}$ is

(a) $10\hat{i} + 2\hat{j} - 5\hat{k}$ (b) $-8\hat{i} - 4\hat{j} - \hat{k}$

(c) $8\hat{i} + 4\hat{j} + \hat{k}$ (d) $-10\hat{i} - 2\hat{j} - 5\hat{k}$

Learning Plus

1. If the vector $\vec{b}$ is collinear with the vector $\vec{a} = (2\sqrt{2}, -1, 4)$ and $|\vec{b}| = 10$, then

(a) $\vec{a} \pm \vec{b} = 0$ (b) $\vec{a} \pm 2\vec{b} = 0$

(c) $2\vec{a} \pm \vec{b} = 0$ (d) $3\vec{a} \pm \vec{b} = 0$

2. The vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC. The length of the median through A is

(a) $\sqrt{18}$ (b) $\sqrt{72}$

(c) $\sqrt{33}$ (d) $\sqrt{288}$

3. The vertices of a triangle are A (1, 1, 2), B(4, 3, 1) and C(2, 3, 5). A vector representing the internal bisector of the angle A is :

(a) $\hat{i} + \hat{j} + 2\hat{k}$ (b) $2\hat{i} - 2\hat{j} + \hat{k}$

(c) $2\hat{i} + 2\hat{j} - \hat{k}$ (d) $2\hat{i} + 2\hat{j} + \hat{k}$

4. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b} + \vec{c}$, $\vec{b}$ to $\vec{c} + \vec{a}$ and $\vec{c}$ to $\vec{a} + \vec{b}$. Then $|\vec{a} + \vec{b} + \vec{c}|$

(a) $2\sqrt{5}$ (b) $2\sqrt{2}$

(c) $10\sqrt{5}$ (d) $5\sqrt{2}$

5. The value of a, for which the points A, B, C with position vectors $2\hat{i} - \hat{j} - \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $a\hat{i} - 3\hat{j} - \hat{k}$ respectively are the vertices of a right angled triangle with $C = \pi/2$ are

(a) –2 and –1 (b) –2 and 1

(c) 2 and –1 (d) 2 and 1

6. A particle is acted upon by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ which displace it from a point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. The workdone in standard units by the force is given by

(a) 40 (b) 30

(c) 25 (d) 15

7. If $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j}$, $\vec{c} = \hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{n}$ be a unit vector such that $\vec{b}.\vec{n} = 0$, $\vec{a}.\vec{n} = 0$ then value of $|\vec{c}.\vec{n}|$ is

(a) 1 (b) 3

(c) 5 (d) 2

8. If $|\vec{a}| = 5$, $|\vec{a} - \vec{b}| = 8$ and $|\vec{a} + \vec{b}| = 10$, then $|\vec{b}|$ is equal to

(a) 1 (b) $\sqrt{57}$

(c) 3 (d) 2

9. Angle between diagonals of a parallelogram whose side are represented by $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - \hat{k}$

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{2}\right)$

(c) $\cos^{-1}\left(\frac{4}{9}\right)$ (d) $\cos^{-1}\left(\frac{5}{9}\right)$

10. If $\vec{u} = \vec{a} - \vec{b}$, $\vec{v} = \vec{a} + \vec{b}$ and $|\vec{a}| = |\vec{b}| = 2$, then $|\vec{u} \times \vec{v}|$ is equal to

(a) $\sqrt{2(16 - (\vec{a}.\vec{b})^2)}$ (b) $2\sqrt{(16 - (\vec{a}.\vec{b})^2)}$

(c) $2\sqrt{(4 - (\vec{a}.\vec{b})^2)}$ (d) $\sqrt{2(4 - (\vec{a}.\vec{b})^2)}$

11. If the unit vectors $\hat{e}_1$ and $\hat{e}_2$ are inclined at an angle 2θ and $|\vec{e}_1 - \vec{e}_2| < 1$, then for $\theta \in [0, \pi]$, θ may lie in the interval :

(a) $\left[0, \frac{\pi}{6}\right)$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

(c) $\left(\frac{5\pi}{6}, \pi\right]$ (d) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

12. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle of $\cos^{-1}\frac{11}{14}$ and doubled in magnitude, then it becomes $4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$. The value of 'x' is

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$

(c) $\frac{1}{3}$ (d) 2

13. Given the three vectors $\vec{a} = -2\hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + 5\hat{j}$ and $\vec{c} = 4\hat{i} + 4\hat{j} - 2\hat{k}$. The projection of the vector $3\vec{a} - 2\vec{b}$ on the vector $\vec{c}$ is

(a) 11 (b) –11

(c) 13 (d) –13

14. If $\overrightarrow{OA} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\overrightarrow{OB} = \hat{i} + 3\hat{j} + \hat{k}$, then the area of the triangle OAB is

(a) $\sqrt{15}$ (b) $3\sqrt{5}$

(c) $\frac{3}{2}\sqrt{10}$ (d) $\frac{5\sqrt{5}}{3}$

15. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $(\vec{a} \wedge \vec{b}) = \frac{\pi}{2}$, $\vec{a} \cdot \vec{c} = 4$, then

(a) $[\vec{a}\,\vec{b}\,\vec{c}]^2 = |\vec{a}|$ (b) $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|$

(c) $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ (d) $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|^2$

16. Vector of length 3 unit which is perpendicular to $\hat{i} + \hat{j} + \hat{k}$ and lies in the plane of $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} - 3\hat{j}$

(a) $\frac{3}{\sqrt{6}}(\hat{i} - 2\hat{j} + \hat{k})$ (b) $\frac{3}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k})$

(c) $\frac{3}{\sqrt{114}}(8\hat{i} - 7\hat{j} - \hat{k})$ (d) $\frac{3}{\sqrt{114}}(-7\hat{i} + 8\hat{j} - \hat{k})$

17. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero vectors such that $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the acute angle between the vectors $\vec{b}$ and $\vec{c}$, then sinθ equals is

(a) $\frac{1}{3}$ (b) $\frac{\sqrt{2}}{3}$

(c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

18. If $\hat{a}$, $\hat{b}$, $\hat{c}$ be the unit vectors such that $\hat{b}$ is not parallel to $\hat{c}$ and $\hat{a} \times (2\hat{b} \times \hat{c}) = \hat{b}$, then the angle that $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$ are respectively:

(a) $\frac{\pi}{3}$ & $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ & $\frac{2\pi}{3}$

(c) $\frac{\pi}{2}$ & $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$ & $\frac{\pi}{3}$

19. If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a} \parallel (\vec{b} \times \vec{c})$, then $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ is equal to

(a) $\vec{a}^2(\vec{b} \cdot \vec{c})$ (b) $\vec{b}^2(\vec{a} \cdot \vec{c})$

(c) $\vec{c}^2(\vec{a} \cdot \vec{b})$ (d) 0

20. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent vectors, then which one of the following set of vectors is linearly dependent?

(a) $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$

(b) $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$

(c) $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$

(d) $\vec{a}-\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$

21. The value of $[(\vec{a} + 2\vec{b} - \vec{c}), (\vec{a} - \vec{b}), (\vec{a} - \vec{b} - \vec{c})]$ is equal to the box product

(a) $[\vec{a}\,\vec{b}\,\vec{c}]$ (b) $2[\vec{a}\,\vec{b}\,\vec{c}]$

(c) $3[\vec{a}\,\vec{b}\,\vec{c}]$ (d) $4[\vec{a}\,\vec{b}\,\vec{c}]$

22. Let $\vec{r}$ be a vector perpendicular to $\vec{a} + \vec{b} + \vec{c}$, where $[\vec{a}\ \vec{b}\ \vec{c}] = 2$. If $\vec{r} = \ell(\vec{b} \times \vec{c}) + m(\vec{c} \times \vec{a}) + n(\vec{a} \times \vec{b})$, then $(\ell + m + n)$ is equal to

(a) 2 (b) 1

(c) 0 (d) 3

23. Given the vertices A (2, 3, 1), B(4, 1, –2), C(6, 3, 7) & D(–5, –4, 8) of a tetrahedron. The length of the altitude drawn from the vertex D is

(a) 7 (b) 9

(c) 11 (d) 13

24. Let $\vec{a} = x\hat{i} + 12\hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} + 2x\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{k}$. If the ordered set $[\vec{b}\ \vec{c}\ \vec{a}]$ is left handed, then

(a) $x \in (2, \infty)$ (b) $x \in (-\infty, -3)$

(c) $x \in (-3, 2)$ (d) $x \in \{3, -2\}$

25. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\vec{a} + 2\vec{b} + 3\vec{c}$, $\lambda\vec{b} + 4\vec{c}$ and $(2\lambda - 1)\vec{c}$ are non-coplanar for
(a) all values of λ
(b) all except one value of λ
(c) all except two values of λ
(d) non value of λ

26. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three non-coplanar vectors, then $(\vec{u} + \vec{v} - \vec{w}) \cdot [(\vec{u} - \vec{v}) \times (\vec{v} - \vec{w})]$ equals
(a) 0 (b) $\vec{u} \cdot (\vec{v} \times \vec{w})$
(c) $\vec{u} \cdot \vec{w} \times \vec{v}$ (d) $3\vec{u} \cdot \vec{v} \times \vec{w}$

27. If $\vec{b}$ and $\vec{c}$ are any two perpendicular unit vectors and $\vec{a}$ is any vector, then
$(\vec{a} \cdot \vec{b})\vec{b} + (\vec{a} \cdot \vec{c})\vec{c} + \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|^2}(\vec{b} \times \vec{c})$ is equal to
(a) $\vec{a}$ (b) $\vec{b}$
(c) $\vec{c}$ (d) $2\vec{a}$

28. The set of values of 'm' for which the vectors
$\vec{a} = m\hat{i} + (m+1)\hat{j} + (m+8)\hat{k}$,
$\vec{b} = (m+3)\hat{i} + (m+4)\hat{j} + (m+5)\hat{k}$
and $\vec{c} = (m+6)\hat{i} + (m+7)\hat{j} + (m+8)\hat{k}$
are non-coplanar is
(a) R (b) R – {1}
(c) R – {1, 2} (d) ϕ

29. Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is
(a) $-\hat{i} + \hat{j} + 2\hat{k}$ (b) $3\hat{i} - \hat{j} + \hat{k}$
(c) $3\hat{i} + \hat{j} - \hat{k}$ (d) $\hat{i} - \hat{j} - \hat{k}$

30. If A(1, 1, 1), C(2, 1, 2), the vector equation of the line $\overrightarrow{AB}$ is $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + t(6\hat{i} - 3\hat{j} + 2\hat{k})$ and d is the shortest distance of the point C from $\overrightarrow{AB}$, then
(a) B(6, –3, 2) (b) B(5, –4, 1)
(c) $d = \sqrt{2}$ (d) $d = \sqrt{6}$

31. The distance between the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ is
(a) 10/3 (b) 3/10
(c) $\frac{10}{3\sqrt{3}}$ (d) 10/9

32. If line $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r} \cdot (3\hat{i} - 2\hat{j} - m\hat{k}) = 14$, then the value of m is
(a) 2
(b) –2
(c) 0
(d) can not be predicted with these informations

33. If the acute angle that the vector, $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ makes with the plane of the two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$ is $\cot^{-1}\sqrt{2}$ then
(a) $\alpha(\beta + \gamma) = \beta\gamma$ (b) $\beta(\gamma + \alpha) = \gamma\alpha$
(c) $\gamma(\alpha + \beta) = \alpha\beta$ (d) $\alpha\beta + \beta\gamma + \gamma\alpha = 0$

34. For a non zero vector $\vec{A}$ If the equations $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ hold simultaneously, then
(a) $\vec{A}$ is perpendicular to $\vec{B} - \vec{C}$
(b) $\vec{A} = \vec{B}$
(c) $\vec{B} = \vec{C}$
(d) $\vec{C} = \vec{A}$

Advanced Level Multiconcept Questions

1. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then the vectors $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ are:
(a) collinear (b) linearly independent
(c) perpendicular (d) parallel

2. The vector $\vec{c}$, directed along the external bisector of the angle between the vectors $\vec{a} = 7\hat{i} - 4\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$ with $|\vec{c}| = 5\sqrt{6}$, is
(a) $\frac{5}{3}(\hat{i} - 7\hat{j} - 2\hat{k})$ (b) $\frac{5}{3}(\hat{i} + 7\hat{j} - 2\hat{k})$
(c) $\frac{5}{3}(-\hat{i} + 7\hat{j} - 2\hat{k})$ (d) $\frac{5}{3}(-\hat{i} + 7\hat{j} + 2\hat{k})$

3. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a
(a) parallelogram, which is neither a rhombus nor a rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

4. If $\vec{z}_1 = a\hat{i} + b\hat{j}$ and $\vec{z}_2 = c\hat{i} + d\hat{j}$ are two vectors in $\hat{i}$ and $\hat{j}$ system, where $|\vec{z}_1| = |\vec{z}_2| = r$ and $\vec{z}_1 \cdot \vec{z}_2 = 0$, then $\vec{w}_1 = a\hat{i} + c\hat{j}$ and $\vec{w}_2 = b\hat{i} + d\hat{j}$ satisfy

(a) $|\vec{w}_1| = r$ (b) $|\vec{w}_2| = r$

(c) $\vec{w}_1 \cdot \vec{w}_2 = 0$ (d) none of these

5. A line passes through a point A with position vector $3\hat{i} + \hat{j} - \hat{k}$ and parallel to the vector $2\hat{i} - \hat{j} + 2\hat{k}$. If P is a point on this line such that AP = 15 units, then the position vector of the point P is/are

(a) $13\hat{i} + 4\hat{j} - 9\hat{k}$ (b) $13\hat{i} - 4\hat{j} + 9\hat{k}$

(c) $7\hat{i} - 6\hat{j} + 11\hat{k}$ (d) $7\hat{i} + 6\hat{j} - 11\hat{k}$

6. If $\vec{r} = \hat{i} + 5\hat{j} + 5\hat{k} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$

and $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 6$

are the equations of a line and plane respectively, then which of the following is false ?

(a) line is perpendicular to the plane

(b) line lies in the plane

(c) line is parallel to the plane but does not lie in the plane

(d) line cuts the plane is one point only

7. Unit vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. A unit vector $\vec{d}$ is perpendicular to them. If $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \frac{1}{6}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$, and the angle between $\vec{a}$ and $\vec{b}$ is 30°, then $\vec{c}$ is

(a) $(\hat{i} - 2\hat{j} + 2\hat{k})/3$ (b) $(\hat{i} - 2\hat{j} + 2\hat{k})/3$

(c) $(-2\hat{i} - 2\hat{j} - \hat{k})/3$ (d) $(-\hat{i} + 2\hat{j} - 2\hat{k})/3$

8. Let $\vec{p} = 2\hat{i} + 3\hat{j} - a\hat{k}$, $\vec{q} = b\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$. If $\vec{p}, \vec{q}, \vec{r}$ are coplanar and $\vec{p} \cdot \vec{q} = 20$, then a and b have the values

(a) 1, 3 (b) 9, 7

(c) 5, 5 (d) 13, 9

9. If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{b} = y\hat{i} + z\hat{j} + x\hat{k}$ and $\vec{c} = z\hat{i} + x\hat{j} + y\hat{k}$, then $\vec{a} \times (\vec{b} \times \vec{c})$ is

(a) parallel to $(y - z)\hat{i} + (z - x)\hat{j} + (x - y)\hat{k}$

(b) orthogonal to $\hat{i} + \hat{j} + \hat{k}$

(c) orthogonal to $(y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$

(d) orthogonal to $x\hat{i} + y\hat{j} + z\hat{k}$

10. If $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors satisfying the condition $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$, then

(a) $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs

(b) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{a}|^2$

(c) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{c}|^2$

(d) $|\vec{b}| = |\vec{c}|$

11. Let $\vec{b} = 4\hat{i} + 3\hat{j}$. Let $\vec{c}$ be a vector perpendicular to $\vec{b}$ and it lies in the XY-plane. A vector in the XY-plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$ is

(a) $2\hat{i} + \hat{j}$ (b) $\hat{i} - 2\hat{j}$

(c) $\frac{1}{5}(2\hat{i} + 11\hat{j})$ (d) $2\hat{i} - \hat{j}$

12. Two adjacent sides of a parallelogram ABCD are given by $\vec{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\vec{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by

(a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$

(c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

Comprehension – 1 (13 to 15)

If A, B and C are three non collinear points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively and plane ABC is not passing through the origin.

13. The vectors $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ are

(a) parallel vectors

(b) coplanar vectors

(c) non coplanar vectors

(d) linearly dependent vectors

14. If $|\vec{b}| = 1, |\vec{c}| = 1, |\vec{a}| = 7$ and $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) = \frac{1}{2}\vec{a}$, where $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then $|\vec{a} \times \vec{c}|$ is equal to

(a) 7 (b) $\frac{7\sqrt{3}}{2}$

(c) $\frac{7}{2}$ (d) $\frac{7}{\sqrt{2}}$

15. Equation of the plane ABC can be given as

(a) $\vec{r}.(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}) = 0$

(b) $[\vec{r}\,\vec{a}\,\vec{b}] + [\vec{r}\,\vec{b}\,\vec{c}] + [\vec{r}\,\vec{c}\,\vec{a}] + [\vec{c}\,\vec{b}\,\vec{a}] = 0$

(c) $(\vec{r}-\vec{a}).(\vec{a}\times\vec{b}) = 0$

(d) $[\vec{r}\,\vec{a}\,\vec{b}] + [\vec{r}\,\vec{b}\,\vec{c}] + [\vec{r}\,\vec{c}\,\vec{a}] + [\vec{a}\,\vec{b}\,\vec{c}] = 0$

Comprehension – 2 (16 to 18)

$\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = \hat{i}-\hat{j}+\hat{k}$ and non zero vector $\vec{c}$ follow the relation $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$

16. Then vector $\vec{c}$ may be given as

(a) $4\hat{i}+2\hat{j}+4\hat{k}$ (b) $4\hat{i}-2\hat{j}+4\hat{k}$

(c) $\hat{i}+\hat{j}+\hat{k}$ (d) $\hat{i}-4\hat{j}+\hat{k}$

17. Volume of parallopiped whose adjacent sides are given by $\vec{a}$, $\vec{b}$, $\vec{b}\times\vec{c}$ (use the $\vec{c}$ of previous question)

(a) 18 (b) 54

(c) 12 (d) 36

18. Bisector of obtuse angle between the vector $\vec{b}$ and $\vec{c}$ is

(a) $(2+\sqrt{3})\hat{i}+(1-\sqrt{3})\hat{j}+(2+\sqrt{3})\hat{k}$

(b) $(2+\sqrt{3})\hat{i}+(1-\sqrt{3})\hat{j}-(2+\sqrt{3})\hat{k}$

(c) $(2-\sqrt{3})\hat{i}+(\sqrt{3}+1)\hat{j}+(2-\sqrt{3})\hat{k}$

(d) $(2+\sqrt{3})\hat{i}-(1-\sqrt{3})\hat{j}+(2+\sqrt{3})\hat{k}$

19. Match the column:

Column - I	Column - I
(a) If V_1, V_2, V_3 are the volumes of parallelopiped, triangular prism and tetrahedron respectively. The three coterminus edges of all three figures are the vectors $\hat{i}-\hat{j}-6\hat{k}$, $\hat{i}-\hat{j}+4\hat{k}$ and $2\hat{i}-5\hat{j}+3\hat{k}$, then	(P) $2V_1+3V_3=5V_2$
(b) If V_1, V_2, V_3 are the volumes of parallelopiped, triangular prism and tetrahedron respectively. The three coterminus edges of all three figures are the vectors $-2\hat{i}+3\hat{j}-3\hat{k}$, $4\hat{i}+5\hat{j}-3\hat{k}$ and $6\hat{i}+2\hat{j}-3\hat{k}$, then	(Q) $V_1+V_2+V_3 = 60$
	(R) $V_1+3V_3 = 3V_2$
(c) If V_1, V_2, V_3 are the volumes of parallelopiped, triangular prism and tetrahedron respectively. The three coterminus edges of all three figures are the vectors $-3\hat{i}+\hat{j}+\hat{k}$, $4\hat{i}+2\hat{j}+4\hat{k}$ and $2\hat{i}+2\hat{j}$, then	(S) $V_1+V_2+V_3 = 50$
	(T) $V_1 : V_2 : V_3 = 6 : 3 : 1$

20. Match the column:

Column – I	Column – II
(a) Given two vectors $\vec{a} = 2\hat{i}-3\hat{j}+6\hat{k}$, $\vec{b} = -2\hat{i}+2\hat{j}-\hat{k}$ and $\lambda = \frac{\text{the projection of } \vec{a} \text{ on } \vec{b}}{\text{the projection of } \vec{b} \text{ on } \vec{a}}$, then the value of 3λ is	(p) 1
(b) If $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$, $\vec{c} = 3\hat{i}+\hat{j}$ and $\vec{a}+P\vec{b}$ is normal to $\vec{c}$, then P is equal	(q) 7
(c) Let $\vec{a}, \vec{b}, \vec{c}$ be three non zero vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, then $\lambda(\vec{b}\times\vec{a})+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}$, where λ is equal to	(r) 5
(d) $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is equal to	(s) 2

NUMERICAL BASED QUESTIONS

21. If the vectors $\vec{r}_1 = \lambda x\hat{i}+\hat{j}, \vec{r}_2 = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{r}_3 = 2\hat{i}+\hat{j}+x\hat{k}(\lambda>0)$ are coplanar vectors for only one real x then λ is

22. Let $\vec{a},\vec{b},\vec{c}$ be three non-coplanar vectors and $\vec{p},\vec{q},\vec{r}$ are vectors defined by the relations $\vec{p} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}, \vec{q} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}, \vec{r} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$, then the value of the expression $(\vec{a}+\vec{b}).\vec{p}+(\vec{b}+\vec{c}).\vec{q}+(\vec{c}+\vec{a}).\vec{r}$ is equal to

23. The area of the parallelogram having diagonals $\vec{a} = 3i+j-2k$ and $\vec{b} = i-3j+4k$ is $a\sqrt{b}$ then find value of a/b

24. Given vectors $\vec{a} = 3\hat{i}-6\hat{j}-\hat{k}$, $\vec{b} = \hat{i}+4\hat{j}-3\hat{k}$ and $\vec{c} = 3\hat{i}-4\hat{j}-12\hat{k}$ then absolute value of projection of $\hat{a}\times\hat{b}$ on vector $\vec{c}$ is

25. The value of k for which vectors $\vec{a} = \hat{i}-\hat{j}$ and $\vec{b} = -2\hat{i}+k\hat{j}$ are collinear is

26. The vectors $\vec{a}, \vec{b}$ are non collinear. If the vectors $\vec{a}+(x+1)\vec{b}$ and $(2x-3)\vec{b}-\vec{a}$ are collinear then the value of x is

27. In the vectors $\hat{i}+3\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+4\hat{k}$ and $3\hat{i}+2\hat{j}+x\hat{k}$ are coplanar, then the value of x = ?

28. If θ is angle between the vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ then $\cos\theta$ =

29. If $|\vec{a}|=2$, $|\vec{b}|=3$, $\vec{c}=\vec{a}+3\vec{b}$ and $(\vec{a},\vec{b})=\frac{\pi}{3}$ then $|\vec{c}|^2$ =

30. If $\alpha=2\hat{i}+3\hat{j}-\hat{k}$, $\beta=-\hat{i}+2\hat{j}-4\hat{k}$, $\gamma=\hat{i}+\hat{j}+\hat{k}$ then absolute value of $(\alpha\times\beta).(\alpha\times\gamma)$ =

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$.

[JEE (Main) 2016]

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

2. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}|=3$, $|(\vec{a}\times\vec{b})\times\vec{c}|=3$ and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30°. Then $\vec{a}.\vec{c}$ is equal to:

[JEE (Main) 2017]

(a) $\frac{1}{8}$ (b) $\frac{25}{8}$

(c) 2 (d) 5

3. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b}=24$, then $|\vec{u}|^2$ is equal to:

[JEE Main 2018]

(a) 315 (b) 256

(c) 84 (d) 336

4. Let $\vec{a}=\hat{i}-j,\vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}$ be a vector such that $\vec{a}\times\vec{c}+\vec{b}=0$ and $\vec{a}\cdot\vec{c}=4$, the $|\vec{c}|^2$ is equal to:

[JEE Main-2019 (January)]

(a) $\frac{19}{2}$ (b) 9

(c) 8 (d) $\frac{17}{2}$

5. Let S be the set of all triangle in the xy – plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is:

[JEE Main-2019 (January)]

(a) 9 (b) 18

(c) 32 (d) 36

6. The equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is:

[JEE Main-2019 (January)]

(a) $x+2y-2z=0$ (b) $x-2y+z=0$

(c) $5x+2y-4z=0$ (d) $3x+2y-3z=0$

7. Let $\vec{a}=\hat{i}+\hat{j}+\sqrt{2}\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+\sqrt{2}\hat{k}$ and $\vec{c}=5\hat{i}+\hat{j}+\sqrt{2}\hat{k}$ be three vectors such that the projection vector $\vec{b}$ on $\vec{a}$ is $\vec{a}$. IF $\vec{a}+\vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to: *[JEE Main-2019 (January)]*

(a) $\sqrt{22}$ (b) 4

(c) $\sqrt{32}$ (d) 6

8. The plane passing through the point (4, – 1, 2) and parallel to the lines $\frac{x+2}{3}=\frac{y-2}{-1}=\frac{z+1}{2}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-4}{3}$ also passes through the point:

[JEE Main-2019 (January)]

(a) (1, 1 , – 1) (b) (1, 1 ,1)

(c) (–1, –1 , – 1) (d) (–1, –1 , 1)

9. Let $\vec{a} = 2\hat{i} + \lambda_1\hat{j} + 3\hat{k}$, $\vec{b} = 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k}$ and $\vec{c} = 3\hat{i} + 6\hat{j} + (\lambda_3 - 1)\hat{k}$ be three vectors such that $\vec{b} = 2\vec{a}$ and $\vec{a}$ is perpendicular to $\vec{c}$. Then a possible value of $(\lambda_1, \lambda_2, \lambda_3)$ is:

[JEE Main-2019 (January)]

(a) (1, 3, 1) (b) $\left(-\frac{1}{2}, 4, 0\right)$

(c) $\left(\frac{1}{2}, 4, -2\right)$ (d) (1, 5, 1)

10. Let $\vec{\alpha} = (\lambda - 2)\vec{a} + \vec{b}$ and $\vec{\beta} = (4\lambda - 2)\vec{a} + 3\vec{b}$ be two given vectros where $\vec{a}$ and $\vec{b}$ are non collinear. The value of λ for which vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear, is:

[JEE Main-2019 (January)]

(a) –4 (b) –3

(c) 4 (d) 3

11. The plane which bisects the line segment joining the points (–3, –3, 4) and (3, 7, 6) at right angles, passes through which one of hte following points?

[JEE Main-2019 (January)]

(a) (–2, 3, 5) (b) (4, –1, 7)

(c) (2, 1, 3) (d) (4, 1, –2)

12. Let $\sqrt{3}\hat{i} + j, \hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ respecively be the position vectros of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is: [JEE Main-2019 (January)]

(a) 4 (b) 3

(c) 2 (d) 1

13. Let $\vec{a} = \hat{i} + 2\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + \lambda\hat{j} + 4\hat{k}$ and $\vec{c} = 2\hat{i} + 4\hat{j} + (\lambda^2 - 1)\hat{k}$ be coplanar vectors. Then the non – zero vector $\vec{a} \times \vec{c}$ is:

[JEE Main-2019 (January)]

(a) $-10\hat{i} - 5\hat{j}$ (b) $-14\hat{i} - 5\hat{j}$

(c) $-14\hat{i} + 5\hat{j}$ (d) $-10\hat{i} + 5\hat{j}$

14. A tetrahedron has vertices P(1, 2, 1), Q(2, 1, 3), R(–1, 1, 2) and O(0, 0, 0). The angle between the faces OPQ and PQR is:

[JEE Main-2019 (January)]

(a) $\cos^{-1}\left(\frac{17}{31}\right)$ (b) $\cos^{-1}\left(\frac{19}{35}\right)$

(c) $\cos^{-1}\left(\frac{9}{32}\right)$ (d) $\cos^{-1}\left(\frac{7}{31}\right)$

15. The sum of the distinct real values of μ, for which the vectors, $\mu\hat{i} + \hat{j} + \hat{k}, \hat{i} + \mu\hat{j} + \hat{k}, \hat{i} + \hat{j} + \mu\hat{k}$ are co-planar, is:

[JEE Main-2019 (January)]

(a) –1 (b) 0

(c) 1 (d) 2

16. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors, out of which vectors $\vec{b}$ and $\vec{c}$ are non-parallel. If α and β are the angles which vector $\vec{a}$ makes with vectors $\vec{b}$ and $\vec{c}$ respectively and $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, then $|\alpha - \beta|$ is equal to:

[JEE Main-2019 (January)]

(a) 30° (b) 90°

(c) 60° (d) 45°

17. The length of the perpendicular from the point (2, –1, 4) on the straight line, $\frac{x+3}{10} = \frac{y-2}{-7} = \frac{z}{1}$ is:

[JEE Main-2019 (April)]

(a) less than 2

(b) greater than 3 but less than 4

(c) greater than 4

(d) greater than 2 but less than 3

18. The magnitude of the projection of the vector $2\hat{i} + 3\hat{j} + \hat{k}$ on the vector perpendicular to the plane contining the vectors $\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$, is:

[JEE Main-2019 (April)]

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{\frac{3}{2}}$

(c) $\sqrt{6}$ (d) $3\sqrt{6}$

19. Let $\vec{a} = 3\hat{i} + 2\hat{j} + x\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, for some real x. Then $|\vec{a} \times \vec{b}| = r$ is possible if:

[JEE Main-2019 (April)]

(a) $3\sqrt{\frac{3}{2}} < r < 5\sqrt{\frac{3}{2}}$ (b) $0 < r \le \sqrt{\frac{3}{2}}$

(c) $\sqrt{\frac{3}{2}} < r \le 3\sqrt{\frac{3}{2}}$ (d) $r \ge 5\sqrt{\frac{3}{2}}$

20. Let $\vec{\alpha} = 3\hat{i} + \hat{j}$ and $\vec{\beta} = 2\hat{i} - \hat{j} + 3\hat{k}$. If $\vec{\beta} = \vec{\beta}_1 - \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1 \times \vec{\beta}_2$ is equal to: [JEE Main-2019 (April)]

(a) $-3\hat{i} + 9\hat{j} + 5\hat{k}$ (b) $3\hat{i} - 9\hat{j} - 5\hat{k}$

(c) $\frac{1}{2}(-3\hat{i} + 9\hat{j} + 5\hat{k})$ (d) $\frac{1}{2}(3\hat{i} - 9\hat{j} + 5\hat{k})$

21. If a unit vector $\vec{a}$ makes angles $\pi/3$ with $\hat{i}$, $\pi/4$ with $\hat{j}$ and $\theta \in (0, \pi)$ with $\hat{k}$, then a value of θ is: [JEE Main-2019 (April)]

(a) $\frac{5\pi}{12}$ (b) $\frac{5\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

22. The distance of the point having position vector $-\hat{i} + 2\hat{j} + 6\hat{k}$ from the straight line passing through the point (2, 3, – 4) and parallel to the vector, $6\hat{i} + 3\hat{j} - 4\hat{k}$ is: [JEE Main-2019 (April)]

(a) 7 (b) $4\sqrt{3}$

(c) $2\sqrt{13}$ (d) 6

23. Let A(3, 0, –1), B(2, 10, 6) and C(1, 2, 1) be the vertices of a triangle and M be the midpoint of AC. If G divides BM in the ratio, 2 : 1, then cos (∠GOA) (O being the origin) is equal to: [JEE Main-2019 (April)]

(a) $\frac{1}{\sqrt{30}}$ (b) $\frac{1}{6\sqrt{10}}$

(c) $\frac{1}{\sqrt{15}}$ (d) $\frac{1}{2\sqrt{15}}$

24. If the volume of parallelopiped formed by the vectors $\hat{i} + \lambda\hat{j} + \hat{k}$, $\hat{j} + \lambda\hat{k}$ and $\lambda\hat{i} + \hat{k}$ is minimum, then l is equal to: [JEE Main-2019 (April)]

(a) $\sqrt{3}$ (b) $-\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $-\sqrt{3}$

25. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ be two vectors. If a vector perpendicular to both the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ has the magnitude 12 then one such vector is: [JEE Main-2019 (April)]

(a) $4(2\hat{i} + 2\hat{j} - \hat{k})$ (b) $4(-2\hat{i} - 2\hat{j} - \hat{k})$

(c) $4(2\hat{i} - 2\hat{j} - \hat{k})$ (d) $4(2\hat{i} + 2\hat{j} + \hat{k})$

26. Let $\alpha \in R$ and the three vectors $\vec{a} = \alpha\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \alpha\hat{k}$ and $\vec{c} = \alpha\hat{i} - 2\hat{j} + 3\hat{k}$. Then the set $S = \{\alpha : \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar$\}$ [JEE Main-2019 (April)]

(a) is singleton
(b) Contains exactly two numbers only one of which is positive
(c) Contains exactly two positive numbers
(d) is empty

27. A vector $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$ $(\alpha, \beta \in R)$ lies in the plane of the vectors, $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + 4\hat{k}$. If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then [JEE Main-2020 (January)]

(a) $\vec{a} \cdot \hat{k} + 4 = 0$ (b) $\vec{a} \cdot \hat{k} + 2 = 0$

(c) $\vec{a} \cdot \hat{i} + 1 = 0$ (d) $\vec{a} \cdot \hat{i} + 3 = 0$

28. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. If $\lambda = \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ and $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$, then the ordered pair, $(\lambda, \vec{d})$ is equal to: [JEE Main-2020 (January)]

(a) $\left(\frac{3}{2}, 3(\vec{a} \times \vec{c})\right)$ (b) $\left(-\frac{3}{2}, 3(\vec{c} \times \vec{b})\right)$

(c) $\left(\frac{3}{2}, 3(\vec{b} \times \vec{c})\right)$ (d) $\left(-\frac{3}{2}, 3(\vec{a} \times \vec{b})\right)$

29. Let the volume of a parallelopiped whose coterminous edges are given by $\vec{u} = \hat{i} + \hat{j} + \lambda\hat{k}$, $\vec{v} = \hat{i} + \hat{j} + 3\hat{k}$ and $\vec{w} = 2\hat{i} + \hat{j} + \hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\vec{u}$ and $\vec{w}$, then cos θ can be [JEE Main-2020 (January)]

(a) $\frac{5}{3\sqrt{3}}$ (b) $\frac{7}{6\sqrt{6}}$

(c) $\frac{5}{7}$ (d) $\frac{7}{6\sqrt{3}}$

30. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is [JEE Main-2020 (January)]

(a) $2\sqrt{30}$ (b) 3

(c) $\frac{7\sqrt{30}}{2}$ (d) $3\sqrt{30}$

31. Let $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ be two vectors. If $\vec{c}$ is a vector such that $\vec{b} \times \vec{c} = \vec{b} \times \vec{a}$ and $\vec{c} \cdot \vec{a} = 0$, then $\vec{c} \cdot \vec{b}$ is equal to [JEE Main-2020 (January)]

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$

(c) – 1 (d) $\frac{-3}{2}$

32. The projection of the line segment joining the points (1, −1, 3) and (2, −4, 11) on the line joining the points (−1, 2, 3) and (3, −2, 10) is : [JEE Main-2020 (January)]

33. If the vectors, $\vec{p} = (a+1)\hat{i} + a\hat{j} + a\hat{k}$, $\vec{q} = a\hat{i} + (a+1)\hat{j} + a\hat{k}$ and $\vec{r} = a\hat{i} + a\hat{j} + (a+1)\hat{k}$, $(a \in R)$ are coplanar and $3(\vec{p}\cdot\vec{q})^2 - \lambda|\vec{r}\times\vec{q}|^2 = 0$ then the value of λ is :

[JEE Main-2020 (January)]

34. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = \sqrt{3}, |\vec{b}| = 5, \vec{b}\cdot\vec{c} = 10$ and angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to the vectors $\vec{b}\times\vec{c}$, then $\left|\vec{a}\times(\vec{b}\times\vec{c})\right|$ is equal to_____.

[JEE Main-2020 (January)]

35. Let the position vectors of points 'A' and 'B' be $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}+\hat{j}+3\hat{k}$, respectively. A point 'P' divides the line segment AB internally in the ratio $\lambda : 1$ $(\lambda > 0)$. If O is the origin and $\overrightarrow{OB}.\overrightarrow{OP} - 3|\overrightarrow{OA}\times\overrightarrow{OP}|^2 = 6$, then λ is equal to ____.

[JEE Main-2020 (September)]

36. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $|\vec{a}-\vec{b}|^2 + |\vec{a}-\vec{c}|^2 = 8$. Then $|\vec{a}+2\vec{b}|^2 + |\vec{a}+2\vec{c}|^2$ is equal to [JEE Main-2020 (September)]

37. Let a plane P contain two lines

$\vec{r} = \hat{i} + \lambda(\hat{i}+\hat{j}), \lambda \in R$ and

$\vec{r} = -\hat{j} + \mu(\hat{j}-\hat{k}), \mu \in R$

If $Q(\alpha, \beta, \gamma)$ is the foot of the perpendicular drawn form the point M(1, 0, 1) to P, then $3(\alpha+\beta+\gamma)$ equals____.

[JEE Main-2020 (September)]

38. The lines $\vec{r} = (\hat{i}-\hat{j}) + \ell(2\hat{i}+\hat{k})$ and

$\vec{r} = (2\hat{i}-\hat{j}) + m(\hat{i}+\hat{j}-\hat{k})$

[JEE Main-2020 (September)]

(a) do not intersect for any values of ℓ and m

(b) intersect for all values of ℓ and m

(c) intersect when $\ell = 2$ and $m = \frac{1}{2}$

(d) intersect when $\ell = 1$ and $m = 2$

39. If $\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$, then the value of

$|\hat{i}\times(\vec{a}\times\hat{i})|^2 + |\hat{j}(\vec{a}\times\hat{j})|^2 + |\hat{k}\times(\vec{a}\times\hat{k})|^2$

is equal to____.

[JEE Main-2020 (September)]

40. Let x_0 be the point of local maxima of

$f(x) = \vec{a}.(\vec{b}\times\vec{c})$, where

$\vec{a} = x\hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = -2\hat{i} + x\hat{j} - \hat{k}$ $\vec{c} = 7\hat{i} - 2\hat{j} + x\hat{k}$. Then the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ at $x = x_0$ is

[JEE Main-2020 (September)]

(a) 14 (b) −30

(c) −4 (d) −22

41. Let the vectores $\vec{a}, \vec{b}, \vec{c}$ be such that $|\vec{a}| = 2, |\vec{b}| = 4$ and $|\vec{c}| = 4$. If the projection of $\vec{b}$ on $\vec{a}$ is equal to the projection of $\vec{c}$ on $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{c}$, then the value of $\left|\vec{a}+\vec{b}-\vec{c}\right|$ is ——.

[JEE Main-2020 (September)]

42. If the volume of a parallelopiped, whose coterminus edges are given by the vectors

$\vec{a} = \hat{i}+\hat{j}+n\hat{k}, \vec{b} = 2\hat{i}+4\hat{j}-n\hat{k}$ and $\vec{c} = \hat{i}+n\hat{j}+3\hat{k}$ $(n \geq 0)$, is 158 cu.units, then :

[JEE Main-2020 (September)]

(a) $n = 7$ (b) $\vec{b}\cdot\vec{c} = 10$

(c) $n = 9$ (d) $\vec{a}\cdot\vec{c} = 17$

43. If $\vec{x}$ and $\vec{y}$ be two non-zero vectors such that $|\vec{x}+\vec{y}| = |\vec{x}|$ and $2\vec{x} + \lambda\vec{y}$ is perpendicular to $\vec{y}$ then the value of λ is _____. [JEE Main-2020 (September)]

44. A plane P meets the coordinate axes at A, B and C respectively. The centroid of ΔABC is given to be (1, 1, 2). Then the equation of the line through this centroid and perpendicular to the plane P is :

[JEE Main-2020 (September)]

(a) $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{1}$

(b) $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-2}{2}$

(c) $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$

(d) $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-2}{1}$

45. If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}|\vec{a}+\vec{b}| + |\vec{a}-\vec{b}|$ is _____.

[JEE Main-2020 (September)]

46. Let $\vec{a}=\hat{i}+2\hat{j}-\hat{k}, \vec{b}=\hat{i}-\hat{j}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{a}=\vec{c}\times\vec{a}$ and $\vec{r}\cdot\vec{b}=0$, then $\vec{r}\cdot\vec{a}$ is equal to

[JEE Main-2021 (February)]

47. Let three vectors $\vec{a},\vec{b}$ and $\vec{c}$ be such that is coplanar with $\vec{a}$ and $\vec{b}, \vec{a}\cdot\vec{c}=7$ and $\vec{b}$ is perpendicular to $\vec{c}$, where $\vec{a}=-\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+\hat{k}$, then the value of $2|\vec{a}+\vec{b}+\vec{c}|^2$ is

[JEE Main-2021 (February)]

48. Let $\vec{a}=\hat{i}+\alpha\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\alpha\hat{j}+\hat{k}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $8\sqrt{3}$ square units, then $\vec{a}\cdot\vec{b}$ is equal to

[JEE Main-2021 (February)]

49. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$. If the vector $\vec{x}$ is perpendicular to $(3\hat{i}+2\hat{j}-\hat{k})$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$ is equal to

[JEE Main-2021 (March)]

50. If $\vec{a}=\alpha\hat{i}+\beta\hat{j}+3\hat{k}$
$\vec{b}=-\beta\hat{i}-\alpha\hat{j}-\hat{k}$ and
$\vec{c}=\hat{i}-2\hat{j}-\hat{k}$

such that $\vec{a}\cdot\vec{b}=1$ and $\vec{b}\cdot\vec{c}=-3$, then $\frac{1}{3}((\vec{a}\times\vec{b})\cdot\vec{c})$ is equal to

[JEE Main-2021 (March)]

JEE-ADVANCED PREVIOUS YEARS

1. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unit vectors satisfying $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2=9$, then $|2\hat{a}+5\hat{b}+5\hat{c}|$ is

[IIT JEE 2012]

2. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}|=\sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is

[IIT JEE 2012]

(a) 0 (b) 3
(c) 4 (d) 8

3. Let $\overrightarrow{PR}=3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ}=\hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT}=\hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

[JEE Advanced-2013]

(a) 5 (b) 20
(c) 10 (d) 30

4. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k}: a,b,c\in\{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is

[JEE Advanced-2013]

5. Match List I with List II and select the correct answer using the code given below the lists:

[JEE Advanced-2013]

List - I	List - II
P. Volume of parallelopiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	1. 100
Q. Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2. 30
R. Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	3. 24
S. Area of a paralelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4. 60

Codes :

	P	Q	R	S
(a)	4	2	3	1
(b)	2	3	1	4
(c)	3	4	1	2
(d)	1	4	3	2

6. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is

[JEE Advanced-2014]

7.

List-I		List-II	
P.	Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm \frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals	1.	1
Q.	Let $A_1, A_2, \ldots\ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of nsides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, $k = 1, 2, \ldots, n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a_k} \times \vec{a_{k+1}})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a_k}.\vec{a_{k+1}})\right\rvert$, then the minimum value of n is	2.	2
R.	If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	3.	8
S.	Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	4.	9

[JEE Advanced-2014]

Codes :

	P	Q	R	S
(a)	4	3	2	1
(b)	2	4	3	1
(c)	4	3	1	2
(d)	2	4	1	3

8. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is(are) true?

(a) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$

(b) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$

(d) $\vec{a}.\vec{b} = -72$

9.

Column- I		Column-II	
(a)	In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(b)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is (are)	(Q)	2
(c)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are)	(R)	3
(d)	Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5,q, b is an arithmetic progression, then the value(s) of $\lvert q - a\rvert$ is (are)	(S)	4
		(T)	5

[JEE Advanced-2015]

10. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a nonzero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then

[JEE Advanced-2014]

(*a*) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$

(*b*) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$

(*c*) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$

(*d*) $\vec{a} = (\vec{a}.\vec{y})(\vec{z} - \vec{y})$

11.

Column- I		**Column-II**
(*a*) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(P)	1
(*b*) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(Q)	2
(*c*) In R^2, let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overline{OX}$ and $\overline{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(R)	3
(*d*) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \lvert\alpha x - 1\rvert + \lvert\alpha x - 2\rvert + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(S)	5
	(T)	6

[JEE Advanced-2015]

12. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r}), (\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is

[JEE Advanced-2015]

13. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\hat{\omega} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\hat{u} \times \vec{v}| = 1$ and $\hat{w} \cdot (\hat{u} \times \vec{v}) = 1$. Which of the following statement(s) is(are) correct?

[JEE Advanced-2016]

(*a*) There is exactly one choice for such

(*b*) There are infinitely many choices for such

(*c*) If lies in the *xy*-plane then $|u_1| = |u_2|$

(*d*) If lies in the *xz*-plane then $2|u_1| = |u_3|$

14. Consider a pyramid OPQRS located in the first octat $(x \geq 0, y \geq 0, z \geq 0)$ with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then

[JEE Advanced-2016]

(*a*) the acute angle between OQ and OS is $\frac{\pi}{3}$

(*b*) the equation of the plane containing the triangle OQS is $x - y = 0$

(*c*) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(*d*) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

15. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$\overrightarrow{OP}.\overrightarrow{OQ}+\overrightarrow{OR}.\overrightarrow{OS}=\overrightarrow{OR}.\overrightarrow{OP}+\overrightarrow{OQ}.\overrightarrow{OS}=\overrightarrow{OQ}.\overrightarrow{OR}+\overrightarrow{OP}.\overrightarrow{OS}$

Then the triangle PQR has S as its

[JEE Advanced-2017]

(a) centroid (b) orthocenter
(c) incentre (d) circumcentre

16. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b}=0$. For some $x, y \in \mathbb{R}$, let $\vec{c}=x\vec{a}+y\vec{b}+(\vec{a}\times\vec{b})$. If $|\vec{c}|=2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ____

[JEE Advanced-2018]

17. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origion O such that S lies on the diagonal OT. If $\vec{p}=\overrightarrow{SP},\vec{q}=\overrightarrow{SQ},\vec{r}=\overrightarrow{SR}$, $\vec{t}=\overrightarrow{ST}$ and the value of $\left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right|$ is ....

[JEE Advanced-2018]

18. Three lines are given by

$\vec{r}=\lambda\hat{i},\lambda\in R$

$\vec{r}=\mu(\hat{i}+\hat{j}),\mu\in R$ and

$\vec{r}=\nu(\hat{i}+\hat{j}+\hat{k}),\nu\in R$

Let the lines cut the plane $x+y+z=1$ at the points A, B and C respectively. If the area of the triangle ABC is D then the value of $(6\Delta)^2$ equals

[JEE Advanced-2019]

19. Let $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ be two vectors. Consider a vector. $\vec{c}=\alpha\vec{a}+\beta\vec{b},\alpha,\beta\in R$. If the projection of $\vec{c}$ on the vector $(\vec{a}+\vec{b})$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c}-(\vec{a}\times\vec{b})\right).\vec{c}$ equals

[JEE Advanced-2019]

20. In a triangle PQR, let $\vec{a}=\overrightarrow{QR},\vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|=3,|\vec{b}|=4$ and $\frac{\vec{a}.(\vec{c}-\vec{b})}{\vec{c}.(\vec{a}-\vec{b})}=\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$, then the value of $|\vec{a}\times\vec{b}|^2$ is:

[JEE(Advanced)-2020]

21. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ}=a\hat{i}+b\hat{j}$ and $\overrightarrow{PS}=a\hat{i}-b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w}=\hat{i}+\hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}|+|\vec{v}|=|\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE?

[JEE(Advanced)-2020]

(a) $a+b=4$
(b) $a-b=2$
(c) The length of the diagonal PR of the parallelogram PQRS is 4
(d) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

ANSWER KEY

Topicwise Questions

1. (*d*)	**2.** (*a*)	**3.** (*a*)	**4.** (*c*)	**5.** (*a*)	**6.** (*d*)	**7.** (*d*)	**8.** (*a*)	**9.** (*b*)	**10.** (*b*)
11. (*a*)	**12.** (*a*)	**13.** (*b*)	**14.** (*c*)	**15.** (*b*)	**16.** (*b*)	**17.** (*c*)	**18.** (*a*)	**19.** (*d*)	**20.** (*b*)
21. (*b*)	**22.** (*a*)	**23.** (*c*)	**24.** (*a*)	**25.** (*c*)	**26.** (*a*)	**27.** (*a*)	**28.** (*d*)	**29.** (*a*)	**30.** (*b*)
31. (*b*)	**32.** (*c*)	**33.** (*b*)	**34.** (*a*)	**35.** (*b*)					

Learning Plus

1. (*c*)	**2.** (*c*)	**3.** (*d*)	**4.** (*d*)	**5.** (*d*)	**6.** (*a*)	**7.** (*c*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)
11. (*a*)	**12.** (*d*)	**13.** (*b*)	**14.** (*c*)	**15.** (*d*)	**16.** (*d*)	**17.** (*d*)	**18.** (*d*)	**19.** (*a*)	**20.** (*b*)
21. (*c*)	**22.** (*c*)	**23.** (*c*)	**24.** (*c*)	**25.** (*c*)	**26.** (*b*)	**27.** (*a*)	**28.** (*a*)	**29.** (*c*)	**30.** (*c*)
31. (*c*)	**32.** (*a*)	**33.** (*a*)	**34.** (*c*)						

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/NUMERICAL

1. (*a*, *d*) **2.** (*a*, *c*) **3.** (*a*) **4.** (*a*, *b*, *c*) **5.** (*b*, *d*) **6.** (*a*, *c*, *d*) **7.** (*a*, *d*) **8.** (*a*, *d*) **9.** (*a*, *b*, *c*, *d*)
10. (*a*, *b*, *c*) **11.** (*c*, *d*) **12.** (*b*) **13.** (*c*) **14.** (*a*) **15.** (*b*) **16.** (*a*) **17.** (*d*) **18.** (*c*)
19. (*a*)–P, R, S, T; (*b*)–P, R, T; (*c*)–P, Q, R, T **20.** (*a*) → (q), (*b*) → (r), (*c*) → (s), (*d*) → (p) **21.** (*b*)
22. (*c*) **23.** (1.67) **24.** (14) **25.** (2) **26.** (0.67) **27.** (2) **28.** (0.167) **29.** (103) **30.** (74)

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*d*)	**2.** (*c*)	**3.** (*d*)	**4.** (*a*)	**5.** (*d*)	**6.** (*b*)	**7.** (*d*)	**8.** (*b*)	**9.** (*b*)	**10.** (*a*)
11. (*d*)	**12.** (*d*)	**13.** (*d*)	**14.** (*b*)	**15.** (*a*)	**16.** (*a*)	**17.** (*b*)	**18.** (*b*)	**19.** (*d*)	**20.** (*c*)
21. (*c*)	**22.** (*a*)	**23.** (*c*)	**24.** (*c*)	**25.** (*c*)	**26.** (*d*)	**27.** (*b*)	**28.** (*d*)	**29.** (*d*)	**30.** (*d*)
31. (*b*)	**32.** (8)	**33.** (1)	**34.** (30)	**35.** (0.8)	**36.** (2)	**37.** (5)	**38.** (*a*)	**39.** (18)	**40.** (*d*)
41. (6)	**42.** (*b*)	**43.** (01.00)	**44.** (*a*)	**45.** (4)	**46.** (12)	**47.** (75)	**48.** (2)	**49.** (486)	**50.** (2)

JEE-ADVANCED PREVIOUS YEARS

1. (3) **2.** (*c*) **3.** (*c*) **4.** (5) **5.** (*c*) **6.** (4) **7.** (*a*) **8.** (*a*, *c*, *d*)
9. (*a*) – PQ, (*b*) – PQ, (*c*) – PQST, (*d*) QT **10.** (*a*, *b*, *c*) **11.** (*a*) – PRS, (*b*) – P, (*c*) – PQ, (*d*) – ST
12. Bonus **13.** (*b*, *c*) **14.** (*b*, *c*, *d*) **15.** (*b*) **16.** (3) **17.** (0.5) **18.** (0.75) **19.** (18.00) **20.** (108.00)
21. (*a*, *c*)

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29 THREE DIMENTIONAL GEOMETRY

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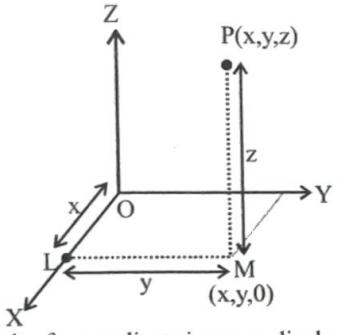
COORDINATES OF A POINT IN SPACE

In two dimensional geometry, magnitude of x & y coordinates are perpendicular distances from y & x axis respectively

But in case of three dimensional geometry it is understood in different way.



(*a*) Magnitude of x-coordinate is perpendicular distance from y-z plane, similarly magnitude of y & z coordinates are perpendicular distances from x-z and x-y plane

(*b*) If a point lies on x-axis then its coordinates are (x, 0, 0) and similarly on y-axis & z-axis the coordinates are (0, y, 0) and (0, 0, z)

Remark : The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

DISTANCE BETWEEN TWO POINTS

Let P and Q be two given points in space. Let the co-ordinates of the points P and Q be $(x_1, y_1 z_1)$ and (x_2, y_2, z_2) respectively with respect to a set OX, OY, OZ of rectangular axes. The position vectors of the points P and Q are given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= (x_2 - x_1)\hat{i} - (y_2 - y_1)\hat{j} - (z_2 - z_1)\hat{k}$.

$\therefore$ $PQ = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

TRAIN YOUR BRAIN

Example- 1: Show by using distance formula that the points (4, 5, –5), (0, –11, 3) and (2, –3, –1) are collinear.

Sol. Let $A \equiv (4, 5, -5), B \equiv (0, -11, 3), C \equiv (2, -3, -1)$.

$AB = \sqrt{(4-0)^2 + (5+11)^2 + (-5-3)^2} = \sqrt{336} = \sqrt{4 \times 84} = 2\sqrt{84}$

$BC = \sqrt{(0-2)^2 + (-11+3)^2 + (3+1)^2} = \sqrt{84}$

$AC = \sqrt{(4-2)^2 + (5+3)^2 + (-5+1)^2} = \sqrt{84}$

$BC + AC = AB$

Hence points A, B, C are collinear and C lies between A and B.

SECTION FORMULA

To determine the co-ordinates of a point R which divides the joining of two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m_1 : m_2$. Let OX, OY, OZ be a set of rectangular axes.

The position vectors of the two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ ....(1) and

$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ ....(2)



Also if the co-ordinates of the point R are (x, y, z), then

$\overrightarrow{OR} = x\hat{i} + y\hat{j} + z\hat{k}$. .....(3)

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

Hence $m_2 \overrightarrow{PR} = m_1 \overrightarrow{RQ}$ or $m_2(\overrightarrow{OR} - \overrightarrow{OP}) = m_1(\overrightarrow{OQ} - \overrightarrow{OR})$

or $\overrightarrow{OR} = \dfrac{m_1\overrightarrow{OQ} + m_2\overrightarrow{OP}}{m_1 + m_2}$

or $x\hat{i} + y\hat{j} + z\hat{k} = \dfrac{(m_1x_2 + m_2x_1)\hat{i} + (m_1y_2 + m_2y_1)\hat{j} + (m_1z_2 + m_2z_1)\hat{k}}{(m_1 + m_2)}$

[Using (1), (2) and (3)]

Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \; y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2},$$

$$z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

(for external division take –ve sign)

Remark: The middle point of the segment PQ is obtained by putting $m_1 = m_2$. Hence the co-ordinates of the middle point of PQ are

$$\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right)$$

CENTROID OF A TRIANGLE

Let ABC be a triangle. Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let AD be a median of the ΔABC. Thus D is the mid point of BC.

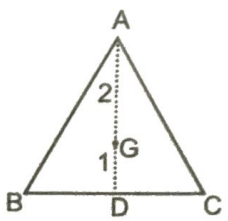
$\therefore$ The co-ordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right)$$

Now if G is the centroid of ΔABC, then G divides AD in the ratio 2 : 1. Let the co-ordinates of G be (x, y, z). Then

$$x = \frac{2.\left(\frac{x_2 + x_3}{2}\right) + 1.x_1}{2 + 1}, \text{ or } x = \frac{x_1 + x_2 + x_3}{3}.$$

Similarly $y = \dfrac{1}{3}(y_1 + y_2 + y_3)$, $z = \dfrac{1}{3}(z_1 + z_2 + z_3)$.



CENTROID OF A TETRAHEDRON

Let ABCD be a tetrahedron, the co-ordinates of whose vertices are (x_r, y_r, z_r), r = 1, 2, 3, 4.

Let G_1 be the centroid of the face ABC of the tetrahedron. Then the co-ordinates of G_1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

The fourth vertex D of the tetrahedron does not lie in the plane of ΔABC. We know from statics that the centroid of the tetrahedron divides the line DG_1 in the ratio 3 : 1. Let G be the centroid of the tetrahedron and if (x, y, z) are its co-ordinates, then

$$x = \frac{3.\frac{x_1 + x_2 + x_3}{3} + 1.x_4}{3 + 1} \text{ or } x = \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

Similarly $y = \dfrac{1}{4}(y_1 + y_2 + y_3 + y_4)$, $z = \dfrac{1}{4}(z_1 + z_2 + z_3 + z_4)$.

TRAIN YOUR BRAIN

Example 2: Find the ratio in which the xy-plane divides the join of (–3, 4, –8) and (5, –6, 4). Also find the point of intersection of the line with the plane.

Sol. Let the xy-plane (i.e., z = 0 plane) divide the line joining the points (–3,4, –8) and (5, –6, 4) in the ratio $\mu : 1$, in the point R. Therefore, the co-ordinates of the point R are

$$\left(\frac{5\mu - 3}{\mu + 1}, \frac{-6\mu + 4}{\mu + 1}, \frac{4\mu - 8}{\mu + 1}\right) \quad \ldots(1)$$

But on xy-plane, the z co-ordinate of R is zero

$\therefore$ $(4\mu - 8)/(\mu + 1) = 0$,

or $\mu = 2$. Hence $\mu : 1 = 2 : 1$. Thus the required ratio is 2 : 1.

Again putting $\mu = 2$ in (1), the co-ordinates of the point R become (7/3, –8/3, 0).

Example-3: Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Sol. Given $A \equiv (2, 3, 4)$, $B \equiv (-1, 2, -3)$, $C \equiv (-4, 1, -10)$.

A (2, 3, 4) B (–1, 2, –3)

Let C divide AB internally in the ratio k : 1, then

$$C \equiv \left(\frac{-k + 2}{k + 1}, \frac{2k + 3}{k + 1}, \frac{-3k + 4}{k + 1}\right)$$

$\therefore$ $\dfrac{-k + 2}{k + 1} = -4 \Rightarrow 3k = -6 \Rightarrow k = -2$

For this value of k, $\dfrac{2k + 3}{k + 1} = 1$, and $\dfrac{-3k + 4}{k + 1} = -10$

Since $k < 0$, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

DIRECTION COSINES OF A LINE

If α, β, γ are the angles which a given directed line makes with the positive directions of the axes. of x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines (briefly written as d.c.'s) of the line. These d.c.'s are usually denote by ℓ, m, n.

Let AB be a given line. Draw a line OP parallel to the line AB and passing through the origin O. Measure angles α, β, γ, then $\cos\alpha, \cos\beta, \cos\gamma$ are the d.c.'s of the line AB. It can be easily seen that ℓ, m, n, are the direction cosines of a line if and only if $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector in the direction of that line.

- Clearly OP'(i.e. the line through O and parallel to BA) makes angle $180° - \alpha, 180° - \beta, 180° - \gamma$ with OX, OY and OZ respectively. Hence d.c.'s of the line BA are $\cos(180° - \alpha), \cos(180° - \beta), \cos(180° - \gamma)$ i.e., are $-\cos\alpha, -\cos\beta, -\cos\gamma$. i.e. $-\hat{i}, -\hat{j}, -\hat{k}$
- If the length of a line OP through the origin O be r, then the co-ordinates of P are $(\ell r, mr, nr)$ where ℓ, m, n are the d c.'s of OP.
- If ℓ, m, n are direction cosines of any line AB, then they will satisfy $\ell^2 + m^2 + n^2 = 1$.

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

DIRECTION RATIOS OF A LINE

If the direction cosines ℓ, m, n of a given line be proportional to any three numbers a, b, c respectively, then the numbers a, b, c are called direction ratios (briefly written as d.r.'s of the given line.

Note : That d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$

Relation between direction cosines and direction ratios

Let a, b, c be the direction ratios of a line whose d.c.'s are ℓ, m, n. From the definition of d.r.'s. we have $\ell/a = m/b = n/c = k$ (say). Then $\ell = ka, m = kb, n = kc$. But $\ell^2 + m^2 + n^2 = 1$.

$\therefore\ k^2(a^2 + b^2 + c^2) = 1$, or $k^2 = 1/(a^2 + b^2 + c^2)$

or $k = \pm\dfrac{1}{\sqrt{(a^2 + b^2 + c^2)}}$.

$\ell = \pm\dfrac{a}{\sqrt{(a^2 + b^2 + c^2)}}, m = \pm\dfrac{b}{\sqrt{(a^2 + b^2 + c^2)}},$

$n = \pm\dfrac{c}{\sqrt{(a^2 + b^2 + c^2)}}$

Remark. Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then ka, kb, kc are also direction ratios of that line where k is any non-zero real number. Moreover if a, b, c are direction ratios of a line, then $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector parallel to that line.

TRAIN YOUR BRAIN

Example 4: Find the direction cosines ℓ, m, n of the two lines which are connected by the relation $\ell + m + n = 0$ and $mn - 2n\ell - 2\ell m = 0$.

Sol. The given relations are $\ell + m + n = 0$ or $\ell = -m - n$ ....(1)

and

$mn - 2n\ell - 2\ell m = 0$ ....(2)

Putting the value of ℓ from (1) in the relation (2), we get

$mn - 2n(-m-n) - 2(-m-n)m = 0$ or $2m^2 + 5mn + 2n^2 = 0$

or $(2m + n)(m + 2n) = 0$.

$\therefore\ \dfrac{m}{n} = -\dfrac{1}{2}$ and -2.

From (1), we have $\dfrac{\ell}{n} = \dfrac{-m-n}{n} = -\dfrac{m}{n} - 1$ ....(3)

Now when $\dfrac{m}{n} = -\dfrac{1}{2}$, (3) given $\dfrac{\ell}{n} = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$.

$\therefore\ \dfrac{m}{1} = \dfrac{n}{-2}$ and $\dfrac{\ell}{1} = \dfrac{n}{-2}$

i.e. $\dfrac{\ell}{1} = \dfrac{m}{1} = \dfrac{n}{-2} = \dfrac{\sqrt{(\ell^2 + m^2 + n^2)}}{\sqrt{\{1^2 + 1^2 + (-2)^2\}}} = \dfrac{1}{\sqrt{6}}$

$\therefore$ The d.c.'s of one line are $\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}$.

Again when $\dfrac{m}{2} = -2$, (3) given $\dfrac{\ell}{n} = 2 - 1 = 1$.

i.e. $\dfrac{\ell}{1} = \dfrac{m}{-2} = \dfrac{n}{1} = \dfrac{\sqrt{(\ell^2 + m^2 + n^2)}}{\sqrt{\{1^2 + (-2)^2 + 1^2\}}} = \dfrac{1}{\sqrt{6}}$

$\therefore$ The d.c.'s of the other line are $\dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$.

- To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the another line whose d.c.'s are l, m, n.

Let O be the origin. Then $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

$\therefore\ \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Now the unit vector along the line whose d.c.'s are

$l, m, n = l\hat{i} + m\hat{j} + n\hat{k}$.

$\therefore$ projection of PQ on the line whose d.c.'s are l, m, n

$= [(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}] \cdot (l\hat{i} + m\hat{j} + n\hat{k})$

$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

- The angle q between these two lines is given by cos q

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

If l_1, m_1, n_1 and l_2, m_2, n_2 are two sets of real numbers, then

$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

Now, we have

$\sin^2 q = 1 - \cos^2 q = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2 =$

$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

$$= \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & n_1 \\ \ell_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2$$

- Condition for lines to be perpendicularity

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0.$

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

- Condition for lines to be paralled

$$\Rightarrow \frac{l_1}{l_2}, \frac{m_1}{m_2}, \frac{n_1}{n_2} \quad \Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Direction cosines of axes

Since the positive x-axis makes an angle 0°, 90°, 90° with axes of x, y and z axes respectively,

So D.C.'s of x-axis are 1, 0, 0. Similarly

D.C.'s of y-axis are 0, 1, 0.

D.C.'s of z-axis are 0, 0, 1.

Example 5: Show that the lines whose d.c.'s are satisfies by $\ell + m + n = 0$ and $2mn + 3\ell n - 5\ell m = 0$ are at right angles.

Sol. From the first relation, we have

$\ell = -m - n.$ ....(1)

Putting this value of ℓ in the second relation, we have

$2mn + 3(-m - n)n - 5(-m - n)m = 0$ or

$5m^2 + 4mn - 3n^2 = 0$ or $5(m/n)^2 + 4(m/n) - 3 = 0$ ....(2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the d,c,'s of the two lines. Then the roots of (2) are m_1/n_1 and m_2/n_2.

$\therefore$ product of the roots $= \frac{m_1}{n_1}\cdot\frac{m_2}{n_2} = -\frac{3}{5}$

or $\frac{m_1m_2}{3} = \frac{n_1n_2}{-5}$. ....(3)

Again from (1), $n = -\ell - m$ and putting this value of n in the second given relation, we have

$2m(-\ell - m) + 3\ell(-\ell - m) - 5\ell m = 0$

or $3(\ell/m)^2 + 10(\ell/m) + 2 = 0.$

$\therefore \frac{\ell_1}{m_1}\cdot\frac{\ell_2}{m_2} = \frac{2}{3}$ or $\frac{\ell_1\ell_2}{2} = \frac{m_1m_2}{3}$

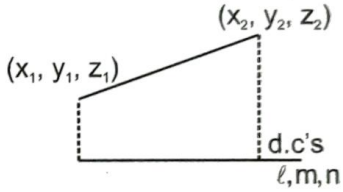
From (3) and (4) we have $\frac{\ell_1\ell_2}{2} = \frac{m_1m_2}{3} \frac{n_1n_2}{-5} = k$ (say)

$\therefore \ell_1\ell_2 + m_1m_2 + n_1n_2 = (2 + 3 - 5)k = 0 \,.\, k = 0.$

$\Rightarrow$ The lines are at right angles.

PROJECTION OF A LINE SEGMENT ON A LINE

(i) If the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$



(ii) Vector form : projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a} \,.\, \hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

In the above case we can consider

$\overrightarrow{PQ}$ as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ in place of $\vec{a}$ and

$\ell\hat{i} + m\hat{j} + n\hat{k}$ in place of $\vec{b}$.

(iii) $\ell|\vec{r}|, m|\vec{r}|$ & $n|\vec{r}|$ are the projection of $\vec{r}$ in OX, OY & OZ axes.

(iv) $\vec{r} = |\vec{r}|(\ell\hat{i} + m\hat{j} + n\hat{k})$

Example 6: Find the angle between the lines whose direction cosines are $-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$.

Sol. Let θ be the required angle, then

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2} \Rightarrow \theta = 120°,$$

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PLANE

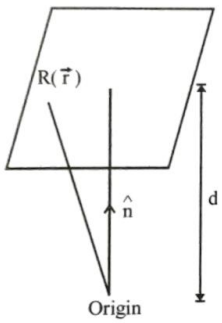
(i) General equation of degree one in x, y, z i.e. $ax + by + cz + d = 0$ represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

where a, b, c are the direction ratios of the normal to the plane.

(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.

(iv) Normal form of the equation of plane :



$(\vec{r} - d\hat{n}).\hat{n} = 0$ Projection of $\vec{r}$ on $\hat{n} = d$

$\vec{r}.\hat{n} = d$ .... (i) $(d > 0)$

$\hat{n} \rightarrow$ unit vector pointing from origin towards plane & perpendicular to it.

$d \rightarrow$ perpendicular distance of the plane from the origin.

$d\hat{n} \rightarrow$ p.v. of foot of perpendicular from origin in plane.

i.e. from (i) $\ell x + my + nz = d$ (cartesian form)

(v) Parametric equation of plane :

Vectors $\vec{r} - \vec{a}.\vec{p}$ and $\vec{q}$ are coplanar. Hence $\vec{r} - \vec{a} = \lambda\vec{p} + \mu\vec{q}$

or $\vec{r} = \vec{a} + \lambda\vec{p} + \mu\vec{q}$ denotes a plane containing the point $\vec{a}$ and is parallel to two non collinear vectors $\vec{p}$ and $\vec{q}$.

Note that (i) can be obtained form (ii) by taking dot with $\vec{p} \times \vec{q}$ in (ii)

(vi) Equation of a Plane through three points.

The equation of the plane through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(vii) Equation of a plane if the length of the perpendicular from the origin on the plane is 'p' and d.c's of the perpendiculars as ℓ, m, n is $\ell x + my + nz = p$

(viii) **Angle between two planes:**

Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$; where θ is angle between their normals

Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

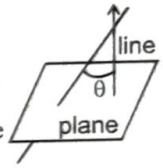
parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

and **Coincident** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(ix) Angle between a plane and a line is the complement of the angle between the normal to the plane and the line. If

Line : $\vec{r} = \vec{a} + \lambda\vec{b}$
Plane : $\vec{r}.\vec{n} = d$

then $\cos(90 - \theta) = \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$.



where θ is the angle between the line and normal to the plane.

(x) Length of the $\perp^{ar}$ from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is $p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

(xi) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

(xii) Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by $\left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \pm\left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$ of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.

(xiii) Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

(xiv) Equation of y-z plane is $x = 0$

Equation of z-x plane is $y = 0$

Equation of x-y plane is $z = 0$

Equation of plane parallel to x-y plane at a distance c is $z = c$ or $z = -c$

Equation of plane parallel to y-z plane at a distance c is $x = c$ or $x = -c$

Equation of plane parallel to z-x plane at a distance c is $y = c$ or $y = -c$

Example 7: Find the equation to the plane through the three points $(0, -1, -1)$, $(4, 5, 1)$ and $(3, 9, 4)$.

Sol. The equation of any plane passing through the point $(0, -1, -1)$ is given by

$a(x-0)+b\{y-(-1)\}+c\{z-(-1)\}=0$

or $ax+b(y+1)+c(z+1)=0$....(1)

If the plane (1) passes through the point $(4, 5, 1)$, we have

$4a+6b+2c=0$ ....(2)

If the plane (1) passes through the point $(3, 9, 4)$, we have

$3a+10b+5c=0$ ....(3)

Now solving the equations (2) and (3), we have

$$\frac{a}{30-20}=\frac{b}{6-20}=\frac{c}{40-18}=\lambda \text{ (say)}.$$

$\therefore$ $a=10\lambda, b=-14\lambda, c=22\lambda$.

Putting these value of a, b, c in (1), the equation of the required plane is given by

$\lambda[10x-14(y+1)+22(z+1)]=0$

or $10x-14(y+1)+22(z+1)=0$ or $5x-7y+11z+4=0$.

Example 8: Find the equation of the plane through $(1, 0, -2)$ and perpendicular to each of the planes

$2x+y-z-2=0$ and $x-y-z-3=0$.

Sol. The equation of any plane through the point $(1, 0, -2)$ is

$a(x-1)+b(y-0)+c(z+2)=0$. ...(1)

If the plane (1) is perpendicular to the planes $2x+y-z-2=0$ and $x-y-z-3=0$, we have

$a(2)+b(1)+c(-1)=0$ i.e., $2a+b-c=0$, ...(2)

and $a(1)+b(-1)+c(-1)=0$ i.e., $a-b-c=0$. ...(3)

Adding the equation (2) and (3), we have $c=\frac{3}{2}a$.

Subtracting (3) from (2), we have $b=-\frac{1}{2}a$.

Putting the values of b and c in (1), the equation of the required plane is given by

$$a(x-1)-\frac{1}{2}ay+\frac{3}{2}a(z+2)=0$$

or $2x-2-y+3z+6=0$ or $2x-y+3z+4=0$.

Example 9: Find the equation of the plane passing through the line of intersection of the planes $2x-7y+4z=3$, $3x-5y+4z+11=0$, and the point $(-2, 1, 3)$

Sol. The equation of any plane through the line of intersection of the given plane is

$(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0$. ...(1)

If the plane (1) passes through the point $(-2, 1, 3)$, then substituting the co-ordinates of this point in the equation (1), we have

$\{2(-2)-7(1)+4(3)-3\}+\lambda\{3(-2)-5(1)+4(3)+11\}=0$

or $(-2)+\lambda(12)=0$ or $\lambda=1/6$.

Putting this value of λ in (1), the equation of the required plane is

$(2x-7y+4z-3)+(1/6)(3x-5y+4z+11)=0$

or $15x-47y+28z=7$.

Example 10: Find the image of the point P $(3, 5, 7)$ in the plane $2x+y+z=0$.

Sol. Given plane is $2x+y+z=0$.... (1)

$P\equiv(3, 5, 7)$

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R then PQ $\perp$ plane (1)

Let $R\equiv(2r+3, r+5, r+7)$

Since R lies on plane (1)

$\therefore$ $2(2r+3)+r+5+r+7=0$

or, $6r+18=0$

$\therefore$ $r=-3$

$\therefore$ $R\equiv(-3, 2, 4)$

Let $Q\equiv(\alpha, \beta, \gamma)$

Since R is the middle point of PQ

$$\therefore -3=\frac{\alpha+3}{2} \Rightarrow \alpha=-9$$

$$2=\frac{\beta+5}{2} \Rightarrow \beta=-1$$

$$4=\frac{\gamma+7}{2} \Rightarrow \gamma=1$$

$\therefore$ $Q=(-9, -1, 1)$.

Example 11: Find the distance between the planes $2x-y+2z=4$ and $6x-3y+6z=2$.

Sol. Given planes are $2x-y+2z-4=0$ .....(1)

and $6x-3y+6z-2=0$ .....(2)

We find that $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Hence planes (1) and (2) are parallel.

Plane (2) may be written as $2x-y+2z-\frac{2}{3}=0$ ... (3)

$\therefore$ Required distance between the planes

$$=\frac{\left|4-\frac{2}{3}\right|}{\sqrt{2^2+(-1)^2+2^2}}=\frac{10}{3.3}=\frac{10}{9}$$

Example 12: Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Sol. Given planes are $x + y + z - 6 = 0$ .....(1)

and $2x + 3y + 4z + 5 = 0$ .....(2)

Given point is P (1, 1, 1).

Equation of any plane through the line of intersection of planes (1) and (2) is

$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ .....(3)

If plane (3) passes through point P, then

$1 + 1 + 1 - 6 + k(2 + 3 + 4 + 5) = 0$

or, $k = \frac{3}{14}$

From (3) required plane is $20x + 23y + 26z - 69 = 0$

Example 13: Find the planes bisecting the angles between planes $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Sol. Given planes are $-2x - y - 2z + 9 = 0$ .....(1)

and $3x - 4y + 12z + 13 = 0$ .....(2)

Equations of bisecting planes are

$$\frac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm \frac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$$

or, $13[-2x - y - 2z + 9] = \pm 3(3x - 4y + 12z + 13)$

or, $35x + y + 62z = 78$, ...(3)

[Taking +ve sign]

and $17x + 25y - 10z = 156$ .....(4)

[Taking – ve sign]

Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)(3) + (-1)(-4) + (-2)(12)$

$= -6 + 4 - 24 = -26 < 0$

$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$

$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example 14: If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Sol. Given planes are $x - cy - bz = 0$ .....(1)

$cx - y + az = 0$ .....(2)

$bx + ay - z = 0$ .....(3)

Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as

$x - cy - bz + \lambda(cx - y + az) = 0$

or, $x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0$ .....(4)

If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$\therefore$ $\frac{1 + c\lambda}{b} = \frac{-(c + \lambda)}{a} = \frac{-b + a\lambda}{-1}$

(i) (ii) (iii)

From (i) and (ii), $a + ac\lambda = -bc - b\lambda$

or, $\lambda = -\frac{(a + bc)}{(ac + b)}$ .....(5)

From (ii) and (iii),

$c + \lambda = -ab + a^2\lambda$

or $\lambda = \frac{-(ab + c)}{1 - a^2}$ .....(6)

From (5) and (6), we have $\frac{-(a + bc)}{ac + b} = \frac{-(ab + c)}{(1 - a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$

or, $a^2 + b^2 + c^2 + 2abc = 1$.

STRAIGHT LINE

(i) Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines ℓ, m, n are

$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = \lambda$, or

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$, where *a*, *b*, *c* are d.r. of normal of plane

and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2)

$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda$

(ii) Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

- Procedure to convert unsymmetrical Form of staight line to Symmetrical Form

Let the direction ratios of the line of intersection (AB) of two planes

$a_1x + b_1y + c_1z + d_1 = 0$ .....(i)

and $a_2x + b_2y + c_2z + d_2 = 0$ .....(ii)

are a, b, c.

Direction ratios of normal to plane (i) are a_1, b_1, c_1 and

Direction ratios of normal to plane (ii) are a_2, b_2, c_2

Line AB lies in both the plane (i) and (ii)

Hence $aa_1 + bb_1 + cc_1 = 0$ and $aa_2 + bb_2 + cc_2 = 0$

these two will give the proportional values of a, b, c.

Let the line AB cuts the xy plane at $(x_1, y_1, 0)$

Hence $a_1x_1 + b_1y_1 = d_1$ and $a_2x_1 + b_2y_1 = d_2$

This will give a point on the line AB

$\therefore$ equation of AB is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - 0}{c}$

Note: 3 planes $a_rx + b_ry + c_rz = d_r$, $r = 1, 2, 3$

- Can intersect at a point ≡ system of equation in 3 variables having unique solution.
- Can intersect coaxially ≡ system of equation in 3 variables having infinite solution.
- May not a common point ≡ system of equation in 3 variables having no solution.

(iii) General equation of the plane containing the line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is

$A(x - x_1) + B(y - y_1) + c(z - z_1) = 0$

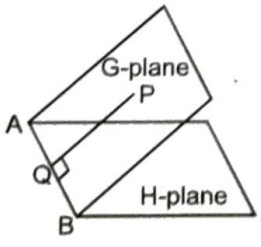
where $A\ell + bm + cn = 0$.

(iv) Coplanarity of two lines: If the lines $\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are coplanar /intersect. If

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

(v) Line of Greatest Slope

AB is the line of intersection of G-plane and H is the h orizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersection of the given plane with any horizontal plane.



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Example 15: Show that the distance of the point of intersection of the line $\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is 13.

Sol. The equation of the given line are $\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{12} = r$ (say). ....(1)

The co-ordinates of any point on the line (1) are $(3r + 2, 4r - 1, 12r + 2)$. If this point lies on the plane $x - y + z = 5$, we have $3r + 2 - (4r - 1) + 12r + 2 = 5$, or $11r = 0$, or $r = 0$.

Putting this value of r, the co-ordinates of the point of intersection of the line (1) and the given plane are (2, −1, 2).

$\therefore$ The required distance = distance between the points (2, −1, 2) and (−1, −5, −10)

$= \sqrt{(2 + 1)^2 + (-1 + 5)^2 + (2 + 10)^2}$

$= \sqrt{(9 + 16 + 144)} = \sqrt{(169)} = 13$

Example 16: Find the equation of the plane through the line $3x - 4y + 5z = 10, 2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

Sol. The equation of the given line are

$3x - 4y + 5z = 10, 2x + 2y - 3z = 4$ ...(1)

The equation of any plane through the line (1) is

$(3x - 4y + 5z - 10) + \lambda(2x + 2y - 3z - 4) = 0$

or $(3 + 2\lambda)x + (-4 + 2\lambda)y + (5 - 3\lambda)z - 10 - 4\lambda = 0$. ...(2)

The plane (1) will be parallel to the line $x = 2y = 3z$

i.e. $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$ if

$(3 + 2\lambda).6 + (-4 + 2\lambda).3 + (5 - 3\lambda).2 = 0$

or $\lambda(12 + 6 - 6) + 18 - 12 + 10 = 0$ or $\lambda = -\frac{4}{3}$.

Putting this value of λ in (2), the required equation of the plane is given by

$\left(3 - \frac{8}{3}\right)x + \left(-4 - \frac{8}{3}\right)y + (5 + 4)z - 10 + \frac{16}{3} = 0$

or $x - 20y + 27z = 14$.

Example 17: Find the equation of a plane passing through the line $\frac{x - 1}{1} = \frac{y - 2}{1} = \frac{z - 2}{-2}$ and making an angle of 30° with the plane $x + y + z = 5$.

Sol. The equation of the required plane is

$(x - y + 1) + \lambda(2y + z - 6) = 0$

$\Rightarrow x + (2\lambda - 1)y + \lambda z + 1 - 6\lambda = 0$

Since it makes an angle of 30° with $x + y + z = 5$

$\Rightarrow \frac{|1 + (2\lambda - 1) + \lambda|}{\sqrt{3}.\sqrt{1 + \lambda^2 + (2\lambda - 1)^2}} = \frac{\sqrt{3}}{2}$

$\Rightarrow |6\lambda| = 3\sqrt{5\lambda^2 - 4\lambda + 2}$

$\Rightarrow 4\lambda^2 = 5\lambda^2 - 4\lambda + 2$

$\Rightarrow \lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = (2 \pm \sqrt{2})$

$\Rightarrow (x - y + 1) + (2 \pm \sqrt{2})(2y + x - 6) = 0$ are two required planes.

Example 18: Find the image of the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ in the plane $x+2y+z=12$

Sol. Any point on the given line is $2r+1, -r-1, 4r+3$. If this point lies on the planes,

then $2r+1-2r-2+4r+3=12 \Rightarrow r=\frac{5}{2}$.

Hence the point of intersection of the given line and that of the plane is $\left(6,-\frac{7}{2},13\right)$.

Also a point on the line is $(1,-1,3)$.

Let (α,β,γ) be its image in the given plane. In such a case

$\frac{\alpha-1}{1}=\frac{\beta+1}{2}=\frac{\gamma-3}{1}=\lambda$

$\Rightarrow \alpha=\lambda+1, \beta=2\lambda-1, \gamma=\lambda+3$. Now the midpoint of the image and the point $(1,-1,3)$ lies on the plane i.e. $\left(1+\frac{\lambda}{2},\lambda-1,3+\frac{\lambda}{2}\right)$ lies in the plane

$\Rightarrow \lambda=\frac{10}{3}$. Hence the image of $(1,-1,3)$ is $\left(\frac{8}{3},\frac{7}{3},\frac{14}{3}\right)$.

Hence the equation of the required line is

$$\frac{x-6}{\frac{10}{3}}=\frac{y+\frac{7}{2}}{\frac{-35}{6}}=\frac{z-13}{\frac{25}{3}}$$

or $\frac{x-6}{4}=\frac{y+\frac{7}{2}}{-7}=\frac{z-13}{10}$.

- **Foot, length and equation of perpendicular from a point to a line :**

(i) Cartesian form : Let equation of the line be

$\frac{x-a}{\ell}=\frac{y-b}{m}=\frac{z-c}{n}=r$ (say).....(i)

and A (a, b, g) be the point.

Any point on line (i) is $P(\ell r+a, mr+b, nr+c)$ .....(ii)

If it is the foot of the perpendicular from A on the line, then AP is perpendicular to the line. So $\ell(\ell r+a-\alpha)+m(mr+b-\beta)+n(nr+c-\gamma)=0$ i.e. $r=(\alpha-a)\ell+(\beta-b)m+(\gamma-c)n$ since $\ell^2+m^2+n^2=1$. Putting this value of r in (ii), we get the foot of perpendicular from point A on the given line. Since foot of perpendicular P is known, then the length of perpendicular is given by AP $=\sqrt{(\ell r+a-\alpha)^2+(mr+b-\beta)^2+(nr+c-\gamma)^2}$ the equation of perpendicular is given by $=\frac{y-\beta}{mr+b-\beta}=\frac{z-\gamma}{nr+c-\gamma}$

(ii) Vector Form : Equation of a line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ is parallel to $\vec{b}_1\times\vec{b}_2$. So the vector equation of such a line is $\vec{r}=\vec{\alpha}+\lambda(\vec{b}_1\times\vec{b}_2)$.

Position vector $\vec{\beta}$ of the image of a point $\vec{\alpha}$ in a straight line $\vec{r}=\vec{a}+\lambda\vec{b}$ is given by $\vec{\beta}=2\vec{a}-\left[\frac{2(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b}-\vec{\alpha}$. Position vector of the foot of the perpendicular on line is $\vec{f}=\vec{a}-\left[\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b}$. The equation of the perpendicular is $\vec{r}=\vec{\alpha}+\mu\left[(\vec{a}-\vec{\alpha})-\left(\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right)\vec{b}\right]$.

To find image of a point w. r. t a line

Let $L\equiv\frac{x-x_2}{a}=\frac{y-y_2}{b}=\frac{z-z_2}{c}$ is a given line

Let (x', y', z') is the image of the point $P(x_1, y_1, z_1)$ with respect to the line L. Then

(i) $a(x_1-x')+b(y_1-y')+c(z_1-z')=0$

(ii) $\frac{\frac{x_1+x'}{2}-x_2}{a}=\frac{\frac{y_1+y'}{2}-y_2}{b}=\frac{\frac{z_1+z'}{2}-z_2}{c}=\lambda$

from (ii) get the value of x', y', z' in terms of λ as $x'=2a\lambda+2x_2-x_1$,

$y'=2b\lambda+2y_2-y_1$, $z'=2c\lambda+2z_2-z_1$

now put the values of x', y', z' in (i) get λ and resubtitute the value of λ to to get (x' y' z').

TRAIN YOUR BRAIN

Example 19: Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$.

Sol. Given line is $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$ ..... (1)

$P\equiv(2,-3,1)$

Co-ordinates of any point on line (1) may be taken as $Q\equiv(2r-1,3r+3,-r-2)$

Direction ratios of PQ are $2r-3, 3r+6, -r-3$

Direction ratios of AB are 2, 3, – 1

Since PQ $\perp$ AB

$\therefore\ 2(2r-3)+3(3r+6)-1(-r-3)=0$

or, $14r+15=0 \qquad \therefore r=\dfrac{-15}{14}$

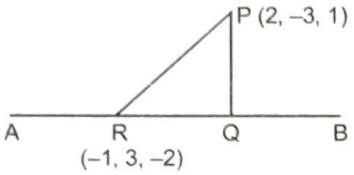
$\therefore\ Q\equiv\left(\dfrac{-22}{7},\dfrac{-3}{14},\dfrac{-13}{14}\right) \qquad \therefore PQ=\sqrt{\dfrac{531}{14}}$ units.

2nd Method :

Given line is $\dfrac{x+1}{2}=\dfrac{y-3}{3}=\dfrac{z+2}{-1}$

$P\equiv(2,-3,1)$

Direction ratios of line (1) are $\dfrac{2}{\sqrt{14}},\dfrac{3}{\sqrt{14}},-\dfrac{1}{\sqrt{14}}$



RQ = length of projection of RP on AB

$=\left|\dfrac{2}{\sqrt{14}}(2+1)+\dfrac{3}{\sqrt{14}}(-3-3)-\dfrac{1}{\sqrt{14}}(1+2)\right|=\dfrac{15}{\sqrt{14}}$

$PR^2=3^2+6^2+3^2=54$

$\therefore\ PQ=\sqrt{PR^2-RQ^2}=\sqrt{54-\dfrac{225}{14}}=\sqrt{\dfrac{531}{14}}$ units.

Example 20: Find the equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x+2y-4z=0$ and $2x-y+2z=0$.

Sol. Equation of the plane through the lines $x+2y-4z=0$ and $2x-y+2z=0$ is

$x+2y-4z+\lambda(2x-y+2z)=0$ ...(1)

If (1, 1, 1) lies on this plane, then $-1+3\lambda=0$

$\Rightarrow \lambda=\dfrac{1}{3}$, so that the plane becomes

$3x+6y-12z+2x-y+2z=0$

$\Rightarrow x+y-2z=0$ ....(2)

Also (1) will be perpendicular to (2)

if $1+2\lambda+2-\lambda-2(-4+2\lambda)=\Rightarrow\lambda=\dfrac{11}{3}$.

$\Rightarrow$ Equation of plane perpendicular to (2) is

$5x-y+2z=0.$ ...(3)

Therefore the equation of line through (1, 1, 1) and perpendicular to the given line is parallel to the normal to the plane (3). Hence the required line is

$\dfrac{x-1}{5}=\dfrac{y-1}{-1}=\dfrac{z-1}{2}$

2nd Method:

Solving the equation of planes $x+2y-4z=0$ and $2x-y+2z=0$, we get $\dfrac{x}{0}=\dfrac{y}{-10}=\dfrac{z}{-5}$ ...(1)

Any point P on the line (1) can be written as $(0,-10\lambda,-5\lambda)$.

Direction ratios of the line joining P and Q(1, 1, 1) is $(1,1,+10\lambda,1+5\lambda)$.

Line PQ is perpendicular to line (1)

$\Rightarrow 0(1)-10(1+10\lambda)-5(1+5\lambda)=0$

$\Rightarrow 0-10-100\lambda-5-25x=0$

or $125\lambda+15=0\Rightarrow=\dfrac{-15}{125}=\dfrac{-3}{25}\Rightarrow P=\left(0,\dfrac{6}{5},\dfrac{3}{5}\right)$

Direction ratios of $PQ=\left(-1,\dfrac{1}{5},\dfrac{-2}{5}\right)$.

Hence equations of lien are $\dfrac{x-1}{5}=\dfrac{y-1}{-1}=\dfrac{z-1}{2}$.

Example 21: Find the shortest distance (S.D.) between the lines $\dfrac{x-3}{3}=\dfrac{y-8}{-1}=\dfrac{z-3}{1},\dfrac{x+3}{-3}=\dfrac{y+7}{2}=\dfrac{z-6}{4}$.

Find also its equations and the points in which it meets the given lines.

Sol. The equations of the given lines are

$(x-3)/3=(y-8)/-1=(z-3)/1=r_1$ (say) ...(1)

and $(x+3)/(-3)=(y+7)/2=(z-6)/4=r_2$ (say) ...(2)

Any point on line (1) is $(3r_1+3,-r_1+8,r_1+3)$, say P. ...(3)

any point on line (2) is $(-3r_2-3,2r_2-7,4r_2+6)$, say Q. ..(4)

The d.r.'s of the line PQ are $(-3r_2-3)-(3r_1+3)$, $(2r_2-7)-(-r_1+8),(4r_2+6)-(r_1+3)$

or $-3r_2-3r_1-6,2r_2+r_1-15,4r_2-r_1+3.$ ...(5)

Let the line PQ be the lines of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have

$3(-3r_2-3r_1-6)-1(2r_2+r_1-15)+1.(4r_2-r_1+3)=0$

and $-3(-3r_2-3r_1-6)+2.(2R_2+r_1-15)+4(4r_2-r_1+3)=0$

or $-7r_2-11r_1=0$ and $11r_2+7r_1=0$.

Solving these equations, we get $r_1=r_2=0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have P(3, 8, 3), Q(–3, –7, 6)

And the d.r.'s of PQ (the line of S.D.) are –6, –15, 3 or –2, –5, 1.

The length of S.D. = the distance between the points P and $Q=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}=3\sqrt{30}$.

Now the line PQ of shortest distance is the line passing through P(3, 8, 3) and having d.r.'s –2, –5, 1 and hence its equations are given by

$\dfrac{x-3}{-2}=\dfrac{y-8}{-5}=\dfrac{z-3}{1}$ or $\dfrac{x-3}{2}=\dfrac{y-8}{5}=\dfrac{z-3}{1}$.

Example 22: Find the equation of the line of intersection of planes $4x+4y-5z=12, 8x+12y-13z=32$ in the symmetric form.

Sol. Given planes are $4x+4y-5z-12=0$ .....(1)

and $8x+12y-13z-32=0$ .....(2)

Let ℓ, m, n be the direction ratios of the line of intersection: then

$4\ell + 4m - 5n = 0$

and $8\ell + 12m - 13n = 0$ ..... (3)

$\therefore \quad \frac{\ell}{-52+60} = \frac{m}{-40+52} = \frac{n}{48-32}$

or, $\frac{\ell}{8} = \frac{m}{12} = \frac{n}{16}$

or, $\frac{\ell}{2} = \frac{m}{3} = \frac{n}{4}$

Hence direction ratios of line of intersection are 2, 3, 4.

Here $4 \neq 0$, therefore line of intersection is not parallel to xy-plane.

Let the line of intersection meet the xy-plane at P $(\alpha, \beta, 0)$.

Then P lies on planes (1) and (2)

$\therefore \quad 4\alpha + 4\beta - 12 = 0$

or, $\alpha + \beta - 3 = 0$ ..... (4)

and $8\alpha + 12\beta - 32 = 0$

or, $2\alpha + 3\beta - 8 = 0$ ..... (5)

Solving (5) and (6), we get

$\frac{\alpha}{-8+9} = \frac{\beta}{-6+8} = \frac{1}{3-2}$

or, $\frac{\alpha}{1} = \frac{\beta}{2} = \frac{1}{1}$

$\therefore \quad \alpha = 1, \beta = 2$

Hence equation of line of intersection in symmetrical form is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-0}{4}$.

Example 23: Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Sol. Equation of any plane passing through (1, 2, 0) may be taken as

$a(x-1) + b(y-2) + c(z-0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point $(-3, 1, 2)$ on line (2) lies in plane (1)

$\therefore \quad a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get

$\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13} = k$ (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6(x-1) + 2(y-2) + 13(z-0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example 24: Find the distance of the point $(1, 0, -3)$ from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Sol. Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

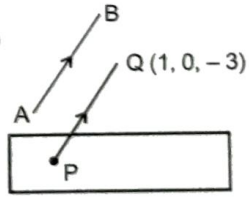
Equation of a line passing through the point Q$(1, 0, -3)$ and parallel to line (2) is

$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r.$ ..... (3)

Co-ordinates of any point on line (3) may be taken as

P$(2r+1, 3r, -6r-3)$

If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),



$\therefore \quad (2r+1) - (3r) - (-6r-3) = 9$

$r = 1$

or, P $\equiv (3, 3, -9)$

Distance between points Q$(1, 0, -3)$ and P$(3, 3, -9)$

$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-3))^2}$

$= \sqrt{4+9+36} = 7.$

Distance between two points and section formula

1. Distance of the point (1, 2, 3) from the co-ordinate axes are

(a) 13, 10, 5 (b) $\sqrt{13}, \sqrt{10}, \sqrt{5}$

(c) $\sqrt{5}, \sqrt{13}, \sqrt{10}$ (d) $\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}}$

2. The projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3 is

(a) $\frac{10}{7}$ (b) $\frac{22}{7}$

(c) $\frac{18}{7}$ (d) $\frac{14}{7}$

3. Points (1, 1, 1), (–2, 4, 1), (–1, 5, 5) and (2, 2, 5) are the vertices of a

(a) Rectangle (b) Square
(c) Parallelogram (d) Trapezium

Direction cosines & ratio

4. If the direction ratios of a line are 1, –3, 2, then the direction cosines of the line are

(a) $\frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{2}{14}$ (b) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(c) $\frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (d) $\frac{-1}{14}, \frac{-2}{\sqrt{14}}, \frac{-3}{14}$

5. If α, β, γ be the direction angles of a vector and $\cos\alpha = \frac{14}{15}, \cos\beta = \frac{1}{3}$ then $\cos\gamma =$

(a) $\pm\frac{2}{15}$ (b) $\frac{1}{5}$

(c) $\pm\frac{1}{15}$ (d) $\pm\frac{1}{5}$

Lines & angle between lines and Concepts of line

6. The number of straight lines that are equally inclined to the three dimensional co-ordinate axes, is

(a) 2 (b) 4
(c) 6 (d) 8

7. The angle between the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is

(a) $\cos^{-1}\frac{1}{5}$ (b) $\cos^{-1}\frac{1}{3}$

(c) $\cos^{-1}\frac{1}{2}$ (d) $\cos^{-1}\frac{1}{4}$

8. The length of the perpendicular from point (1, 2, 3) to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(a) 5 (b) 6
(c) 7 (d) 8

9. The perpendicular distance of the point (2, 4, –1) from the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ is

(a) 3 (b) 5
(c) 7 (d) 9

10. The straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{-2}$ are

(a) Parallel lines (b) Intersecting at 60°
(c) Skew lines (d) Intersecting at right angle

11. If direction ratios of two lines are 5, –12, 13 and –3, 4, 5 then the angle between them is

(a) $\cos^{-1}(1/65)$ (b) $\cos^{-1}(2/65)$
(c) $\cos^{-1}(3/65)$ (d) $\pi/2$

12. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

(a) $\sqrt{30}$ (b) $2\sqrt{30}$

(c) $5\sqrt{30}$ (d) $3\sqrt{30}$

13. The point of intersection of the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}, \frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4}$ is

(a) $21, \frac{5}{3}, \frac{10}{3}$ (b) (2, 10, 4)

(c) (–3, 3, 6) (d) (5, 7, –2)

14. The direction ratios of the line $x - y + z - 5 = 0 = x - 3y - 6$ are

(a) 3, 1, –2 (b) 2, –4, 1

(c) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (d) $\frac{2}{\sqrt{14}}, \frac{-4}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

15. The line joining the points (3, 5, –7) and (–2, 1, 8) meets the yz-plane at point

(a) $\left(0, \frac{13}{5}, 2\right)$ (b) $\left(2, 0, \frac{13}{5}\right)$

(c) $\left(0, 2, \frac{13}{5}\right)$ (d) (2, 2, 0)

Plane & its concepts

16. The direction cosines of the normal to the plane $x + 2y - 3z + 4 = 0$ are

(a) $-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (b) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(c) $-\frac{1}{14}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (d) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}$

17. The equation of the plane which is parallel to y-axis and cuts off intercepts of length 2 and 3 from x-axis and z-axis is

(a) $3x + 2z = 1$ (b) $3x + 2z = 6$

(c) $2x + 3z = 6$ (d) $3x + 2z = 0$

18. If a plane cuts off intercepts –6, 3, 4 from the co-ordinate axes, then the length of the perpendicular from the origin to the plane is

(a) $\frac{1}{\sqrt{61}}$ (b) $\frac{13}{\sqrt{61}}$

(c) $\frac{12}{\sqrt{29}}$ (d) $\frac{5}{\sqrt{41}}$

19. If the product of distances of the point (1, 1, 1) from the origin and the plane $x - y + z + k = 0$ be 5, then $k =$

(a) -2 (b) -3

(c) 4 (d) 7

20. The equation of the plane which is parallel to the plane $x - 2y + 2z = 5$ and whose distance from the point (1, 2, 3) is 1, is

(a) $x - 2y + 2z = 3$ (b) $x - 2y + 2z + 3 = 0$

(c) $x - 2y + 2z = 6$ (d) $x - 2y + 2z + 6 = 0$

21. If the planes $3x - 2y + 2z + 17 = 0$ and $4x + 3y - kz = 25$ are mutually perpendicular, then $k =$

(a) 3 (b) -3

(c) 9 (d) -6

22. The equation of a plane which cuts equal intercetps of unit length on the coordinate axes, is

(a) $x + y + z = 0$ (b) $x + y + z = 1$

(c) $x + y - z = 1$ (d) $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$

23. The equation of the plane through (2, 3, 4) and parallel to the plane $x + 2y + 4z = 5$ is

(a) $x + 2y + 4z = 10$ (b) $x + 2y + 4z = 3$

(c) $x + y + 2z = 2$ (d) $x + 2y + 4z = 24$

24. If the plane $x - 3y + 5z = d$ passes through the point (1,2,4), then the lengths of intercepts cut by it on the axes of x, y, z are respectively

(a) 15, –5, 3 (b) 1, –5, 3

(c) –15, 5, –3 (d) 1, –6, 20

25. The intercepts of the plane $5x - 3y + 6z = 60$ on the co-ordinate axes are

(a) (10, 20, – 10) (b) (10, – 20, 12)

(c) (12, – 20, 10) (d) (12, 20, – 10)

26. The equation of the plane through the point (1, 2, 3) and parallel to the plane $x + 2y + 5z = 0$ is

(a) $(x - 1) + 2(y - 2) + 5(z - 3) = 0$

(b) $x + 2y + 5z = 14$

(c) $x + 2y + 5z = 6$

(d) $x + 2y + 5z = 3$

27. The equation of the plane passing through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and the origin is

(a) $3x + 2y + z + 1 = 0$ (b) $3x + 2y + z = 0$

(c) $2x + 3y + z = 0$ (d) $x + y + z = 0$

28. The distance of the plane $6x - 3y + 2z - 14 = 0$ from the origin is

(a) 2 (b) 1

(c) 14 (d) 8

29. The equation of the plane passing through (1, 1, 1) and (1, –1, –1) and perpendicular to $2x - y + z + 5 = 0$ is

(a) $2x + 5y + z - 8 = 0$ (b) $x + y - z - 1 = 0$

(c) $2x + 5y + z + 4 = 0$ (d) $x - y + z - 1 = 0$

30. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ parallel to x-axis is

(a) $y - 3z + 6 = 0$ (b) $3y - z + 6 = 0$

(c) $y + 3z + 6 = 0$ (d) $3y - 2z + 6 = 0$

31. The point of intersection of the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z+2}{3}$ and the plane is $2x + 3y + z = 0$

(a) (0, 1, –2) (b) (1, 2, 3)

(c) (–1, 9, –25) (d) $\left(\frac{-1}{11}, \frac{9}{11}, \frac{-25}{11}\right)$

32. The equation of the plane passing through the points (3,2,2) and (1,0,–1) and parallel to the line $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z-2}{3}$, is

(a) $4x-y-2z+6=0$

(b) $4x-y+2z+6=0$

(c) $4x-y-2z-6=0$

(d) $-3x+2z+5=0$

33. The line $\frac{x+3}{3}=\frac{y-2}{-2}=\frac{z+1}{1}$ and the plane $4x+5y+3z-5=0$ intersect at a point

(a) (3, 1, –2) (b) (3, –2, 1)

(c) (2, –1, 3) (d) (–1, –2, –3)

34. The equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point (0, 7, –7) is

(a) $x+y+z=1$ (b) $x+y+z=2$

(c) $x+y+z=0$ (d) $x+y+z=3$

35. The angle between the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and the plane $3x+2y-3z=4$ is

(a) 45° (b) 0°

(c) $\cos^{-1}\left(\frac{24}{\sqrt{29}\sqrt{22}}\right)$ (d) 90°

Learning Plus

1. If the sum of the squares of the distances of a point from the three coordinate axes be 36, then its distance from the origin is

(a) 6 (b) $3\sqrt{2}$

(c) $2\sqrt{3}$ (d) $6\sqrt{2}$

2. The locus of a point P which moves such that $PA^2-PB^2=2k^2$ where A and B are (3, 4, 5) and (–1, 3, –7) respectively is

(a) $8x+2y+24z-9+2k^2=0$

(b) $8x+2y+24z-2k^2=0$

(c) $8x+2y+24z+9+2k^2=0$

(d) $8x+2y+24z+9-2k^2=0$

3. Which of the following set of points are non- collinear

(a) (1, –1, 1), (–1, 1, 1), (0, 0, 1)

(b) (1, 2, 3), (3, 2, 1), (2, 2, 2)

(c) (–2,4, –3), (4, –3, –2), (–3, –2, 4)

(d) (2, 0, –1), (3, 2, –2), (5, 6, –4)

4. A line makes angles α, β, γ with the coordinate axes. If $\alpha+\beta=90°$, then $\gamma=$

(a) 0 (b) 90°

(c) 180° (d) 45°

5. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

6. A line makes the same angle θ with each of the x and z-axis. If the angle β, which it makes with y-axis is such that $\sin^2\beta=3\sin^2\theta$, then $\cos^2\theta$ equals

(a) 2/3 (b) 1/5

(c) 3/5 (d) 2/5

7. If $\left(\frac{1}{2},\frac{1}{3},n\right)$ are the direction cosines of a line, then the value of n is

(a) $\frac{\sqrt{23}}{6}$ (b) $\frac{23}{6}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

8. Let O be the origin and P be the point at a distance 3 units from origin. If D.r.'s of OP are (1, –2, –2), then co-ordinates of P is given by

(a) 1, –2, –2

(b) 3, –6, –6

(c) 1/3, –2/3, –2/3

(d) 1/9, –2/9, –2/9

9. The coordinates of the points A, B, C, D are (4, α, 2), (5, –3, 2), (β, 1, 1) & (3, 3, –1). Line AB would be perpendicular to line CD when

(a) $\alpha=-1, \beta=-1$ (b) $\alpha=1, \beta=2$

(c) $\alpha=2, \beta=1$ (d) $\alpha=2, \beta=2$

10. Equation of the angle bisector of the angle between the lines

$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$ & $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{-1}$ is:

(a) $\frac{x-1}{2}=\frac{y-2}{2}; z-3=0$

(b) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

(c) $x-1=0; \frac{y-2}{1}=\frac{z-3}{1}$

(d) $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{3}$

11. The straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$ are

(a) Parallel lines
(b) Intersecting at 60°
(c) Skew lines
(d) Intersecting at right angle

12. The non zero value of 'a' for which the lines $2x-y+3z+4=0=ax+y-z+2$ and $x-3y+z=0=x+2y+z+1$ are co-planar is

(a) –2 (b) 4
(c) 6 (d) 0

13. If the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \frac{x-1}{3}=\frac{y-2}{-1}=\frac{z-3}{4}$ and $\frac{x+k}{3}=\frac{y-1}{2}=\frac{z-2}{h}$ are concurrent then

(a) $h=-2, k=-6$ (b) $h=\frac{1}{2}, k=2$

(c) $h=6, k=2$ (d) $h=2, k=\frac{1}{2}$

14. Let the points A(a, b, c) and B(a′, b′, c′) be at distances r and r′ from origin. The line AB passes through origin when

(a) $\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$ (b) $aa'+bb'+cc'=rr'$

(c) $aa'+bb'+cc'=r^2+r'^2$ (d) $\frac{2a'}{a}=\frac{b'}{b}=\frac{2c'}{c}$

15. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x=y+a=z$ and $x+a=2y=2z$. The co-ordinates of each of the points of intersection are given by

(a) (3a, 3a, 3a), (a, a, a) (b) (3a, 2a, 3a), (a, a, a)
(c) (3a, 2a, 3a), (a, a, 2a) (d) (2a, 3a, 3a,), (2a, a, a)

16. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if

(a) k = 0 or –1 (b) k = 1 or –1
(c) k = 0 or –3 (d) k = 3 or –3

17. Angle between the pair of lines $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+1}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

(a) $\cos^{-1}\left(\frac{13}{9\sqrt{38}}\right)$ (b) $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(c) $\cos^{-1}\left(\frac{4}{\sqrt{38}}\right)$ (d) $\cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{19}}\right)$

18. The shortest distance between the z-axis and the line, $x+y+2z-3=0, 2x+3y+4z-4=0$ is

(a) 1 (b) 2
(c) 3 (d) 4

19. The equation of a plane which passes through (2, –3, 1) & is perpendicular to the line joining the points (3, 4, –1) & (2, –1, 5) is given by:

(a) $x+5y-6z+19=0$ (b) $x-5y+6z-19=0$
(c) $x+5y+6z+19=0$ (d) $x-5y-6z-19=0$

20. The locus represented by $xy+yz=0$ is

(a) A pair of perpendicular lines
(b) A pair of parallel lines
(c) A pair of parallel planes
(d) A pair of perpendicular planes

21. The equation of the plane passing through the point (1, –3, –2) and perpendicular to planes $x+2y+2z=5$ and $3x+3y+2z=8$, is

(a) $2x-4y+3z-8=0$ (b) $2x-4y-3z+8=0$
(c) $2x-4y+3z+8=0$ (d) $2x-4y-3z-8=0$

22. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is

(a) $x^2+y^2+z^2-x-2y-3z=0$
(b) $x^2+2y^2+3z^2-x-2y-3z=0$
(c) $x^2+4y^2+9z^2+x+2y+3=0$
(d) $x^2+y^2+z^2+x+2y+3z=0$

23. The distance of the point (–1, –5, –10) from the point of intersection of the line, $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane, $x-y+z=5$, is

(a) 10 (b) 11
(c) 12 (d) 13

24. The distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line, $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is

(*a*) 1 (*b*) 6/7

(*c*) 7/6 (*d*) 2/3

25. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. Locus of the point common to the planes through A, B, C and parallel to coordinate plane, is

(*a*) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ (*b*) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(*c*) $ax+by+cz=1$ (*d*) $ax+by+cz=0$

26. Two systems of rectangular axes have same origin. If a plane cuts them at distances a, b, c and a_1, b_1, c_1 from the origin, then

(*a*) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}$

(*b*) $\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a_1^2}-\frac{1}{b_1^2}+\frac{1}{c_1^2}$

(*c*) $a^2+b^2+c^2=a_1^2+b_1^2+c_1^2$

(*d*) $a^2-b^2+c^2=a_1^2+b_1^2+c_1^2$

27. The angle between the plane $2x-y+z=6$ and a plane perpendicular to the planes $x+y+2z=7$ and $x-y=3$ is

(*a*) $\pi/4$ (*b*) $\pi/3$

(*c*) $\pi/6$ (*d*) $\pi/2$

28. If the angle θ between the line $\frac{x+1}{1}=\frac{y-1}{2}=\frac{z-2}{2}$ and the plane $2x-y+\sqrt{\lambda}z+4=0$ is such that $\sin\theta=\frac{1}{3}$. The value of λ is

(*a*) $-\frac{4}{3}$ (*b*) $\frac{3}{4}$

(*c*) $-\frac{3}{5}$ (*d*) $\frac{5}{3}$

29. The equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$, is

(*a*) $7x-8y+3z-25=0$

(*b*) $7x-8y+3z+25=0$

(*c*) $-7x+8y-3z+5=0$

(*d*) $7x-8y-3z+5=0$

30. The length and foot of the perpendicular from the point (7, 14, 5) to the plane $2x+4y-z=2$, are

(*a*) $\sqrt{21},(1,2,8)$ (*b*) $3\sqrt{21},(3,2,8)$

(*c*) $21\sqrt{3},(1,2,8)$ (*d*) $3\sqrt{21},(1,2,8)$

31. The equation of the plane passing through the points (0, 1, 2) and (–1, 0, 3) and perpendicular to the plane $2x+3y+z=5$ is

(*a*) $3x-4y+18z+32=0$

(*b*) $3x+4y-18z+32=0$

(*c*) $4x+3y-17z+31=0$

(*d*) $4x-3y+z+1=0$

32. A point moves so that the sum of the squares of its distances from the six faces of a cube given by $x=\pm1, y=\pm1, z=\pm1$ is 10 units. The locus of the point is

(*a*) $x^2+y^2+z^2=1$ (*b*) $x^2+y^2+z^2=2$

(*c*) $x+y+z=1$ (*d*) $x+y+z=2$

33. The coplanar points A, B, C, D are (2 – x, 2, 2), (2, 2 – y, 2), (2, 2, 2 – z) and (1, 1, 1) respectively. Then

(*a*) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

(*b*) $x+y+z=1$

(*c*) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=1$

(*d*) $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1$

34. The direction ratios of a normal to the plane through (1, 0, 0), (0, 1, 0), which makes an angle of $\pi/4$ with the plane $x+y=3$ are

(*a*) $(1,\sqrt{2},1)$ (*b*) $(1,1,\sqrt{2})$

(*c*) (1, 1, 2) (*d*) $(\sqrt{2},1,1)$

35. The line, $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ intersects the curve $xy=c^2, z=0$ then c is equal to

(*a*) ±1 (*b*) $\pm\frac{1}{3}$

(*c*) $\pm\sqrt{5}$ (*d*) $\pm\frac{1}{2}$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/NUMERICAL

1. The equations of the planes through the origin which are parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}$ and at a distant $\frac{5}{3}$ from it is/are

(*a*) $2x+2y+z=0$ (*b*) $x+2y+2z=0$
(*c*) $2x-2y+z=0$ (*d*) $x-2y+2z=0$

2. If the edges of a rectangular parallelopiped are 3, 2, 1 then the angle between a pair of diagonals is given by

(*a*) $\cos^{-1}\frac{6}{7}$ (*b*) $\cos^{-1}\frac{3}{7}$

(*c*) $\cos^{-1}\frac{2}{7}$ (*d*) none of these

3. The equation of the line $x+y+z-1=0, 4x+y-2z+2=0$ written in the symmetrical form is

(*a*) $\frac{x+1}{1}=\frac{y-2}{-2}=\frac{z-0}{1}$

(*b*) $\frac{x}{1}=\frac{y}{-2}=\frac{z}{1}$

(*c*) $\frac{x+1/2}{1}=\frac{y-1}{-2}=\frac{z-1/2}{1}$

(*d*) $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z-2}{2}$

4. The acute angle that the vector $2\hat{i}-2\hat{j}+\hat{k}$ makes with the plane contained by the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+2\hat{k}$ is given by

(*a*) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (*b*) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(*c*) $\tan^{-1}\left(\sqrt{2}\right)$ (*d*) $\cot^{-1}\left(\sqrt{2}\right)$

5. The planes $2x-3y-7z=0, 3x-14y-13z=0$ and $8x-31y-33z=0$
(*a*) pass through origin
(*b*) intersect in a common line
(*c*) form a triangular prism
(*d*) none of these

6. The base of the pyramid AOBC is an equilateral triangle OBC with each side equal to $4\sqrt{2}$. 'O' is the origin of reference, AO is perpendicular to the plane of Δ OBC and $|\vec{AO}|=2$. Then the cosine of the angle between the skew straight lines one passing through A and the mid point of OB and the other passing through O and the mid point of BC is :

(*a*) $-\frac{1}{\sqrt{2}}$ (*b*) 0

(*c*) $\frac{1}{\sqrt{6}}$ (*d*) $\frac{1}{\sqrt{2}}$

7. If the length of perpendicular drawn from origin on a plane is 7 units and its direction ratios are $-3, 2, 6$, then that plane is
(*a*) $-3x+2y+6z-7=0$ (*b*) $-3x+2y+6z-49=0$
(*c*) $3x-2y-6z-49=0$ (*d*) $-3x+2y-6z-49=0$

8. Let a perpendicular PQ be drawn from P(5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-2}{8}=\frac{z-6}{-5}$ where Q is foot of perpendicular.

Then
(*a*) Q is (9, 13, –15)
(*b*) PQ = 14
(*c*) the equation of plane containing PQ and the given line is $9x-4y-z-14=0$
(*d*) None of these

9. Let a perpendicular PQ be drawn from P (5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$ where Q is the foot of perpendicular, then
(*a*) Q is (9, 13, – 15)
(*b*) PQ = 14
(*c*) the equation of plane containing PQ and the given line is $9x-4y-z-14=0$
(*d*) none of these

Comprehension – 1 (Q. 10 to 12)

Let L_1 and L_2 be the lines whose equation are $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ respectively. A and B are two points on L_1 and L_2 respectively such that AB is perpendicular both the lines L_1 and L_2.

10. Shortest distance between the lines L_1 and L_2 is

(*a*) $\sqrt{30}$ (*b*) $2\sqrt{30}$

(*c*) $3\sqrt{30}$ (*d*) none of these

11. Co-ordinates of the point A are

(*a*) (1, 8, 2) (*b*) (3, 8, 3)

(*c*) (–3, 8, 3) (*d*) none of these

12. Co-ordinates of the point B are

(*a*) (–3, –7, 6) (*b*) (2, 7, 6)

(*c*) (1, 6, 3) (*d*) none of these

Comprehension – 2 (Q. 13 to 15)

Let two planes $P_1 : 2x - y + z = 2$ and $P_2 : x - 2y - z = 3$ are given.

On the basis of the above information, answer the following questions.

13. The equation of the plane through the intersectioin of P_1 and P_2 and the point (3, 2, 1) is

(*a*) $3x - y + 2z - 9 = 0$ (*b*) $x - 3y + 2z + 1 = 0$

(*c*) $2x - 3y + z - 1 = 0$ (*d*) $4x - 3y + 2z - 8 = 0$

14. Equation of the plane which passes through the point (– 1, 3, 2) and is perpendicular to each of the planes P_1 and P_2 is

(*a*) $x + 3y - 5z + 2 = 0$ (*b*) $x + 3y + 5z - 18 = 0$

(*c*) $x - 3y - 5z + 20 = 0$ (*d*) $x - 3y + 5z = 0$

15. The equation of the acute angle bisector of planes P_1 and P_2 is

(*a*) $x - 3y + 2z + 1 = 0$ (*b*) $3x + y - 5 = 0$

(*c*) $x + 3y - 2z + 1 = 0$ (*d*) $3x + z + 7 = 0$

16. Match the column:

Column – I	**Column – II**
(*a*) The plane XOZ divides the join of (1, –1, 5) and (2, 3, 4) in the ratio $\lambda : 1$ then λ is	(p) 7
(*b*) Algebraic sum of intercepts made by the plane $x + 3y - 4z + 6 = 0$ on the axes is	(q) 0
(*c*) Cosine of the angle between the two planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ is	(r) $\frac{1}{3}$
(*d*) The distance of the point P(3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 8 = 0$	(s) $-\frac{13}{2}$

17.

Column - I	**Column - II**
(*a*) If acute and obtuse angle bisectors of the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ are represented by A and O, then	(P) A: $32x + 13y - 3 = 0$
(*b*) If acute and obtuse angle bisectors of the planes $x - 2y + 2z - 3 = 0$ and $2x - 3y + 6x + 8 = 0$ are represented by A and O, then	(Q) O: $x - 5y - 4z - 45 = 0$
(*c*) The acute and obtuse angle bisectors of the planes $2x + y - 2z + 3 = 0$ and $6x + 2y - 3z - 8 = 0$ are repesented by A and O, then	(R) A: $23x - 13y + 32z + 45 = 0$
	(S) O: $4x - y + 5z - 45 = 0$
	(T) A: $13x - 23y + 32z + 3 = 0$

NUMERICAL VALUE BASED

18. Lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, if k is equals to

19. If the line $\frac{x-3}{2}=\frac{y+5}{k}=\frac{z+1}{2k}$ is parallel to the plane $6x + 8y + 2z - 4 = 0$, then absolute value of k is

20. The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes is

21. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

22. Two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=z$ intersect at a point if k is equal to

23. A line with positive direction cosines passes through the point P(2, 1, 2) and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals $\sqrt{b}$,then find value of b/2

24. The value of k such that $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$ is

25. A variable plane at a distance of the one unit from the origin cuts the coordinates axes at A, B and C. If the centroid D (x,y, z) of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is

26. If the controid of the tetrahedron OABC where A,B,C are the points (a, 2, 3), (1, b, 2) and (2,1,c) be (1, 2, 3) then the point (a,b,c) is at distance $5\sqrt{\lambda}$, from origin, then λ must be equal to

27. The equation of a plane which bisects the line joining (1, 5, 7) and (–3, 1, –1) is $x + y + 2z = \lambda$ then λ must be

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is:

[JEE (Main) 2016]

(a) $3\sqrt{10}$ (b) $10\sqrt{3}$

(c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$

2. If the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$, lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to:

[JEE (Main) 2016]

(a) 26 (b) 18

(c) 5 (d) 2

3. If the image of the point P(1, –2, 3) in the plane $2x + 3y - 4z + 22 = 0$ measured parallel to line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to :

[JEE (Main) 2017]

(a) $6\sqrt{5}$ (b) $3\sqrt{5}$

(c) $2\sqrt{42}$ (d) $\sqrt{42}$

4. The distance of the point (1, 3 – 7) form the plane passing through the point (1, – 1, – 1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is:

[JEE (Main) 2017]

(a) $\frac{10}{\sqrt{74}}$

(b) $\frac{20}{\sqrt{74}}$

(c) $\frac{10}{\sqrt{83}}$

(d) $\frac{5}{\sqrt{83}}$

5. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x + y + z = 7$ is :

[JEE Main - 2018]

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) $\sqrt{\frac{2}{3}}$ (d) $\frac{2}{\sqrt{3}}$

6. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is : [JEE Main - 2018]

(a) $\frac{1}{3\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{4\sqrt{2}}$

7. The equation of line passing through $(-4, 1, 3)$, parallel to the plane $x + 2y - z - 5 = 0$ and intersecting the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z-2}{-1}$ is :

[JEE Main - 2019 (January)]

(a) $\frac{x+4}{2} = \frac{y+3}{1} = \frac{z+1}{4}$

(b) $\frac{x+4}{1} = \frac{y-3}{1} = \frac{z-1}{3}$

(c) $\frac{x+4}{3} = \frac{y-3}{-1} = \frac{z-1}{1}$

(d) $\frac{x+4}{-1} = \frac{y-3}{1} = \frac{z-1}{1}$

8. The plane through the intersection of the plane $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to y-axis also pass through the point : [JEE Main - 2019 (January)]

(a) $(-3, 0, -1)$ (b) $(-3, 1, 1)$

(c) $(3, 3, -1)$ (d) $(3, 2, 1)$

9. If the line $x = ay + b$, $z = cy + d$ and $x = a'z + b'$, $y = c'y + d'$ are perpendicular, then:

[JEE Main - 2019 (January)]

(a) $cc' + a + a' = 0$

(b) $aa' + c + c' = 0$

(c) $ab' + bc' + 1 = 0$

(d) $bb' + cc' + 1 = 0$

10. Let A be a point on the line $\hat{r} = (1 - 3\mu)\hat{i} + (\mu - 1)\hat{j} + (2 + 5\mu)\hat{k}$ and B (3, 2, 6) be a point in the space. Then the value of μ which the vector $\overrightarrow{AB}$ is parallel to the plane $x - 4y + 3z = 1$ is:

[JEE Main - 2019 (January)]

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{4}$

11. On which of the following lines lies the point of intersection of the line, $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane, $x + y + z = 2$? [JEE Main - 2019 (January)]

(a) $\frac{x+3}{3} = \frac{4-y}{3} = \frac{z+1}{-2}$

(b) $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-5}{-1}$

(c) $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+4}{-5}$

(d) $\frac{x-2}{2} = \frac{y-3}{2} = \frac{z+3}{3}$

12. Two lines $\frac{x-3}{1} = \frac{y+1}{3} = \frac{z-6}{-1}$ and $\frac{x+5}{7} = \frac{y-2}{-6} = \frac{z-3}{4}$ intersect at the point R. The refelection of R in the xy – plane has coordinats : [JEE Main - 2019 (January)]

(a) $(2, -4, -7)$ (b) $(2, 4, 7)$

(c) $(2, -4, 7)$ (d) $(-2, 4, 7)$

13. If the point $(2, \alpha, \beta)$ lies on the plane which passes through the points $(3, 4, 2)$ and $(7, 0, 6)$ and is perpendicular to he plane $2x - 5y = 15$, then $2\alpha - 3\beta$ is equal to :

[JEE Main - 2019 (January)]

(a) 12 (b) 7

(c) 5 (d) 17

14. The plane containing the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-1}{3}$ and also containing its projection on the plane $2x + 3y - z = 5$, contains which one of the following points?

[JEE Main - 2019 (January)]

(a) $(2, 2, 0)$ (b) $(-2, 2, 2)$

(c) $(0, -2, 2)$ (d) $(2, 0, -2)$

15. The direction ratios of normal to the plane through the points $(0, -1, 0)$ and $(0, 0, 1)$ and making an angle $\frac{\pi}{4}$ with the plane $y - z + 5 = 0$:-[JEE Main - 2019 (January)]

(a) $2, -1, 1$ (b) $1, \sqrt{2}, -\sqrt{2}$

(c) $\sqrt{2}, 1, -1$ (d) $2\sqrt{3}, 1, -1$

16. The perpendicular distance from the origin to the plane containing the two lines, $\frac{x+2}{3} = \frac{y-2}{5} = \frac{z+5}{7}$ and $\frac{x-1}{1} = \frac{y-4}{4} = \frac{z+4}{7}$, is :

[JEE Main - 2019 (January)]

(1) $11\sqrt{6}$ (b) $\frac{11}{\sqrt{6}}$

(c) 11 (d) $6\sqrt{11}$

17. If an angle between the line, $\frac{x+1}{2}=\frac{y-2}{1}=\frac{z-3}{-2}$ and the plane, $x-2y-kz=3$ is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$, then a value of k is: [JEE Main - 2019 (January)]

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{3}{5}}$

(c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

18. Let S be the set of all real values of λ such that a plane passing through the points $(-\lambda^2,1,1)$, $(1-\lambda^2,1)$ and $(1,1,-\lambda^2)$ also passes through the point $(-1,-1,1)$. Then S is equal to [JEE Main - 2019 (January)]

(a) $\{\sqrt{3}\}$ (b) $\{\sqrt{3},-\sqrt{3}\}$

(c) $\{1,-1\}$ (d) $\{3,-3\}$

19. The equation of a plane containing the line of intersection of the planes $2x-y-4=0$ and $y+2z-4=0$ and passing through the point (1, 1, 0) is : [JEE Main - 2019(April)]

(a) $x+3y+z=4$ (b) $x-y-z=0$

(c) $x-3y-2z=-2$ (d) $2x-z=2$

20. The vector equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$ is: [JEE Main - 2019(April)]

(a) $\vec{r}\times(\hat{i}+\hat{k})+2=0$ (b) $\vec{r}.(\hat{i}-\hat{k})-2=0$

(c) $\vec{r}.(\hat{i}-\hat{k})+2=0$ (d) $\vec{r}\times(\hat{i}-\hat{k})+2=0$

21. If a point R(4, y, z) lies on the line segment joining the points P(2, –3, 4) and Q(8, 0, 10), then the distance of R from the origin is: [JEE Main - 2019(April)]

(a) $2\sqrt{14}$ (b) 6

(c) $\sqrt{53}$ (d) $2\sqrt{21}$

22. A plane passing through the points (0, –1, 0) and (0, 0, 1) and making an angle $\frac{\pi}{4}$ with the plane $y-z+5=0$, also passes through the point. [JEE Main - 2019(April)]

(a) $(-\sqrt{2},1,-4)$ (b) $(\sqrt{2},1,4)$

(c) $(\sqrt{2},-1,4)$ (d) $(-\sqrt{2},-1,-4)$

23. If the line, $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-2}{3}$ meets the plane, $x+2y+3z=15$ at a point P, then the distance of P from the origin is [JEE Main - 2019(April)]

(a) $\frac{9}{2}$ (b) $2\sqrt{5}$

(c) $\frac{\sqrt{5}}{2}$ (d) $\frac{7}{2}$

24. The vertices B and C of a ΔABC lie on the line, $\frac{x+2}{3}=\frac{y-1}{0}=\frac{z}{4}$ such that BC = 5 units. Then the area (in sq. units) of this triangle, given that the point A(1, –1, 2), is :- [JEE Main - 2019(April)]

(a) $2\sqrt{34}$ (b) $\sqrt{34}$

(c) 6 (d) $5\sqrt{17}$

25. Let P be the plane, while contains the line of intersection of the planes, $x+y+z-6=0$ and $2x+3y+z+5=0$ and it is perpendicualr to the xy-plane. Then the distance of the point (0, 0, 256) from P is equal to :- [JEE Main - 2019(April)]

(a) $63\sqrt{5}$ (b) $205\sqrt{5}$

(c) $17/\sqrt{5}$ (d) $11/\sqrt{5}$

26. A perpendicular is drawn from a point on the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{1}$ to the plane $x+y+z=3$ such that the foot of the perpendicular Q also lies on the plane $x-y+z=3$. Then the co-ordinates of Q are : [JEE Main - 2019(April)]

(a) (2, 0, 1) (b) (4, 0, –1)

(c) (–1, 0, 4) (d) (1, 0, 2)

27. If the plane $2x-y+2z+3=0$ has the distance $\frac{1}{3}$ and $\frac{2}{3}$ units from the planes $4x-2y+4z+\lambda=0$ and $2x-y+2z+\mu=0$, respecctively, then the maximum value of $\lambda+\mu$ is equal to: [JEE Main - 2019(April)]

(a) 15 (b) 5

(c) 13 (d) 9

28. If the length of the perpendicular from the point $(\beta,0,\beta)$ $(\beta\neq 0)$ to the line, $\frac{x}{1}=\frac{y-1}{0}=\frac{z+1}{1}$ is $\sqrt{\frac{3}{2}}$, then β is equal to : [JEE Main - 2019(April)]

(a) –1 (b) 2

(c) –2 (d) 1

29. If Q(0, –1, –3) is the image of the point P in the plane $3x - y + 4z = 2$ and R is the point (3, –1, –2) then the area (in sq. units) of ΔPQR is : [JEE Main - 2019 (April)]

(a) $\frac{\sqrt{65}}{2}$ (b) $\frac{\sqrt{91}}{4}$

(c) $2\sqrt{13}$ (d) $\frac{\sqrt{91}}{2}$

30. If the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the plane $2x + 3y - z + 13 = 0$ at a point P and the plane $3x + y + 4z = 16$ at a point Q, then PQ is equal to : [JEE Main - 2019(April)]

(a) $2\sqrt{14}$ (b) $\sqrt{14}$

(c) $2\sqrt{7}$ (d) 14

31. The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ $\vec{r} = (\hat{i} + \hat{j}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$ is : [JEE Main - 2019(April)]

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{3}$ (d) 3

32. A plane which bisects the angle between the two given planes $2x - y + 2z - 4 = 0$ and $x + 2y + 2z - 2 = 0$, passes through the point : [JEE Main - 2019(April)]

(a) (2, 4, 1) (b) (2, –4, 1)

(c) (1, 4, –1) (d) (1, –4, 1)

33. Let P be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then the image of R in the plane P is [JEE Main-2020 (January)]

(a) (6, 5, –2) (b) (4, 3, 2)

(c) (6, 5, 2) (d) (3, 4, –2)

34. If the foot of the perpendicular drawn from the point (1, 0, 3) on a line passing through (α, 7, 1) is $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, then α is equal to ___. [JEE Main-2020 (January)]

35. If the system of linear equations, [JEE Main-2020 (January)]

$x + y + z = 6$

$x + 2y + 3z = 10$

$3x + 2y + \lambda z = \mu$

has more than two solutions, then $\mu - \lambda^2$ is equal to.

36. The mirror image of the point (1, 2, 3) in a plane is $\left(\frac{-7}{3}, \frac{-4}{3}, \frac{-1}{3}\right)$. Which of the following points lies on this plane? [JEE Main-2020 (January)]

(a) (–1, –1, –1) (b) (1, –1, 1)

(c) (–1, –1, 1) (d) (1, 1, 1)

37. If the distance between the plane, $23x - 10y - 2z + 48 = 0$ and the plane containing the lines $\frac{x+1}{2} = \frac{y-3}{4} = \frac{z+1}{3}$ and $\frac{x+3}{2} = \frac{y+2}{6} = \frac{z-1}{\lambda} (\lambda \in R)$ is equal to $\frac{k}{\sqrt{633}}$, then k is equal to___. [JEE Main-2020 (January)]

38. A plane passing through the point (3, 1, 1) contains two lines whose direction ratios are 1, –2, 2 and 2, 3, –1 respectively. If this plane also passes through the point (α, –3, 5), then α is equal to : [JEE Main-2020 (September)]

(a) 5 (b) 10

(c) –10 (d) –5

39. Let $A = \{X = [x, y, z]^T : PX = 0 \text{ and } x^2 + y^2 + z^2 = 1\}$, where P $= \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & -4 \\ 1 & 9 & -1 \end{bmatrix}$, then the set A : [JEE Main-2020 (September)]

(a) is an empty set.

(b) contains more than two elements.

(c) contains exactly two elements.

(d) is a singleton.

40. The plane passing through the points(1,2,1), (2,1,2) and parallel to the line, $2x = 3y$, $z = 1$ also passes through the point : [JEE Main-2020 (September)]

(a) (0, 6, –2) (b) (–2, 0, 1)

(c) (0, –6, 2) (d) (2, 0 –1)

41. The plane which bisects the line joining, the points (4, –2, 3) and (2, 4, –1) at right angles also passes through the point: [JEE Main-2020 (September)]

(a) (4, 0, 1) (b) (0, –1, 1)

(c) (0, 1, –1) (d) (4, 0, –1)

42. The foot of the perpendicular drawn from the point (4, 2, 3) to the line joining the points (1, –2, 3) and (1, 1, 0) lies on the plane : [JEE Main-2020 (September)]

(a) $x - 2y + z = 1$ (b) $x + 2y - z = 1$

(c) $x - y - 2y = 1$ (d) $2x + y - z = 1$

43. The distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is

[JEE Main-2020 (September)]

(a) 7 (b) $\frac{7}{5}$

(c) 1 (d) $\frac{1}{7}$

44. If the equation of a plane P, passing through the intersection of the planes $x+4y-z+7=0$ and $3x+y+5z=8$ is $ax+by+6z=15$ for some $a, b \in R$, then the distance of the point (3, 2, –1) from the plane P is _____.

[JEE Main-2020 (September)]

45. If for some $\alpha \in R$, this lines

$L_1: \frac{x+1}{2}=\frac{y-2}{-1}=\frac{z-1}{1}$ and

$L_2: \frac{x+2}{\alpha}=\frac{y+1}{5-\alpha}=\frac{z+1}{1}$ are coplanar, then the line L_2 passes through the point :

[JEE Main-2020 (September)]

(a) (10, 2, 2) (b) (2, –10, –2)

(c) (10, –2, –2) (d) (–2, 10, 2)

46. If (a, b, c) is the image of the point (1, 2, –3) in the line, $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$, then a + b + c is equal to :

[JEE Main-2020 (September)]

(a) 2 (b) 1

(c) 3 (d) –1

47. Let L denote the line in the xy-plane with x and y intercepts as 3 and 1 respectively. Then the image of the point (–1, –4) in this line is :

[JEE Main-2020 (September)]

(a) $\left(\frac{29}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{29}{5}, \frac{11}{5}\right)$

(c) $\left(\frac{8}{5}, \frac{29}{5}\right)$ (d) $\left(\frac{11}{5}, \frac{28}{5}\right)$

48. The shortest distance between the lines $\frac{x-1}{0}=\frac{y+1}{-1}=\frac{z}{1}$ and $x+y+z+1=0, 2x-y+z+3=0$ is :

[JEE Main-2020 (September)]

(a) $\frac{1}{\sqrt{2}}$ (b) 1

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}$

49. Suppose the line $\frac{x-2}{\alpha}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies on the plane $x+3y-2z+\beta=0$. Then $(\alpha+\beta)$ is equal to

[JEE Main-2021 (August)]

50. The square of the distance of the point of intersection of the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+1}{6}$ and the plane $2x-y+z=6$ from the point $(-1,-1,2)$ is

[JEE Main-2021 (August)]

51. Let Q be the foot of the perpendicular from the point $P(7,-2,13)$ on the plane containing the lines $\frac{x+1}{6}=\frac{y-1}{7}=\frac{z-3}{8}$ and $\frac{x-1}{3}=\frac{y-2}{5}=\frac{z-3}{7}$. Then $(PQ)^2$, is equal to

[JEE Main-2021 (August)]

52. The distance of the point $P(3,4,4)$ from the point of intersection of the line joining the points. $Q(3,-4,-5)$ and $R(2,-3,1)$ and the plane $2x+y+z=7$, is equal to

[JEE Main-2021 (July)]

53. If the shortest distance between the lines $\vec{r_1}=\alpha\hat{i}+2\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+2\hat{k}), \lambda \in R, \alpha>0$ and $\vec{r_2}=-4\hat{i}-\hat{k}+\mu(3\hat{i}-2\hat{j}-2\hat{k}), \mu \in R$ is 9, then α is equal to

[JEE Main-2021 (July)]

54. Let P be a plane passing through the points $(1,0,1),(1,-2,1)$ and $(0,1,-2)$. Let a vector $\vec{a}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ be such that $\vec{a}$ is parallel to the plane P, perpendicular to $(\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{a}\cdot(\hat{i}+\hat{j}+2\hat{k})=2$, then $(\alpha-\beta+\gamma)^2$ equals

[JEE Main-2021 (July)]

55. Let P be a plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+5}{2}$ and parallel to the line $\frac{x-1}{4}=\frac{y-2}{-3}=\frac{z+5}{7}$. If the point $(1,-1,\alpha)$ lies on the plane P, then the value of $|5\alpha|$ is equal to

[JEE Main-2021 (March)]

56. Let the mirror image of the point $(1,3,a)$ with respect to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})-b=0$ be $(-3,5,2)$. Then, the value of $|a+b|$ is equal to

[*JEE Main-2021 (March)*]

57. The equation of the planes parallel to the plane $x-2y+2z-3=0$ which are at unit distance from the point $(1,2,3)$ is $ax+by+cz+d=0$. If $(b-d)=K(c-a)$, then the positive value of K is

[*JEE Main-2021 (March)*]

58. Let the plane $ax+by+cz+d=0$ bisect the line joining the points $(4,-3,1)$ and $(2,3,-5)$ at the right angles. If \$a, b, c, d\$ are integers, then the minimum value of $\left(a^2+b^2+c^2+d^2\right)$ is

[*JEE Main-2021 (March)*]

59. Let P be an arbitrary point having sum of the squares of the distances from the planes 0 and $x-2y+z=0$, equal to 9. If the locus of the point P is $x^2+y^2+z^2=9$, then the value of $I-n$ is equal

[*JEE Main-2021 (March)*]

60. If the distance of the point $(1,-2,3)$ from the plane $x+2y-3z+10=0$ measured parallel to the line, $\frac{x-1}{3}=\frac{2-y}{m}=\frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of $|m|$ is equal to

[*JEE Main-2021 (March)*]

61. Let $(\lambda,2,1)$ be a point on the plane which passes through the point $(4,-2,2)$. If the plane is perpendicular to the line joining the points $(-2,-21,29)$ and $(-1,-16,23)$, then $\left(\frac{\lambda}{11}\right)^2-\frac{4\lambda}{11}-4$ is equal to

[*JEE Main-2021 (February)*]

JEE-ADVANCED PREVIOUS YEARS

1. The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1, –1, 4) with the plane $5x-4y-z=1$. If S is the foot of the perpendicular drawn from the point T(2, 1,4) to QR, then the length of the line segment PS is

[*IIT JEE-2012*]

(*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\sqrt{2}$

(*c*) 2 (*d*) $2\sqrt{2}$

2. The equation of a plane passing through the line of intersection of the planes x + 2y + 3z = 2 and x – y + z = 3 and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is

[*IIT JEE-2012*]

(*a*) $5x-11y+z=17$

(*b*) $\sqrt{2}x+y=3\sqrt{2}-1$

(*c*) $x+y+z=\sqrt{3}$

(*d*) $x-\sqrt{2}y=1-\sqrt{2}$

3. If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

[*IIT JEE-2012*]

(*a*) $y+2z=-1$ (*b*) $y+z=-1$

(*c*) $y-z=-1$ (*d*) $y-2z=-1$

4. Perpendicular are drawn from points on the line $\frac{x+2}{2}=\frac{y+1}{-1}=\frac{z}{3}$ to the plane x + y + z = 3. The feet of perpendiculars lie on the line

[*JEE Advanced-2013*]

(*a*) $\frac{x}{5}=\frac{y-1}{8}=\frac{z-2}{-13}$ (*b*) $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{-5}$

(*c*) $\frac{x}{4}=\frac{y-1}{3}=\frac{z-2}{-7}$ (*d*) $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

5. A line l passing through the origin is perpendicular to the lines

$l_1:(3+t\hat{i})+(-1+2t)\hat{j}+(4+2t)\hat{k},\ -\infty<t<\infty$

$l_2:(3+2s\hat{i})+(3+2s)\hat{j}+(2+s)\hat{k},\ -\infty<s<\infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)

[*JEE Advanced-2013*]

(*a*) $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ (*b*) (–1, ,–1, 0)

(*c*) (1, 1, 1) (*d*) $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$

6. Two lines $L_1: x=5,\ \frac{y}{3-\alpha}=\frac{z}{-2}$ and $L_2: x=\alpha,\ \frac{y}{-1}=\frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[*JEE Advanced-2013*]

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

7. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2. Match List - I with List- II and select the correct answer using the code given below the lists

[JEE Advanced-2013]

List- I	List- II
P. a =	1. 13
Q. b =	2. –3
R. c =	3. 1
S. d =	4. –2

Codes :

	P	Q	R	S
(a)	3	2	4	1
(b)	1	3	4	2
(c)	3	2	1	4
(d)	2	4	1	3

8. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x$, $z = 1$ and $y = -x$, $z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

[JEE Advanced-2014]

(a) $\sqrt{2}$ (b) 1

(c) -1 (d) $-\sqrt{2}$

9. In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relation is (are) true ? *[JEE Advanced-2015]*

(a) $2\alpha + \beta + 2\gamma + 2 = 0$ (b) $2\alpha - \beta + 2\gamma + 4 = 0$

(c) $2\alpha + \beta - 2\gamma - 10 = 0$ (d) $2\alpha - \beta + 2\gamma - 8 = 0$

10. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. LetM be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

[JEE Advanced-2015]

(a) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (b) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$

(c) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (d) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

11. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

[JEE Advanced-2016]

(a) $x + y - 3z = 0$ (b) $3x + z = 0$

(c) $x - 4y + 7z = 0$ (d) $2x - y = 0$

12. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is *[JEE Advanced-2017]*

(a) $14x + 2y - 15z = 1$ (b) $-14x + 2y + 15z = 3$

(c) $14x - 2y + 15z = 27$ (d) $14x + 2y + 15z = 31$

13. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. The, which of the following statement(s) is(are) TRUE ?

[JEE Advanced-2018]

(a) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1.

(b) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(c) Thee acute angle between P_1 and P_2 is 60°.

(d) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intrsection of P_1 and P_2 then the distance of the point (2, 1, 1) from the plane P_2 is $\frac{2}{\sqrt{3}}$.

14. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then length of PR is .... *[JEE Advanced-2018]*

15. Let L_1 and L_2 denotes the lines

$\vec{r} = \hat{i} + \lambda(-1 + 2\hat{j} + 2\hat{k}), \lambda \in R$ and

$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in R$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe (s) L_3 ? *[JEE Advanced -2019]*

(a) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(b) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(c) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(d) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

16. Three lines

$L_1: \vec{r} = \lambda\hat{i}, \lambda \in R$,

$L_2: \vec{r} = \vec{k} + \mu\hat{j}, \mu \in R$ and

$L_3: \vec{r} = \hat{i} + \hat{j} + \nu\hat{k}, \nu \in R$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ?

[JEE Advanced -2019]

(*a*) $\hat{k} + \hat{j}$ (*b*) $\hat{k}$

(*c*) $\hat{k} + \frac{1}{2}\hat{j}$ (*d*) $\hat{k} - \frac{1}{2}\hat{j}$

17. Let L_1 and L_2 be the following straight line.

$L_1: \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}$ and $L_2: \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$

Suppose the straight line

$L: \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$

lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

[JEE(Advanced) - 2020]

(*a*) $\alpha - \gamma = 3$

(*b*) $l + m = 2$

(*c*) $\alpha - \gamma = 1$

(*d*) $l + m = 0$

18. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point $(3, 2, -1)$ is the mirror image of the point $(1, 0, -1)$ with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/ are TRUE?

[JEE(Advanced) - 2020]

(*a*) $\alpha + \beta = 2$

(*b*) $\delta - \gamma = 3$

(*c*) $\delta + \beta = 4$

(*d*) $\alpha + \beta + \gamma = \delta$

ANSWER KEY

Topicwise Questions

1. (*b*)	2. (*b*)	3. (*b*)	4. (*a*)	5. (*a*)	6. (*b*)	7. (*a*)	8. (*c*)	9. (*c*)	10. (*d*)
11. (*a*)	12. (*d*)	13. (*a*)	14. (*a*)	15. (*a*)	16. (*a,d*)	17. (*b*)	18. (*c*)	19. (*c*)	20. (*d*)
21. (*a*)	22. (*b*)	23. (*d*)	24. (*a*)	25. (*c*)	26. (*a*)	27. (*b*)	28. (*a*)	29. (*b*)	30. (*a*)
31. (*d*)	32. (*d*)	33. (*b*)	34. (*c*)	35. (*b*)					

Learning Plus

1. (*b*)	2. (*c*)	3. (*c*)	4. (*b*)	5. (*d*)	6. (*c*)	7. (*a*)	8. (*a*)	9. (*a*)	10. (*a*)
11. (*d*)	12. (*a*)	13. (*d*)	14. (*a*)	15. (*b*)	16. (*c*)	17. (*b*)	18. (*b*)	19. (*a*)	20. (*d*)
21. (*a*)	22. (*a*)	23. (*d*)	24. (*a*)	25. (*a*)	26. (*a*)	27. (*d*)	28. (*d*)	29. (*b*)	30. (*d*)
31. (*d*)	32. (*b*)	33. (*a*)	34. (*b*)	35. (*c*)					

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/NUMERICAL

1. (*a, b*)	2. (*a, b, c*)	3. (*a,b*)	4. (*b,d*)	5. (*a, b*)	6. (*a, d*)	7. (*b,c*)	8. (*b,c*)	9. (*b, c*)
10. (*c*)	11. (*b*)	12. (*a*)	13. (*b*)	14. (*c*)	15. (*a*)	16. (*a*)→(r),(*b*)→(s),(*c*)→(q),(*d*)→(p)		
17. (*a*) - R;(*b*) Q ; (*c*) - S, P		18. [0000]	19. [0001]	20. [0600]	21. [1.33]	22. [4.5]	23. [1.5]	24. [0007]
25. [0009]	26. [0003]	27. [0008]						

JEE Mains & Advanced Past Years Questions

JEE-MAINS PREVIOUS YEARS

1. (*b*)	2. (*d*)	3. (*c*)	4. (*c*)	5. (*c*)	6. (*a*)	7. (*c*)	8. (*d*)	9. (*b*)	10. (*a*)
11. (*c*)	12. (*a*)	13. (*b*)	14. (*d*)	15. (*b, c*)	16. (*b*)	17. (*a*)	18. (*b*)	19. (*b*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*a*)	24. (*b*)	25. (*d*)	26. (*a*)	27. (*c*)	28. (*a*)	29. (*d*)	30. (*a*)
31. (*a*)	32. (*b*)	33. (*a*)	34. (4)	35. [13]	36. (*b*)	37. [3]	38. (*a*)	39. (*c*)	40. (*b*)
41. (*d*)	42. (*d*)	43. (*c*)	44. (3)	45. (*b*)	46. (*a*)	47. (*d*)	48. (*c*)	49. [7]	50. [61]
51. [96]	52. [7]	53. [6]	54. [81]	55. [38]	56. [1]	57. [4]	58. [28]	59. [0]	60. [2]
61. [8]									

JEE-ADVANCED PREVIOUS YEARS

1. (*a*)	2. (*a*)	3. (*bc*)	4. (*d*)	5. (*bd*)	6. (*ad*)	7. (*a*)	8. (*c*)	9. (*bd*)	10. (*ab*)
11. (*c*)	12. (*d*)	13. (*cd*)	14. (8)	15. (*abd*)	16. (*cd*)	17. (*a,b*)	18. (*a,b,c*)		

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PROBABILITY

INTRODUCTION

The theory of probability originated from the game of chance and gambling. In days of old, gamblers used to gamble in a gambling house with a die to win the amount fixed among themselves. They were always desirous to get the prescribed number on the upper face of a die when it was thrown on a board. Shakuni of Mahabharat was perhaps one of them. People started to study the subject of probability from the middle of seventeenth century. The mathematicians Huygens, Pascal Fermat and Bernoulli contributed a lot to this branch of Mathematics. A.N. Kolmogorow proposed the set theoretic model to the theory of probability.

We study the occurrence of events which are equally likely in probability. Applying the laws of probability we obtain a formula $w_t = \frac{N_t}{N_0} = e^{-\lambda t}$ in radio activity, where w_t is the probability of an atom not to disintegrate in time t, N_t is the number of atoms that have not disintegrated in time t and N_0 is the number of atoms involved and λ is the characteristics constant of the radioactive element. The mathematician Boltzmann in his theorem in classical statistics related entropy to probability. The application of probability theory in quantum mechanics is of greater importance. Here the probability of finding an electron in a state with certain quantum numbers in an element of given volume in the vicinity of a point is determined. This branch of mathematics has applications in applied sciences and statistics also when we plan and organize production. The methods of this theory help in solving the diverse problems of natural science as a result of which its study becomes essential now.

Probability gives us a measure for likelihood that something will happen. However it must be appreciated that probability can never predict the number of times that on occurrence actually happens. But being able to quantify the likely occurrence of an event is important because most of the decisions that affect our daily lives are based on likelihoods and not on absolute certainties.

DEFINITIONS

(a) Experiment : An action or operation resulting in two or more outcome, which are unpredictable in advance.

e.g. (i) tossing of a coin
(ii) throwing a dice
(iii) drawing a card

(b) Sample space : A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point often there will be more than one sample space that can describes outcomes of an experiment, but there is usually only one that will provide the most information. If a sample space has a finite number of points it is called finite sample space and infinite sample space if it has infinite number of points.

e.g. (i) In toss of a coin, $S = \{H, T\}$ where H and T are sample points representing a head and a tail respectively.
(ii) In throw of a die, $S = \{1, 2, 3, 4, 5, 6\}$ where the numbers are the sample points representing the six faces.

TRAIN YOUR BRAIN

Example 1: A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Sol. Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is $S = \{HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6\}$.
Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, T_i means tail on the coin and the number i on the die.

Example 2: Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Sol. In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on till head is obtained. Hence, the desired sample space is $S = \{H, TH, TTH, TTTH, TTTTH, ...\}$

(c) Trial : When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a trial and the outcomes

are called cases. The number of times the experiment is repeated is called the number of trials.

For example:

- One toss of coin is a trial when coin is tossed 5 times.
- One throw of a die is a trial when the die is thrown 4 times.

(d) **Event :** A subset of sample space, i.e. a set of some of possible outcomes of a random experiment is called as event.

e.g. Scoring a six on the throw of a dice

(e) **Simple event :** Each sample point in the sample space is called an elementary event or simple event. For example occurrence of head in throw of a coin is simple event.

(f) **Sure event :** The set containing all sample points is a sure event as in the throw of a die the occurrence of natural number less than 7, is a sure event.

(g) **Null event :** The set which does not contain any sample point.

(h) **Mixed/compound event :** A subset of sample space S containing more than one element is called a mixed event or a compound event.

(i) **Compliment of an event :** Let S be the sample space and E be an event then E^c or $\overline{E}$ represents complement of event E which is a subset containing all sample points in S which are not in E. It refers to the non occurrence of event E.

(i) tossing a coin and the coin landing up head.

(ii) scoring a six on the throw of a die,

(iii) winning the first prize in a raffle.

(iv) being felt a hand of four cards which are all clubs.

In every case it is set of some or all possible outcomes of the experiment. Therefore event (A) is subset of sample space (S). If outcome of an experiment is an element of A we say that event A has occurred.

- An event consisting of a single point of S is called a simple or elementary event.
- If an event has more than one simple point it is called compound event.
- ϕ is called impossible event and S (sample space) is called sure event.

(j) **Mutually Exclusive Events :** Two events are said to be Mutually Exclusive (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other. If A and B are two mutually exclusive events then $P(A \cap B) = 0$.

Consider, for example, choosing numbers at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

If, Event A is the selection of a prime number,

Event B is the selection of an odd number,

Event C is the selection of an even number,

then A and C are mutually exclusive as none of the numbers in this set is both prime and even. But A and B are not mutually exclusive as some numbers are both prime and odd (viz. 3, 5, 7, 11).

(k) **Equally Likely Events :** Events are said to be Equally Likely when each event is as likely to occur as any other event.

(l) **Exhaustive Events :** Events A, B, C.....L are said to be Exhaustive Event if no event outside this set can result as an outcome of an experiment. For example, if A and B are two events defined on a sample space S, then A and B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.

Comparative study of Equally likely, Mutually Exclusive and Exhaustive events :

Experiment	Events	E/L	M/E	Exhaustive
1. Throwing of a die	A : throwing an odd face {1, 3, 5} B :throwing a composite {4, 6}	No	Yes	No
2. A ball is drawn from an urn containing 2W, 3R and 4G balls	E_1 : getting a W ball E_2 : getting a R ball E_3 : getting a G ball	No	Yes	Yes
3. Throwing a pair of dice	A : Throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more {46, 64, 55, 56, 65, 66}	Yes	No	No
4. From a well shuffled pack of cards, a card is drawn	E_1 : getting a heart E_2 : getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5. From a well shuffled pack of a cards, a card is drawn	A = getting a heart B = getting a face card	No	No	No

CLASSICAL DEFINITION OF PROBABILITY

If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by $P(A) = m/n$.

Note : **(i)** $0 \le P(A) \le 1$

(ii) $P(A) + P(\bar{A}) = 1$, Where $\bar{A}$ = Not A.

(iii) If x cases are favourable to A and y cases are favourable to $\bar{A}$ then $P(A) = \frac{x}{(x+y)}$ and $P(\bar{A}) = \frac{y}{(x+y)}$. We say that Odds In Favour Of A are x : y and odds against A are y : x.

TRAIN YOUR BRAIN

Example 3: Words are formed with the letters of the word PEACE. Find the probability that 2E's come together

Sol. Total number of words which can be formed with the letters P, E, A, C, E $= \frac{\lfloor 5}{\lfloor 2} = 60$.

Number of words in which 2 E's come together

$= \lfloor 4 = 24$ $\therefore$ required probability $= \frac{24}{60} = \frac{2}{5}$.

Example 4: A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

Sol. n(s) = the total number of ways of drawing 4 balls out of total 9 balls : ${}^9C_4$

A : Drawing 2 red and 2 green balls $n(A) = {}^5C_2 \times {}^4C_2$

$$\therefore P(A) = \frac{n(A)}{n(s)} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{\frac{5\times4\times4\times3}{2\times2}}{\frac{9\times8\times7\times6}{4\times3\times2}} = \frac{10}{21}$$

Example 5: If the letters of INTERMEDIATE are arranged, then the probability no two E's occur together is

Sol. I → 2, N → 1, T → 2, E → 3, R → 1, M → 1, D → 1, A → 1 (3E's Rest 9)

First arrange rest of letters $= \frac{9!}{2!2!}$

Now 3E's can be placed into place in ${}^{10}C_3$ ways so

favourable cases $= \frac{9!}{2!2!} \times {}^{10}C_3 = 3 \times 10!$

Total cases $= \frac{12!}{2!2!3!}$.

Probability $= \frac{3\times10!\times2!\times2!\times3!}{12!} = \frac{6}{11}$

Example 6: From a group of 10 persons consisting of 5 lawyers, 3 doctors and 2 engineers, four persons are selected at random. The probability that the selection contains at least one of each category is

Sol. $n(S) = {}^{10}C_4 = 210$; $n(E)$

$= {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2$

$= 105$.

$\therefore$ $P(E) = \frac{105}{210} = \frac{1}{2}$.

THE PROBABILITY THAT AN EVENT DOES NOT HAPPEN

If, in a possibility space of n equally likely occurrences, the number of times an event A occurs is r, there are n - r occasions when A does not happen. 'The event A does not happen' is denoted by $\bar{A}$ (and is read as 'not A')

$$\text{Thus } P(A) = \frac{r}{n} \text{ and } P(\bar{A}) = \frac{n-r}{n} = 1 - \frac{r}{n}$$

$$\text{i.e. } P(\bar{A}) = 1 - P(A) \text{ or } P(A) + P(\bar{A}) = 1$$

This relationship is most useful in the 'at least one' type of problem, as is illustrated below.

Example : If four cards are drawn at random from a pack of fifty-two playing cards find the probability that at least one of them is an ace.

If A is a combination of four cards containing at least one ace. (i.e. either one ace, or two aces, or three aces or four aces) then $\bar{A}$ is a combination of four cards containing no aces.

Now $P(\bar{A})$

$$= \frac{\text{Number of combinations of four cards with no aces}}{\text{Total number of combinations of four cards}}$$

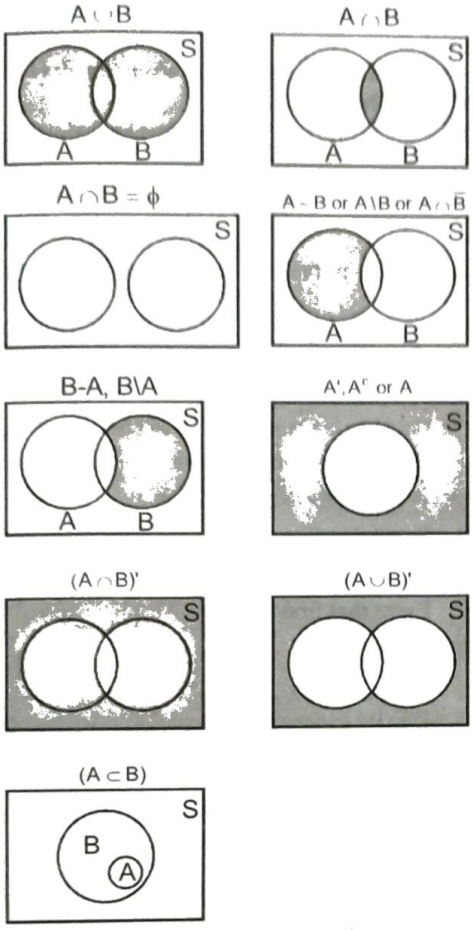
$= {}^{48}C_4 / {}^{52}C_4 = 0.72$

Using $P(A) + P(\bar{A}) = 1$ we have $P(A)$

$= 1 - P(\bar{A}) = 1 - 0.72 = 0.28$

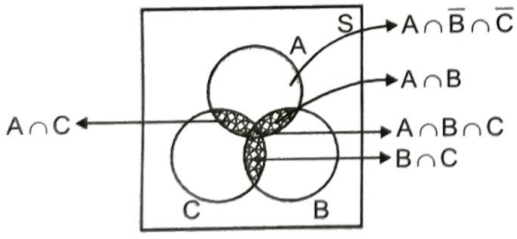
VENN DIAGRAMS

A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and union and intersection are indicated by overlapping circles. Let S is the sample space of an experiment and A, B, C are three events corresponding to it



e.g.: Let us now conduct an experiment of tossing a pair of dice.

Two events defined on the experiment are



A : getting a doublet

$\{11, 22, 33, 44, 55, 66\}$

B : getting total score of 10 or more

$\{64, 46, 55, 56, 65, 66\}$

ADDITION THEOREM

$A \cup B = A + B =$ A or B denotes occurrence of at least A or B.

For 2 events A & B :

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note : (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(This is known as generalized addition theorem)

$P(A + B)$	$P(A \cap \bar{B}) + P(A \cap B) + P(B \cap A)$
$P(A) + P(B \cap A)$	$1 - P(A \cap B)$
$P(A \text{ or } B)$	$1 - P(\bar{A} \cap \bar{B})$
$P(B) + P(A \cap \bar{B})$	
P (occurrence of atleast one A or B)	

(b) $P(A \cup B) = P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(A \cap B) = 1 - P(\bar{A} \cap \bar{B})$

(c) Opposite of "atleast A or B" is neither A nor B

i.e. $\bar{A} + \bar{B} = 1 - (A \text{ or } B) = \bar{A} \cap \bar{B}$

Note that $P(A + B) + P(\bar{A} \cap \bar{B}) = 1$.

(d) If A & B are mutually exclusive then

$P(A \cup B) = P(A) + P(B)$.

(e) For any two events A & B, P (exactly one of A, B occurs)

$= P(A \cap \bar{B}) + P(B \cap \bar{A}) = P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - P(A \cap B) = P(\bar{A} \cup \bar{B}) - P(\bar{A} \cap \bar{B})$

(f) De Morgan's Law : If A & B are two subsets of a universal set U, then

(i) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

(ii) $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ &

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

TRAIN YOUR BRAIN

Example 7: A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

Sol. Let A = event that we get a white ball, B = event that we get a black ball

So, the events are mutually exclusive

$$P(A) = \frac{{}^6C_1}{{}^{15}C_1}, \quad P(B) = \frac{{}^5C_1}{{}^{15}C_1}$$

$$\text{So } P(A + B) = P(A) + P(B) = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$$

Example 8: Three numbers are chosen at random without replacement from 1, 2, 3, ..., 10. The probability that the minimum of the chosen numbers is 4 or their maximum is 8, is

Sol. The probability of 4 being the minimum number $= \frac{{}^6C_2}{{}^{10}C_3}$

(because, after selecting 4 any two can be selected from 5, 6, 7, 8, 9, 10). The probability of 8 being the maximum number $= \frac{{}^7C_2}{{}^{10}C_3}$ The probability of 4 being the minimum number and 8 being the maximum number $= \frac{3}{{}^{10}C_3}$

$\therefore$ the required probability

$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{{}^6C_2}{{}^{10}C_3} + \frac{{}^7C_2}{{}^{10}C_3} - \frac{3}{{}^{10}C_3} = \frac{11}{40}$$

CONDITIONAL PROBABILITY

Let A and B be two events such that $P(A) > 0$. Then $P(B|A)$ denote the conditional probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we led to the definition.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

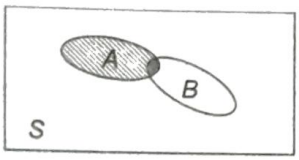
which is called conditional probability of B given A.

MULTIPLICATION THEOREM

$P(A \cap B) = P(A)\ P(B \mid A)$ which is called compound probability or multiplication theorem. It says the probability that both A and B occur is equal to the probability that A occur times the probability that B occurs given that A has occurred.

Note : For any three events A_1, A_2, A_3 we have $P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid (A_1 \cap A_2))$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad \text{where } P(A) \neq 0)$$



The shaded portion shows the favourable region and lined portion shows the reduced sample space.

TRAIN YOUR BRAIN

Example 9: Two dice are thrown. Find the probability that the numbers appeared has a sum of 8 if it is known that the second dice always exhibits 4.

Sol. Let A be the event of occurrence of 4 always on the second die

$= \{(1,4), (2,4), (3,4)(4,4), (5,4), (6,4)\}$; $\therefore\ n(A) = 6$

and B be the event of occurrence of such numbers on both dice whose sum is $8 = \{(4,4)\}$.

Thus, $A \cap B = A \cap \{(4,4)\} = \{(4,4)\}$

$$\therefore\ n(A \cap B) = 1\ \therefore\ P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6}$$

or $$\frac{P(A \cap B)}{P(A)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Example 10: A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag ?

Sol. Let A be the event of drawing first ball white and B be the event of drawing second ball blue.

Here A and B are dependent events.

$$P(A) = \frac{6}{16},\ P(B|A) = \frac{7}{15} \Rightarrow P(AB) = P(A).\ P(B|A)$$

$$= \frac{6}{16} \times \frac{7}{15} = \frac{7}{40}$$

Example 11: A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Sol. E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$$\therefore\ P(E_1) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \text{ and } P(E_2) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$$

$$\Rightarrow\ P(E) = P(E_1) + P(E_2) = 2 \times \frac{4}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} = \frac{6}{35}$$

INDEPENDENT EVENTS

Two events A & B are said to be independent if occurrence or non occurrence of one does not effect the probability of the occurrence or non occurence of other.

(a) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be dependent or Contingent. For two independent events A and B

$P(A \cap B) = P(A).\ P(B)$. Often this is taken as the definition of independent events.

(b) Three events A, B & C are independent if & only if all the following conditions hold ;

$P(A \cap B) = P(A).\ P(B)$; $P(B \cap C) = P(B).\ P(C)$

$P(C \cap A) = P(C).\ P(A)$ & $P(A \cap B \cap C) = P(A).\ P(B).\ P(C)$

i.e., they must be pairwise as well as mutually independent.

Similarly for n events A_1, A_2, A_3, ... A_n to be independent, the number of these conditions is equal to ${}^nC_2 + {}^nC_3 + \ldots\ldots\ldots + {}^nC_n = 2^n - n - 1$.

Note : Independent events are not in general mutually exclusive & vice versa.

Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments.

Similarly , m events A_1 , ... , A_m are called independent if

$P(A_1 \cap .... \cap A_m) = P(A_1) ..... P(A_m)$

as well as for every k different events

$A_{j_1}, A_{j_2}, ..., A_{j_k}$

$P(A_{j_1} \cap A_{j_2} \cap .... \cap A_{j_k}) = P(A_{j_1}) P(A_{j_2}) .... P(A_{j_k})$

where $k = 2, 3, \ldots, m-1$.

Example 12: The probability that an anti aircraft gun can hit an enemy plane at the first, second and third shot are 0.6, 0.7 and 0.1 respectively. The probability that the gun hits the plane is

Sol. Let the events of hitting the enemy plane at the first, second and third shot are respectively A, B and C. Then as given $P(A) = 0.6, P(B) = 0.7, P(C) = 0.1$

Since events A, B, C are independent, so $P(A \cup B \cup C) + P(\overline{A \cap B \cap C}) = 1$

$\Rightarrow P(A \cup B \cup C)$

$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$ Required probability =

$P(A \cup B \cup C) = 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$

$= 1 - (1 - 0.6)(1 - 0.7)(1 - 0.1))$

$= 1 - (0.4)(0.3)(0.9) = 1 - 0.108 = 0.892$

Example 13: *A* and *B* are two independent witnesses in a case. The probability that A will speak the truth is 3/4; and that of *B* is 4/5. In what percentage of cases are they likely to contradict each other in stating the same fact?

Sol. If *E* is the event of their contradicting each other then

$E = (A \cap \overline{B}) \cup (\overline{A} \cap B)$ also $(A \cap \overline{B})$ and $(\overline{A} \cap B)$ are two mutually exclusive events.

$P(E) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$

$\frac{3}{4} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{1}{4} = \frac{7}{20}$

$\therefore$ in 35% of the cases they are likely to contradict each other.

Example 14: The independent probabilities that, *A*, *B* and *C* solve a Mathematical problem are $\frac{1}{3}, \frac{1}{3}$ and $0\frac{1}{4}$ respectively. Find the probability that just two of them only solve the problem.

Sol. Given that $P(A) = \frac{1}{3}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$ and *A*, *B*, *C* are independent events.

The problem gets solved by any two of them solving but the third one fails.

Required probability

$= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$

$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{4}$

$= \frac{3}{36} + \frac{2}{36} + \frac{2}{36} = \frac{7}{36}$

Example 15: If two events A and B are such that $P(\overline{A}) = 0.3$, $P(B) = 0.4$ and $P(A\overline{B}) = 0.5$ then $P(B|(A \cup \overline{B}))$ equals

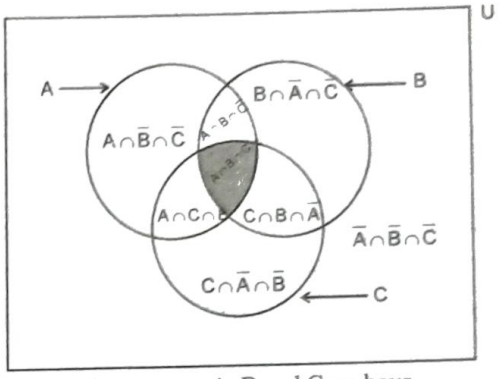
Sol. We have $P(B|A \cup \overline{B}) = \frac{P[B \cap (A \cup \overline{B})]}{P(A \cup \overline{B})}$

$= \frac{P[(B \cap A) \cup (B \cap \overline{B})]}{P(A) + P(\overline{B}) - P(A \cap B)}$

$= \frac{P(AB)}{P(A) + P(\overline{B}) - P(AB)} = \frac{P(A) - P(A\overline{B})}{P(A) + P(\overline{B}) - P(AB)}$

$= \frac{0.7 - 0.5}{0.7 + 0.6 - 0.5} = \frac{0.2}{0.8} = \frac{1}{4}$

PROBABILITY OF THREE EVENTS



For any three events A, B and C we have

(a) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(b) P (at least two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(c) P (exactly two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(d) P (exactly one of A, B, C occurs)
$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Note : If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive. i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$. However the converse of this is not true.

BINOMIAL PROBABILITY DISTRIBUTION

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a trial. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases this probability will not change from one trial to the next (as in tossing a coin or die.) Such trials are then said to be independent

and are often called Bernoulli trials after James Bernoulli who investigated them at the end of the seventeenth century.

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then q = 1 – p is the probability that the event will fail to happen in any single trial (called the probability of failure). The probability that the event will happen exactly x times in n trials (i.e., n successes and n – x failures will occur) is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \quad ...(i)$$

where the random variable X denotes the number of successes in n trials and x = 0, 1, ...n.

e.g.: The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{6!}{2!\,4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete probability function (i) is often called the binomial distribution since for x = 0, 1, 2, ...n, it corresponds to successive terms in the binomial expansion $(q + p)^n$

$$= q^n + \binom{n}{1} q^{n-1}p + \binom{n}{2} q^{n-2}p^2 + ... + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The special case of a binomial distribution with n = 1 is also called the Bernoulli distribution.

TRAIN YOUR BRAIN

Example 16: If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) atleast six heads

(iii) atmost six heads

Sol. The repeated tosses of a coin are Bernoulli trials. Let X denotes the number of heads in an experiment of 10 trials. Clearly, X has the binomial distribution with n = 10 and p $= \frac{1}{2}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, x = 0, 1, 2, ...n

Here $n = 10, p = \frac{1}{2}, q = 1 - p = \frac{1}{2}$.

Therefore $P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now

(i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4! 2^{10}} = \frac{105}{512}$

(ii) P (atleast six heads) = P (X ≥ 6)

= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6! \times 4!}\right) + \left(\frac{10!}{7! \times 3!}\right) + \left(\frac{10!}{8! \times 2!}\right) + \left(\frac{10!}{9! \times 1!}\right) + \left(\frac{10!}{10!}\right)\right] \frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P (at most six heads) = P (X ≤ 6)

= P(X = 0) + P(x = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)

$$\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}$$

$$+ {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10} = \frac{848}{1024} = \frac{53}{64}$$

Example 17: A coin is tossed 7 times. Each time a man calls head. The probability that he wins the toss on more than three occasions is

Sol. The man has to win at least 4 times then required probability

$$= {}^7C_4\left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^3 + {}^7C_5 \cdot \left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6 \cdot \frac{1}{2} + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= ({}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7)\cdot\frac{1}{2^7} = \frac{64}{2^7} = \frac{1}{2}$$

TOTAL PROBABILITY THEOREM

Let an event A occurs with one of the n mutually exclusive & exhaustive events $B_1, B_2, B_3, ... B_n$

then $A = AB_1 + AB_2 + AB_3 + ...+AB_n$

$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n)$

$$= \sum_{i=1}^{n} P(AB_i)$$

$P(A) = P(B_1)\,P(A \mid B_1) + P(B_2)\,P(A \mid B_2) + ...+ P(B_n)\,P(A \mid B_n) = \sum P(B_i)\,P(A \mid B_i)$

TRAIN YOUR BRAIN

Example 18: A purse contains 4 copper and 3 silver coins and another purse contains 6 copper and 2 silver coins. One coin is drawn from any one of these two purses. The probability that it is a copper coin is

Sol. Let A ≡ event of selecting first purse
B ≡ event of selecting second purse
C ≡ event of drawing a copper coin
Then given event has two disjoint cases : AC and BC
$\therefore$ $P(C) = P(AC + BC) = P(AC) + P(BC)$

$= P(A)P(C) + P(B)P(C) = \frac{1}{2}.\frac{4}{7} + \frac{1}{2}.\frac{6}{8} = \frac{37}{56}$

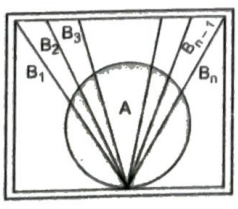
Example 19: Three groups A, B, C are contesting for positions on the Board of Directors of a Company. The probabilities of their winning are 0.5, 0.3, 0.2 respectively. If the group A wins, the probability of introducing a new product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5 respectively. Find the probability that the new product will be introduced.

Sol. Given $P(A) = 0.5$, $P(B) = 0.3$ and $P(C) = 0.2$
$\therefore$ $P(A) + P(B) + P(C) = 1$ then events A, B, C are exhaustive.
If P(E) = Probability of introducing a new product, then as given
$P(E|A) = 0.7$, $P(E|B) = 0.6$ and $P(E|C) = 0.5$.
$\therefore$ $P(E) = P(A).P(E|A) + P(B).P(E|B) + P(C).P(E|C)$
$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5$
$= 0.35 + 0.18 + 0.10 = 0.63$

BAYE'S THEOREM

If an event A can occur only with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots B_n$ & the probabilities $P(A|B_1)$, $P(A|B_2)$ ..... $P(A|B_n)$ are known then $P(B_i|A) = \frac{P(B_i).P(A|B_i)}{\sum_{i=1}^{n} P(B_i).P(A|B_i)}$.

Explanation : A ≡ event what we have ; B_i ≡ event what we want ; $B_2, B_3, \ldots B_n$, are alternative event.
Now, $P(AB_i) = P(A).P(B_i|A) = P(B_i).P(A|B_i)$



$$P(B_i|A) = \frac{P(B_i).P(A|B_i)}{P(A)} = \frac{P(B_i).P(A|B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

$$\text{or } P(B_i|A) = \frac{P(B_i).P(A|B_i)}{\sum P(B_i).P(A|B_i)}$$

TRAIN YOUR BRAIN

Example 20: Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ?

Sol. Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$. Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1) = P$ (a gold coin from box I) $= \frac{2}{2} = 1$

$P(A|E_2) = P$ (a gold coin from box II) $= 0$

$P(A|E_3) = P$ (a gold coin from box III) $= \frac{1}{2}$

Now, the probability that the other coin in the box is of gold
= the probability that gold coin is drawn from the box I = $P(E_1|A)$ By Baye's theorem, $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

Example 21: An unbiased coin is tossed. If the result is head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, ... 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?

Sol. Let E_1 be the event "toss results in a head",
E_2 be the event "toss results in a tail"
A be the event "the noted number is 7 or 8"

We have $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A|E_1) = P(7) + P(8) = \frac{6}{36} + \frac{5}{36} = \frac{11}{36}$

Since

$7 = \{1+6, 2+5, 3+4, 4+3, 5+2, 6+1\}$ and

$8\{2+6, 3+5, 4+4, 5+3, 6+2\}$

$$P(A \mid E_2) = \frac{2}{11}$$

We know that

$$P(A) = P(A|E_1) \times P(E_1) + P(A|E_2) \times P(E_2)$$

$$\frac{11}{36} \times \frac{1}{2} + \frac{2}{11} \times \frac{1}{2} = \frac{193}{792}$$

Example 22: A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4; and the probability that it contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement until all the defective are found. What is the probability that the testing procedure stops at the twelfth testing?

Sol. Suppose A is the event that the testing procedure ends at the twelfth testing.

A_1 = {the event that the lot contains 2 defective}

A_2 = {the event that the lot contains 3 defective}

$P(A_1) = 0.4;\ P(A_2) = 0.6,$

$P(A) = P(A_1)\,P(A|A_1) + P(A_2)\,P(A|A_2)$

$$= 0.4\left\{\frac{{}^{18}C_{10} \cdot {}^{2}C_{1}}{{}^{20}C_{11}} \cdot \frac{1}{9}\right\} + 0.6\left\{\frac{{}^{17}C_{9} \cdot {}^{3}C_{2}}{{}^{20}C_{11}} \cdot \frac{1}{9}\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10} \cdot \frac{|18}{|10|18} \cdot \frac{|11|9}{|20} \cdot 2 + \frac{6}{10} \cdot \frac{|17}{|9|18} \cdot \frac{|11|9}{|20} \cdot 3\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10} \times \frac{11 \times 9}{19 \times 20} \cdot 2 + \frac{6}{10} \times \frac{9 \times 10 \times 11}{18.19.20} \cdot 3\right\}$$

$$= \frac{44}{1900} + \frac{55}{1900} = \frac{99}{1900}$$

PROBABILITY THROUGH STATISTICAL (STOCHATIC) TREE DIAGRAM

These tree diagrams are generally drawn by economist and give a simple approach to solve a problem.

TRAIN YOUR BRAIN

Example 23: A bag initially contains 1 red ball and 2 blue balls. A trial consists of selecting a ball at random noting its colour and replacing it together with an additional ball of the same colour. Given that three trials are made, draw a tree diagram illustrating the various probabilities. Hence or otherwise, find the probability that

(a) atleast one blue ball is drawn

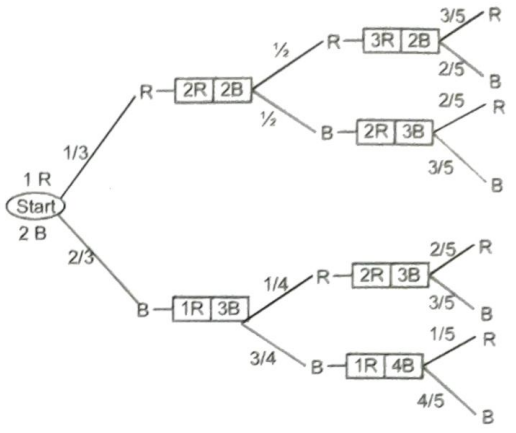
(b) exactly one blue ball is drawn

(c) Given that all three balls drawn are of the same colour find the probability that they are all red.

Sol. $P(A) = 1 - P(RRR) = 1 - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} = 1 - \frac{1}{10} = \frac{9}{10}$

$$P(\text{exactly one Blue}) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5}$$

$$= \frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{3}{15} =$$



$$P(C) = P\left(\frac{RRR}{(RRR \cup BBB)}\right) = \frac{P(RRR)}{P(RRR) + P(BBB)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{4}{10}} = \frac{1}{5}$$

COINCIDENCE TESTIMONY

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B then P (their combined statement is true)

$$= \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}.$$

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B. However if P is the probability of the happening of the event before their statement then P (their combined statement is true)

$$= \frac{p p_1 p_2}{p p_1 p_2 + (1-p)(1-p_1)(1-p_2)}$$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood. If this is not the case and c is the change of their coincidence testimony then the

probability that the statement is true = $p\,p_1\,p_2$

probability that the statement is false

$= (1-p).c\,(1-p_1)\,(1-p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

TRAIN YOUR BRAIN

Example 24: A speaks truth in 75% case and B in 80% cases. What is the probability that they contradict each other in stating the same fact ?

Sol. There are two mutually exclusive cases in which they contradict each other. i.e. $\bar{A}B$ and $A\bar{B}$

Hence required probability

$= P(A\bar{B} + \bar{A}B) = P(A\bar{B}) + P(\bar{A}B)$

$= P(A)P(\bar{B}) + P(\bar{A})P(B) = \frac{3}{4}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{4}{5} = \frac{7}{20}$

Example 25: A die is thrown, a man C gets a prize of Rs. 5 if the die turns up 1 and gets a prize of Rs. 3 if the die turns up 2, else he gets nothing. A man A whose probability of speaking the truth is $\frac{1}{2}$ tells C that the die has turned up 1 and another man B whose probability of speaking the truth is $\frac{2}{3}$ tells C that the die has turned up 2. Find the expectation value of C.

Sol. Let A and B be the events 'A speaks the truth' and 'B speaks the truth' respectively. Then

$P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$.

The experiment consists of three hypothesis

(i) the die turns up 1

(ii) the die turns up 2

(iii) the die turns up 3, 4, 5 or 6

Let these be the events E_1, E_2 and E_3 respectively then

$P(E_1) = P(E_2) = \frac{1}{6}$ and $P(E_3) = \frac{4}{6}$.

Let E be the event that the statements made by A and B agree to the same conclusion.

$\therefore\ P(E/E_1) = P(A).P(\bar{B}) = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6}$

$P(E/E_2) = P(\bar{A}).P(B) = \frac{1}{2}\times\frac{2}{3} = \frac{2}{6}$

$P(E/E_3) = P(\bar{A}).P(\bar{B}) = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6}$

$\therefore\ P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3)P(E/E_3)$

$= \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{2}{6} + \frac{4}{6}\cdot\frac{1}{6} = \frac{7}{36}$

Thus $P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E)} = \frac{1}{7}$

$P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E)} = \frac{2}{7}$

$P(E_3/E) = \frac{P(E_3)P(E/E_3)}{P(E)} = \frac{4}{7}$

$\therefore$ expectation of $C = \frac{1}{7}\times 5 + \frac{2}{7}\times 3 + 0 = \text{Rs. } \frac{11}{7}$

PROBABILITY DISTRIBUTION

(a) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by

$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i$ (Since $\sum p_i = 1$)

(c) Variance of a random variable is given by

$\sigma^2 = \sum(x_i - \mu)^2.p_i \Rightarrow \sigma^2 = \sum p_i x_i^2 - \mu^2$

($\because$ SD $= +\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variate 'X' is given by ; $P(X = r) = {}^nC_r\, p^r q^{n-r}$. The recurrence formula

$\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}\cdot\frac{p}{q}$, is very helpful for quickly computing P(1), P(2), P(3) etc. if P(0) is known.

(e) Mean of BPD = np ; variance of BPD = npq.

(f) If p represents a persons chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM **expectations = pM**

TRAIN YOUR BRAIN

Example 26: A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine

(i) k

(ii) $P(X<3)$

(iii) $P(X>6)$

(iv) $P(0<X<3)$

Hint : Use $\sum P(X) = 1$ to determine k, $P(X<3)$ $= P(0) + P(1) + P(2)$, $P(X>6) = P(7)$ etc.]

Example 27: A pair of dice is thrown 5 times. If getting a doublet is considered as a success, then find the mean and variance of successes.

Sol. In a single throw of a pair of dice, probability of getting a doublet $= \frac{1}{6}$

considering it to be a success, $p = \frac{1}{6}$

$\therefore\ q = 1 - \frac{1}{6} = \frac{5}{6}$

mean $= 5\times\frac{1}{6} = \frac{5}{6}$, variance $= 5\times\frac{1}{6}\cdot\frac{5}{6} = \frac{25}{36}$

Example 28 A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of successes.

Sol. p = probability of getting a total of $9 = \frac{4}{36} = \frac{1}{9}$

$\therefore\ q = 1 - \frac{1}{9} = \frac{8}{9}$ $\quad\therefore$ mean $= np = 4 \times \frac{1}{9} = \frac{4}{9}$

$$\text{variance} = npq = 4 \times \frac{1}{9} \times \frac{8}{9} = \frac{32}{81}$$

GEOMETRICAL PROBABILITY

The following statements are axiomatic :

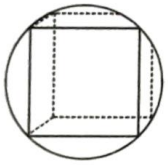
(a) If a point is taken at random on a given straight line AB, the chance that it falls on a particular segment PQ of the line is PQ/AB

(b) If a point is taken at random on the area S which includes an area σ, the chance that the points falls on σ is σ/S.

TRAIN YOUR BRAIN

Example 29: A sphere is circumscribed over a cube. Find the probability that a point lies inside the sphere, lies outside the cube.

Sol. Required probability $= \frac{\text{favorable volume}}{\text{total volume}}$



Clearly if edge length of cube is a radius of sphere will be $\frac{a\sqrt{3}}{2}$

Thus, volume of sphere $= \frac{4}{3}\pi\left(\frac{a\sqrt{3}}{2}\right)^3 = \frac{\pi a^3\sqrt{3}}{2}$

Hence $P = 1 - \frac{1}{\pi\frac{\sqrt{3}}{2}} = 1 - \frac{2}{\pi\sqrt{3}}$

OTHER DEFINITIONS OF PROBABILITY

(a) Axiomatic probability : Axiomatic approach is another way of describing probability of an event, in this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] satisfying the following axioms:

(i) For any event E, $P(E) \geq 0$

(ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, the $P(E \cup F) = P(E) + P(F)$

It follows from (iii) that $P(\phi) = 0$

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots\ldots, \omega_n$ i.e, $S = \{\omega_1, \omega_2, \ldots\ldots\ldots\omega_n\}$

It follows from the axiomatic definition of probability that:

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots\ldots\ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \sum P(\omega_i)$, $\omega_i \in A$

(b) Empirical probability : A method which can be adopted in the example given above is to throw the dart several times (each throw is a trial) and count the number of times you hit the bull's-eye (a success) and the number of times you miss (a failure). Then an empirical value of the probability that you hit the bull's - eye with any one throw is $\frac{\text{number of successes}}{\text{number of successes + number of failures}}$

If the number of throws is small this does not give a particular good estimate but for a large number of throws the result is more reliable.

When the probability of the occurrence of an event A cannot be worked out exactly, an empirical value can be found by adopting the approach described above, that is :

(i) making a large number of trials (i.e. set up an experiment in which the event may, or may not, occur and note the outcome)

(ii) counting the number of times the event does occur, i.e. the number of successes,

(iii) calculating the value of

$$\frac{\text{number of successes}}{\text{number of trials (i.e. successes + failures)}} = \frac{r}{n}$$

The probability of then event A occurring is defined as $P(A) = \lim_{n\to\infty}\left(\frac{r}{n}\right)$

$n \to \infty$ mean that the number of trials is large (but what should be taken as 'large' depends on the problem).

IMPORTANT POINTS

(a) If $A_1 \leq A_2$ then $P(A_1) \leq P(A_2)$

and $P(A_2 - A_1) = P(A_2) - P(A_1)$

(b) If $A = A_1 \cup A_2 \cup \ldots\ldots \cup A_n$ where $A_1, A_2, \ldots\ldots A_n$ are mutually exclusive events then

$P(A) = P(A_1) + P(A_2) + \ldots\ldots P(A_n) = 1$

(c) Let A & B are two events corresponding to sample space S then $P(S|A) = P(A|A) = 1$

(d) Let A and B are two events corresponding to sample space S and F is any other event s.t. $P(F) \neq 0$ then

$$P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F) - (P(A \cap B) \mid F))$$

(e) $P(\overline{A} \mid B) = 1 - P(A \mid B)$

(f) $P(A \cap B) \leq P(A), P(B)$

$$\leq P(A \cup B) \leq P(A) + P(B)$$

TRAIN YOUR BRAIN

Example 30: A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a spade shall win a prize; find their respective chances.

Sol. Let p be the chance of cutting a spade and q be the chance of not cutting a spade from a pack of 52 cards.

Then $p = \frac{^{13}C_1}{^{52}C_1} = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut a spade once and then A cuts a spade at the 4th turn.

Similarly he will cut a spade at the 7th turn when A, B, C fail to cut spade twice, etc.

Hence A's chance of winning the prize

$$= p + q^3p + q^6p + q^9p + \ldots\ldots = \frac{p}{1-q^3} = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^3} = \frac{16}{37}$$

Similarly B's chance

$$= (pq + q^4p + q^7p + \ldots\ldots)$$

$$= q(p + q^3p + q^6p + \ldots\ldots) = \frac{3}{4}.\frac{16}{37} = \frac{12}{37}$$

and C's chance $= \frac{3}{4}$ of B's chance $= \frac{3}{4}.\frac{12}{37} = \frac{9}{37}$

Example 31: In a test an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he copied it, is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

Sol. Let A_1 be the event that the examinee guesses that answer ; A_2 the event that he copies the answer and A_3 the event that he knows the answer. Also let A be the event that he answers correctly. Then as given, we have

$$P(A_1) = \frac{1}{3}, P(A_2) = \frac{1}{6}, P(A_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

[We have assumed here that the events A_1, A_2 and A_3 are mutually exclusive and totally exhaustive.]

Now $P(A \mid A_1) = \frac{1}{4}$, $P(A \mid A_2) = \frac{1}{8}$ (as given)

Again it is reasonable to take the probability of answering correctly given that he knows the answer as 1, that is $P(A \mid A_3) = 1$. We have to find $P(A_3 \mid A)$.

By Baye's theorem, we have

$$P(A_3 \mid A)$$

$$= \frac{P(A_3)P(A \mid A_3)}{P(A_1)P(A \mid A_1) + P(A_2)P(A \mid A_2) + P(A_3)P(A \mid A_3)}$$

$$= \frac{(1/2)}{(1/3)(1/4) + (1/6)(1/8) + 1/2).1} = \frac{24}{29}$$

Example 32: A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as

A = {The first bulb is defective}

B = {The second bulb is non-defective}

C = {The two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent,

(ii) A, B, C are independent.

Sol. We have $P(A) = \frac{50}{100}.1 = \frac{1}{2}$; $P(B) = 1.\frac{50}{100} = \frac{1}{2}$;

$$P(C) = \frac{50}{100}.\frac{50}{100} + \frac{50}{100}.\frac{50}{100} = \frac{1}{2}$$

$A \cap B$ is the event that first bulb is defective and second is non-defective. $\therefore$ $P(A \cap B) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

$A \cap C$ is the event that first bulb is defective and second is non-defective. $\therefore$ $P(A \cap C) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

Similarly $P(B \cap C) = \frac{1}{4}$. Thus we have

$P(A \cap B) = P(A).P(B)$; $P(A \cap C)$

$= P(A).P(C)$; $P(B \cap C) = P(B).P(C)$

$\therefore$ A, B and C are pairwise independent.

There is no element in $A \cap B \cap C$

$\therefore$ $P(A \cap B \cap C) = 0$

$\therefore$ $P(A \cap B \cap C) \neq P(A).P(B).P(C)$

Hence A, B and C are not mutually independent.

Classical Definition of Probability

1. If a dice is thrown twice, then the probability of getting 1 in the first throw only is

(a) $\frac{1}{36}$ (b) $\frac{3}{36}$

(c) $\frac{5}{36}$ (d) $\frac{1}{6}$

2. The probability of getting a total of 5 or 6 in a single throw of 2 dice is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{3}$ (d) $\frac{1}{6}$

3. Two dice are thrown simultaneously. What is the probability of obtaining a multiple of 2 on one of them and a multiple of 3 on the other

(a) $\frac{5}{36}$ (b) $\frac{11}{36}$

(c) $\frac{1}{6}$ (d) $\frac{1}{3}$

4. A coin is tossed 4 times. The probability that at least one head turns up is

(a) $\frac{1}{16}$ (b) $\frac{2}{16}$

(c) $\frac{14}{16}$ (d) $\frac{15}{16}$

5. In a college, 25% of the boys and 10% of the girls offer Mathematics. The girls constitute 60% of the total number of students. If a student is selected at random and is found to be studying Mathematics, the probability that the student is a girl, is

(a) $\frac{1}{6}$ (b) $\frac{3}{8}$

(c) $\frac{5}{8}$ (d) $\frac{5}{6}$

6. If a coin be tossed n times then probability that the head comes odd times is

(a) $\frac{1}{2}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2^{n-1}}$ (d) 2^{n-1}

7. The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact

(a) $\frac{4}{5}$ (b) $\frac{1}{5}$

(c) $\frac{7}{20}$ (d) $\frac{3}{20}$

8. A bag contains 3 red, 7 white and 4 black balls. If three balls are drawn from the bag, then the probability that all of them are of the same colour is

(a) $\frac{6}{71}$ (b) $\frac{7}{81}$

(c) $\frac{10}{91}$ (d) $\frac{7}{91}$

9. The letter of the word 'ASSASSIN' are written down at random in a row. The probability that no two S occur together is

(a) $\frac{1}{35}$ (b) $\frac{1}{14}$

(c) $\frac{4}{13}$ (d) $\frac{2}{35}$

10. A bag contains 4 white and 3 red balls. Two draws of one ball each are made without replacement. Then the probability that both the balls are red is

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$

(c) $\frac{3}{7}$ (d) $\frac{4}{7}$

11. If a committee of 3 is to be chosen from a group of 38 people of which you are a member. What is the probability that you will be on the committee

(a) $\binom{38}{3}$ (b) $\binom{37}{2}$

(c) $\binom{37}{2}\Big/\binom{38}{3}$ (d) $\frac{666}{8436}$

Addition and multiplication theorem

12. If A and B are two events such that $P(A)=0.4, P(A+B)=0.7$ and $P(AB)=0.2$, then $P(B)=$

(a) 0.1 (b) 0.3

(c) 0.5 (d) 0.4

13. A card is drawn at random from a pack of cards. The probability of this card being a red or a queen is

(a) $\frac{1}{13}$ (b) $\frac{1}{26}$

(c) $\frac{1}{2}$ (d) $\frac{7}{13}$

14. If A and B are two events such that $P(A\cup B)+P(A\cap B)=\frac{7}{8}$ and $P(A)=2P(B)$, then $P(A)=$

(a) $\frac{7}{12}$ (b) $\frac{7}{24}$

(c) $\frac{5}{12}$ (d) $\frac{17}{24}$

15. The probability that at least one of A and B occurs is 0.9. If A and B occur simultaneously with probability 0.3, then $P(A')+P(B')=$

(a) 0.9 (b) 0.2

(c) 0.8 (d) 1.2

16. Let A and B are two events and $P(A')=0.3$ $P(B)=0.4, P(A\cap B')=0.5$, then $P(A\cup B')$ is

(a) 0.5 (b) 0.8

(c) 1 (d) 0.1

Conditional probability

17. If A and B are two events such that $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$ and $P(A\cap B)=\frac{1}{5}$, then $P\left(\frac{\bar{B}}{\bar{A}}\right)=$

(a) $\frac{37}{40}$ (b) $\frac{37}{45}$

(c) $\frac{23}{40}$ (d) $\frac{26}{40}$

18. If $4P(A)=6P(B)=10P(A\cap B)=1$, then $P\left(\frac{B}{A}\right)=$

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$

(c) $\frac{7}{10}$ (d) $\frac{19}{60}$

19. A pair has two children. If one of them is boy, then the probability that other is also a boy, is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{3}$ (d) $\frac{1}{6}$

20. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

(a) $\frac{5}{17}$ (b) $\frac{12}{17}$

(c) $\frac{17}{30}$ (d) $\frac{3}{5}$

21. In a certain town, 40% of the people have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. If a person selected at random from the town, has brown hair, the probability that he also has brown eyes, is

(a) $\frac{1}{5}$ (b) $\frac{3}{8}$

(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Independent events and their Probability

22. A and B are two independent events. The probability that both A and B occur is $\frac{1}{6}$ and the probability that neither of them occurs is $\frac{1}{3}$. Then the probability of the two events are respectively

(a) $\frac{1}{2}$ and $\frac{1}{3}$ (b) $\frac{1}{5}$ and $\frac{1}{6}$

(c) $\frac{1}{2}$ and $\frac{1}{6}$ (d) $\frac{2}{3}$ and $\frac{1}{4}$

23. The probability of solving a question by three students are $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ respectively. Probability of question is being solved will be

(a) $\frac{33}{48}$ (b) $\frac{35}{48}$

(c) $\frac{31}{48}$ (d) $\frac{37}{48}$

24. Three coins are tossed. If one of them shows tail, then the probability that all three coins show tail, is

(*a*) $\frac{1}{7}$ (*b*) $\frac{1}{8}$

(*c*) $\frac{2}{7}$ (*d*) $\frac{1}{6}$

25. For two events A and B, if $P(A)=P\left(\frac{A}{B}\right)=\frac{1}{4}$ and $P\left(\frac{B}{A}\right)=\frac{1}{2}$, then

(*a*) A and B are independent

(*b*) $P\left(\frac{A'}{B}\right)=\frac{3}{4}$

(*c*) $P\left(\frac{B'}{A'}\right)=\frac{1}{2}$

(*d*) All of these

Binomial Distributions of Probability

26. In a box containing 100 eggs, 10 eggs are rotten. The probability that out of a sample of 5 eggs none is rotten if the ampling is with replacement is

(*a*) $\left(\frac{1}{10}\right)^5$ (*b*) $\left(\frac{1}{5}\right)^5$

(*c*) $\left(\frac{9}{5}\right)^4$ (*d*) $\left(\frac{9}{10}\right)^5$

27. The items produced by a firm are supposed to contain 5% defective items. The probability that a sample of 8 items will contain less than 2 defective items, is

(*a*) $\frac{27}{20}\left(\frac{19}{20}\right)^7$ (*b*) $\frac{533}{400}\left(\frac{19}{20}\right)^6$

(*c*) $\frac{153}{20}\left(\frac{1}{20}\right)^7$ (*d*) $\frac{35}{16}\left(\frac{1}{20}\right)^6$

28. The probability that a man can hit a target is $\frac{3}{4}$. He tries 5 times. The probability that he will hit the target at least three times is

(*a*) $\frac{291}{364}$ (*b*) $\frac{371}{464}$

(*c*) $\frac{471}{502}$ (*d*) $\frac{459}{512}$

29. The probability that a student is not a swimmer is 1/5. What is the probability that out of 5 students, 4 are swimmers

(*a*) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ (*b*) $\left(\frac{4}{5}\right)^4\frac{1}{5}$

(*c*) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4\times{}^5C_4$ (*d*) ${}^5C_1\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^4$

30. An experiment succeeds twice as often as it fails. Find the probability that in 4 trials there will be at least three success

(*a*) $\frac{4}{27}$ (*b*) $\frac{8}{27}$

(*c*) $\frac{16}{27}$ (*d*) $\frac{24}{27}$

Expectations

31. A die is tossed thrice. If getting a four is considered a success, then the mean and variance of the probability distribution of the number of successes are

(*a*) $\frac{1}{2},\frac{1}{12}$ (*b*) $\frac{1}{6},\frac{5}{12}$

(*c*) $\frac{5}{6},\frac{1}{2}$ (*d*) $\frac{1}{2},\frac{5}{12}$

32. A die is tossed twice. Getting a number greater than 4 is considered a success. Then the variance of the probability distribution of the number of successes is

(*a*) $\frac{2}{9}$ (*b*) $\frac{4}{9}$

(*c*) $\frac{1}{3}$ (*d*) $\frac{5}{9}$

33. A die is tossed thrice. A success is getting 1 or 6 on a toss. The mean and the variance of number of successes

(*a*) $\mu=1,\sigma^2=2/3$ (*b*) $\mu=2/3,\sigma^2=1$

(*c*) $\mu=2,\sigma^2=2/3$ (*d*) None of these

Baye's theorem

34. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. The probability that it is actually a six, is

(*a*) $\frac{3}{8}$ (*b*) $\frac{1}{5}$

(*c*) $\frac{3}{4}$ (*d*) $\frac{3}{6}$

35. A bag X contains 2 white and 3 black balls and another bag Y contains 4 white and 2 black balls. One bag is selected at random and a ball is drawn from it. Then the probability for the ball chosen be white is

(*a*) $\frac{2}{15}$ (*b*) $\frac{7}{15}$

(*c*) $\frac{8}{15}$ (*d*) $\frac{14}{15}$

Learning Plus

1. A 9 digit number using the digits 1, 2, 3, 4, 5, 6, 7, 8 & 9 is written randomly without repetetion. The probability that the number will be divisible by 9 is:

   (*a*) 1/9 (*b*) 1/2
   (*c*) 1 (*d*) $9!/9^9$

2. A bag contains 3 white, 3 black and 2 red balls. One by one three balls are drawn without replacing them. The probability that the third ball is red, is

   (*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$
   (*c*) $\frac{2}{3}$ (*d*) $\frac{1}{4}$

3. Two dies are rolled simultaneously. The probability that the sum of the two numbers on the top faces will be atleast 10 is:

   (*a*) 1/6 (*b*) 1/12
   (*c*) 1/18 (*d*) 1/16

4. A quadratic equation is chosen from the set of all quadratic equations which are unchanged by squaring their roots. The chance that the chosen equation has equal roots is

   (*a*) 1/2 (*b*) 1/3
   (*c*) 1/4 (*d*) 2/3

5. 5 persons entered the lift cabin on the ground floor of an 8 floor building. Suppose that each of them independently and with equal probability, can leave the cabin at any other floor, starting from the first. The probability that all 5 persons leave at different floors is

   (*a*) $\left(\frac{5}{8}\right)^5$ (*b*) $\frac{{}^8C_5}{8^5}$
   (*c*) $\frac{5!}{8^5}$ (*d*) $\frac{{}^8C_5 5!}{8^5}$

6. There are ten prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top ten contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8th contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C, is

   (*a*) 1/4 (*b*) 1/3
   (*c*) 1/12 (*d*) 1/10

7. The chance that a 13 card combination from a pack of 52 playing cards is dealt to a player in a game of bridge, in which 9 cards are of the same suit, is

   (*a*) $\frac{4\,.\,{}^{13}C_9\,.\,{}^{39}C_4}{{}^{52}C_{13}}$ (*b*) $\frac{4!\,.\,{}^{13}C_9\,.\,{}^{39}C_4}{{}^{52}C_{13}}$
   (*c*) $\frac{{}^{13}C_9\,.\,{}^{39}C_4}{{}^{52}C_{13}}$ (*d*) $\frac{2!\,.\,{}^{13}C_9\,.\,{}^{39}C_4}{{}^{52}C_{13}}$

8. In drawing of a card from a well shuffled ordinary deck of playing cards the events 'card drawn is spade' and 'card drawn is an ace' are

   (*a*) Mutually exclusive
   (*b*) Equally likely
   (*c*) Forming an exhaustive system
   (*d*) Both event does not related to each other

9. A is one of the 6 horses entered for a race and is to be ridden by one of two jockeys B or C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win, if C rides A, his chance is trebled. Then the odds against his winning are

   (*a*) 5/13 (*b*) 18/5
   (*c*) 13/18 (*d*) 13/5

10. A pair of dice is thrown. If total of numbers turned up on both the dies is 8, then the probability that the number then up on the second die is 5' is

    (*a*) $\frac{5}{36}$ (*b*) $\frac{1}{6}$
    (*c*) $\frac{1}{5}$ (*d*) $\frac{2}{5}$

11. A bowl has 6 red marbles and 3 green marbles. The probability that a blind folded person will draw a red marble on the second draw from the bowl without replacing the marble from the first draw, is

    (*a*) 2/3 (*b*) 1/4
    (*c*) 5/12 (*d*) 5/8

12. Three numbers are chosen at random without replacement from {1, 2, 3 ...., 10}. The probability that the minimum of the chosen numbers is 3 or their maximum is 7 is

    (*a*) 1/2 (*b*) 1/3
    (*c*) 1/4 (*d*) 11/40

13. If atleast one child in a family with 3 children is a boy then the probability that 2 of the children are boys, is

    (*a*) 3/7 (*b*) 4/7
    (*c*) 1/3 (*d*) 3/8

14. If two of the 64 squares are chosen at random on a chess board, the probability that they have a side in common is

    (*a*) 1/9 (*b*) 1/18
    (*c*) 2/7 (*d*) 1/3

15. Two numbers a and b are selected from the set of natural number then the probability that $a^2 + b^2$ is divisible by 5 is

    (*a*) 9/25 (*b*) 7/18
    (*c*) 11/36 (*d*) 17/81

16. If a, b and c are three numbers (not necessarily different) chosen randomly and with replacement from the set {1, 2, 3, 4, 5}, the probability that (ab + c) is even, is

(a) $\frac{50}{125}$ (b) $\frac{59}{125}$

(c) $\frac{64}{125}$ (d) $\frac{75}{125}$

17. An ordinary cube has four blank faces, one face marked 2 another marked 3. Then the probability of obtaining a total of exactly 12 in 5 throws, is

(a) $\frac{5}{1296}$ (b) $\frac{5}{1944}$

(c) $\frac{5}{2592}$ (d) $\frac{5}{1967}$

18. The probability that the three cards drawn from a pack of 52 cards are all red is

(a) $\frac{1}{17}$ (b) $\frac{3}{19}$

(c) $\frac{2}{19}$ (d) $\frac{2}{17}$

19. A basket contains 5 apples and 7 oranges and another basket contains 4 apples and 8 oranges. One fruit is picked out from each basket. Find the probability that the fruits are both apples or both oranges

(a) $\frac{24}{144}$ (b) $\frac{56}{144}$

(c) $\frac{68}{144}$ (d) $\frac{76}{144}$

20. The probabilities of events, $A \cap B$, A, B & $A \cup B$ are respectively in A.P. with probability of second term equal to the common difference. Therefore the events A and B are

(a) compatible
(b) independen
(c) such that one of them must occur
(d) such that one is twice as likely as the other

21. Out of 13 applicants for a job, there are 5 women and 8 men. It is desired to select 2 persons for the job. The probability that at least one of the selected persons will be a woman is

(a) 25/39 (b) 14/39
(c) 5/13 (d) 10/13

22. A fair die is tossed. If the number is odd, find the probability that it is prime is

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{3}$

23. Odds in favour of A's speaking the truth are 1 : 2 and odds against B's speaking the truth are 1 : 3. A die is thrown. Both A and B assert that on the die 4 has turned up. Then the probability of the truth of their assertion is

(a) $\frac{1}{30}$ (b) $\frac{1}{24}$

(c) $\frac{11}{360}$ (d) $\frac{15}{17}$

24. Two buses A and B are scheduled to arrive at a town central bus station at noon. The probability that bus A will be late is 1/5. The probability that bus B will be late is 7/25. The probability that the bus B is late given that bus A is late is 9/10. Then the probabilities

(i) neither bus will be late on a particular day and
(ii) bus A is late given that bus B is late, are respectively

(a) 2/25 and 12/28 (b) 18/25 and 22/28
(c) 7/10 and 18/28 (d) 12/25 and 2/28

25. A & B are two independent events such that $P(\bar{A}) = 0.7, P(\bar{B}) = a$ & $P(A \cup B) = 0.8$, then a is

(a) 5/7 (b) 2/7
(c) 1 (d) 1/7

26. A & B having equal skill, are playing a game of best of 5 points. After A has won two points & B has won one point, the probability that A will win the game is:

(a) 1/2 (b) 2/3
(c) 3/4 (d) 4/5

27. Before a race the chance of three runners, A, B, C were estimated to be proportional to 5, 3, 2, but during the race A meets with an accident which reduces his chance to 1/3. If the respective chance of B and C are P(2) and P(3) then

(a) $P(B) = \frac{4}{15}$ (b) $P(C) = \frac{4}{15}$

(c) $P(C) = \frac{2}{5}$ (d) $P(A) = \frac{4}{15}$

28. Two cards are drawn from a well shuffled pack of 52 playing cards one by one. If A : the event that the second card drawn is an ace and B : the event that the first card drawn in an ace card. then which of the following is true ?

(a) $P(A) = \frac{4}{17}$; $P(B) = \frac{1}{13}$

(b) $P(A) = \frac{1}{13}$; $P(B) = \frac{1}{13}$

(c) $P(A) = \frac{1}{13}$; $P(B) = \frac{1}{17}$

(d) $P(A) = \frac{16}{221}$; $P(B) = \frac{4}{51}$

29. In a series of 3 independent trials the probability of exactly 2 success is 12 times as large as the probability of 3 successes. The probability of a success in each trial is:
(a) 1/5 (b) 2/5
(c) 3/5 (d) 4/5

30. A bag contains 2 white & 4 black balls. A ball is drawn 5 times, each being replaced before another is drawn. The probability that atleast 4 of the balls drawn are white is:
(a) 4/81 (b) 10/243
(c) 11/243 (d) 11/81

31. From an urn containing 3 red & 2 white balls, a man is to draw 2 balls at random without replacement, being promised 20 paise for each red ball he draws & 10 paise for each white one. His expectation is:
(a) 24 paise (b) 30 paise
(c) 23 paise (d) 35 paise

32. A & B throw with one dice for a stake of Rs. 99/- which is to be won by the player who first throws 4. If A has the first throw then their respective expectations of rupees are:
(a) 50 & 49 (b) 54 & 45
(c) 45 & 54 (d) 44 & 45

33. Let A consists of 3G and 2D articles. Lot B consists of 4G and 1D article. A new lot C is formed by taking 3 articles from A and 2 from B. The probability that an article chosen at random from C is defective, is
(a) 1/3 (b) 2/5
(c) 8/25 (d) 4/25

34. A box has four dice in it. Three of them are fair dice but the fourth one has the number five on all of its faces. A die is chosen at random from the box and is rolled three times and shows up the face five on all the three occasions. The chance that the die chosen was a rigged die, is
(a) $\frac{216}{217}$ (b) $\frac{215}{219}$
(c) $\frac{216}{219}$ (d) $\frac{215}{217}$

35. A card is drawn from a well shuffled ordinary deck of 52-playing cards. The card drawn is found to be a spade. Then the probability that the card is an ace, is
(a) $\frac{1}{13}$ (b) $\frac{1}{52}$
(c) $\frac{1}{4}$ (d) $\frac{1}{26}$

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/ NUMERICAL

1. A student appears for tests I, II & III. The student is successful if he passes either in tests I & II or tests I & III. The probabilities of the student passing in the tests I, II & III are p, q &1/2 respectively. If the probability that the student is successful is 1/2, then:
(a) $p = 1, q = 0$
(b) $p = 2/3, q = 1/2$
(c) $p = 3/5, q = 2/3$
(d) There are infinitely many values of p & q.

2. A pair of fair dice having six faces numbered from 1 to 6 are thrown once, suppose two events E and F are defined as
E : Product of the two numbers appearing is divisible by 5.
F : At least one of the dice shows up the face one.
Then the events E and F are
(a) mutually exclusive
(b) independent
(c) neither independent nor mutually exclusive
(d) are equiprobable

3. If M & N are any two events, then which one of the following represents the probability of the occurance of exactly one of them ?
(a) $P(M) + P(N) - 2P(M \cap N)$
(b) $P(M) + P(N) - P(M \cap N)$
(c) $P(\overline{M}) + P(\overline{N}) - 2P(\overline{M} \cap \overline{N})$
(d) $P(M \cap \overline{N}) + P(\overline{M} \cap N)$

4. Let $0 < P(A) < 1, 0 < P(B) < 1$ & $P(A \cup B) = P(A) + P(B) - P(A).P(B)$, then:
(a) $P(B/A) = P(B) - P(A)$
(b) $P(A^c \cup B^c) = P(A^c) + P(B^c)$
(c) $P((A \cup B)^c) = P(A^c).P(B^c)$
(d) $P(A/B) = P(A)$

5. A student has to match three historical events i.e. Dandi March, Quit India Movement and Mahatma Gandhi's assasination with the years 1948, 1930 and 1942. The student has no knowledge of the correct answers and decides to match the events and years randomly. Let E_i : $(0 \le i \le 3)$ denote the event that the student gets exactly i correct answer, then
(a) $P(E_0) + P(E_3) = P(E_1)$ (b) $P(E_0).P(E_1) = P(E_3)$
(c) $P(E_0 \cap E_1) = P(E_2)$ (d) $P(E_0) + P(E_1) + P(E_3) = 1$

6. If A & B are two events such that $P(B) \neq 1$, $\bar{B}$ denotes the event complementary to B, then

(a) $P(A \mid \bar{B}) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

(b) $P(A \cap B) \geq P(A) + P(B) - 1$

(c) $P(A) \gtrless P(A|B)$ according as $P(A|\bar{B}) \gtrless P(A)$

(d) $P(A|\bar{B}) + P(\bar{A}|\bar{B}) = 1$

7. If E_1 and E_2 are two events such that $P(E_1) = 1/4$, $P(E_2|E_1) = 1/2$ and $P(E_1|E_2) = 1/4$ then

(a) E_1 and E_2 are independent

(b) E_1 and E_2 are exhaustive

(c) E_2 is twice as likely to occur as E_1

(d) probabilities of the events $E_1 \cap E_2$, E_1 and E_2 are in GP.

8. If M & N are independent events such that $0 < P(M) < 1$ & $0 < P(N) < 1$, then:

(a) M & N are mutually exclusive

(b) M & $\bar{N}$ are independent

(c) $\bar{M}$ & $\bar{N}$ are independent

(d) $P(M/N) + P(\bar{M}/N) = 1$

9. An unbiased coin is tossed n times. Let X denote the number of times head occurs. If $P(X = 4)$, $P(X = 5)$ and $P(X = 6)$ are in AP, then the value of n can be

(a) 7 (b) 10

(c) 12 (d) 14

Comprehension – 1 (No. 10 to 12)

There are n urns each containing $(n + 1)$ balls such that the ith urn contains i white balls and $(n + 1 - i)$ red balls. Let u_i be the event of selecting i^{th} urn, $i = 1, 2, 3, \ldots\ldots, n$ and w denotes the event of getting a white ball.

10. If $P(u_i) \propto i$ where $i = 1, 2, 3, \ldots\ldots, n$ then $\lim_{n\to\infty} P(w)$ is equal to

(a) 1 (b) 2/3

(c) 3/4 (d) 1/4

11. If $P(u_i) = c$, where c is a constant then $P(u_n/w)$ is equal to

(a) $\dfrac{2}{n+1}$ (b) $\dfrac{1}{n+1}$

(c) $\dfrac{n}{n+1}$ (d) $\dfrac{1}{2}$

12. If n is even and E denotes the event of choosing even numbered urn ($P(u_i) = \dfrac{1}{n}$), then the value of P(w/E), is

(a) $\dfrac{n+2}{2(n+1)}$ (b) $\dfrac{n+2}{2(n+1)}$

(c) $\dfrac{n}{n+1}$ (d) $\dfrac{1}{n+1}$

13. If n positive integers taken at random and multiplied together, then the chance that the last digit of the product would be

Column – I

(a) If the probability that units digit in square of an even integer is 4 is p, then the value of 5p is

(b) If A and B are independent events and $P(A \cap B) = \dfrac{1}{6}$, $P(\bar{A}) = \dfrac{2}{3}$, then $6P(B/A) =$

(c) Among 2 children, a child may equally be a boy or a girl if the probability that exactly one of them is a boy is p, then 6p =

(d) A boy has 20% chance of hitting at a target. Let p denote the probability of hitting the target for the first time at the nth trial.

If p satisfies the inequality $625p^2 - 175p + 12 < 0$, then value of n is

Column – II

(p) 1 (q) 2

(r) 3 (s) 4

14. Match the column:

Column – I

(a) A pair of dice is thrown. If total of numbers turned up on both the dies is 8, then the probability that the number then up on the second die is 5' is

(b) A box contains 4 white and 3 black balls. Two balls aredrawn successively and is found that second ball is white,then the probability that Ist ball is also white is

(c) A biased coin with probability p, $0 < p < 1$ of heads is tossed until a head appears for the first time. If the probabilitythat the number of tosses required is even is 2/5, then p equals

(d) A coin whose faces are marked 3 and 5 is tossed 4 times : what is the probability that the sum of the numbers thrown being less, than 15?

Column – II

(p) 5/16 (q) 1/3

(r) $\dfrac{1}{2}$ (s) $\dfrac{1}{5}$

NUMERICAL VALUE BASED

15. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14. Then the probability that neither A nor B occurs is

16. A bag contains 3 red and 3 white balls. Two balls are drawn one by one. The probability that they are of different colours is.

17. The probability that an anti aircraft gun can hit an enemy plane at the first, second and third shot are 0.6, 0.7 and 0.1 respectively. The probability that the gun hits the plane is

18. The probability that atleast one of the events A and B happens is 0.6. If probability of their simultaneous happening is 0.2, then $P(\overline{A}) + P(\overline{B})$ is

19. One hundred identical coins, each with probability, p, of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then the value of p is

20. If the letter of the word SUCCESS are arranged, then the probability that similar letters occurs together is

21. What is the standard deviation of the random variable whose p.m.f.'s are given below.

X = x	1	2	3
P(X = x)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

22. Calculate the expected value of the sum of two numbers obtained when two fair dice are rolled.

23. An anti-aircraft gun can take a maximum of four shots at any plane moving away from it. The probabilities of hitting the plane at the 1st, 2nd, 3rd and 4th shots are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that at least one shot hits the plane ?

24. A box contains 15 transistors, 5 of which are defective. An inspector takes out one transistor at random, examines it for defects and replaces it. After it has replaced another inspector does the same thing and then so does a third inspector. The probabuility that atleast one of the inspectors finds a defective transistor, is equal to

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true?

[*JEE Main-2016*]

(*a*) E_1 and E_2 are independent
(*b*) E_2 and E_3 are independent
(*c*) E_1 and E_3 are independent
(*d*) E_1, E_2 and E_3 are independent

2. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one by one, with replacement, then the variance of the number of green balls drawn is :

[*JEE Main-2017*]

(*a*) $\frac{6}{25}$ (*b*) $\frac{12}{5}$
(*c*) 6 (*d*) 4

3. If two different numbers are taken form the set {0, 1, 2, 3, ..., 10), then the probability that their sum as well as absolute diference are both multiple of 4, is -

[*JEE Main-2017*]

(*a*) $\frac{7}{55}$ (*b*) $\frac{6}{55}$

(*c*) $\frac{12}{55}$ (*d*) $\frac{14}{45}$

4. For three events A, B and C, P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs) = P(Exactly one of C or A occurs) = $\frac{1}{4}$ and P(All the three events occur simultaneously) = $\frac{1}{16}$. Then the probability that at least one of the events occurs, is :

[*JEE Main-2017*]

(*a*) $\frac{3}{16}$ (*b*) $\frac{7}{32}$

(*c*) $\frac{7}{16}$ (*d*) $\frac{7}{64}$

5. A bag contains 4 red and 6 blacks balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is:

[*JEE Main-2018*]

(*a*) $\frac{2}{5}$ (*b*) $\frac{1}{5}$

(*c*) $\frac{3}{4}$ (*d*) $\frac{3}{10}$

6. Two cards are drawn successively with replacement from a well shuffled deck of 52 cards. Let X denote the random variable of number of aces obtained in the two drawn cards. Then $P(X=1)+P(X=2)$ equals :

[JEE Main - 2019 (January)]

(a) $\frac{49}{169}$ (b) $\frac{52}{169}$

(c) $\frac{24}{169}$ (d) $\frac{25}{169}$

7. An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a secon ball is drawn at random from it. The probability that the second ball is red, is:

[JEE Main - 2019 (January)]

(a) $\frac{26}{49}$ (b) $\frac{32}{49}$

(c) $\frac{27}{49}$ (d) $\frac{21}{49}$

8. An unbiased coin is tossed. If outcome is a head then a pair of unbiased dice is rolled and the sum of the numbers obtained on the is noted. If the toss of the coin results in tail then a card from a well-shuffled pack of nine cards numbered 1, 2, 3,.....,9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is: [JEE Main - 2019 (January)]

(a) $\frac{13}{36}$ (b) $\frac{15}{72}$

(c) $\frac{19}{72}$ (d) $\frac{19}{36}$

9. If the probability of hitting a target by a shooter, in any shot, is $\frac{1}{3}$, then the minimum number of independent shots at the target required by him so that the probability of hitting the target at least once is greater than $\frac{5}{6}$, is :

[JEE Main - 2019 (January)]

(a) 3 (b) 6

(c) 5 (d) 4

10. Let $S=\{1,2,....20\}$. A subset B is S is said to be "nice", if the sum of the elements of B is 203. Then the probability that a randomly chose n subset of S is 'nice' is:

[JEE Main - 2019 (January)]

(a) $\frac{7}{2^{20}}$ (b) $\frac{5}{2^{20}}$

(c) $\frac{4}{2^{20}}$ (d) $\frac{6}{2^{20}}$

11. In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to:

[JEE Main - 2019 (January)]

(a) $\frac{200}{6^5}$ (b) $\frac{150}{6^5}$

(c) $\frac{225}{6^5}$ (d) $\frac{175}{6^5}$

12. In a game, a man wins Rs. 100 if he gets 5 ot 6 on a throw of a fair die and loses Rs. 50 for getting any other number on the die. If he decides to throw the die either till he gets a five of a six or to a maximum of three throws, then his expected gain/loss (in rupees) is :

[JEE Main - 2019 (January)]

(a) $\frac{400}{9}$ loss (b) 0

(c) $\frac{400}{3}$ gain (d) $\frac{400}{3}$ loss

13. In a class of 60 students, 40 opted for NCC, 30 opted for NSS and 20 opted for both NCC and NSS. If one of these students is selected at random, then the probability that the student selected has opted neither for NCC nor for NSS is : [JEE Main - 2019 (January)]

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) $\frac{5}{6}$

14. Let A and B be two non-null events such that $A \subset B$. Then, which of the following statements is always correct ?

[JEE Main - 2019(April)]

(a) $P(A|B)=1$ (b) $P(A|B)=P(B)-P(A)$

(c) $P(A|B) \le P(A)$ (d) $P(A|B) \ge P(A)$

15. The minimum number of times one has to toss a fair coin so that the probability of observing at least one head is at least 90% is [JEE Main - 2019(April)]

(a) 5 (b) 3

(c) 2 (d) 4

16. Four persons can hit a target correctly with probabilities $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ and $\frac{1}{8}$ respectively. If all hit at the target independently, then the probability that the target would be hit is [JEE Main - 2019(April)]

(a) $\frac{25}{192}$ (b) $\frac{1}{192}$

(c) $\frac{25}{32}$ (d) $\frac{7}{32}$

17. Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls is : [JEE Main - 2019(April)]

(a) $\frac{1}{11}$ (b) $\frac{1}{17}$

(c) $\frac{1}{10}$ (d) $\frac{1}{12}$

18. Minimum number of times a fair coin must be tossed so that the probability of getting at least one head is more than 99% is : [JEE Main - 2019(April)]

(a) 5 (b) 6

(c) 7 (d) 8

19. If three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is : [JEE Main - 2019(April)]

(a) $\frac{3}{10}$ (b) $\frac{1}{10}$

(c) $\frac{3}{20}$ (d) $\frac{1}{5}$

20. Let a random variable X have a binomial distribution with mean 8 and variance 4. If $P(x \le 2) = \frac{k}{2^{16}}$, then k is equal to : [JEE Main - 2019(April)]

(a) 17 (b) 1

(c) 121 (d) 137

21. For and initial screening of an admission test, a candidates is given fifty problems to solve. If the probability that the candidate can solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problems is : [JEE Main - 2019(April)]

(a) $\frac{316}{25}\left(\frac{4}{5}\right)^{48}$ (b) $\frac{54}{5}\left(\frac{4}{5}\right)^{49}$

(c) $\frac{164}{25}\left(\frac{1}{5}\right)^{48}$ (d) $\frac{201}{5}\left(\frac{1}{5}\right)^{49}$

22. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for k = 3, 4, 5 otherwise X takes the value – 1. Then the expected value of X is. [JEE Main-2020 (January)]

(a) $\frac{3}{16}$ (b) $\frac{-1}{8}$

(c) $\frac{-3}{16}$ (d) $\frac{1}{8}$

23. In a workshop there are five machines and the probability of any one of them to be out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to: [JEE Main-2020 (January)]

(a) $\frac{17}{2}$ (b) $\frac{17}{4}$

(c) $\frac{17}{8}$ (d) 4

24. Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Then, which of the following is TRUE? [JEE Main-2020 (January)]

(a) $P\left(\frac{A'}{B'}\right) = \frac{1}{3}$ (b) $P\left(\frac{A}{B}\right) = \frac{2}{3}$

(c) $P\left(\frac{A}{B'}\right) = \frac{1}{3}$ (d) $P\left(\frac{A}{A \cup B}\right) = \frac{1}{4}$

25. Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is [JEE Main-2020 (January)]

(a) 0.02 (b) 0.10

(c) 0.01 (d) 0.20

26. In an box, there are 20 cards, out of which 10 are labelled as A and the remaining 10 are labelled as B. Cards are drawn at random, one after the other and with replacement, till a second A-card is obtained. The probability that the second A-card appears before the third B-card is : [JEE Main-2020 (January)]

(a) $\frac{13}{16}$ (b) $\frac{15}{16}$

(c) $\frac{9}{16}$ (d) $\frac{11}{16}$

27. A random variable X has the following probability distribution:

X:	1	2	3	4	5
P(x):	K^2	2k	K	2K	$5K^2$

Then $P(x > 2)$ is equal to : [JEE Main-2020 (January)]

(a) $\frac{23}{36}$ (b) $\frac{7}{12}$

(c) $\frac{1}{36}$ (d) $\frac{1}{6}$

28. If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is :

[*JEE Main-2020 (January)*]

(*a*) $\frac{946}{2^{10}}$ (*b*) $\frac{965}{2^{10}}$

(*c*) $\frac{946}{2^{10}}$ (*d*) None of these

29. Let E^C denote the complement of an event E. Let E_1, E_2 and E_3 be any pairwise independent events with $P(E_1)>0$ and $P(E_1 \cap E_2 \cap E_3)=0$. Then $P(E_2^C \cap E_3^C / E_1)$ is equal to :

[*JEE Main-2020 (September)*]

(*a*) $P(E_3^C)-P(E_2^C)$ (*b*) $P(E_2^C)+P(E_3)$

(*c*) $P(E_3)-P(E_2^C)$ (*d*) $P(E_3^C)-P(E_2)$

30. Box I contains 30 cards numbered 1 to 30 and Box II contains 20 cards numbered 31 to 50. A box is selected at random and a card is drawn from it. The number on the card is found to be a non-prime number. The probability that the card was drawn from Box I is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{4}{17}$ (*b*) $\frac{2}{3}$

(*c*) $\frac{2}{5}$ (*d*) $\frac{8}{17}$

31. The probability that a randomly chosen 5-digit number is made from exactly two digits is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{150}{10^4}$ (*b*) $\frac{134}{10^4}$

(*c*) $\frac{121}{10^4}$ (*d*) $\frac{135}{10^4}$

32. A die is thrown two times and the sum of the scores appearing on the die is observed to be a multiple of 4. Then the conditional probability that the score 4 has appeared atleast once is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{4}$

(*c*) $\frac{1}{8}$ (*d*) $\frac{1}{9}$

33. In a game two players A and B take turns in throwing a pair of fair dice starting with player A and total of scores on the two dice, in each throw is noted. A wins the game if he throws a total of 6 before B throws a total of 7 and B wins the game if he throws a total of 7 before A throws a total of six. The game stops as soon as either of the players wins. The probability of A winning the game is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{5}{31}$ (*b*) $\frac{31}{61}$

(*c*) $\frac{5}{6}$ (*d*) $\frac{30}{61}$

34. Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{13}}$ is equal to _____.

[*JEE Main-2020 (September)*]

35. The probability of a man hitting a target is $\frac{1}{10}$. The least number of shots required, so that the probability of his hitting the target at least once is greater than $\frac{1}{4}$, is _____.

[*JEE Main-2020 (September)*]

36. Four fair dice are thrown independently 27 times. Then the expected number of times, at least two dice show up a three or a five, is _____.

[*JEE Main-2020 (September)*]

37. The probabilities of three events A, B and C are given by $P(A)=0.6$, $P(B)=0.4$ and $P(C)=0.5$. If $P(A \cup B)=0.8$, $P(A \cap C)=0.3$, $P(A \cap B \cap C)=0.2$, $P(B \cap C)=\beta$ and $P(A \cup B \cup C)=\alpha$, where $0.85 \le \alpha \le 0.95$, then β lies in the interval :

[*JEE Main-2020 (September)*]

(*a*) [0.25, 0.35] (b) [0.35, 0.36]

(*c*) [0.36, 0.40] (*d*) [0.20, 0.25]

38. Out of 11 consecutive natural numbers if three numbers are selected at random (without repetition), then the probability that they are in A.P. with positive common difference, is :

[*JEE Main-2020 (September)*]

(*a*) $\frac{5}{101}$ (*b*) $\frac{10}{99}$

(*c*) $\frac{5}{33}$ (*d*) $\frac{15}{101}$

39. Let in a Binomial distribution, consisting of 5 independent trials, probabilities of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to:

[*JEE Main-2021 (March)*]

(*a*) $\frac{40}{243}$ (*b*) $\frac{128}{625}$

(*c*) $\frac{80}{243}$ (*d*) $\frac{32}{625}$

40. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to:

[*JEE Main-2021 (March)*]

(*a*) $\frac{1}{18}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{9}$ (d) $\frac{1}{27}$

JEE-ADVANCED
PREVIOUS YEARS

1. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denotes respectively the events that the engines E_1 E_2 and E_3 are functioning. Which of the following is (are) true ?

[IIT JEE-2012]

(a) $P\left[X_1^c \mid x\right] = \frac{3}{16}$

(b) P[Exactly two engines of the ship are functioning $|X] = \frac{7}{8}$

(c) $P[X \mid X_2] = \frac{5}{16}$

(d) $P[X \mid X_1] = \frac{7}{16}$

2. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1,2,3,4,5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is [IIT JEE-2012]

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$

(c) $\frac{125}{216}$ (d) $\frac{127}{216}$

3. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}$, $P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct? [IIT JEE-2012]

(a) $P(X \cup Y) = \frac{2}{3}$

(b) X and Y are independent

(c) X and Y are not independent

(d) $P(X^c \cap Y) = \frac{1}{3}$

4. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is [JEE Advanced-2013]

(a) $\frac{235}{256}$ (b) $\frac{21}{256}$

(c) $\frac{3}{256}$ (d) $\frac{253}{256}$

5. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\, p = \alpha\beta$ and $(\beta - 3\gamma)\, p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1).

Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$

[JEE Advanced-2013]

Comprehension –1 (No. 6 & 7)

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

6. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is [JEE Advanced-2013]

(a) $\frac{82}{648}$ (b) $\frac{90}{648}$

(c) $\frac{558}{648}$ (d) $\frac{566}{648}$

7. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is [JEE Advanced-2013]

(a) $\frac{116}{181}$ (b) $\frac{126}{181}$

(c) $\frac{65}{181}$ (d) $\frac{55}{181}$

8. Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is [JEE Advanced-2014]

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Comprehension – 2 (No. 9 & 10)

Box 1 contains three cards bearing numbers 1, 2, 3; box 2 contains five cards bearing numbers 1, 2, 3, 4, 5; and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the ith box, i = 1, 2, 3.

[JEE Advanced-2014]

9. The probability that $x_1 + x_2 + x_3$ is odd, is

(a) $\frac{29}{105}$ (b) $\frac{53}{105}$

(c) $\frac{57}{105}$ (d) $\frac{1}{2}$

10. The probability that x_1, x_2, x_3 are in an arithmetic progression, is

(a) $\frac{9}{105}$ (b) $\frac{10}{105}$

(c) $\frac{11}{105}$ (d) $\frac{7}{105}$

11. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is

Comprehension – 3 (No. 12 & 13)

Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II. [*JEE Advanced-2015*]

12. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(a) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$
(b) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
(c) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$
(d) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

13. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are)

(a) $n_1 = 4$ and $n_2 = 6$ (b) $n_1 = 2$ and $n_2 = 3$
(c) $n_1 = 10$ and $n_2 = 20$ (d) $n_1 = 3$ and $n_2 = 6$

14. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the actory turn out to be defective. It is known that P (computer turns out to be defective given that it is produced in plant T_1) $= 10P$ (computer turns out to be defective given that it is produced in plant T_2) where $P(E)$ denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is [*JEE Advanced-2016*]

(a) $\frac{36}{73}$ (b) $\frac{47}{79}$

(c) $\frac{78}{93}$ (d) $\frac{75}{83}$

Comprehension – 4 (No. 15 & 16)

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games.

[*JEE Advanced-2016*]

15. $P(X > Y)$ is

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$

(c) $\frac{1}{2}$ (d) $\frac{7}{12}$

16. $P(X = Y)$ is

(a) $\frac{11}{36}$ (b) $\frac{1}{3}$

(c) $\frac{13}{36}$ (d) $\frac{1}{2}$

17. Let X and Y be two events such that $P(X) = \frac{1}{3}, P(X \mid Y) = \frac{1}{2}$ and $P(Y \mid X) = \frac{2}{5}$. Then

[*JEE Advanced-2017*]

(a) $P(Y) = \frac{4}{15}$ (b) $P(X' \mid Y) = \frac{1}{2}$

(c) $P(X \cup Y) = \frac{2}{5}$ (d) $P(X \cap Y) = \frac{1}{5}$

18. Three randomly chose nonnegative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is [*JEE Advanced-2017*]

(a) $\frac{1}{2}$ (b) $\frac{36}{55}$

(c) $\frac{6}{11}$ (d) $\frac{5}{11}$

Comprehension – 5 (No. 19 & 20)

There are five students S_1, S_2, S_4 and S_5 in a music class and for them thre are five sets R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is alotted to the students S_i, i = 1, 2, 3, 4, 5. But, on the examination day, the five students are randomly allotted the five seats.

19. The probability that, on that examination day, the student S_1 gets the previously allotted seart R_1 and NONE of the remaining students gets the seat previously allotted to him/her is - [JEE Advanced-2018]

(a) $\frac{3}{40}$ (b) $\frac{1}{8}$

(c) $\frac{7}{40}$ (d) $\frac{1}{5}$

20. For i = 1, 2, 3, 4, let T_i denote the event that the students S_i and S_{i+1} do NOT sit adjacent to each other on the day of the examination. Then the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is - [JEE Advanced-2018]

(a) $\frac{1}{15}$ (b) $\frac{1}{10}$

(c) $\frac{7}{60}$ (d) $\frac{1}{5}$

21. Let |X| denote the number of elements in set X. Let $S = \{1,2,3,4,5,6\}$ be a sample space, where wach element is equally likely to occur. If A and B are indepenent events with S, then the number of ordered pairs (A,B) such that $1 \le |B| < |A|$, equals [JEE Advanced-2019]

22. Let S be the sample space of all 3×3 matrices with entries from the set $\{0,1\}$. Let the events E_1 and E_2 be given by

$E_1 = \{A \in S\} : \det A = 0\}$ and

$E_2 = \{A \in S :$ sum of entries of A is $7\}$.

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals___ [JEE Advanced-2019]

23. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contants 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}, \frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ? [JEE Advanced -2019]

(a) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(b) Probability that the chosen ball is green equals $\frac{39}{80}$

(c) Probability that the chosen ball is green, given that the selected bag is B_3 equals $\frac{3}{8}$

(d) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\frac{5}{13}$

24. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently, Then probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is [JEE(Advanced) - 2020]

(a) $\frac{40}{81}$ (b) $\frac{20}{81}$

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

25. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is _____ [JEE(Advanced) - 2020]

26. Two fair dice, each with faces numbered 1,2,3,4,5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of $14p$ is _____ [JEE(Advanced) - 2020]

ANSWER KEY

Topicwise Questions

1. (*c*)	**2.** (*b*)	**3.** (*c*)	**4.** (*d*)	**5.** (*b*)	**6.** (*a*)	**7.** (*c*)	**8.** (*c*)	**9.** (*b*)	**10.** (*a*)
11. (*c*)	**12.** (*c*)	**13.** (*d*)	**14.** (*a*)	**15.** (*c*)	**16.** (*b*)	**17.** (*a*)	**18.** (*a*)	**19.** (*c*)	**20.** (*b*)
21. (*b*)	**22.** (*a*)	**23.** (*a*)	**24.** (*a*)	**25.** (*d*)	**26.** (*d*)	**27.** (*a*)	**28.** (*d*)	**29.** (*a*)	**30.** (*c*)
31. (*d*)	**32.** (*b*)	**33.** (*a*)	**34.** (*a*)	**35.** (*c*)					

Learning Plus

1. (*c*)	**2.** (*d*)	**3.** (*a*)	**4.** (*a*)	**5.** (*d*)	**6.** (*a*)	**7.** (*a*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)
11. (*a*)	**12.** (*d*)	**13.** (*a*)	**14.** (*b*)	**15.** (*a*)	**16.** (*b*)	**17.** (*c*)	**18.** (*d*)	**19.** (*d*)	**20.** (*d*)
21. (*a*)	**22.** (*a*)	**23.** (*d*)	**24.** (*c*)	**25.** (*b*)	**26.** (*c*)	**27.** (*b*)	**28.** (*b*)	**29.** (*a*)	**30.** (*c*)
31. (*c*)	**32.** (*b*)	**33.** (*c*)	**34.** (*c*)	**35.** (*a*)					

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/COLUMN/NUMERICAL

1. (*a,b,c,d*)	**2.** (*c,d*)	**3.** (*a,c,d*)	**4.** (*c,d*)	**5.** (*a,b,c,d*)	**6.** (*a,b,c,d*)	**7.** (*a,c,d*)	**8.** (*b,c,d*)	**9.** (*a,d*)	**10.** (*b*)
11. (*a*)	**12.** (*b*)	**13.** (*a*) → (q), (*b*) → (r), (*c*) → (r), (*d*) → (r)				**14.** (*a*) → (s),(*b*) → (r),(*c*) → (q), (*d*) → (p)			
15. [0.39]	**16.** [0.6]	**17.** [0.892]	**18.** [1.2]	**19.** [0.5049]	**20.** [0.057]	**21.** [0.2]	**22.** [0007]	**23.** [0.6976]	**24.** [0.703]

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*a*)	**7.** (*b*)	**8.** (*c*)	**9.** (*c*)	**10.** (*b*)
11. (*d*)	**12.** (*b*)	**13.** (*a*)	**14.** (*d*)	**15.** (*d*)	**16.** (*c*)	**17.** (*a*)	**18.** (*c*)	**19.** (*b*)	**20.** (*d*)
21. (*b*)	**22.** (*d*)	**23.** (*c*)	**24.** (*c*)	**25.** (*b*)	**26.** (*d*)	**27.** (*a*)	**28.** (*d*)	**29.** (*d*)	**30.** (*d*)
31. (*d*)	**32.** (*d*)	**33.** (*d*)	**34.** (8)	**35.** (3)	**36.** (11)	**37.** (*a*)	**38.** (*c*)	**39.** (*d*)	**40.** (*c*)

JEE-ADVANCED PREVIOUS YEARS

1. (*bd*)	**2.** (*a*)	**3.** (*ab*)	**4.** (*a*)	**5.** (6)	**6.** (*a*)	**7.** (*d*)	**8.** (*a*)	**9.** (*b*)	**10.** (*c*)
11. (8)	**12.** (*ab*)	**13.** (*cd*)	**14.** (*c*)	**15.** (*b*)	**16.** (*c*)	**17.** (*ab*)	**18.** (*c*)	**19.** (*a*)	**20.** (*c*)
21. (422.00)	**22.** (0.50)	**23.** (*bc*)	**24.** (*b*)	**25.** (6)	**26.** (8.00)				

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31 MATHEMATICAL REASONING

time/real prayasians

STATEMENTS

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language "A sentence is called a mathematically acceptable statement if it is either true or false but not both". A statement is assumed to be either true or false. A true statement is known as a valid statement and a false statement is known as an invalid statement.

Note: A statement can not be both true and false at the same time.

Note:

(*i*) Imperative sentences (expresses a request or command), exclamatory sentences (expresses some strong feeling), Interrogative sentences (asks some questions) are not considered as a statement in mathematical language.

(*ii*) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.

TRUTH TABLE

Truth table is that which gives truth values of compound statements.

It has a number of rows and columns. The number of rows depend upon the number of simple statements.

Note that for n statements, there are 2^n rows.

(*i*) Truth table for single statement p:
Number of rows = $2^1 = 2$

p
T
F

(*ii*) Truth table for two statements p and q:
Number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(*iii*) Truth table for three statements p, q and r.
Number of rows = $2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Compound Statements

A compound statement is a statement which is made up of two or more statements. In this case, each statement is called a component statement.

e.g. All rational numbers are real and all real numbers are complex numbers.

The component statements are

p : all rational numbers are real

q : all real numbers are complex numbers

NEGATION (OR DENIAL)

The denial of a statement is called the negative of the statement and denoted as ~.

e.g. p : All integers are rational numbers".

$\sim p$: At least one integer is not a rational number.

While forming the negation of a statement, phrases like, "it is not the case" or "It is false that" are also used

e.g. p : Everyone in Germany speaks German.

$\sim p$: It is false that everyone in Germany speaks German.

If p is true then $\sim p$ must be false and if p is false then $\sim p$ must be true

Truth Table (~ p)		
p	~ p	~ (~ p)
T	F	T
F	T	F
Rule ~ is true only when p is false.		

It may be noticed that $\sim(\sim p) = p$. Also p and $\sim p$ are contrary.

Important: It may be observed that negation is not a binary operation, it is a unary operation i.e. a modifier.

1. $\sim p$ is true if p is false.
2. $\sim p$ is false if p is true.

Conjunction

Simple statements when combined by the word "and" ($\wedge$), the resulting compound statement is called a conjunction denoted as $p \wedge q$.

e.g. A point occupies a position and its location can be determined.

The component statement are

p : A point occupies a position

q : Its location can be determined

Both statements are true.

Important: Do not think that a statement with "And" is always a compound statement.

e.g. A mixture of alcohol and water can be separated by chemical methods.

(Here "And" refers to two things).

Note:

(*i*) The compound statement with 'And' is true if all its component statements are true.

(*ii*) The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

The following truth table shows the truth values of $p \wedge q$ (p and q) and $q \wedge p$ (q and p).

Truth Table ($p \wedge q$, $q \wedge p$)			
p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F
Rule : $p \wedge q$ is true only when p and q are true			

Remark: The above truth table shows that $p \wedge q = q \wedge p$.

Disjunction or Alternation

Simple statements p and q are combined by the connective '*OR*' ($\vee$), then the compound statements denoted as $p \vee q$ (p or q) so formed is called a disjunction. e.g. Two lines in a plane either intersect at one point or they are parallel.

Sometimes we use the connective 'either....or....' to obtain $p \vee q$ and real $p \vee q$ as 'either p or q'.

Note:

(*i*) A compound statement with an 'Or' is true when one component statement is true or both the component statement are true.

(*ii*) A compound statement with an 'Or' is false when both the component statements are false.

Truth Table ($p \vee q$, $q \vee p$)			
p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F
Rule : $p \vee q$ is false only when p and q are false			

Important: e.g. A student who has taken Biology or Chemistry can apply for M.Sc. microbiology program.

This means that student who have taken both Biology and Chemistry or only Biology or only Chemistry can apply for the microbiology program. This is example of **inclusive "Or"**. In this case truth table is same as $p \vee q$.

Important: e.g. Student can take French or Sanskrit as their third language.

This means that student have to choose only one subject from French and Sanskrit. It exclude the case when one student can choose both subject. This is case of **exclusive "Or"**. This is represented as $p \veebar q$ or $p \otimes q$. Truth table for exclusive or is as follows.

Truth Table ($p \veebar q$)		
p	q	$p \veebar q$ or $p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F
Rule : $p \vee q$ is true only when one of p and q is true and the other is false		

Quantifiers:

Quantifiers are phrases like "There exists" and "for all".

Negation of Quantifiers

(*i*) P = There exist a number which is equal to its square.
$\sim P$ = There does not exist a number which is not equal to its square.

(*ii*) P = For every real number x, x is less than $x + 1$.
$\sim P$ = There exist a number for which x is not less than $x + 1$.

Implication:

There are three types of implications :

(*i*) "If..... then"

(*ii*) "Only if"

(*iii*) "If and only if"

(1) "If ..... then" type of compound statement is called **conditional statement**.

The statement 'if p then q' is denoted by $p \to q$ (to be read as 'p implies q') or by $p \Rightarrow q$. Note that $p \to q$ also means

(*i*) p is sufficient for q (*ii*) q is necessary for p

(*iii*) p lead to q (*iv*) q if p

(*v*) q when p (*vi*) if p, then q

e.g. p : a number is a multiple of 9

q : a number is a multiple of 3.

Then $p \to q$ is false only when p is true and q is false. Truth table for $p \to q$ is as follows.

Truth Table ($p \to q$, $q \to p$)			
p	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T
Rule : p → q is false only when p is true and q is false.			

(2) "If and only if" type of compound statement is called **Biconditional or equivalence** or **'double implication'**. Symbolically 'p iff q' is represented by $p \leftrightarrow q$ or by $p \Leftrightarrow q$.

(*i*) p is a necessary and sufficient condition for q.

(*ii*) q is necessary and sufficient condition for p.

(*iii*) If p then q and if q then p

(*iv*) q if and only if p.

The following are other illustrations which actually do not appear to be so but they in fact are biconditional.

(*i*) If you work hard only then you can succeed.

(*ii*) You can go on leave only if your boss permits. The truth table for biconditional is as follows :

Truth Table ($p \leftrightarrow p$, $q \leftrightarrow p$)			
p	*q*	*p ↔ q*	*q ↔ p*
T	*T*	*T*	*T*
T	*F*	*F*	*F*
F	*T*	*F*	*F*
F	*F*	*T*	*T*
Rule : p ↔ q is true only when both p and q have the same truth value.			

Contrapositive and Converse: Contrapositive and converse are certain other statements which can be formed from a given statement with "if ..... then".

Contrapositive of $p \to q$ is $\sim q \to \sim p$

e.g. If a number is multiple of 6 then it is multiple of 2.

Contrapositive: If a number is not multiple of 2 then it is not multiple of 6.

Converse of $p \to q$ is $q \to p$

e.g. If the angles of a triangle are equal then it is equilateral triangle.

Converse is if triangle is equilateral then angles of triangle are equal.

Truth Table ($p \to q$)				
p	q	p → q	Contrapositive ($\sim q \to \sim p$)	Converse ($q \to p$)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Note: Truth table for $p \to q$ is same as its contrapositive.

Negation of implication:

If p and q are two statements, then

$\sim (p \Rightarrow q) = p \wedge \sim q \quad [\because p \Rightarrow q \equiv \sim p \wedge q]$

Proof:

p	q	p ⇒ q	~ (p ⇒ q)
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	~ q	p ∧ ~ q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Negation of Biconditional statement or equivalence theorem:

If p and q are two statement, then

$\sim (p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$

Tautologies and fallacies:

The compound statements (or propositions) which are true for any truth value of their components are called **'Tautologies'**.

The negation of tautology is called a **fallacy** or a **contradiction** i.e. a proposition which is false for any truth value of their components is called a fallacy.

Note:

(*i*) $p \vee q$ is true if at least one of p and q is true.

(*ii*) $p \veebar q$ is true if exactly one of p and q is true and the other is false.

(*iii*) $p \wedge q$ is true if both p and q are true

(*iv*) A tautology is always true.

(*v*) A fallacy is always false.

Algebra of statements:

Statements satisfy many laws some of which are given below:

1. **Idempotent Laws:** If p is any statement then
   (*i*) $p \vee p = p$ (*ii*) $p \wedge p = p$
2. **Associative Laws:** If p, q, r are any three statements, then
   (*i*) $p \vee (q \vee r) = (p \vee q) \vee r$
   (*ii*) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
3. **Commutative Laws:** If p, q are any two statements, then
   (*i*) $p \vee q = q \vee p$
   (*ii*) $p \wedge q = q \wedge p$
4. **Distributive Laws:** If p, q, r are any three statements, then
   (*i*) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
   (*ii*) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
5. **Identity Laws:** If p is any statement, t is tautology and c is a contradiction, then
   (*i*) $p \vee t = t$ (*ii*) $p \wedge t = p$
   (*iii*) $p \vee c = p$ (*iv*) $p \wedge c = c$
6. **Complement Laws:** It t is a tautology, c is a contradiction and p is any statement, then
   (*i*) $p \vee (\sim p) = t$ (*ii*) $p \wedge (\sim p) = c$
   (*iii*) $\sim t = c$ (*iv*) $\sim c = t$
7. **Involution Law:** If p is any statement, then $\sim (\sim p) = p$.
8. **De-Morgan's Laws:** If p and q are two statements, then
   (*i*) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$
   (*ii*) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

Topicwise Questions

1. Which of the following is not a statement ?
(*a*) Every set is a finite set
(*b*) 8 is less than 6
(*c*) Where are you going ?
(*d*) The sum of interior angles of a triangles is 180 degrees

2. Which of the following is a statement ?
(*a*) May you live long !
(*b*) May God bless you !
(*c*) The sun is a star
(*d*) Hurrah ! We have won the match

3. Which is a statement ?
(*a*) $x+1=6$ (*b*) $5 \in N$
(*c*) $x+y<12$ (*d*) $x+y>12$

4. Negation of "$2+3=5$ and $8<10$" is
(*a*) $2+3\neq 5$ or $8<10$ (*b*) $2+3=5$ or $8 \not< 10$
(*c*) $2+3\neq 5$ or $8 \not< 10$ (*d*) $2+3=5$ or $8<10$

5. If p : Ram is smart.
q : Ram is intelligent
Then, the symbolic form of Ram is smart and intelligent, is
(*a*) $(p \wedge q)$ (*b*) $(p \vee q)$
(*c*) $(p \wedge \sim q)$ (*d*) $(p \vee \sim q)$

6. Negation of "Manu is in class X or Anu is in class XII" is
(*a*) Manu is not in class X but Anu is in class XII
(*b*) Manu is not is class X but Anu is not in class XII
(*c*) Manu is not in class X and Anu is not in class XII
(*d*) Manu is in class X and Anu is in class XII

7. Truth value of the statement "if p then q" is false when
(*a*) p is true, q is true (*b*) p is true, q is false
(*c*) p is false, q is true (*d*) p is false, q is false

8. Let p and q be two statements.
Then, $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is a
(*a*) tautology
(*b*) contradiction
(*c*) neither tautology nor contradiction
(*d*) either tautology or contradiction

9. Let $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then, this law is known as
(*a*) Commutative law
(*b*) Associative law
(*c*) De-Morgan's law
(*d*) Distributive law

10. The false statement in the following is
(*a*) $p \wedge (\sim p)$ is a contradiction
(*b*) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
(*c*) $\sim(\sim p) \Leftrightarrow p$ is a tautology
(*d*) $p \vee (\sim p)$ is a tautology

11. A compound sentence formed by two simple statements p and q using connective 'or' is called
(*a*) conjunction (*b*) disjunction
(*c*) implication (*d*) division

12. The contrapositive of $p \Rightarrow \sim q$ is
(*a*) $\sim p \Rightarrow q$ (*b*) $\sim q \Rightarrow p$
(*c*) $q \Rightarrow \sim p$ (*d*) $\sim q \Rightarrow \sim p$

13. If $p = \Delta ABC$ is equilateral and q = each angle is 60°. Then, symbolic form of statement
(*a*) $p \vee p$ (*b*) $p \wedge q$
(*c*) $p \Rightarrow q$ (*d*) $p \Leftrightarrow q$

14. If p = He is intelligent
q = He is strong
Then, symbolic form of statement.
"It is wrong that he is intelligent or strong," is
(*a*) $\sim p \vee \sim p$ (*b*) $\sim (p \wedge q)$
(*c*) $\sim p \wedge \sim q$ (*d*) $p \vee \sim q$

15. If p and q are two statements, then statement $p \Rightarrow q \wedge \sim q$ is
(*a*) tautology (*b*) contradiction
(*c*) neither tautology nor contradiction
(*d*) either tautology or contradiction

16. If p and q are two statements, then
$\sim (p \wedge q) \vee \sim (q \Leftrightarrow p)$
(*a*) tautology (*b*) contradiction
(*c*) neither tautology nor contradiction
(*d*) either tautology or contradiction

17. If p : A man is happy
q : A man is rich
Then, the statement, "If a man is not happy, then he is not rich" is written is
(*a*) $\sim p \rightarrow \sim q$ (*b*) $\sim q \rightarrow p$
(*c*) $\sim q \rightarrow \sim p$ (*d*) $q \rightarrow \sim p$

18. $\sim (p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
(*a*) $\sim p$ (*b*) p
(*c*) q (*d*) $\sim q$

19. The compound statement $p \rightarrow (\sim p \vee q)$ is false, then the truth values of p and q are respectively
(*a*) T, T (*b*) T, F
(*c*) F, T (*d*) F, F

20. $\sim [\sim p \wedge (p \Leftrightarrow q)] \equiv$
(*a*) $p \vee q$ (*b*) $q \wedge p$
(*c*) T (*d*) F

JEE Mains & Advanced Past Years Questions

JEE MAIN
PREVIOUS YEARS

1. The Boolean expression $(p \wedge \sim q) \vee q \wedge (\sim p \wedge q)$ is equivalent to: [JEE Main-2016]
   (a) $p \wedge q$ (b) $p \vee q$
   (c) $p \vee \sim q$ (d) $\sim p$

2. The following statement $(p \rightarrow q) \rightarrow [(\sim p \rightarrow q) \rightarrow q]$ is: [JEE Main-2017]
   (a) a tauology (b) equivalent to $\sim p \rightarrow q$
   (c) equivalent to $p \rightarrow \sim q$ (d) a fallacy

3. The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to: [JEE Main-2018]
   (a) p (b) q
   (c) $\sim q$ (d) $\sim p$

4. If the Boolean expression $(p \oplus q) \wedge (\sim p \Theta q)$ is equivalent to $p \wedge q$, where $\oplus, \Theta \in \{\wedge, \vee\}$, then the ordered pair $(\oplus, \Theta)$ is: [JEE Main-2019 (January)]
   (a) $(\wedge, \vee)$ (b) $(\vee, \vee)$
   (c) $(\wedge, \vee)$ (d) $(\wedge, \wedge)$

5. The logical statement $[\sim(\sim p \vee q) \vee (p \wedge r) \wedge (\sim q \wedge r)]$ is equivalent to: [JEE Main-2019 (January)]
   (a) $(p \wedge r) \wedge \sim q$ (b) $(\sim p \wedge \sim q) \wedge r$
   (c) $\sim p \vee r$ (d) $(p \wedge \sim q) \vee r$

6. Consider the following three statements:
   P : 5 is a prime number.
   Q : 7 is a factor of 192
   R : L.C.M. of 5 and 7 is 35.
   Then the truth value of which one of the following statements is true? [JEE Main-2019 (January)]
   (a) $(\sim P) \vee (Q \wedge R)$ (b) $(P \wedge Q) \vee (\sim R)$
   (c) $(\sim P) \wedge (\sim Q \wedge R)$ (d) $P \vee (\sim Q \wedge R)$

7. Contrapositive of the statement "If two numbers are not equal, then their squares are not equals" is: [JEE Main-2019 (January)]
   (a) If the squares of two numbers are not equal, then the numbers are equal
   (b) If the squares of two numbers are equal, then the numbers are not equal
   (c) If the squares of two numbers are equal, then numbers are equal
   (d) If the squares of two numbers are not equal, then the numbers are not equal

8. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology? [JEE Main-2019 (January)]
   (a) $(p \wedge r) \rightarrow (p \vee r)$ (b) $(p \vee r) \rightarrow (p \wedge r)$
   (c) $p \wedge r$ (d) $p \vee r$

9. The Boolean expression $((p \wedge q) \wedge (p \wedge \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to: [JEE Main-2019 (January)]
   (a) $p \wedge q$ (b) $p \wedge (\sim q)$
   (c) $(\sim p) \wedge (\sim q)$ (d) $p \wedge (\sim q)$

10. The expression $\sim(\sim p \rightarrow q)$ is logically equivalent to: [JEE Main-2019 (January)]
    (a) $\sim p \wedge \sim q$ (b) $p \wedge \sim q$
    (c) $\sim p \wedge q$ (d) $p \wedge q$

11. The contrapoistive of the statement "If you are born in India, they you are a citizen of India", is: [JEE Main-2019 (April)]
    (a) If you are born in India, then you are not a citizen of India.
    (b) If you are not a citizen of India, then you are not born in India.
    (c) If you are a citizen of India, then you are born in India.
    (d) If you are not born in India, then you are not a citizen of India.

12. Which one of the following statements is not a tautology? [JEE Main-2019 (April)]
    (a) $(p \wedge q) \rightarrow p$ (b) $(p \wedge q) \rightarrow (\sim p) \wedge q$
    (c) $p \rightarrow (p \vee q)$ (d) $(p \wedge q) \rightarrow (p \wedge (\sim q))$

13. For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is [JEE Main-2019 (April)]
    (a) $p \wedge q$ (b) $p \leftrightarrow q$
    (c) $\sim p \vee \sim q$ (d) $\sim p \wedge \sim q$

14. If $P \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively: [JEE Main-2019 (April)]
    (a) F, T, T (b) T, F, F
    (c) T, T, F (d) F, F, F

15. Which one of the following Boolean expressions is a tautology? [JEE Main-2019 (April)]
    (a) $(p \vee q) \wedge (\sim p \vee \sim q)$ (b) $(p \wedge q) \vee (p \vee \sim q)$
    (c) $(p \vee q) \wedge (p \vee \sim q)$ (d) $(P \vee q) \vee (p \vee \sim q)$

16. The negation of the boolean expression $\sim s \vee (\sim r \wedge s)$ is equivalent to: [JEE Main-2019 (April)]
    (a) r (b) $s \wedge r$
    (c) $s \vee r$ (d) $\sim s \wedge \sim r$

17. If the truth value of the statement $p \rightarrow (q \vee r)$ is false (F), then the truth values of the statements p, q, r are respectively: [JEE Main-2019 (April)]
(a) F, T, T (b) T, F, F
(c) T, T, F (d) T, F, T

18. The Boolean expression $\sim (p \Rightarrow (\sim q)$ is [JEE Main-2019 (April)]
(a) $(\sim p) \Rightarrow q$ (b) $p \vee q$
(c) $q \Rightarrow \sim p$ (d) $p \wedge$

19. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to [JEE Main-2020 (January)]
(a) p (b) q
(c) $\sim p$ (d) $\sim q$

20. Let A, B, C and D be four non-empty sets. The contrapositive statement of " If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$ " is: [JEE Main-2020 (January)]
(a) If $A \not\subseteq C$, , then $A \subseteq B$ and $B \subseteq D$
(b) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \subseteq D$
(c) If $A \subseteq C$, then $B \subset A$ or $D \subset B$
(d) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

21. Which one of the following is a tautology? [JEE Main-2020 (January)]
(a) $(P \wedge (P \rightarrow Q)) \rightarrow Q$ (b) $(P \wedge (P \vee Q))$
(c) $Q \rightarrow (P \wedge (P \rightarrow Q))$ (d) $(P \vee (P \wedge Q))$

22. Which of the following statement is a tautology ? [JEE Main-2020 (January)]
(a) $\sim (p \vee \sim q) \rightarrow p \wedge q$ (b) $\sim (p \vee \sim q) \rightarrow p \vee q$
(c) $\sim (p \wedge \sim q) \rightarrow p \vee q$ (d) $p \vee (\sim q) \rightarrow p \wedge q$

23. Negation of the statement: [JEE Main-2020 (January)]
$\sqrt{5}$ is an integer or 5 is irrational is:
(a) $\sqrt{5}$ is not an integer or 5 is not irrational
(b) $\sqrt{5}$ is an integer and 5 is irrational
(c) $\sqrt{5}$ is irrational or 5 is an integer.
(d) $\sqrt{5}$ is not an integer and 5 is not irrational

24. If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively: [JEE Main-2020 (January)]
(a) F, F (b) F, T
(c) T, F (d) T, T

25. Which of the following is a tautology? [JEE Main-2020 (September)]
(a) $(\sim p) \wedge (p \vee q) \rightarrow q$ (b) $(\sim q) \vee (p \wedge q) \rightarrow q$
(c) $(p \rightarrow q) \wedge (q \rightarrow p)$ (d) $(q \rightarrow p) \vee \sim (p \rightarrow q)$

26. The contrapositive of the statement "If I reach the station in time, then I will catch the train" is: [JEE Main-2020 (September)]
(a) If I will catch the train, then I reach the station in time
(b) If I do not reach the station in time, then I will catch the train
(c) If I do not reach the station in time, then I will not catch the train
(d) If I will not catch the train, then I do not reach the station in time

27. Let p, q, r be three statements such that the truth value of $(p \wedge q) \rightarrow (\sim q \vee r)$ is F. Then the truth values of p, q, r are respectively: [JEE Main-2020 (September)]
(a) T, F, T (b) F, T, F
(c) T, T, T (d) T, T, F

28. The proposition $p \rightarrow \sim (p \wedge \sim q)$ is equivalent to: [JEE Main-2020 (September)]
(a) q (b) $(\sim p) \wedge q$
(c) $(\sim p) \vee (\sim q)$ (d) $(\sim p) \vee q$

29. Given the following two statements: [JEE Main-2020 (September)]
$(S_1) : (q \vee p) \rightarrow (p \leftrightarrow \sim q)$ is a tautology:
$(S_2) : \sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy. Then:
(a) only (S_1) is correct.
(b) both (S_1) and (S_2) are correct.
(c) only (S_2) is correct.
(d) both (S_1) and (S_2) are not correct.

30. The statement $(p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow (p \vee q))$ is [JEE Main-2020 (September)]
(a) a tautology
(b) equivalent to $(p \vee q) \wedge (\sim p)$
(c) a contradiction
(d) equivalent to $(p \wedge q) \vee (\sim q)$

31. The negation of the Boolean expression $x \leftrightarrow \sim y$ is equivalent to: [JEE Main-2020 (September)]
(a) $(x \wedge \sim y) \wedge (\sim x \wedge \sim y)$
(b) $(x \wedge y) \vee (\sim x \wedge \sim y)$
(c) $(x \wedge y) \wedge (\sim x \vee \sim y)$
(d) $(\sim x \wedge y) \vee (\sim x \wedge \sim y)$

32. Consider the statement : "For an integer n, if $n^3 - 1$ is even, then n is odd." The contrapositive statement of this statement is: [JEE Main-2020 (September)]
(a) For an integer n, if n is odd, then $n^3 - 1$ is even
(b) For an integer n, if n is even, then $n^3 - 1$ is even.
(c) For an integer n, if n is even, then $n^3 - 1$ is odd.
(d) For an integer n, if $n^3 - 1$ is not even, then n is not odd.

33. The negation of the Boolean expression $p \vee (\sim p \wedge q)$ is equivalent to: [JEE Main-2020 (September)]
(a) $\sim p \vee q$ (b) $p \wedge \sim q$
(c) $\sim p \vee \sim q$ (d) $\sim p \wedge \sim q$

34. The statement among the following that is a tautology is [JEE Main-2021 (February)]

(a) $A \wedge (A \vee B)$ (b) $B \rightarrow [A \wedge (A \rightarrow B)]$

(c) $A \vee (A \wedge B)$ (d) $[A \wedge (A \rightarrow B)] \rightarrow B$

35. The negation of the statement $\sim p \wedge (p \vee q)$ is : [JEE Main-2021 (February)]

(a) $\sim p \wedge q$ (b) $p \wedge \sim q$

(c) $\sim p \vee q$ (d) $p \vee \sim q$

36. For the statement p and q, concsider the following compound statements: [JEE Main-2021 (February)]

(a) $(\sim q \wedge (p \rightarrow q) \rightarrow \sim P$

(b) $((P \vee Q)) \wedge \sim P) \rightarrow P$

Then which of the following statements is correct ?

(a) (a) is a tautology but not (b)

(b) (a) and (b) both are not tautologies.

(c) (a) and (b) both are tautologies.

(d) (b) is a tautology but not (a).

37. The statement $A \rightarrow (B \rightarrow A)$ is equivalent to: [JEE Main-2021 (February)]

(a) $A \rightarrow (A \wedge B)$ (b) $A \rightarrow (A \vee B)$

(c) $A \rightarrow (A \rightarrow B)$ (d) $A \rightarrow (A \leftrightarrow B)$

38. The contrapositive of the statement " If you will work, you will earn money" is: [JEE Main-2021 (February)]

(a) If you will not earn money, you will not work

(b) You will earn money, if you will not work

(c) If you will earn money, you will work

(d) To earn money, you need to work

39. Let $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$ be two logical expressions Then : [JEE Main-2021 (February)]

(a) F_1 is not a tautology but F_2 is a tautology

(b) F_1 is a tautology but F_2 is not a tautology

(c) F_1 and F_2 both are a tautologies

(d) Both F_1 and F_2 are not tautologies

40. Which of the following Boolean expression is a tautology? [JEE Main-2021 (March)]

(a) $(p \wedge q) \vee (p \vee q)$ (b) $(p \wedge q) \vee (p \rightarrow q)$

(c) $(p \wedge q) \wedge (p \rightarrow q)$ (d) $(p \wedge q) \rightarrow (p \rightarrow q)$

41. If the B oolean expressions $(p \Rightarrow q) \Rightarrow (q * (\sim p))$ is a tuatology, then the Boolean expressions $p * (\sim q)$ is equivalent to [JEE Main-2021 (March)]

(a) $q \Rightarrow p$ (b) $\sim q \Rightarrow p$

(c) $p \Rightarrow \sim q$ (d) $p \Rightarrow q$

42. If the Boolean expressions $(p \wedge q) * (p \otimes q)$ is a tautology then $*$ and $\otimes$ are respectively given by [JEE Main-2021 (March)]

(a) $\rightarrow, \rightarrow$ (b) $\wedge, \vee$

(c) $\vee, \rightarrow$ (d) $\wedge, \rightarrow$

43. If P and Q are two statement, then which of the follwing compound statement is a tautology? [JEE Main-2021 (March)]

(a) $(P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$

(b) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$

(c) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$

(d) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

ANSWER KEY

Topicwise Questions

1. (c)	2. (c)	3. (b)	4. (c)	5. (a)	6. (c)	7. (b)	8. (c)	9. (d)	10. (b)	11. (b)	12. (c)
13. (d)	14. (c)	15. (c)	16. (c)	17. (a)	18. (a)	19. (b)	20. (a)				

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. (b)	2. (a)	3. (d)	4. (a)	5. (a)	6. (d)	7. (c)	8. (b)	9. (c)	10. (a)	11. (b)	12. (d)
13. (d)	14. (b)	15. (d)	16. (b)	17. (c)	18. (d)	19. (c)	20. (d)	21. (a)	22. (b)	23. (d)	24. (d)
25. (a)	26. (d)	27. (d)	28. (d)	29. (d)	30. (a)	31. (c)	32. (c)	33. (d)	34.(d)	35. (d)	36. (c)
37. (b)	38. (a)	39. (a)	40. (d)	41. (a)	42. (a)	43. (b)					

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32 STATISTICS

time/real progressions

DATA

(*i*) Ungrouped data, raw data or individual series.

(*ii*) Discrete frequency or ungrouped data.

Definition: Data consist of n distinct values $x_1, x_2, \ldots x_n$ occuring with frequency $f_1, f_2, \ldots, f_n$ respectively. This data in tabular form is called discrete frequency distribution.

(*iii*) Continuous frequency or grouped data:

Definition: A continuous frequency distribution is a series in which the data are classified into different class intervals without gaps along with their respective frequencies.

MEASURES OF CENTRAL TENDENCY

Averages are generally, the central part of the distribution and therefore, they are also called the measures of **Central Tendency.**

It can be divided into two groups:

(*a*) Mathematical Average: (Mean)

(*i*) Arithmetic mean or mean

(*ii*) Geometric mean

(*iii*) Harmonic mean

(*b*) Positional Average:

(*i*) Median

(*ii*) Mode or positional average

ARITHMETIC MEAN

1. **Individual observation or unclassified data :**

If $x_1, x_2, \ldots x_n$ be n observations, then their arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

2. **Arithmetic mean of discrete frequency distribution :**

Let $x_1, x_2, \ldots, x_n$ be n observation and let $f_1, f_2, \ldots, f_n$ be their corresponding frequencies, then their mean is

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Short cut method: If the values of x or (and) f are large the calculation of arithmetic mean by the previous formula used, is quite tedious and time consuming. In such case we take the deviation from an arbitrary point A.

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

where A = Assumed mean

$d_i = x_i - A$ = deviation for each term

Step deviation method: Sometimes during the application of shortcut method of finding the mean, the deviation d_i are divisible by a common number h (say). In such case the calculation is reduced to a great extent by taking

$$u_i = \frac{x_i - A}{h}, \; i = 1, 2, \ldots \ldots n$$

$$\therefore \quad \text{mean } \bar{x} = A + h\left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)$$

3. **Combined mean:** If $\bar{x}_1, \bar{x}_2, \ldots \ldots, \bar{x}_k$ are the mean of k series of sizes $n_1, n_2, \ldots, n_k$ respectively then the mean of the composite series is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots \ldots n_k\bar{x}_k}{n_1 + n_2 + \ldots \ldots + n_k}$$

4. **Properties of Arithmetic Mean:**

(*i*) In a statistical data, the sum of the deviation of items from $A.M.$ is always zero.

i.e. $\sum_{i=1}^{n}(x_i - \bar{x}) = 0 \quad \text{or} \quad \sum_{i=1}^{n} x_i - n\bar{x}$

or $\quad n\bar{x} - n\bar{x} = 0 \qquad \left(\because \bar{x} = \frac{\sum x_i}{n}\right)$

(*ii*) In a statistical data, the sum of squares of the deviation of items from $A.M.$ is least i.e.

$\sum_{i=1}^{n}(x_i - \bar{x})^2$ is least

(*iii*) If each of the n given observation be doubled, then their mean is doubled.

(iv) If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$. The mean of $ax_1, ax_2, \ldots, ax_n$ is $a\bar{x}$ where a is any number different from zero.

(v) Arithmetic mean is independent of origin i.e. it is not effected by any change in origin.

TRAIN YOUR BRAIN

Example 1: If n persons donate each rupees $1, 2, 4, 8, \ldots\ldots 2^{n-1}$ respectively then the mean donation per person is

Sol. $\bar{x} = \dfrac{1+2+2^2+2^3+\ldots\ldots+2^{n-1}}{n}$ this is a $G.P.$ with common ratio as 2,

$$= \frac{(2^n-1)}{n.(2-1)} = \frac{2^n-1}{n} \text{ rupees}$$

Example 2: Find the mean of first n natural numbers whose frequencies are equal to the corresponding numbers.

Sol. Here $x_1 = 1, x_2 = 2, x_3 = 3 \ldots x_n = n$

also $f_1 = 1, f_2 = 2, f_3 = 3 \ldots f_n = n$

so $\bar{X} = \dfrac{x_1f_1 + x_2f_2 + x_3f_3 + \ldots + x_nf_n}{f_1 + f_2 + \ldots + f_n}$

$$= \frac{1.1 + 2.2 + 3.3 + \ldots + n.n}{1 + 2 + 3 \ldots + n}$$

$$= \frac{n(n+1)(2n+1)\cdot 2}{6\cdot n(n+1)} = \frac{2n+1}{3}$$

GEOMETRIC MEAN

1. **Individual data :** If $x_1, x_2, x_3, \ldots\ldots, x_n$ are n values of a variate x, none of them being zero, then the geometric mean G is defined as-

$$G = (x_1 x_2 x_3 \ldots\ldots x_n)^{1/n}$$

or $G = \text{antilog}\left(\dfrac{\log x_1 + \log x_2 + \ldots\ldots + \log x_n}{n}\right)$

or $G = \text{antilog}\left(\dfrac{1}{n}\sum_{i=1}^{n}\log x_i\right)$

2. **Geometric Mean of grouped data :**

Let $x_1, x_2, \ldots\ldots, x_n$ be n observation and let $f_1, f_2, \ldots\ldots, f_n$ be their corresponding frequency then their Geometric Mean is

$$G = \left(x_1^{f_1} x_2^{f_2} \ldots\ldots x_n^{f_n}\right)^{1/N} \text{ where } N = \sum_{i=1}^{n} f_i$$

$$\therefore G = \text{antilog}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{\sum_{i=1}^{n} f_i}\right)$$

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Example 3: The geometric mean of the numbers $7, 7^2, 7^3, \ldots\ldots, 7^n$ is

Sol. $GM = \sqrt[n]{7.7^2, 7^3 \ldots\ldots 7^n}$

$$= \left[7^{1+2+3+\ldots+n}\right]^{1/n} = \left[7^{n(n+1)/2}\right]^{1/n} = 7^{n+1/2}$$

HARMONIC MEAN

Harmonic mean is reciprocal of mean of reciprocal.

1. **Individual observation:** The $H.M.$ of $x_1, x_2, \ldots\ldots, x_n$ of n observation is given by

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots + \frac{1}{x_n}} \quad \text{i.e. } H = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$$

2. **H.M. of grouped data:** Let $x_1, x_2, \ldots\ldots, x_n$ be n observation and let $f_1, f_2, \ldots\ldots, f_n$ be their corresponding frequency then $H.M.$ is

$$H = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

Relation between A.M., G.M., and H.M.

$A.M. \geq G.M. \geq H.M.$

Equality sign holds only when all the observations in the series are same.

MEDIAN

1. **Median of an individual series :**

Let n be the number of observations-

(i) arrange the data in ascending or descending order.

(ii) **(a) If n is odd then-**

Median (M) = Value of $\left(\dfrac{n+1}{2}\right)^{th}$ observation

(b) If n is even then

Median (M) = mean of $\left(\dfrac{n}{2}\right)^{th}$ and $\left(\dfrac{n}{2}+1\right)^{th}$ observation

i.e. $M = \dfrac{\left(\frac{n}{2}\right)^{th} \text{ observation} + \left(\frac{n}{2}+1\right)^{th} \text{ observation}}{2}$

2. **Median of the discrete frequency distribution:**

Algorithm to find the median :

Step-I: Find the cumulative frequency $(C.F.)$

Step-II: Find $\frac{N}{2}$, where $N = \sum_{i=1}^{n} f_i$

Step-III: See the cumulative frequency ($C.F.$) just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

Step-IV: The value obtained in step III is the median.

3. **Median of grouped data or continuous series:**

Let the no. of observations be n

(*i*) Prepare the cumulative frequency table

(*ii*) Find the median class *i.e.* the class in which the $\left(\frac{N}{2}\right)^{\text{th}}$ observation lies

(*iii*) The median value is given by the formulae

$$\text{Median } (M) = \ell + \left[\frac{\left(\frac{N}{2}\right) - F}{f}\right] \times h$$

N = total frequency = Σf_i

ℓ = lower limit of median class

f = frequency of the median class

F = cumulative frequency of the class preceding the median class

h = class interval (width) of the median class

4. **Properties of Median:**

(A) The sum of the absolute value of deviations of the items from median is minimum

(B) It is a positional average and it is not influenced by the position of the items.

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Example 4: The median of the items 6, 10, 4, 3, 9, 11, 22, 18, is

(*a*) 9 (*b*) 10 (*c*) 9.5 (*d*) 11

Sol. Let s arrange the items in ascending order 3, 4, 6, 9, 10, 11, 18, 22.

In this data the number of items is $n = 8$, which is even.

$\therefore$ Median = M = average of $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms.

= Average of $\left(\frac{8}{2}\right)$ th and $\left(\frac{8}{2}+1\right)$ th terms

= Average of 4th and 5th terms

$= \frac{9+10}{2} = \frac{19}{2} = 9.5$

Example 5: In a class of 20 students, the marks obtained by them in a test are grouped as follows

Marks	No. of students
0	1
1	3
4	2
5	6
7	2
9	3
11	1
15	1
16	1
	20

Find the median marks.

Sol.

x	f	Σf
0	1	1
1	3	4
4	2	6
5	6	12
7	2	14
9	3	17
11	1	18
15	1	19
16	1	20

Here $n = 20$, so $\frac{n}{2} = 10$ and $\frac{n}{2}+1 = 11$

Size corresponding to $\frac{n}{2}$ and $\left(\frac{n}{2}+1\right)$ is both '5'

Hence median is 5.

Example 6: The marks obtained by 60 students in a certain examination are given below

Marks	No. of Students
10 – 20	04
20 – 30	05
30 – 40	11
40 – 50	06
50 – 60	05
60 – 70	08
70 – 80	09
80 – 90	07
90 – 100	05

Calculate the median mark.

Sol. We calculate the cumulative frequency from the following table:

Marks – Obtained	Frequency (No. of students) (f)	(Cumulative Frequency) (f)
10 - 20	04	04
20 - 30	05	09
30 - 40	11	20
40 - 50	06	26
50 - 60	05	31
60 - 70	08	39
70 - 80	09	48
80 - 90	07	55
90 – 100	05	60
	$n = 60$	

Here we have $n = 60$, $\frac{n}{2} = 30$

We see that 30 lies in the class 50 - 60 with cumulative frequency 31.

Thus the median class 50 - 60

$\therefore$ l = lower limit of the median class i.e. the class in which $\frac{n}{2}$ th or $\frac{n+1}{2}$ th observation lies = 50

f = frequency of the median class = 05

$F_1 = 26$ (cumulative frequency of the class preceding the median class)

h = class size = 60 – 50 = 10

$$\text{Median} = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times h$$

$$= 50 + \frac{30 - 26}{5} \times 10 = 50 + 8 = 58$$

MODE

Mode is that value in a series which occurs most frequently. In a frequency distribution, mode is that variate which has the maximum frequency.

1. **Computation of Mode:**

   Mode for individual series: In the case of individual series, the value which is repeated maximum number of times is the mode of the series.

2. **Mode for grouped data (discrete frequency distribution series):** In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.

3. **Mode for continuous frequency distribution:**

   (*i*) First find the model class i.e. the class which has maximum frequency. The model class can be determined either by inspecting or with the help of grouping data.

   (*ii*) The mode is given by the formula

$$\text{Mode} = \ell + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h$$

   where

   $\ell \rightarrow$ lower limit of the model class

   $h \rightarrow$ width of the model class

   $f_m \rightarrow$ frequency of the model class

   $f_{m-1} \rightarrow$ frequency of the class preceding model class

   $f_{m+1} \rightarrow$ frequency of the class succeeding model class

4. **Properties of Mode:** It is not effected by presence of extremely large or small items.

   Emperical relation

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Example 7: Calculate the mode from the following data:

Marks	No. of Students
0 - 10	04
10 - 20	12
20 - 30	24
30 - 40	36
40 - 50	20
50 - 60	16
60 - 70	08
70 - 80	05

Sol.

Value (Class)	Frequency
0 - 10	04
10 - 20	12
20 - 30	24
30 - 40	36
40 - 50	20
50 - 60	16
60 - 70	08
70 - 80	05
	$n = 125$

We choose the highest frequency class as the modal class.

So here 30 - 40 is the modal class.

Now

l = lower limit of modal class = 30

h = width of the modal class = 10

f_1 = frequency of the class preceding the modal class = 24

f_2 = frequency of the class following the modal class = 20

f = frequency of the modal class = 36

$$\text{So mode} = l + \frac{(f - f_1)}{(2f - f_1 - f_2)} \cdot h$$

$$= 30 + \frac{36 - 24}{72 - 24 - 20} \cdot 10$$

$$= 30 + \frac{12}{28} \cdot 10$$

$$= 30 + \frac{30}{7} = 34.28$$

Relation between Mean, Median and Mode

(*a*) **Symmetric Distribution** A distribution is a symmetric distribution if values of mean, mode and median coincide.

Mean = Model = Median

(*b*) **Skewed Distribution** A distribution which is not symmetric is called a skewed distribution. In a moderately asymmetric the interval between the mean and median is approximately one-third of the interval between the mean and the mode. We have the following empirical relation between them.

Mean - Mode = 3(Mean - Median)

$\Rightarrow$ Mode = 3Median - 2Mean (Approximately)

MEASURES OF DISPERSION

Dispersion is the measure of the variations. The degree to which numerical data tend to spread about an average value is called the dispersion of the data.

The measures of dispersion commonly used are:

(*i*) Range

(*ii*) Mean Deviation

(*iii*) Standard Deviation

(*iv*) Variance

(*i*) **Range:** The range is the difference between the greatest and the least observations of the distribution. Thus if A and B are the greatest and the smallest observations respectively in a distribution, then its range = $A - B$.

The co-efficient of range (or scatter) = $\frac{A - B}{A + B}$

(*ii*) **Mean Deviation:** Mean deviation is defined as the arithmetic mean of the absolute deviations of all the values taken about any central value.

(*a*) **Mean deviation of individual observations:**

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable x, then the mean deviation from a number A is given by

$$M.D. = \frac{1}{n}\sum_{i=1}^{n} |x_i - A|$$

$$= \frac{1}{n}\sum |d_i|, \text{ where } d_i = x_i - A$$

(*b*) **Mean deviation of a discrete frequency distribution:**

If $x_1, x_2, \ldots\ldots, x_n$ are n observation with frequencies $f_1, f_2, \ldots\ldots, f_n$, then mean deviation from a number A is given by -

$$\text{Mean Deviation} = \frac{1}{N}\sum f_i |x_i - A|$$

where $N = \sum_{i=1}^{n} f_i$

(*c*) **Mean deviation of a grouped or continuous frequency distribution:**

For calculating mean deviation of a continuous frequency distribution the procedure is same as for a discrete frequency distribution. The only difference is that here we have to obtain the mid-point of the various classes and take the deviations of these mid points from the given central value (median or mean)

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Example 8: For an *A.P.* 2, 5, 8, ... , 21 terms are given. Find the mean deviation of *A.P.* from its mean.

Sol. $\Sigma x_i = 2 + 5 + 8 \ldots\ldots\ldots + 21$ terms

$$= \frac{21}{2}[4 + 20 \times 3] = 21 \times 32 = 672$$

$$\therefore \bar{x} = \frac{672}{21} = 32$$

$$\Sigma |x_i - \bar{x}| = 2d[1 + 2 + 3 + \ldots\ldots\ldots + 10]$$

$$= 2 \times 3\left[\frac{10 \times 11}{2}\right] = 330$$

$$\therefore \text{M.D.} = \frac{330}{21} = \frac{110}{7}$$

VARIANCE AND STANDARD DEVIATION

The variance of a variate x is the arithmetic mean of the squares of all deviations of x from the arithmetic mean of the observations and is denoted by var (x) or σ^2

The positive square root of the variance of a variate x is known as standard deviation i.e. standard deviation = $+\sqrt{\text{var}(x)}$

(*i*) **Variance of Individual observations :**

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable x, then by definition

$$\text{var}(x) = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right] = \sigma^2 \qquad \ldots(i)$$

$$\text{or} \quad \text{var}(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \qquad \ldots(ii)$$

If the values of variable x are large, the calculation of variance from the above formulae is quite tedious and time consuming. In that case, we take deviation from an arbitrary point A (say) then

$$\text{var}(x) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2 \quad \ldots(iii)$$

(ii) Variance of a discrete frequency distribution:

If $x_1, x_2, \ldots, x_n$ are n observations with frequencies $f_1, f_2, \ldots, f_n$ then $\text{var}(x) = \frac{1}{N}\left\{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2\right\}$ ...(i)

or $\text{var}(x) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$ ...(ii)

If the values of x or f are large, we take the deviations of the values of variable x from an arbitrary point A (say)

$\therefore\ d_i = x_i - A;\ i = 1, 2, \ldots n$

$$\therefore\ \text{Var}(x) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2 \quad \ldots(iii)$$

where $N = \sum_{i=1}^{n} f_i$

Sometime $d_i = x_i - A$ are divisible by a common number h (say)

then $u_i = \frac{x_i - A}{h} = \frac{d_i}{h}$, $i = 1, 2, \ldots, n$

$$\text{then var}(x) = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right] \quad \ldots(iv)$$

(iii) Variance of a grouped or continuous frequency distribution:

In a grouped or continuous frequency distribution any of the formulae discussed in discrete frequency distribution can be used.

PROPERTIES OF VARIANCE

1. If $x_1, x_2, x_3, \ldots, x_n$ be n values of a variable X. If these values are changed to $x_1 + a, x_2 + a, \ldots x_n + a$, where $a \in R$, then the variance remains unchanged.
2. If $x_1, x_2, \ldots, x_n$ values of a variable X and let 'a' be a non-zero real number. Then, the variance of the observation $ax_1, ax_2, \ldots, ax_n$ is a^2 Var(X)

COMBINED STANDARD DEVIATION

If there are two sets of observations containing n_1 and n_2 items with respective mean $\bar{x}_1$ & $\bar{x}_2$ and standard deviation s_1 and s_2, then the mean $\bar{x}$ and the standard deviations of $n_1 + n_2$ observations, taken together, are

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1(\sigma_1^2 + d_1'^2) + n_2(\sigma_2^2 + d_2'^2)\right]$$

where $d_1' = \bar{x} - \bar{x}_1$, $d_2' = \bar{x} - \bar{x}_2$

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Example 9: Find the variance of first n natural numbers

Sol. We have

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$$= \frac{1}{n}(1^2 + 2^2 + \ldots + n^2) - \left(\frac{1}{n}(1 + 2 + \ldots + n)\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2 - 1}{12}$$

ANALYSIS OF FREQUENCY DISTRIBUTIONS

The measure of variability which is independent of units is called coefficient of variation (C.V.) The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\bar{X}} \times 100$$

where σ and $\bar{X}$ are the standard deviation and mean of the data.

For comparing the variability of two series, we calculate the coefficient of variation for each series the series having greater C.V. is said to be more variable or conversely less consistent, less uniform less stable or less homogeneous than the other and the series having lesser C.V. is said to be more consistent (or homogeneous) than the other.

MEAN AND OF VARIANCE OF BINOMIAL DISTRIBUTION

If the frequencies of the values 0, 1, 2, ..., n of a variate are represented by the following coefficients of a binomial :

$$q^n, {}^nC_1 q^{n-1}p, {}^nC_2 q^{n-2}p^2, \ldots, p^n$$

where p is the probability of the success of the experiment (variate), q is the probability of its failure and $p + q = 1$ i.e. distribution is a binomial distribution then:

$$P(x = r) = {}^nC_r q^{n-r} p^r$$

$$\text{mean } \bar{x} = \sum p_i x_i = np$$

$$\text{Variance } \sigma^2 = npq = \bar{x}q$$

Topicwise Questions

ARITHMETIC MEAN

1. If the mean of 3, 4, x, 7, 10 is 6, then the value of x is-

(a) 4 (b) 5

(c) 6 (d) 7

2. The *A.M.* of the series 1, 2, 4, 8, 16,....., 2^n is-

(a) $\frac{2^n-1}{n}$ (b) $\frac{2^{n+1}-1}{n+1}$

(c) $\frac{2^n+1}{n}$ (d) $\frac{2^n-1}{n+1}$

3. Mean of the first n terms of the *A.P.* $a+(a+d)+(a+2d)$ + ... is

(a) $a+\frac{nd}{2}$ (b) $a+\frac{(n-1)d}{2}$

(c) $a+(n-1)d$ (d) $a+nd$

4. If the mean of n observations $1^2, 2^2, 3^2,...., n^2$ is $\frac{46n}{11}$, then n is equal to

(a) 11 (b) 12

(c) 23 (d) 22

5. Following table shows the weight of 12 students:

Weight (in kgs.)	67	70	72	73	75
No. of students	4	3	2	2	1

Find their mean weight

(a) 70.25 kg. (b) 70.50 kg

(c) 70.75 kg. (d) None of these

6. A factory employs 100 workers of whom 60 work in the first shift and 40 work in the second shift. The average wage of all the 100 workers is Rs.38. If the average wage of 60 workers of the first shift is Rs.40, then the average wage of the remaining 40 workers of the second shift is-

(a) 35 (b) 40

(c) 45 (d) None of these

7. If $\bar{x}$ is the mean of a set of n observations $x_1, x_2, x_3,..., x_n$ then $\sum_{i=1}^{n}(x_i-\bar{x})$ is equal to

(a) *M.D.* about mean (b) *S.D.*

(c) 0 (d) None of these

8. The mean of a set of observations is $\bar{x}$. If each observation is divided by α, $\alpha \neq 0$, and then is increased by 10 then the mean of the new set is-

(a) $\frac{\bar{x}}{\alpha}$ (b) $\frac{\bar{x}+10}{\alpha}$

(c) $\frac{\bar{x}+10\alpha}{\alpha}$ (d) $a\bar{x}+10$

9. If the mean of first n natural numbers is equal to $\frac{n+7}{3}$, then n is equal to -

(a) 10 (b) 11

(c) 12 (d) None of these

10. The mean of first three terms is 14 and mean of next two terms is 18. The mean of all the five terms is-

(a) 14.5 (b) 15.0

(c) 15.2 (d) 15.6

11. The weighted *AM* of first n natural numbers whose weight are equal to the corresponding numbers is equal to

(a) $2n+1$ (b) $\frac{1}{2}(2n+1)$

(c) $\frac{1}{3}(2n+1)$ (d) $\frac{2n+1}{6}$

12. The mean of distribution, in which the values of X are 1, 2,....., n the frequency of each being unity is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n}{2}$

(c) $\frac{n+1}{2}$ (d) $\frac{n(n-1)}{2}$

GEOMETRIC MEAN & HARMONIC MEAN

13. Geometric mean of the numbers $2, 2^2, 2^3, ..., 2^n$ is

(a) $2^{2/n}$ (b) $2^{n/2}$

(c) $2^{\frac{n-1}{2}}$ (d) $2^{\frac{n+1}{2}}$

14. The harmonic mean of 4, 8, 16 is-

(a) 6.4 (b) 6.7

(c) 6.85 (d) 7.8

MEDIAN

15. The median of 10, 14, 11, 9, 8, 12, 6 is

(*a*) 10 (*b*) 12

(*c*) 14 (*d*) 11

16. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}, \alpha - \frac{5}{2}, \alpha - 3, \alpha - 2, \alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha + 5 (\alpha > 0)$, then the median is –

(*a*) $\alpha - \frac{5}{4}$ (*b*) $\alpha - \frac{1}{2}$

(*c*) $\alpha - 2$ (*d*) $\alpha + \frac{5}{4}$

17. The median of the items 6, 10, 4, 3, 9, 11, 22, 18 is-

(*a*) 9 (*b*) 10

(*c*) 9.5 (*d*) 11

18. Find the median from the following distribution

Marks(x)	0–10	10–20	20–30	30–40	40–50	50–60
No. of students	10	20	30	50	40	30

is

(*a*) 30 (*b*) 45

(*c*) 36 (*d*) 40

MODE

19. Mode of the distribution

Marks	4	5	6	7	8
No. of students	3	5	10	6	1

is

(*a*) 6 (*b*) 10

(*c*) 8 (*d*) None of these

20. The mode of the following distribution

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of student	2	18	30	45	35	20	6	3

is

(*a*) 36 (*b*) 37.3

(*c*) 38 (*d*) 39

21. If median = (mode + 2 mean)*M*, then *M* is equal to-

(*a*) 3 (*b*) 1/3

(*c*) 2 (*d*) None of these

DISPERSION

22. The standard deviation of a variate *x* is σ. The standard deviation of the variable $\frac{ax+b}{c}$; *a*, *b*, *c* are constants, is-

(*a*) $\left(\frac{a}{c}\right)\sigma$ (*b*) $\left|\frac{a}{c}\right|\sigma$

(*c*) $\left(\frac{a^2}{c^2}\right)\sigma$ (*d*) None of these

23. The scores of a batsman in ten innings are: 38, 70, 48, 34, 42, 55, 63, 46, 54, 44, then mean deviation about the median is:

(*a*) 8.4 (*b*) 8.5

(*c*) 8.6 (*d*) 8.8

24. Mean deviation from the mean for the observations –1, 0, 4 is

(*a*) $\sqrt{\frac{14}{3}}$ (*b*) $\frac{2}{3}$

(*c*) 2 (*d*) None of these

25. The *S.D.* of 7 scores 1, 2, 3, 4, 5, 6, 7 is

(*a*) 4 (*b*) 2

(*c*) $\sqrt{7}$ (*d*) None of these

26. Let σ be the standard deviation of *n* observations. Each of the *n* observations is multiplied by a constant *c*. Then the standard deviation of the resulting numbers is-

(*a*) σ (*b*) $|c|.\sigma$

(*c*) $\sigma\sqrt{c}$ (*d*) None of these

27. The mean of 100 items is 50 and their *SD* is 4, the sum of all the items and also the sum of the squares of the items is

(*a*) 5000, 251600 (*b*) 4000, 251600

(*c*) 5000, 26100 (*d*) 3000, 26100

28. The mean and *SD* of distributions of 100 and 150 items are 50, 5 and 40, 6 respectively. The mean and *SD* of all the 250 items taken together are

(*a*) 44, 6.46 (*b*) 42, 7.46

(*c*) 44, 7.46 (*d*) 42, 6.46

29. The variance of first *n* natural numbers is

(*a*) $\frac{n^2+1}{12}$ (*b*) $\frac{n^2-1}{12}$

(*c*) $\frac{(n+1)(2n+1)}{6}$ (*d*) $\frac{(n+1)(2n+1)}{2}$

30. *MD* of the series $a, a+d, a+2d, \ldots\ldots, a+2nd$ from its mean is

(*a*) $\frac{(n+1)d}{2n+1}$ (*b*) $\frac{nd}{2n+1}$

(*c*) $\frac{n(n+1)d}{2n+1}$ (*d*) $\frac{(2n+1)d}{n(n+1)}$

Learning Plus

1. The mean weight of 9 items is 15. If one more item is added to the series, the mean becomes 16. The value of 10th item is

(*a*) 35 (*b*) 30

(*c*) 25 (*d*) 20

2. In the frequency distribution of the discrete data given below, the frequency k against value 0 is missing.

Variable x:	0	1	2	3	4	5
Frequency f:	k	20	40	40	20	4

If the mean is 2.5, then the missing frequency k will be

(*a*) 0 (*b*) 1

(*c*) 3 (*d*) 4

3. If mean of n item is $\bar{x}$. If each rth item is increased by $2r$. Then new mean will be

(*a*) $\bar{x}$ (*b*) $\bar{x}+\frac{n}{2}$

(*c*) $\bar{x}+\frac{n+2}{2}$ (*d*) $\bar{x}+n+1$

4. The *SD* of 15 items is 6 and if each item decreases by 1, then standard deviation will be

(*a*) 5 (*b*) 7

(*c*) $\frac{91}{15}$ (*d*) 6

5. The sum of squares of deviations for 10 observations taken from mean 50 is 250. The coefficient of variation is

(*a*) 10% (*b*) 40%

(*c*) 50% (*d*) None of these

6. Mean of n items is $\bar{x}$. If these n items are increased by $1^2, 2^2, 3^2, \ldots, n^2$ successively, then mean gets increased by

(*a*) $\frac{(n+1)(2n+1)}{6}$ (*b*) $\frac{n(n+1)(2n+1)}{6}$

(*c*) $\frac{n^2}{2}$ (*d*) Remains same

7. The mean value of the median and mean of the odd divisors of 360 is

(*a*) 13 (*b*) 7

(*c*) 6 (*d*) 10

8. A sample of 35 observations has the mean 80 and *s.d.* as 4. A second sample of 65 observation from the same population has mean 60 and *s.d.* 3. The *s.d.* of the combined sample is

(*a*) 5.85 (*b*) 5.58

(*c*) 10.12 (*d*) None of these

9. The mean and *S.D.* of the marks of 200 candidates were found to be 40 and 15 respectively. Later, it was discovered that a score of 40 was wrongly read as 50. The correct mean and *S.D.* respectively are

(*a*) 14.98, 39.95 (*b*) 39.95, 16.12

(*c*) 39.95, 224.5 (*d*) None of these

10. The average of n numbers $x_1, x_2, x_3, \ldots\ldots, x_n$ is M. If x_1 is replaced by x, then the new average is

(*a*) $M-x_1+x$ (*b*) $\frac{(n-1)M-x_1+x}{n}$

(*c*) $\frac{(n+1)M-x_1+x}{n}$ (*d*) $\frac{nM-x_1+x}{n}$

11. The variance of α, β and γ is 9, then variance of $5\alpha, 5\beta$ and 5γ is

(*a*) 45 (*b*) 9/5

(*c*) 5/9 (*d*) 225

12. The mean of n items is $\bar{x}$. If each item is successively increased by $3, 3^2, 3^3, \ldots\ldots, 3^n$, then new mean equals

(*a*) $\bar{x}+\frac{3^{n+1}}{n}$ (*b*) $\bar{x}+3\frac{(3^n-1)}{2n}$

(*c*) $\bar{x}+\frac{3^n}{n}$ (*d*) $\bar{x}+3\frac{(3^n-1)}{2n}$

13. In a class of 100 students there are 70 boys whose average marks in a subject are 75. If the average marks of the complete class is 72, then what is the average of the girls?

(*a*) 73 (*b*) 65

(*c*) 68 (*d*) 74

14. The median of a set of 9 distinct observation is 20.5. If each of the largest 4 observations of the set is increased by 2, then the median of the new set

(*a*) is increased by 2

(*b*) is decreased by 2

(*c*) is two times the original median

(*d*) remains the same as that of the original set

15. In an experiment with 15 observations on x, the following results were available

$\sum x^2 = 2830$, $\sum x = 170$

One observation that was 20, we found to be wrong and was replaced by the correct value 30. Then, the corrected variance is

(*a*) 78.00 (*b*) 188.66

(*c*) 177.33 (*d*) 8.33

16. In a series of $2n$ observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the observations is 2, then $|a|$ equals

(a) $\frac{1}{n}$ (b) $\sqrt{2}$

(c) 2 (d) $\frac{\sqrt{2}}{n}$

17. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$. The standard deviation of observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be

(a) $\sqrt{10}+20$ (b) $\sqrt{10}+10$

(c) $\sqrt{10}$ (d) None of these

18. Median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3, \ldots\ldots, {}^{2n}C_n$ (where n is even) is

(a) ${}^{2n}C_{\frac{n}{2}}$ (b) ${}^{2n}C_{\frac{n+1}{2}}$

(c) ${}^{2n}C_{\frac{n-1}{2}}$ (d) None of these

Advanced Level Multiconcept Questions

NUMERICAL VALUE BASED

1. If the mean of n observation $1^2, 2^2, 3^2, \ldots\ldots, n^2$ is $\frac{46n}{11}$, then n is equal to
2. If the mean deviation of number $1, 1+d, 1+2d, \ldots\ldots, 1+100d$ from their mean is 255, then the d is equal to
3. The means and variance of n observations $x_1, x_2, x_3, \ldots\ldots x_n$ are 5 and 0 respectively. If $\sum_{i=1}^{n} x_i^2 = 400$, then the value of n is equal to
4. The mean of 30 given numbers, when it is given that the mean of 10 of them is 12 and the mean of the remaining 20 is 9, is equal to
5. The algebraic sum of the deviation of 20 observations measured from 30 is 2. Then, mean of observations is
6. If the average of the numbers 148, 146, 144, 142, ... in AP, be 125, then the total numbers in the series will be
7. A group of 10 items has arithmetic mean 6. If the arithmetic mean of 4 of these items is 7.5, then the mean of the remaining items is
8. Mean of 100 observation is 45. If it was later found that two observations 19 and 31 were incorrectly recorded as 91 and 13. The correct mean is
9. In a class of 50 students, 10 have failed and their average marks are 28. The total marks obtained by the entire class are 2800. The average marks of those who have passed, are

JEE Mains & Advanced Past Years Questions

JEE-MAIN PREVIOUS YEARS

1. If the standard deviation of the number 2, 3, a and 11 is 3.5, then which of the following is true? **[JEE Main-2016]**

(a) $3a^2-26a+55=0$ (b) $3a^2-32a+84=0$

(c) $3a^2-34a+91=0$ (d) $3a^2-23a+44=0$

2. If $\sum_{i=1}^{9}(x_i-5)=9$ and $\sum_{i=1}^{9}(x_i-5)^2=45$, then the standard deviation of the 9 items $x_1, x_2 \ldots x_9$ is- **[JEE Main-2018]**

(a) 4 (b) 2

(c) 3 (d) 9

3. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is: **[JEE Main-2019 (January)]**

(a) 16 (b) 22

(c) 20 (d) 18

4. A data consists of n observations : $x_1, x_2, \ldots\ldots x_n$. If $\sum_{i=1}^{n}(x_i+1)^2=9n$ and $\sum_{i=1}^{n}(x_i-1)^2=5n$, then the standard deviation of this data is: **[JEE Main-2019 (January)]**

(a) 5 (b) $\sqrt{5}$

(c) $\sqrt{7}$ (d) 2

5. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is:

[JEE Main-2019 (January)]

(a) 10 : 3 (b) 4 : 9

(c) 5 : 8 (d) 6 : 7

6. If mean and standard deviation of 5 observations x_1, x_2, x_3, x_4, x_5 are 10 and 3, respectively, then the variance of 6 observations $x_1, x_2, ..., x_5$ and - 50 is equal to

[JEE Main-2019 (January)]

(a) 509.5 (b) 586.5

(c) 582.5 (d) 507.5

7. A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X be the number of white balls drawn, then $\left(\frac{\text{mean of } X}{\text{standard deviation of } X}\right)$ is equal to:

[JEE Main-2019 (January)]

(a) 4 (b) $4\sqrt{3}$

(c) $3\sqrt{2}$ (d) $\frac{4\sqrt{3}}{3}$

8. If the sum of the deviations of 50 observations from 30 is 50, then the mean of these observations is:

[JEE Main-2019 (January)]

(a) 30 (b) 51

(c) 50 (d) 31

9. The mean and the variance of five observations are 4 and 5.20, respectively. If three of the observations are 3 , 4 and 4; then the absolute value of the difference of the other two observations, is: [JEE Main-2019 (January)]

(a) 7 (b) 5

(c) 1 (d) 3

10. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is: [JEE Main-2019 (April)]

(a) 40 (b) 49

(c) 48 (d) 45

11. A student scores the following marks in five tests : 45, 54, 41, 57, 43. His score is not known for the sixth test. If the mean score is 48 in the six tests, then the standard deviation of the marks in six tests is [JEE Main-2019 (April)]

(a) $\frac{10}{\sqrt{3}}$ (b) $\frac{100}{\sqrt{3}}$

(c) $\frac{100}{3}$ (d) $\frac{10}{3}$

12. If the standard deviation of the numbers $-1, 0, 1, k$ is $\sqrt{5}$ where $k > 0$, then k is equal to [JEE Main-2019 (April)]

(a) $2\sqrt{\frac{10}{3}}$ (b) $2\sqrt{6}$

(c) $4\sqrt{\frac{5}{3}}$ (d) $\sqrt{6}$

13. The mean and the median of the following ten numbers in increasing order 10, 22, 26, 29, 34, x, 42, 67, 70, y are 42 and 35 respectively, then $\frac{y}{x}$ is equal to:-

[JEE Main-2019 (April)]

(a) $\frac{7}{3}$ (b) $\frac{9}{4}$

(c) $\frac{7}{2}$ (d) $\frac{8}{3}$

14. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is: [JEE Main-2019 (April)]

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is :

(a) 2.8 (b) 3.2

(c) 3.0 (d) 2.5

15. If both the mean and the standard deviation of 50 observations $x_1, x_2, ..., x_{50}$ are equal to 16, then the mean of $(x_1-4)^2, (x_2-4)^2 ..... (x_{50}-4)^2$ is:

[JEE Main-2019 (April)]

(a) 525 (b) 380

(c) 480 (d) 400

16. If the data $x_1, x_2, ..., x_{10}$ is such that the mean of first four of these is 11, the mean of the remaining six is 16 and the sum of squares of all these is 2,000; then the standard deviation of this data is: [JEE Main-2019 (April)]

(a) 4 (b) 2

(c) $\sqrt{2}$ (d) $2\sqrt{2}$

17. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then $(m + n)$ is equal to _____. [JEE Main-2020 (January)]

18. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then $x.y$ is equal to_____. [JEE Main-2020 (January)]

19. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to

[JEE Main-2020 (January)]

(a) - 20 (b) 10

(c) - 10 (d) - 5

20. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is:

[JEE Main-2020 (January)]

(a) 3.99 (b) 4.02

(c) 4.01 (d) 3.98

21. Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$. If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, \ldots\ldots\ldots x_{10} - 3$, then the ordered pair (μ, λ) is equal to:

[JEE Main-2020 (January)]

(a) (3, 3) (b) (3, 6)

(c) (6, 6) (d) (6, 3)

22. If the variance of the terms in an increasing *A.P.*, $b_1, b_2, b_3, \ldots, b_{11}$ is 90, then the common difference of this *A.P.* is _____.

[JEE Main-2020 (September)]

23. Let $X = \{x \in N : 1 \le x \le 17\}$ and $Y = \{ax + b : x \in X\}$ and $a, b \in R, a > 0\}$. If mean and variance of elements of Y are 17 and 216 respectively then $a + b$ is equal to :

[JEE Main-2020 (September)]

(a) 7 (b) –27

(c) 9 (d) –7

24. Let x_i $(1 \le i \le 10)$ be ten observations of a random variable X. If $\sum_{i=1}^{10}(x_i - p) = 3$ and $\sum_{i=1}^{10}(x_i - p)^2 = 9$ where $p \neq 0$ $p \in R$, then the standard deviation of these observations is

[JEE Main-2020 (September)]

(a) $\frac{7}{10}$ (b) $\frac{9}{10}$

(c) $\sqrt{\frac{3}{5}}$ (d) $\frac{4}{5}$

25. For the frequency distribution :

[JEE Main-2020 (September)]

Variate (x) : $x_1 x_2 x_3 \ldots x_{15}$

Frequency (f) : $f_1 f_2 f_3 \ldots f_{15}$

Where $0 < x_1 < x_2 < x_3 < \ldots\ldots < x_{15} = 10$ and

$\sum_{i=1}^{15} f_i > 0$, then standard deviation cannot be :

(a) 1 (b) 6

(c) 2 (d) 4

26. If the variance of the following frequency distribution:

[JEE Main-2020 (September)]

Class :	10 - 20	20 - 30	30 - 40
Frequency :	2	x	2

is 50, then x is equal to _____.

27. The mean and variance of 8 observations are 10 and 13. 5, respectively. If 6 of these observations are 5, 7. 10, 12, 14, 15, then the absolute difference of the remaining two observations is:

[JEE Main-2020 (September)]

(a) 9 (b) 3

(c) 7 (d) 5

28. If the mean and the standard deviation of the data 3, 5, 7, a, b are 5 and 2 respectively, then a and b are the roots of the equation:

[JEE Main-2020 (September)]

(a) $x^2 - 20x + 18 = 0$

(b) $2x^2 - 20x + 19 = 0$

(c) $x^2 - 10x + 18 = 0$

(d) $x^2 - 10x + 19 = 0$

29. The mean and variance of 7 observations are 8 and 16, respectively. If five observations are 2, 4, 10, 12, 14, then the absolute difference of the remaining two observations is:

[JEE Main-2020 (September)]

(a) 2 (b) 4

(c) 3 (d) 1

30. Consider the data on x taking the values 0, 2, 4, 8, ..., 2^n with frequencies ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ respectively. If the mean of this data is $n\frac{728}{2^n}$ then n is equal to _____.

[JEE Main-2020 (September)]

31. If $\sum_{i=1}^{n}(x_i - a) = n$ and $\sum_{i=1}^{n}(x_i - a)^2 = na, (n, a > 1)$

Then the standard deviation of n observations $x_1, x_2, \ldots\ldots x_n$ is :

[JEE Main-2020 (September)]

(a) $a - 1$ (b) $n\sqrt{a-1}$

(c) $\sqrt{n(a-1)}$ (d) $\sqrt{a-1}$

32. Consider three observations a, b and c such that $b = a + c$. If the standard deviation of $a + 2b + 2, c + 2$ is d, then which of the following is true?

[JEE Main-2021]

(a) $b^2 = 3(a^2 + c^2) + 9d^2$

(b) $b^2 = a^2 + c^2 + 3d^2$

(c) $b^2 = 3(a^2 + c^2 + d^2)$

(d) $b^2 = 3(a^2 + c^2) - 9d^2$

ANSWER KEY

Topicwise Questions

1. (*c*) **2.** (*b*) **3.** (*b*) **4.** (*a*) **5.** (*a*) **6.** (*a*) **7.** (*c*) **8.** (*c*) **9.** (*b*) **10.** (*d*) **11.** (*c*) **12.** (*c*)
13. (*d*) **14.** (*c*) **15.** (*a*) **16.** (*a*) **17.** (*c*) **18.** (*c*) **19.** (*a*) **20.** (*a*) **21.** (*b*) **22.** (*b*) **23.** (*c*) **24.** (*c*)
25. (*b*) **26.** (*b*) **27.** (*a*) **28.** (*c*) **29.** (*b*) **30.** (*c*)

Learning Plus

1. (*c*) **2.** (*d*) **3.** (*d*) **4.** (*d*) **5.** (*a*) **6.** (*a*) **7.** (*d*) **8.** (*c*) **9.** (*b*) **10.** (*d*) **11.** (*d*) **12.** (*b*)
13. (*b*) **14.** (*d*) **15.** (*a*) **16.** (*c*) **17.** (*c*) **18.** (*a*)

Advanced Level Multiconcept Questions

MCQ/COMPREHENSION/MATCHING/NUMERICAL

1. [11] **2.** [10.1] **3.** [16] **4.** [10] **5.** [30.1] **6.** [24] **7.** [5] **8.** [44.46] **9.** [63]

JEE Mains & Advanced Past Years Questions

JEE-MAIN
PREVIOUS YEARS

1. (*a*) **2.** (*a*) **3.** (*c*) **4.** (*b*) **5.** (*c*) **6.** (*d*) **7.** (*b*) **8.** (*d*) **9.** (*a*) **10.** (*c*) **11.** (*a*) **12.** (*b*)
13. (*a*) **14.** (*a*) **15.** (*d*) **16.** (*b*) **17.** [18] **18.** [54] **19.** (*a*) **20.** (*a*) **21.** (*a*) **22.** [3] **23.** (*d*) **24.** (*b*)
25. (*b*) **26.** [4] **27.** (*c*) **28.** (*d*) **29.** (*a*) **30.** [06] **31.** (*d*) **32.** (*d*)

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